J. Math. Soc. Japan
Vol. 41, No. 2, 1989

On a question raised by Conway-Norton

By Mong-Lung LANG

(Received Feb. 16, 1987)
(Revised Nov 26, 1987)

0. Introduction.

Let G be a finite group and F be the collection of all modular functions $f(z)$ satisfying:

(1) $f(z)$ is a modular function with respect to a discrete subgroup Γ of $SL_2(\boldsymbol{R})$ of the first kind. (i.e. $f(z)$ is meromorphic on $H^*=H\cup\{\text{cusps of } \Gamma\}$ where H is the upper half plane.)

(2) The genus of Γ is zero and $f(z)$ is a generator of a function field of Γ (i.e. the genus of $\Gamma\backslash H^*$ is zero and $f(z)$ is a generator of a function field of $\Gamma\backslash H^*$).

(3) At $z=i\infty$, $f(z)$ has a Fourier expansion of the form:

$$q^{-1}+a_0+\sum_{n=1}^{\infty} a_n q^n \qquad (q=e^{2\pi iz}).$$

In [2], Conway and Norton have assigned a "Thompson series" of the form:

$$T_\sigma = q^{-1}+H_1(\sigma)q+H_2(\sigma)q^2+\cdots \in F$$

to each element σ of the Fischer-Griess "Monster" group M and conjectured that H_n are characters of M for all n. This remarkable connection between the "Monster" M and modular functions is called *Monstrous Moonshine.*

One of the problem which arose from Conway-Norton paper is that

(*) *For each element σ in $\cdot 0$, is there a class of elements σ_1 in M whose Thompson series T_{σ_1} has a form $\Theta_\sigma(z)/\eta_\sigma(z)$+constant?* (For the definition of $\eta_\sigma(z)$ and $\Theta_\sigma(z)$ see (1.3) and (1.4).)

In [2], Conway and Norton studied elements in $\cdot 0$ of weight 0 and proved that (*) is true for elements of weight 0 (i.e. if σ is of weight 0, then there is a class of elements σ_1 in M whose Thompson series T_{σ_1} has a form $\Theta_\sigma(z)/\eta_\sigma(z)$ +constant). In [6], Kondo and Tasaka studied elements in M_{24} (M_{24} can be naturally embedded in $\cdot 0$) and proved that (*) is true for elements in M_{24}. Recently, Kondo [8] calculated $\Theta_\sigma(z)$ for σ in $2^{12}M_{24}\backslash M_{24}$ and proved that (*)

is false for $\sigma\in 2^{12}M_{24}\setminus M_{24}$ (i.e. there exist some elements $\sigma\in 2^{12}M_{24}\setminus M_{24}$ such that $\Theta_{\sigma}(z)/\eta_{\sigma}(z)$+constant is not a Thompson series T_{σ_1} for any σ_1 in M).

The main purpose of this paper is to (i) calculate $\Theta_{\sigma}(z)$, (ii) show that (*) is false for exactly 15 conjugacy classes of $\cdot 0$, and (iii) find an obstruction to (*). In particular, we will show that if $f_{\sigma}(z)=\Theta_{\sigma}(z)/\eta_{\sigma}(z)$ does not possess a corresponding class in M, then the Riemann surface whose function field is $C(f_{\sigma}(z))$ cannot be realized as $\Gamma_{\sigma}\setminus H^*$ where Γ_{σ} is the fixing group of $f_{\sigma}(z)$ in $SL_2(\boldsymbol{R})$. The main theorems are stated and proved in Theorems 3.2.2, 3.3.3 and 3.4.2.

The author would like to express his gratitude to Professor Harada for his invaluable advice, discussion and encouragement.

Finally, the author wish to thank Professor Kondo and Professor Koike who made several suggestions and corrections on the results in this paper.

1. Frame shapes of Conway group $\cdot 0$.

As usual, we denote by $\cdot 0$ the automorphism group of the Leech lattice L. So $\cdot 0$ has a natural 24-dimensional representation ρ_0 over $\boldsymbol{Q}$ induced by its action on the Leech lattice. We will assign to every element σ (or every conjugacy class) of $\cdot 0$ a Frame shape of degree 24 as follows:

Let $f(x)$=the characteristic polynomial of $\sigma=\det(xI-\rho_0(\sigma))$, then $f(x)$ can be written in the form

$$\prod_t (x^t-1)^{r_t} \qquad \text{where } t\in\boldsymbol{N},\ r_t\in\boldsymbol{Z}.$$

$\prod_t t^{r_t}$ is called the *Frame shape* of σ with respect to ρ_0.

We also refer to the Frame shape of a conjugacy class of $\cdot 0$, as two conjugate elements of $\cdot 0$ having the same Frame shape.

The Frame shape of every conjugacy class of $\cdot 0$ is listed in Table 3. (They are taken from [**7**]. See also [**10**]. These Frame shapes may have been known to other mathematicians too.)

DEFINITION 1.1. Let $\sigma=\prod_t t^{r_t}$ be a Frame shape. We define the following:

(1) $\deg\sigma=\sum_t tr_t$.

(2) $\mathrm{wt}\,\sigma=1/2\sum_t r_t$.

LEMMA 1.2 (Koike [**3**]). *Let $\sigma=\prod_t t^{r_t}$ be a Frame shape of a $\cdot 0$ element, then*

(1) $\deg\sigma=24$,

(2) $\mathrm{wt}\,\sigma\geqq 0$.

We classify every conjugacy class of $\cdot 0$ into the following: (by an abuse of notation, the elements of $\cdot 0$ and their Frame shapes are often identified.)

(1) σ, such that $\mathrm{wt}\,\sigma=0$ (90 of them).

(2) σ, such that $\sigma\in M_{24}$, a subgroup naturally embedded in $\cdot 0$ (21 of them).

(3) σ, such that $\sigma\in 2^{12}M_{24}\setminus M_{24}$ (28 of them).

(4) The remaining 21 conjugacy classes.

DEFINITION 1.3. Let $\eta(z)$ be the Dedekind eta-function

$$\eta(z)=q^{1/24}\prod_{n=1}^{\infty}(1-q^n), \qquad q=e^{2\pi iz}, \qquad z\in H.$$

For a Frame shape $\sigma=\prod_t t^{r_t}$, we put $\eta_\sigma(z)=\prod_t \eta(tz)^{r_t}$.

DEFINITION 1.4 ([**1**]). Let $\{e_1, \cdots, e_{24}\}$ be a natural basis of $\boldsymbol{R}^{24}$, $\sigma=\prod_t t^{r_t}$ a Frame shape in $\cdot 0$. We define the following:

(1) $v(x, y)$; the innner product on $\boldsymbol{R}^{24}$ with $v(e_i, e_j)=2\delta_{ij}$.

(2) $L_\sigma=\{x\in L \mid \sigma(x)=x\}$ where L is the Leech lattice in $\boldsymbol{R}^{24}$ defined in [**1**].

(3) $\Theta_\sigma(z)=\sum_{x\in L_\sigma} q^{1/2v(x,x)}$.

2. Known results for Conway-Norton problem.

In this section we shall state the results obtained by Kondo and Tasaka [**6**]. The following notations are used.

$$\Gamma_0(N)=\left\{\begin{pmatrix} a & b\\ c & d\end{pmatrix}\in SL_2(\boldsymbol{Z}),\ c\equiv 0 \pmod N\right\}$$

$W_{N,e}=$ an Atkin-Lehner involution of $\Gamma_0(N)$

$$=W_e=\begin{pmatrix} ae & b\\ cN & de\end{pmatrix},$$

$a, b, c, d\in\boldsymbol{Z}$, where $e\,\|\,N$, i.e. $e>0$ is a divisor of N with $(e, N/e)=1$ and $\det W_e=e$.

For any h which divides n, we define:

$$\Gamma_0(n\,|\,h)=\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}^{-1}\Gamma_0\left(\frac{n}{h}\right)\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}=\left\{\begin{pmatrix} a & b/h\\ cn & d\end{pmatrix}\in SL_2(\boldsymbol{R})\ a, b, c, d\in\boldsymbol{Z}\right\},$$

for example, $\Gamma_0(8\,|\,2)=\begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix}^{-1}\Gamma_0(4)\begin{pmatrix} 2 & 0\\ 0 & 1\end{pmatrix}$.

Let W_e be an Atkin-Lehner involution of $\Gamma_0\left(\frac{n}{h}\right)$, then

$$\Gamma_0(n\,|\,h)+w_e, \cdots=\left\langle \Gamma_0(n\,|\,h), \begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}^{-1}W_e\begin{pmatrix} h & 0\\ 0 & 1\end{pmatrix}, \cdots\right\rangle.$$

We can simplify our notations as follows:

$N=N-=\Gamma_0(N)$,

$N+e, \cdots=\langle\Gamma_0(N), W_e, \cdots\rangle$,

$N+=\langle\Gamma_0(N)$, all its Atkin-Lehner involutions$\rangle$,

$n|h=n|h-=\Gamma_0(n|h)$,
$n|h+e,\cdots=\Gamma_0(n|h)+w_e,\cdots$.

THEOREM 2.1 (Conway-Norton [2]). *If $\sigma=\prod_t t^{r_t}$ is a Frame shape of weight 0, then (*) is true. Namely, there exists an element σ_1 in the Monster simple group M, such that $\Theta_\sigma(z)/\eta_\sigma(z)=T_{\sigma_1}+constant$, where T_{σ_1} is a Thompson series for σ_1.*

In [6] Kondo and Tasaka have determined $\Theta_\sigma(z)$ explicitly in terms of the classical Jacobi theta functions and the Dedekind eta-function for every $\sigma\in M_{24}$. Furthermore, by using these expressions of $\Theta_\sigma(z)$, they have shown:

If $\sigma\in M_{24}$, then (*) is true.

In [8], Kondo informed us that $\Theta_\sigma(z)$ has been calculated for $\sigma\in 2^{12}M_{24}\setminus M_{24}$ and (*) is true for 20 Frame shapes.

Concerning the Frame shapes σ such that (*) is false, the following problem is very important.

PROBLEM 2.2. *Given $\sigma\in\cdot 0$ such that (*) is false, what is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$ in $SL_2(\boldsymbol{R})$?*

To answer this question, we need the following lemma.

LEMMA 2.3 (M. Koike [4]). *Let σ be elements listed in the following table, then there exist some elements g and g′ in M such that $\Theta_\sigma(z)/\eta_\sigma(z)=t_g+c/t_{g'}$ where c is a constant.*

Frame shape		g	g'	c
6_C	$1^4 2.6^5/3^4$	$6E$	$6D,\ -4$	81
-6_C	$2^5 3^4 6/1^4$	$6E$	$6B,\ 12$	1
10_D	$1^2 2.10^3/5^2$	$10E$	$10C,\ -2$	25
-10_D	$2^3 5^2 10/1^2$	$10E$	$10D,\ 6$	1
12_I	$1^2 4.6^2 12/3^2$	$12I$	$12B,\ -4$	9
-12_I	$2^2 3^2 4.12/1^2$	$12I$	$12H,\ 4$	1
30_D	$1.6.10.15/3.5$	$30G$	$30A,\ -3$	1
-30_D	$2.3.5.30/1.15$	$30G$	$30F,\ 1$	1

Here $t_{g,c}=T_g+c$, c is a constant, and if we do not need to specify the constant term c, we write t_g instead of $t_{g,c}$. Each line of the above table reads as $\Theta_\sigma(z)/\eta_\sigma(z)=t_g+c/t_{g'}$. For example, the first line shows that:

$$\Theta_\sigma(z)/\eta_\sigma(z)=T_{6E}+4+81/(T_{6D}-4)\,.$$

Now the following theorem answers Problem 2.2.

THEOREM 2.4. *Let σ and Γ_σ be elements in $2^{12}M_{24}\setminus M_{24}$ such that* (*) *is false and discrete subgroups in $SL_2(\boldsymbol{R})$ given in the following table, then Γ_σ is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$. Moreover, Γ_σ is of genus zero.*

Frame shape		Γ_σ
6_C	$1^4 2.6^5/3^4$	6−
-6_C	$2^5 3^4 6/1^4$	6−
10_D	$1^2 2.10^3/5^2$	10−
-10_D	$2^3 5^2 10/1^2$	10−
12_I	$1^2 4.6^2 12/3^2$	12−
-12_I	$2^2 3^2 4.12/1^2$	12−
30_D	$1.6.10.15/3.5$	30+15
-30_D	$2.3.5.30/1.5$	30+15

PROOF. We prove the theorem for $\sigma=1^4 2.6^5/3^4$. See [**9**] for the proof of the other 7 cases.

Let $\sigma_1=2^8 3^4/1^4 6^8$, then $\eta_{\sigma_1}=T_{6E}+4$ and the fixing group of η_{σ_1} is $\Gamma_0(6)$.

Applying Lemma 2.3 and the symmetrization formula for T_{6D}: $T_{6D}=T_{6E}-8/(T_{6E}+3)$, we have

$$\Theta_\sigma(z)/\eta_\sigma(z)=\eta_{\sigma_1}+\frac{81}{\eta_{\sigma_1}-8/(\eta_{\sigma_1}-1)-8}$$
$$=\frac{(\eta_{\sigma_1}^3-9\eta_{\sigma_1}^2+81\eta_{\sigma_1}-81)}{(\eta_{\sigma_1}^2-9\eta_{\sigma_1})}.$$

Let Γ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$. Since $\Gamma_0(6)$ fixes η_{σ_1}, $\Gamma_0(6)$ also fixes $\Theta_\sigma(z)/\eta_\sigma(z)$ and thus $\Gamma_0(6)\subseteqq\Gamma$.

On the other hand, from the above equation, we know that η_{σ_1} is a solution of:

$$F(x)=x^3-9x^2+81x-81-\Theta_\sigma(z)/\eta_\sigma(z)(x^2-9x)=0\,.$$

Let f_1 and f_2 be the other two roots of $F(x)$. Since Γ fixes $\Theta_\sigma(z)/\eta_\sigma(z)$, Γ also fixes $F(x)$, this implies that Γ permutes $\{\eta_{\sigma_1}, f_1, f_2\}$. Therefore Γ can be embedded into S_3. Let Γ_0 be the kernel, then Γ/Γ_0 is isomorphic to a subgroup of S_3. So, $[\Gamma:\Gamma_0]\leqq 6$. Since $\Gamma_0(6)$ is a discrete subgroup in $SL_2(\boldsymbol{R})$ of the first kind, Γ is also a discrete subgroup in $SL_2(\boldsymbol{R})$ of the first kind.

cusps (c_i) of $\Gamma_0(6)$	0	1/2	1/3	1/6
$\eta_{\sigma_1}(c_i)$	9	0	1	∞
$\Theta_\sigma/\eta_\sigma(c_i)$	∞	∞	1	∞

Case 1. $[\Gamma:\Gamma_0(6)]=2$. By the above table, we have $[\Gamma_{1/3}:(\Gamma_0(6))_{1/3}]=2$. This implies that $\Gamma_{1/3}=\left\langle A\,|\,A=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}1&1\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-3&1\end{pmatrix}\right\rangle$, and $A(1/6)=4/15$ is equivalent to 1/3. This is a contradiction.

Case 2. $[\Gamma:\Gamma_0(6)]=3$. By the above table, we have $[\Gamma_{1/3}:(\Gamma_0(6))_{1/3}]=3$. This implies that $\Gamma_{1/3}=\left\langle A \mid A=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}1&2/3\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-3&1\end{pmatrix}\right\rangle$, and $A(0)=2/9$ is equivalent to 1/3. This is a contradiction.

Case 3. $[\Gamma:\Gamma_0(6)]=6$. By the above table, we have $[\Gamma_{1/3}:(\Gamma_0(6))_{1/3}]=6$. This implies that $\Gamma_{1/3}=\left\langle A \mid A=\begin{pmatrix}1&0\\3&1\end{pmatrix}\begin{pmatrix}1&1/3\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-3&1\end{pmatrix}\right\rangle$, and $A(0)=2/9$ is equivalent to 1/3. This is a contradiction.

Summing up the above, we have $\Gamma=\Gamma_0(6)$.

3. Conway-Norton problem for the remaining conjugacy classes of ·0.

3.1. Matrix representation of ·0 elements. To give a complete study of Conway-Norton problem for the remaining 21 conjugacy classes (as listed in Theorem 3.1.2), a matrix representation of each element is necessary. To achieve this, we first state a theorem.

THEOREM 3.1.1 (see [**1**]). *·0 is generated by α, β, γ, δ, ε and T, where*

$\alpha=(\infty)(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)$

$\beta=(\infty)(0)(1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14)$

$\gamma=(0,\infty)(1,22)(2,11)(3,15)(4,17)(5,9)(6,19)(7,13)(8,20)(10,16)(12,21)(14,18)$

$\delta=(\infty)(0)(3)(15)(1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)(11,19,22,14,17)$

ε and T are listed in Table 2.

Next, we give a brief explanation of how to obtain matrix representations for the remaining 21 conjugacy classes.

(1) Find a set of representatives of all conjugacy classes of M_{24} expressed as permutations of 24 letters explicitly.

(2) Determine the matrix representation A_i's for each representative. (See Table 2.)

(3) Compute products of T and A_i's.

(4) Study the Frame shape of each matrix computed in (3).

Finally, we state our results in Theorem 3.1.2.

THEOREM 3.1.2. *The matrix representation of the remaining* 21 *conjugacy classes are listed below:*

Frame shape		matrix representation of σ
3_A	$3^9/1^3$	$(A_2TA_{20}T)^3$
5_C	$5^5/1$	$(A_7TA_4)^6$
-6_D	$1^5 3.6^4/2^4$	$(A_6T)^2$

6_F	$3^3 6^3/1.2$	$(A_3 T A_{11})^3$
-6_F	$1.6^6/2^2 3^3$	$(A_7 T)^3$
9_B	$9^3/3$	$(A_2 T A_{20} T)$
9_C	$1^3 9^3/3^2$	$(A_2 T A_{18} T)$
-10_E	$1^3 5.10^2/2^2$	$(A_7 T A_4)^3$
-12_D	$2.3^3 12^3/1.4.6^3$	$(A_3 T A_{10})^3$
12_H	$2^3 6.12^2/1.3.4^2$	$(A_3 T A_8)$
-12_H	$1.2^2 3.12^2/4^2$	$(A_6 T)$
-12_K	$1^3 12^3/2.3.4.6$	*
15_E	$1^2 15^2/3.5$	$(A_5 T)$
-18_B	$1^2 9.18/2.3$	$(A_2 T A_9 T)$
18_C	$1.2.18^2/6.9$	$(A_7 T)$
-18_C	$2^2 9.18/1.6$	$(A_3 T A_{11})$
20_C	$1.2.10.20/4.5$	$(A_5 T A_{10})$
-20_C	$2^2 5.20/1.4$	$(A_1 T A_{17})$
24_F	$1.4.6.24/3.8$	$(A_3 T A_6)$
-24_F	$2.3.4.24/1.8$	$(A_{10} T)$
-30_E	$2.3.5.30/6.10$	$(A_7 T A_4)$

A_i's and T are listed in Table 2.

PROOF. The proof is done by computer.

REMARK 1. The main purpose of finding a matrix representation for each conjugacy classes in $\cdot 0$ is to use the matrix form of each conjugacy classes to determine their theta functions.

REMARK 2. The theta function of $\sigma = 1^3 12^3/2.3.4.6$ can be evaluated without its matrix representation (see 3.2.4).

3.2. Theta series of $\cdot 0$ elements. To calculate the theta function of σ, let us consider the matrix representation of each element: First, for simplicity, we still use σ to denote its matrix representation. Secondly, let V_σ be the eigen space corresponding to 1, then $L_\sigma = L \cap V_\sigma$. Practically, V_σ can be evaluated easily, but $L \cap V_\sigma$ is not so easily determined. To achieve this, we introduce the following:

LEMMA 3.2.1 (see [**6**]). *The Leech lattice L in the Euclidean space $\boldsymbol{R}^{24}$ can be described as disjoint sum in the following way:*

$$L = \bigcup_{X \in \underline{G}} \{(1/2 e_x + L_0) \cup (1/4 e_\Omega + 1/2 e_x + L_1)\}, \qquad \textit{where}$$

(1) $\Omega = \{\infty, 0, 1, \cdots, 22\}$ *is a 24-point set and* $\underline{G} \subset P(\Omega)$ *is the (binary) Golay*

code on Ω. For codes and Golay code, see [**1**].

(2) *Let* $\{e_1, \cdots, e_{24}\}$ *be a natural basis for* $\boldsymbol{R}^{24}$, *then*

$$L_\delta = \{x=\sum x_i e_i \in \boldsymbol{Z}^{24} \mid \sum x_i \equiv \delta \pmod 2\} \qquad \textit{for } \delta=0, 1 .$$

(3) *For a subset* X *of* Ω, *we put* $e_x=\sum_{i\in X} e_i$.

Using the above lemma, we can prove the following theorem.

THEOREM 3.2.2. *Let* σ *and* $\Theta_\sigma(z)$ *be elements in* $\cdot 0$ *and functions defined in the following table, then* $\Theta_\sigma(z)$ *is the theta function of* σ. (A, B, C, D, E *are matrices listed in Table* 1.)

Frame shape		$\Theta_\sigma(z)$
3_A	$3^9/1^3$	$\Theta(z, A)$
5_C	$5^5/1$	$\Theta(z, B)$
-6_D	$1^5 3.\,6^4/2^4$	$\Theta(z, A)$
6_F	$3^3 6^3/1.\,2$	$\Theta(z, C)$
-6_F	$1.\,6^6/2^2 3^3$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$
9_B	$9^3/3$	$\Theta\left(z, \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}\right)$
9_C	$1^3 9^3/3^2$	$\Theta(z, D)$
-10_E	$1^3 5.\,10^2/2^2$	$\Theta(z, B)$
12_D	$2.\,3^3 12^3/1.\,4.\,6^3$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$
12_H	$2^3 6.\,12^2/1.\,3.\,4^2$	$\Theta\left(z, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right)$
-12_H	$1.\,2^2 3.\,12^2/4^2$	$\Theta(z, E)$
-12_K	$1^3 12^3/2.\,3.\,4.\,6$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$
15_E	$1^2 15^2/3.\,5$	$\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$
-18_B	$1^2 9.\,18/2.\,3$	$\Theta\left(z, \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}\right)$
18_C	$1.\,2.\,18^2/6.\,9$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$
-18_C	$2^2 9.\,18/1.\,6$	$\Theta\left(z, \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}\right)$
20_C	$1.\,2.\,10.\,20/4.\,5$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\right)$
-20_C	$2^2 5.\,20/1.\,4$	$\Theta\left(z, \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}\right)$

24_F	1.4.6.24/3.8	$\Theta\left(z, \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}\right)$
-24_F	2.3.4.24/1.8	$\Theta\left(z, \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}\right)$
-30_E	2.3.5.30/6.10	$\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$

PROOF. The proof is done by direct computation.

REMARK. Professor Kondo informed us that,
(1) $A = 3E_6^{-1}$, (2) $B = 5A_4^{-1}$, (3) $C = 3D_4$, and
(4) $\Theta(z, E) = 1/2(\Theta_3(z)\Theta_3(3z)^3 + \Theta_4(z)\Theta_4(3z)^3)$,
where E_6, A_4 and D_4 are Cartan matrices.

We did not find a matrix representation for $\sigma = 1^3 12^3/2.3.4.6$, but it is not necessary, as shown by the following lemma and corollary.

LEMMA 3.2.3. *Let γ, σ be two elements in $\cdot 0$ such that*
(1) $\gamma = \sigma^n$, $n \in \boldsymbol{N}$
(2) $\dim V_\gamma = \dim V_\sigma$
then $\Theta_\gamma(z) = \Theta_\sigma(z)$.

PROOF. By (1) $V_\sigma \subseteqq V_\gamma$; (2) $V_\sigma = V_\gamma$. So $L_\sigma = L_\gamma$. So $\Theta_\gamma(z) = \Theta_\sigma(z)$.

COROLLARY 3.2.4. *For $\sigma = 1^3 12^3/2.3.4.6$, $\Theta_\sigma(z) = \Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$.*

PROOF. Let $\sigma = 1^3 12^3/2.3.4.6$, $\gamma = 1.6^6/2^2 3^3$, then the corollary follows immediately.

The next two sections give important applications to our main theorem.

3.3. The modular functions of $\cdot 0$ elements. The main purpose of this section is to study the modular functions of the remaining elements (as listed in Theorem 3.2.2). A main theorem will be stated and proved in 3.3.3. This proves the conjecture of Koike [**3**].

LEMMA 3.3.1. *Let $M(N) = M(\Gamma_0(N))$ be the set of all functions invariant under $\Gamma_0(N)$, holomorphic on H and meromorphic at the cusps of $\Gamma_0(N)$. Given $f(z)$, $g(z) \in M(N)$ such that $f(z) - g(z)$ is holomorphic at all cusps of $\Gamma_0(N)$. Then $f(z) - g(z) =$ a constant function.*

LEMMA 3.3.2 (see [**5**]). *Let $\sigma = \prod_t t^{r_t}$ be a Frame shape in $\cdot 0$. Assume*
(1) $\sum_t t r_t = 24$,
(2) *$\eta_\sigma(z)$ is invariant under the action of a discrete subgroup Γ of $SL_2(\boldsymbol{R})$ containing $\Gamma_0(N)$ for some N,*
(3) *$\Gamma_\infty = \{\sigma \in \Gamma \mid \sigma(\infty) = \infty\}$ is equal to $\left\langle \begin{pmatrix} 1 & \pm 1 \\ 0 & 1 \end{pmatrix} \right\rangle$,*

(4) *$z=i\infty$ is the unique pole of $\eta_\sigma(z)$ among all inequivalent cusps of Γ. Then $\eta_\sigma(z)$ is a generator of a function field corresponding to Γ. Moreover, Γ is of genus zero.*

Now we are ready to prove the main theorem.

THEOREM 3.3.3. *Let σ, $\Theta_\sigma(z)$, σ_1 and Γ_σ be elements in $\cdot 0$, their theta functions, elements in M and corresponding discrete subgroups in $SL_2(\boldsymbol{R})$ given in the following table, respectively, then*

(1) *If σ_1 appears, then $\Theta_\sigma(z)/\eta_\sigma(z)=T_{\sigma_1}+constant$ and $\Theta_\sigma(z)/\eta_\sigma(z)$ is a generator of a function field corresponding to Γ_σ which is of genus 0, in particular, Γ_σ is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.*

(2) *If x appears, then (*) is false.*

Frame shape		$\Theta_\sigma(z)$	σ_1	Γ_σ
3_C	$3^9/1^3$	$\Theta(z, A)$	$3B$	$3-$
5_B	$5^5/1$	$\Theta(z, B)$	$5B$	$5-$
-6_D	$1^5 3.6^4/2^4$	$\Theta(z, A)$	x	x
6_F	$3^3 6^3/1.2$	$\Theta(z, C)$	$6D$	$6+2$
-6_F	$1.6^6/2^2 3^3$	$\Theta\left(z, \begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}\right)$	$6E$	$6-$
9_B	$9^3/3$	$\Theta\left(z, \begin{pmatrix}6 & 3\\ 3 & 6\end{pmatrix}\right)$	$9B$	$9-$
9_C	$1^3 9^3/3^2$	$\Theta(z, D)$	$9A$	$9+$
-10_E	$1^3 5.10^2/2^2$	$\Theta(z, B)$	x	x
-12_D	$2.3^3 12^3/1.4.6^3$	$\Theta\left(z, \begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}\right)$	$12B$	$12+4$
12_H	$2^3 6.12^2/1.3.4^2$	$\Theta\left(z, \begin{pmatrix}6 & 0\\ 0 & 6\end{pmatrix}\right)$	$12I$	$12-$
-12_H	$1.2^2 3.12^2/4^2$	$\Theta(z, E)$	x	x
-12_K	$1^3 12^3/2.3.4.6$	$\Theta\left(z, \begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}\right)$	$12H$	$12+12$
15_E	$1^2 15^2/3.5$	$\Theta\left(z, \begin{pmatrix}4 & 1\\ 1 & 4\end{pmatrix}\right)$	$15C$	$15+15$
-18_B	$1^2 9.18/2.3$	$\Theta\left(z, \begin{pmatrix}6 & 3\\ 3 & 6\end{pmatrix}\right)$	x	x
18_C	$1.2.18^2/6.9$	$\Theta\left(z, \begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}\right)$	x	x
-18_C	$2^2 9.18/1.6$	$\Theta\left(z, \begin{pmatrix}12 & 6\\ 6 & 12\end{pmatrix}\right)$	x	x
20_C	$1.2.10.20/4.5$	$\Theta\left(z, \begin{pmatrix}4 & 2\\ 2 & 4\end{pmatrix}\right)$	$20F$	$20+20$

-20_C	$2^25.20/1.4$	$\Theta\left(z, \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}\right)$	$20C$	$20+4$
24_F	$1.4.6.24/3.8$	$\Theta\left(z, \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}\right)$	$24I$	$24+24$
-24_F	$2.3.4.24/1.8$	$\Theta\left(z, \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}\right)$	$24C$	$24+8$
-30_E	$2.3.5.30/6.10$	$\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$	x	x

PROOF. We prove the theorem only for $\sigma=1.6^6/2^23^3$ and $\sigma=2.3.5.30/6.10$. See [**9**] for the proof of the other 19 cases.

$\sigma=1.6^6/2^23^3$: From Theorem 3.2.2, $\Theta_\sigma(z)=\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$

	level	weight	character
$\Theta_\sigma(z)$	6	1	$\left(\frac{-}{3}\right)$
$\eta_\sigma(z)$	6	1	$\left(\frac{-}{3}\right)$

where $\left(\frac{-}{3}\right)$ is the Jacobi symbol.

From the above table, we conclude that $f_\sigma(z)=\Theta_\sigma(z)/\eta_\sigma(z)\in M(6)$. To show $f_\sigma(z)-T_{6E}=-3$, where $T_{6E}=\eta_{2^8.3^4/1^4.6^8}-4$. By applying Lemma 3.3.1, it suffices to show $f_\sigma(z)-T_{6E}$ is constant at 0, 1/2, 1/3, 1/6 and equals to -3 at the cusp 1/6.

Since $\deg(\sigma\cdot W_{6,e})=0$ for $e=2, 3, 6$, we have $f_\sigma(z)$ has pole only at $i\infty$. $\Gamma_0(6)\backslash H^*$ is of genus 0 and T_{6E} is the generator of $M(6)$ having a pole at $i\infty$, so $f_\sigma(z)-T_{6E}=$constant. $f_\sigma(z)-T_{6E}=-3$ follows immediately.

$\sigma=2.3.5.30/6.10$: From Theorem 3.2.2, we have $\Theta_\sigma(z)=\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$. Computation shows that $\Theta_\sigma(z)/\eta_\sigma(z)=q^{-1}+0+5q+3q^2+6q^3+\cdots$. Since this does not appear in Table 4 of [**2**], (*) is false.

3.4. Fixing groups of the modular functions. In this section, we will try to determine the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$ for every $\sigma\in\cdot 0$.

Case 1. If (*) is true for some σ, implying that $\Theta_\sigma(z)/\eta_\sigma(z)=T_{\sigma_1}+c$ for some $\sigma_1\in M$, then $\Theta_\sigma(z)/\eta_\sigma(z)$ and T_{σ_1} have the same fixing group and the fixing group of T_{σ_1} is listed in Table 3 of [**2**]. (Groups listed in Table 3 of [**2**] are of genus 0 by Lemma 3.3.2.)

Case 2. If (*) is false. In this case, we have 15 conjugacy classes, the first 8 conjugacy classes are listed in Theorem 2.4 with their fixing groups. The rest 7 elements are $1^53.6^4/2^4$, $1^35.10^2/2^2$, $1.2^23.12^2/4^2$, $1^29.18/2.3$, $1.2.18^2/6.9$,

$2^2 9.18/1.6$ and $2.3.5.30/6.10$. Theorem 3.4.1 will give the fixing group for every one of them.

THEOREM 3.4.1. *Let σ and Γ_σ be elements in $\cdot 0$ and discrete subgroups in $SL_2(\boldsymbol{R})$, given in the following table, respectively, then Γ_σ is the fixing group of σ. Moreover, Γ_σ is of genus zero.*

Frame shape		Γ_σ
-6_D	$1^5 3.6^4/2^4$	6−
-10_E	$1^3 5.10^2/2^2$	10−
-12_H	$1.2^2 3.12^2/4^2$	12−
-18_B	$1^2 9.18/2.3$	18−
18_C	$1.2.18^2/6.9$	18−
-18_C	$2^2 9.18/1.6$	18−
-30_E	$2.3.5.30/6.10$	30+15

PROOF. We prove the theorem for $\sigma=2.3.5.30/6.10$. See [**9**] for the proof of the other 6 cases.

$\sigma=2.3.5.30/6.10$: From Theorem 3.2.2, $\Theta_\sigma(z)=\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$. From Theorem 3.3.3, (for the case $\sigma=1^2 15^2/3.5$) $\Theta_\sigma(z)=\eta_{\sigma_1}(z)-\eta_{\sigma_2}(z)$ where $\sigma_1=3^2 5^2/1.15$ and $\sigma_2=1^2 15^2/3.5$, then $\Theta_\sigma(z)/\eta_\sigma(z)=\eta_{\sigma_3}(z)-\eta_{\sigma_4}(z)$ where $\sigma_3=3.5.6.10/1.2.15.30$ and $\sigma_4=1^2 6.10.15^2/2.3^2 5^2 30$.

Using Table 3 of [**2**], we have

$$\eta_{\sigma_3}=T_{30G}+1+\frac{2}{T_{30G}-2}, \qquad \eta_{\sigma_4}=\frac{T_{30G}-2}{T_{30G}}$$

$$\frac{\Theta_\sigma(z)}{\eta_\sigma(z)}=T_{30G}+1+\frac{2}{T_{30G}-2}-\frac{T_{30G}-2}{T_{30G}} \qquad \text{(Equation 1)}$$

where $T_{30G}=\eta_{3.5/2.30}$ with fixing group $\Gamma_0(30)+W_{15}$.

Let Γ be the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$. Since $\Gamma_0(30)+W_{15}$ fixes T_{30G}, $\Gamma_0(30)+W_{15}$ also fixes $\Theta_\sigma(z)/\eta_\sigma(z)$. Thus, $\Gamma_0(30)+W_{15}\subseteqq\Gamma$. To show $\Gamma=\Gamma_0(30)+W_{15}$, we do the following: From Equation 1 we know T_{30G} is a solution of

$$F(x)=(x^3-2x^2+4x-4)-(\Theta_\sigma(z)/\eta_\sigma(z))x(x-2)=0.$$

Let f_1 and f_2 be the other two roots. Then Γ permutes $\{T_{30G}, f_1, f_2\}$ which implies Γ can be embedded into S_3. Let Γ_0 be the kernel of the action, then Γ/Γ_0 is isomorphic to a subgroup of S_3, so $[\Gamma:\Gamma_0]\leqq 6$. Since $\Gamma_0\subseteqq\Gamma_0(30)+W_{15}$, we also know that $[\Gamma:\Gamma_0(30)+W_{15}]\leqq 6$.

cusps (c_i) of $\Gamma_0(30)+W_{15}$	0	1/2	1/3	1/6
$T_{30G}(c_i)$	2	∞	0	1
$(\Theta_\sigma/\eta_\sigma)(c_i)$	∞	∞	∞	1

The above table shows that if we pick $\sigma\in\Gamma$, then $\sigma(1/6)$ is equivalent to 1/6. Without loss of generality, we may assume that $\sigma(1/6)=1/6$. Computation shows $(\Gamma_0(30)+W_{15})_{1/6}=\left\langle\begin{pmatrix}1&0\\6&1\end{pmatrix}\begin{pmatrix}1&5\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-6&1\end{pmatrix}\right\rangle$.

Case 1. $[\Gamma:\Gamma_0(30)+W_{15}]=2$. Then $[\Gamma_{1/6}:(\Gamma_0(30)+W_{15})_{1/6}]=2$ which implies $\Gamma_{1/6}=\left\langle\begin{pmatrix}1&0\\6&1\end{pmatrix}\begin{pmatrix}1&5/2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-6&1\end{pmatrix}\right\rangle$. Pick $A=\begin{pmatrix}1&0\\6&1\end{pmatrix}\begin{pmatrix}1&5/2\\0&1\end{pmatrix}\begin{pmatrix}1&0\\-6&1\end{pmatrix}$, we have $A(1/3)=13/84$ is equivalent to 1/6. This is a contradiction. So $[\Gamma:\Gamma_0(30)+W_{15}]\neq 2$. Similarly, we can prove that:

Case 2. $[\Gamma:\Gamma_0(30)+W_{15}]=3$ and

Case 3. $[\Gamma:\Gamma_0(30)+W_{15}]=6$ cannot happen.

Summing up the above, we have $\Gamma=\Gamma_0(30)+W_{15}$. In addition, $\Gamma_0(30)+W_{15}$ is of genus zero by Lemma 3.3.2.

THEOREM 3.4.2. *Let σ be a $\cdot 0$ element such that* $(*)$ *is false; then the Riemann surface whose function field is $C(\Theta_\sigma(z)/\eta_\sigma(z))$ cannot be realized as $\Gamma_\sigma\backslash H^*$ where Γ_σ is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$ in $SL_2(\boldsymbol{R})$.*

PROOF. We prove the theorem for $\sigma=2.3.5.30/6.10$. See [**9**] for the proof of the other 14 cases.

$\sigma=2.3.5.30/6.10$: Suppose $C(\Theta_\sigma(z)/\eta_\sigma(z))$ is the function field of the Riemann surface $\Gamma_\sigma\backslash H^*$ then

(1) Γ_σ is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.

(2) Since

$$\Theta_\sigma(z)/\eta_\sigma(z)=(T_{30G}-2)+2/(T_{30G}-2)+3-(T_{30G}-2)/T_{30G}$$

$$=\frac{T_{30G}^3-2T_{30G}^2+4T_{30G}-4}{T_{30G}(T_{30G}-2)},$$

where $30G=3.5/2.30$ then $[C(T_{30G}):C(\Theta_\sigma(z)/\eta_\sigma(z))]=3$.
By (2) and the assumption, we have $[\Gamma_\sigma:\Gamma_0(30)+W_{15}]=3$, which contradicts the fact that $\Gamma_\sigma=\Gamma_0(30)+W_{15}$.

Thus, the Riemann surface whose function field is $C(\Theta_\sigma(z)/\eta_\sigma(z))$, cannot be realized as $\Gamma_\sigma\backslash H^*$.

Table 1.

$$A=\begin{pmatrix} 4 & -1 & 1 & 1 & -2 & 2 \\ -1 & 4 & -1 & 2 & 2 & 1 \\ 1 & -1 & 4 & 1 & -2 & 2 \\ 1 & 2 & 1 & 4 & 1 & 2 \\ -2 & 2 & -2 & 1 & 4 & -1 \\ 2 & 1 & 2 & 2 & -1 & 4 \end{pmatrix} \qquad C=\begin{pmatrix} 6 & 0 & 3 & 3 \\ 0 & 6 & 3 & 3 \\ 3 & 3 & 6 & 3 \\ 3 & 3 & 3 & 6 \end{pmatrix}$$

$$B=\begin{pmatrix} 4 & 1 & 1 & 1 \\ 1 & 4 & -1 & -1 \\ 1 & -1 & 4 & -1 \\ 1 & -1 & -1 & 4 \end{pmatrix} \qquad D=\begin{pmatrix} 4 & 1 & 1 & 2 \\ 1 & 4 & 1 & 2 \\ 1 & 1 & 4 & -1 \\ 2 & 2 & -1 & 4 \end{pmatrix}$$

$$E=\begin{pmatrix} 4 & 2 & 2 & 2 \\ 2 & 4 & 1 & 1 \\ 2 & 1 & 4 & 1 \\ 2 & 1 & 1 & 4 \end{pmatrix}$$

Table 2.

The matrix representation of a permutation A_i listed below can be derived in the following manner: Let $A=(\cdots)\cdots(\cdots,i,j,\cdots)$ be a permutation and $[A]$ be its corresponding matrix, then the $(i+1)$th row of $[A]$ consists of zero except for an entry of 1 in the $(j+1)$th column. The 24th row (column) corresponds to ∞.

$A_1=2^8 1^8=(4,22)(6,7)(8,18)(9,10)(11,12)(13,16)(15,20)(19,21)$

$A_2=1^6 3^6=(0,1,2)(5,14,17)(6,21,19)(8,11,18)(9,20,15)(13,16,22)$

$A_3=1^4 5^4=(1,18,4,2,6)(5,21,20,10,7)(8,16,13,9,12)(11,19,22,14,17)$

$A_4=1^4 4^2 2^2=(4,6,22,7)(8,9,18,10)(11,15,12,20)(13,19,16,21)(3,14)(5,17)$

$A_5=1^3 7^3=(\infty,8,3,0,2,4,9)(14,6,12,17,20,5,1)(7,10,22,19,11,15,18)$

$A_6=1^2 2.\,4.\,8^2=(4,8,6,9,22,18,7,10)(13,12,19,20,16,11,21,15)(3,17,14,5)(\infty,0)$

$A_7=1^2 2^2 3^2 6^2=(0,5,1,14,2,17)(6,16,21,22,19,13)(8,18,11)(9,15,20)(\infty,3)(4,7)$

$A_8=1^2 11^2=(1,2,4,8,16,9,18,13,3,6,12)(5,10,20,17,11,22,21,19,15,7,14)$

$A_9=12^2=(\infty,20,8,21,14,16,0,9,18,6,17,13)(1,10,11,7,3,4,2,15,12,19,5,22)$

$A_{10}=6^4=(\infty,8,14,0,18,17)(1,11,3,2,12,5)(4,15,19,22,10,7)(6,13,20,21,16,9)$

$A_{11}=4^6=(\infty,21,0,6)(1,7,2,19)(3,15,5,10)(4,12,22,11)(8,16,18,13)(9,17,20,14)$

$A_{12}=3^8=(\infty,14,18)(0,17,8)(1,3,12)(2,5,11)(4,19,10)(6,20,16)(7,15,22)(9,13,21)$

$A_{13}=2^{12}=(\infty,0)(1,22)(2,11)(3,15)(4,17)(5,9)(6,19)(7,13)(8,20)(10,16)(12,21)(14,18)$

$A_{14}=2^4 4^2=(\infty,7,3,4)(0,6,14,22)(1,21,17,13)(2,19,5,16)(8,15)(9,11)(10,12)(18,20)$

$A_{15}=2.\,4.\,6.\,12=(0,13,5,6,1,16,14,21,2,22,17,19)(8,9,18,15,11,20)(\infty,4,3,7)(10,12)$

$A_{16}=1.\,2.\,7.\,14=(\infty,14,8,6,3,12,0,17,2,20,4,5,9,1)(7,11,10,15,22,18,19)(16,21)$

$A_{17}=1.\,3.\,5.\,15=(\infty,14,9,19,22,16,1,20,2,5,15,11,18,13,12)(4,6,0,17,10)(7,8,21)$

$A_{18}=1.\,23=(0,1,2,3,4,5,6,7,8,9,10,11,12,13,14,15,16,17,18,19,20,21,22)$

$A_{19}=2^2 10^2=(0,3,1,15,19,12,18,8,4,10)(6,13,14,7,11,16,17,21,22,20)(\infty,9)(2,5)$

$A_{20}=3.\,21=(\infty,7,16,22,8,13,14,4,2,6,1,9,15,20,10,19,11,0,21,17,5)(3,18,12)$

Matrix T. (a = 0.5, b = -0.5)

b	0	0	a	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	a
0	a	0	0	0	0	0	0	0	0	0	0	b	0	0	0	0	0	0	0	0	b	b	0
0	0	a	0	0	0	0	b	0	0	0	b	0	b	0	0	0	0	0	0	0	0	0	0
a	0	0	b	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	a
0	0	0	0	a	0	0	0	0	0	b	0	0	0	0	0	b	b	0	0	0	0	0	0
0	0	0	0	0	a	b	0	0	b	0	0	0	0	0	0	0	0	0	b	0	0	0	0
0	0	0	0	0	b	a	0	0	b	0	0	0	0	0	0	0	0	0	b	0	0	0	0
0	0	b	0	0	0	0	a	0	0	0	b	0	b	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	a	0	0	0	0	0	b	0	0	0	b	0	b	0	0	0
0	0	0	0	0	b	b	0	0	a	0	0	0	0	0	0	0	0	0	b	0	0	0	0
0	0	0	0	b	0	0	0	0	0	a	0	0	0	0	0	b	b	0	0	0	0	0	0
0	0	b	0	0	0	0	b	0	0	0	a	0	b	0	0	0	0	0	0	0	0	0	0
0	b	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	b	b	0
0	0	b	0	0	0	0	b	0	0	0	b	0	a	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	b	0	0	0	0	0	a	0	0	0	b	0	b	0	0	0
a	0	0	a	0	0	0	0	0	0	0	0	0	0	0	b	0	0	0	0	0	0	0	a
0	0	0	0	b	0	0	0	0	0	b	0	0	0	0	0	a	b	0	0	0	0	0	0
0	0	0	0	b	0	0	0	0	0	b	0	0	0	0	0	b	a	0	0	0	0	0	0
0	0	0	0	0	0	0	0	b	0	0	0	0	0	b	0	0	0	a	0	b	0	0	0
0	0	0	0	0	b	b	0	0	b	0	0	0	0	0	0	0	0	0	a	0	0	0	0
0	0	0	0	0	0	0	0	b	0	0	0	0	0	b	0	0	0	b	0	a	0	0	0
0	b	0	0	0	0	0	0	0	0	0	0	b	0	0	0	0	0	0	0	0	a	b	0
0	b	0	0	0	0	0	0	0	0	0	0	b	0	0	0	0	0	0	0	0	b	a	0
a	0	0	a	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	b

Matrix ε. (a = -1)

a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	a	0	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1	0
0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	0	1

Table 3.

The first column gives the Frame shapes of ·0 elements.
The second column gives the theta functions.
The third column gives the Monster elements.
The last column gives the discrete subgroups in $SL_2(\boldsymbol{R})$.
Each line of the table reads as:

(1) If σ_1 appears, then $\Theta_\sigma(z)/\eta_\sigma(z)=T_{\sigma_1}+$constant.

(2) If x appears, then (*) is false.

(3) Γ_σ is the fixing group of $\Theta_\sigma(z)/\eta_\sigma(z)$.

(4) $-n_x=n_x$ if $-n_x$ does not appear.

The following notations are used:

(1) A, B, C, D, E are matrices listed in Table 1.

(2) $E_4(z)=1/2(\Theta_2(z)^8+\Theta_3(z)^8+\Theta_4(z)^8)$

(3) $\Theta_1'(z)=\Theta_2(z)\Theta_3(z)\Theta_4(z)$

(4) $\Theta^{(p)}(z)=\Theta_2(z)\Theta_2(pz)+\Theta_3(z)\Theta_3(pz)$

(5) $\Theta(z,D_4)=1/2(\Theta_3(z)^4+\Theta_4(z)^4)$

(6) $\Psi_i(z)=\Theta_i(2z)$

(7) $\hat{\Psi}_i(z)=\Theta_i(10z)$

(8) $\hat{\Theta}_i=\Theta_i(11z)$

(9) $\Phi(z)=\Theta_2(z)\Theta_3(5z)-\Theta_3(z)\Theta_2(5z)$

Frame shape		$\Theta_\sigma(z)$	σ_1	Γ_σ
1_A	1^{24}	$E_4(z)^3-45/16\Theta_1'(z)^8$	$1A$	$1+$
-1_A	$2^{24}/1^{24}$	1	$2B$	$2-$
2_A	$1^8 2^8$	$E_4(2z)^2+15/256\Theta_2(z)^{16}$	$2A$	$2+$
-2_A	$2^{16}/1^8$	$E_4(2z)$	$2B$	$2-$
2_B	$4^{12}/2^{12}$	1	$4D$	$4\|2-$
2_C	2^{12}	$\Theta_3(2z)^{12}-3/2\Theta_1'(2z)^4$	$4A$	$4+$
3_A	$3^{12}/1^{12}$	1	$3B$	$3-$
-3_A	$1^{12}6^{12}/2^{12}3^{12}$	1	$6B$	$6+6$
3_B	$1^6 3^6$	$\Theta^{(3)}(2z)^6-9/4(\Theta_1'(z)\Theta_1'(3z))^2$	$3A$	$3+$
-3_B	$2^6 6^6/1^6 3^6$	1	$6C$	$6+3$
3_C	$3^9/1^3$	$\Theta(z,A)$	$3B$	$3-$
-3_C	$1^3 6^9/2^3 3^9$	1	$6E$	$6-$
3_D	3^8	$E_4(3z)$	$3C$	$3\|3$
-3_D	$6^8/3^8$	1	$6F$	$6\|3-$
4_A	$4^8/1^8$	1	$4C$	$4-$
-4_A	$1^8 4^8/2^8$	$E_4(2z)$	$4A$	$4+$
4_B	$4^8/2^4$	$\Theta(2z,D_4)$	$4C$	$4-$

4_C	$1^4 2^2 4^4$	$\Theta_3(2z)^{10}-$ $5/4\Theta_2(2z)^4\Theta_4(2z)^2\Theta_4(4z)^4$	$4A$	$4+$
-4_C	$2^6 4^4/1^4$	$\Theta_3(2z)^6-12\eta_\sigma(z)$	$4C$	$4-$
4_D	$2^4 4^4$	$1/4(\Theta_3(2z)^4+\Theta_4(2z)^4)^2$	$4B$	$4\|2+$
4_E	$8^6/4^6$	1	$8F$	$8\|4-$
4_F	4^6	$\Theta_3(4z)^6$	$8B$	$8\|2+$
5_A	$5^6/1^6$	1	$5B$	$5-$
-5_A	$1^6 10^6/2^6 5^6$	1	$10D$	$10+10$
5_B	$1^4 5^5$	$1/2(\Psi_2^4\hat{\Psi}_2^4+\Psi_3^4\hat{\Psi}_3^4+\Psi_4^4\hat{\Psi}_4^4)+$ $3\Psi_2\hat{\Psi}_2\Psi_3\hat{\Psi}_3(2\Psi_2^2\hat{\Psi}_3^2+\Psi_2\hat{\Psi}_2\Psi_3\hat{\Psi}_3+2\Psi_3^2\hat{\Psi}_2^2)$	$5A$	$5+$
-5_B	$2^4 10^4/1^4 5^4$	1	$10B$	$10+5$
5_C	$5^5/1$	$\Theta(z,B)$	$5B$	$5-$
-5_C	$1.10^5/2.5^5$	1	$10E$	$10-$
6_A	$3^4 6^4/1^4 2^4$	1	$6D$	$6+2$
-6_A	$1^4 6^8/2^8 3^4$	1	$6E$	$6-$
6_B	$2^6 12^6/4^6 6^6$	1	$12F$	$12\|2+6$
6_C	$1^4 2.6^5/3^4$	$1/2\sum_{i=2}^4\Theta_i(2z)^5\Theta_i(6z)$	x	$6-$
-6_C	$2^5 3^4 6/1^4$	$1/2\sum_{i=2}^4\Theta_i(2z)\Theta_i(6z)^5$	x	$6-$
6_D	$2.6^5/1^5 3$	1	$6E$	$6-$
-6_D	$1^5 3.6^4/2^4$	$\Theta(z,A)$	x	$6-$
6_E	$1^2 2^2 3^2 6^2$	$(\Theta^{(3)}(2z)\Theta^{(3)}(4z))^2-$ $3/4(\Theta_2(z)\Theta_2(3z)\Theta_4(2z)\Theta_4(6z))^2$	$6A$	$6+$
-6_E	$2^4 6^4/1^2 3^2$	$\Theta^{(3)}(4z)^2$	$6C$	$6+3$
6_F	$3^3 6^3/1.2$	$\Theta(z,C)$	$6D$	$6+2$
-6_F	$1.6^6/2^2 3^3$	$\Theta\left(z,\begin{pmatrix}4&2\\2&4\end{pmatrix}\right)$	$6E$	$6-$
6_G	$2^3 6^3$	$\Theta_3(2z)^3\Theta_3(6z)^3-6\eta_\sigma(z)$	$12A$	$12+$
6_H	$12^4/6^4$	1	$12J$	$12\|6-$
6_I	6^4	$\Theta_3(6z)^4$	$12D$	$12\|3+$
7_A	$7^4/1^4$	1	$7B$	$7-$
-7_A	$1^4 14^4/2^4 7^4$	1	$14C$	$14+14$
7_B	$1^3 7^3$	$\Theta^{(7)}(2z)^3-3/2\Theta_1'(z)\Theta_1'(7z)$	$7A$	$7+$
-7_B	$2^3 14^3/1^3 7^3$	1	$14B$	$14+7$
8_A	$8^4/2^4$	1	$8D$	$8\|2-$
8_B	$2^4 8^4/4^4$	$\Theta_3(4z)^4$	$8B$	$8\|2+$
8_C	$2^2 8^4/1^4 4^2$	1	$8E$	$8-$
-8_C	$1^4 8^4/2^2 4^2$	$\Theta(2z,D_4)$	$8A$	$8+$
8_D	$8^4/4^2$	$\Theta_3(4z)^2$	$8D$	$8\|2-$
8_E	$1^2 2.4.8^2$	$\Theta_3(2z)^3\Theta_3(4z)^3-$ $3/4\Theta_2(2z)^2\Theta_2(4z)\Theta_4(2z)\Theta_4(4z)^2$	$8A$	$8+$

-8_E	$2^3 4.8^2/1^2$	$\Theta_3(4z)\Theta_3(2z)^3-6\eta_\sigma(z)$	$8E$	8−
8_F	$4^2 8^2$	$\Theta(4z, D_4)$	$8C$	8|4+
9_A	$9^3/1^3$	1	$9B$	9−
-9_A	$1^3 18^3/2^3 9^3$	1	$18E$	18+18
9_B	$9^3/3$	$\Theta\left(z, \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}\right)$	$9B$	9−
-9_B	$3.18^3/6.9^3$	1	$18D$	18−
9_C	$1^3 9^3/3^2$	$\Theta(z, D)$	$9A$	9+
-9_C	$2^3 3^2 18^3/1^3 6^2 9^3$	1	$18C$	18+9
10_A	$5^2 10^2/1^2 2^2$	1	$10C$	10+2
-10_A	$1^2 10^4/2^4 5^2$	1	$10E$	10−
10_B	$2^3 20^3/4^3 10^3$	1	$20E$	20|2+10
10_C	$4^2 20^2/2^2 10^2$	1	$20D$	20|2+5
10_D	$1^2 2.10^3/5^2$	$1/2\ \sum_{i=2}^{4}\Theta_i(2z)^3\Theta_i(10z)$	x	10−
-10_D	$2^3 5^2 10/1^2$	$1/2\ \sum_{i=2}^{4}\Theta_i(2z)\Theta_i(10z)^3$	x	10−
10_E	$2.10^3/1^3 5$	1	$10E$	10−
-10_E	$1^3 5.10^2/2^2$	$\Theta(z, B)$	x	10−
10_F	$2^2 10^2$	$1/4(\Theta_3(z)\Theta_3(5z)+\Theta_4(z)\Theta_4(5z))^2$	$20A$	20+
11_A	$1^2 11^2$	$\Theta^{(11)}(2z)^2-1/4(\Theta_2\hat{\Theta}_2-\Theta_3\hat{\Theta}_3+\Theta_4\hat{\Theta}_4)^2$	$11A$	11+
-11_A	$2^2 22^2/1^2 11^2$	1	$22B$	22+11
12_A	$1^4 12^4/3^4 4^4$	1	$12H$	12+12
-12_A	$2^4 3^4 12^4/1^4 4^4 6^4$	1	$12B$	12+4
12_B	$2^2 12^4/4^4 6^2$	1	$12I$	12−
12_C	$6^2 12^2/2^2 4^2$	1	$12G$	12|2+2
12_D	$1.12^3/3^3 4$	1	$12I$	12−
-12_D	$2.3^3 12^3/1.4.6^3$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$	$12B$	12+4
12_E	$4^2 12^2/1^2 3^2$	1	$12E$	12+3
-12_E	$1^2 3^2 4^2 12^2/2^2 6^2$	$\Theta_3(2z)^2\Theta_3(6z)^2-4\eta_\sigma(z)$	$12A$	12+
12_F	$4^3 24^3/8^3 12^3$	1	$24F$	24|4+6
12_G	$4^2 12^2/2.6$	$\Theta_4(2z)\Theta_4(6z)+2\eta_\sigma(z)$	$12E$	12+3
12_H	$2^3 6.12^2/1.3.4^2$	$\Theta\left(z, \begin{pmatrix} 6 & 0 \\ 0 & 6 \end{pmatrix}\right)$	$12I$	12−
-12_H	$1.2^2 3.12^2/4^2$	$\Theta(z, E)$	x	12−
12_I	$1^2 4.6^2 12/3^2$	$\sum_{i=2}^{3}\Theta_i(4z)^3\Theta_i(12z)$	x	12−
-12_I	$2^2 3^2 4.12/1^2$	$\sum_{i=2}^{3}\Theta_i(4z)\Theta_i(12z)^3$	x	12−
12_J	$2.4.6.12$	$\Theta^{(3)}(4z)\Theta^{(3)}(8z)$	$12C$	12|2+
12_K	$2^2 3.12^3/1^3 4.6^2$	1	$12I$	12−
-12_K	$1^3 12^3/2.3.4.6$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$	$12H$	12+12

12_L	$24^2/12^2$	1	$24J$	$24\|12-$
12_M	12^2	$\Theta_3(12z)^2$	$24E$	$24\|6+$
13_A	$13^3/1^2$	1	$13B$	$13-$
-13_A	$1^2 26^2/2^2 13^2$	1	$26B$	$26+26$
14_A	$2^2 28^2/4^2 14^2$	1	$28D$	$28\|2+14$
14_B	$1.2.7.14$	$\Theta^{(7)}(2z)\Theta^{(7)}(4z)-$ $1/2\Theta_2(z)\Theta_2(7z)\Theta_4(2z)\Theta_4(14z)$	$14A$	$14+$
-14_B	$2^2 14^2/1.7$	$\Theta^{(7)}(4z)$	$14B$	$14+7$
15_A	$1^3 15^3/3^3 5^3$	1	$15C$	$15+15$
-15_A	$\frac{2^3 3^3 5^3 30^3}{1^3 6^3 10^3 15^3}$	1	$30A$	$30+6, 10$
15_B	$3^2 15^2/1^2 5^2$	1	$15B$	$15+5$
-15_B	$\frac{1^2 5^2 6^2 30^2}{2^2 3^2 10^2 15^2}$	1	$30D$	$30+5, 6$
15_C	$15^2/3^2$	1	$15D$	$15\|3-$
-15_C	$3^2 30^2/6^2 15^2$	1	$30E$	$30\|3+10$
15_D	$1.3.5.15$	$\Theta^{(3)}(2z)\Theta^{(3)}(10z)-3/2\,\phi(2z)\Phi(6z)$	$15A$	$15+$
-15_D	$\frac{2.6.10.30}{1.3.5.15}$	1	$30C$	$30+3, 5$
15_E	$1^2 15^2/3.5$	$\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$	$15C$	$15+15$
-15_E	$\frac{2^2 3.5.30^2}{1^2 6.10.15^2}$	1	$30G$	$30+15$
16_A	$2^2 16^2/4.8$	$\Theta_3(4z)\Theta_3(8z)$	$16A$	$16\|2+$
16_B	$2.16^2/1^2 8$	1	$16B$	$16-$
-16_B	$1^2 16^2/2.8$	$\Theta_3(4z)^2$	$16C$	$16+$
18_A	$9.18/1.2$	1	$18A$	$18+2$
-18_A	$1.18^2/2^2 9$	1	$18D$	$18-$
18_B	$2.3.18^2/1^2 6.9$	1	$18D$	$18-$
-18_B	$1^2 9.18/2.3$	$\Theta\left(z, \begin{pmatrix} 6 & 3 \\ 3 & 6 \end{pmatrix}\right)$	x	$18-$
18_C	$1.2.18^2/6.9$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 4 \end{pmatrix}\right)$	x	$18-$
-18_C	$2^2 9.18/1.6$	$\Theta\left(z, \begin{pmatrix} 12 & 6 \\ 6 & 12 \end{pmatrix}\right)$	x	$18-$
20_A	$1^2 20^2/4^2 5^2$	1	$20F$	$20+20$
-20_A	$2^2 5^2 20^2/1^2 4^2 10^2$	1	$20C$	$20+4$
20_B	4.20	$\Theta_3(4z)\Theta_3(20z)$	$40B$	$40\|2+$
20_C	$1.2.10.20/4.5$	$\Theta\left(z, \begin{pmatrix} 4 & 2 \\ 2 & 6 \end{pmatrix}\right)$	$20F$	$20+20$

-20_C	$2^2 5.20/1.4$	$\epsilon\left(z, \begin{pmatrix} 10 & 0 \\ 0 & 10 \end{pmatrix}\right)$	$20C$	$20+4$
21_A	$1^2 21^2/3^2 7^2$	1	$21D$	$21+21$
-21_A	$\frac{2^2 3^2 7^2 42^2}{1^2 6^2 14^2 21^2}$	1	$42B$	$42+6, 14$
21_B	$7.21/1.3$	1	$21B$	$21+3$
-21_B	$\frac{1.3.14.42}{2.6.7.21}$	1	$42D$	$42+3, 14$
21_C	3.21	$\Theta^{(7)}(6z)$	$21C$	$21\|3+$
-21_C	$6.42/3.21$	1	$42C$	$42\|3+7$
22_A	2.22	$\Theta_3(2z)\Theta_3(22z)-2\eta_\sigma(z)$	$44AB$	$44+$
23_A	1.23	$\Theta^{(23)}(2z)-2\eta_\sigma(z)$	$23A$	$23+$
-23_A	$2.46/1.23$	1	$46AB$	$46+23$
24_A	$2^2 24^2/6^2 8^2$	1	$24H$	$24\|2+12$
24_B	$\frac{1^2 4.6.24^2}{2.3^2 8^2 12}$	1	$24I$	$24+24$
-24_B	$\frac{2.3^2 4.24^2}{1^2 6.8^2 12}$	1	$24C$	$24+8$
24_C	$8.24/2.6$	1	$24D$	$24\|2+3$
24_D	$12.24/4.8$	1	$24G$	$24\|4+2$
24_E	$2.6.8.24/4.12$	$\Theta_3(4z)\Theta_3(12z)$	$24A$	$24\|2+$
24_F	$1.4.6.24/3.8$	$\Theta\left(z, \begin{pmatrix} 4 & 0 \\ 0 & 6 \end{pmatrix}\right)$	$24I$	$24+24$
-24_F	$2.3.4.24/1.8$	$\epsilon\left(z, \begin{pmatrix} 6 & 0 \\ 0 & 12 \end{pmatrix}\right)$	$24C$	$24+8$
26_A	$2.52/4.26$	1	$52B$	$52\|2+26$
28_A	$4.28/1.7$	1	$28C$	$28+7$
-28_A	$1.4.7.28/2.14$	$\sum_{i=2}^{4}\Theta_i(2z)\Theta_i(14z)$	$28B$	$28+$
28_B	$4.56/8.28$	1	$56BC$	$56\|4+14$
30_A	$\frac{1.2.15.30}{3.5.6.10}$	1	$30F$	$30+2, 15$
-30_A	$\frac{2^2 3.5.30^2}{1.6^2 10^2 15}$	1	$30G$	$30+15$
30_B	$\frac{2.10.12.60}{4.6.20.30}$	1	$60E$	$60\|2+5, 6$
30_C	$6.60/12.30$	1	$60F$	$60\|6+10$
-30_D	$2.3.5.30/1.15$	$1/2\sum_{i=2}^{4}\Theta_i(6z)\Theta_i(10z)$	x	$30+15$
30_D	$1.6.10.15/3.5$	$1/2\sum_{i=2}^{4}\Theta_i(2z)\Theta_i(30z)$	x	$30+15$
30_E	$2.30/3.5$	1	$30G$	$30+15$
-30_E	$2.3.5.30/6.10$	$\Theta\left(z, \begin{pmatrix} 4 & 1 \\ 1 & 4 \end{pmatrix}\right)$	x	$30+15$

33_A	3. 33/1. 11	1	$33A$	33+11
-33_A	$\frac{1.\,6.\,11.\,66}{2.\,3.\,22.\,33}$	1	$66B$	66+6, 11
35_A	1. 35/5. 7	1	$35B$	35+35
-35_A	$\frac{2.\,5.\,7.\,70}{1.\,10.\,14.\,35}$	1	$70B$	70+10, 14
36_A	1. 36/4. 9	1	$36D$	36+36
-36_A	2. 9. 36/1. 4. 18	1	$36B$	36+4
39_A	1. 39/3. 13	1	$39CD$	39+39
-39_A	$\frac{2.\,3.\,13.\,78}{1.\,6.\,26.\,39}$	1	$78BC$	78+6, 26
40_A	2. 40/8. 10	1	$40CD$	40|2+20
42_A	$\frac{4.\,6.\,14.\,84}{2.\,12.\,28.\,42}$	1	$84B$	84|2+6, 14
60_A	$\frac{3.\,4.\,5.\,60}{1.\,12.\,15.\,20}$	1	$60D$	60+12, 15
-60_A	$\frac{1.\,4.\,6.\,16.\,15.\,60}{2.\,3.\,5.\,12.\,20.\,30}$	1	$60C$	60+4, 15

References

[1] J. H. Conway, Three Lectures on Exceptional Groups, Finite Simple Groups, edited by M. B. Powell and G. Higman, Academic Press, N. Y., 1971, pp. 215-247.

[2] J. H. Conway and S. P. Norton, Monstrous Moonshine, Bull. London Math. Soc., **11** (1979), 308-339.

[3] M. Koike, Moonshine of $PSL_2(F_q)$ and the automorphism group of Leech lattice, to appear in Japan. J. Math.

[4] M. Koike, Modular forms and the automorphism group of Leech lattice, pre-print.

[5] K. Harada, Modular functions, Modular forms and Finite groups, Lecture notes at Ohio State University.

[6] T. Kondo and T. Tasaka, The theta functions of sublattices of the Leech lattice, Nagoya Math. J., **101** (1986), 151-179.

[7] T. Kondo, The automorphism group of Leech lattice and elliptic modular functions, J. Math. Soc. Japan, **37** (1985), 337-362.

[8] T. Kondo, The theta functions of sublattices of the Leech lattice II, to appear in J. Fac. Sci., Univ. of Tokyo.

[9] M. L. Lang, On a question raised by Conway-Norton, Ph. D. dissertation, The Ohio State University, 1987.

[10] G. Mason, Finite Groups and Hecke Operators, pre-print.

Mong-Lung LANG
Department of Mathematics
The Ohio State University
Columbus, Ohio 43210
U. S. A.