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On Jacobian fibrations on the Kummer surfaces of the product of non-isogenous elliptic curves

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Introduction.

Let X be a Kummer surface obtained by the minimal resolution of the quotient surface of the product abelian surface $E\times F$ by the inversion automorphism, where E and F are arbitrarily fixed complex elliptic curves which are *not mutually isogenous*. As is well-known, X is an algebraic $K3$ surface.

This paper is concerned with Jacobian fiber space structures on X, i.e., elliptic fiber space structures with a section on X, or in other words, structures as an elliptic curve over $\boldsymbol{C}(\boldsymbol{P}^1)$. By $\mathcal{J}_X$ we denote the set of all Jacobian fibrations of X.

Let us recall that any elliptic fibration of X is given by the morphism $\Phi_{|\Theta|}: X\to\boldsymbol{P}^1$ defined by the complete linear system $|\Theta|$ which contains a divisor having the same type as a non-multiple singular fiber of an elliptic surface. By definition, an irreducible curve C is a section of $\Phi_{|\Theta|}$ if and only if C satisfies $C\cdot\Theta=1$. We note that every section of $\Phi_{|\Theta|}$ is a nodal curve, i.e., a non-singular rational curve whose self-intersection number is -2. The group $\mathrm{Aut}(X)$ acts on $\mathcal{J}_X$ in an obvious manner; $f: \Phi_{|\Theta|}\to\Phi_{|f(\Theta)|}$ for $f\in\mathrm{Aut}(X)$.

By Sterk [**12**], the orbit space $\mathcal{J}_X/\mathrm{Aut}(X)$ is finite, i.e., the number of non-isomorphic Jacobian fibrations of X is finite.

The purpose of this paper is to describe all Jacobian fibrations of X modulo isomorphism, or saying more clearly, to find a minimal complete set of representatives of the orbit space $\mathcal{J}_X/\mathrm{Aut}(X)$.

As a first consequence of this paper, we see that $\mathcal{J}_X$ is divided into eleven $\mathrm{Aut}(X)$-stable subsets $\mathcal{J}_1, \cdots, \mathcal{J}_{11}$ by types of the singular fibers, and the Mordell-Weil group of its member is calculated for each $\mathcal{J}_m (m=1, \cdots, 11)$ as follows (Table A, Theorem (2.1) in §2). Here, for example, by $2\mathrm{I}_8+8\mathrm{I}_1$ we mean two singular fibers of type I_8 (Kodaira's notation) and eight singular fibers of type I_1.

We note that there exist infinitely many nodal curves on X since X has a Jacobian fibration whose Mordell-Weil group is an infinite group by Table A. From this fact we can construct *infinitely many* Jacobian fibrations of X.

Table A.

	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$	$\mathcal{J}_5$	$\mathcal{J}_6$	$\mathcal{J}_7$	$\mathcal{J}_8$	$\mathcal{J}_9$	$\mathcal{J}_{10}$	$\mathcal{J}_{11}$
Type of the singular fibers	$2\mathrm{I}_8+8\mathrm{I}_1$	$\mathrm{I}_4+\mathrm{I}_{12}+8\mathrm{I}_1$	$2\mathrm{IV}^*+a\mathrm{I}_1+b\mathrm{II}$ $a+2b=8$	$4\mathrm{I}_0^*$	$\mathrm{I}_6^*+6\mathrm{I}_2$	$2\mathrm{I}_2^*+4\mathrm{I}_2$	$\mathrm{I}_4^*+2\mathrm{I}_0^*+2\mathrm{I}_1$	$\mathrm{III}^*+\mathrm{I}_2^*+3\mathrm{I}_2+\mathrm{I}_1$ or $\mathrm{III}^*+\mathrm{I}_2^*+2\mathrm{I}_2+\mathrm{III}$	$\mathrm{II}^*+2\mathrm{I}_0^*+a\mathrm{I}_1+b\mathrm{II}$ $a+2b=2$	$\mathrm{I}_8^*+\mathrm{I}_0^*+a\mathrm{I}_1+b\mathrm{II}$ $a+2b=4$	$2\mathrm{I}_4^*+a\mathrm{I}_1+b\mathrm{II}$ $a+2b=4$
Mordell-Weil group	$\boldsymbol{Z}^2\oplus\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}^2\oplus\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}^4$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	$\boldsymbol{Z}/2\boldsymbol{Z}$	$\boldsymbol{Z}/2\boldsymbol{Z}$	{id}	{id}	{id}

Let us note that X is isomorphic to one of the following:

(i) $\mathrm{Km}(E_{\sqrt{-1}}\times E_{(-1+\sqrt{-3})/2})$, (ii) $\mathrm{Km}(E_\rho\times E_{(-1+\sqrt{-3})/2})$,

(iii) $\mathrm{Km}(E_{\sqrt{-1}}\times E_{\rho'})$, (iv) $\mathrm{Km}(E_\rho\times E_{\rho'})$,

where E_ξ is the elliptic curve whose period is ξ in the period domain $H/SL_2(\boldsymbol{Z})$ and ρ and ρ' are elements of $H/SL_2(\boldsymbol{Z})$ which are neither $\sqrt{-1}$ nor $(-1+\sqrt{-3})/2$.

As a second consequence of this paper, we calculate the number of non-isomorphic Jacobian fibrations of X as follows.

Table B.

Type	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$	$\mathcal{J}_5$	$\mathcal{J}_6$	$\mathcal{J}_7$	$\mathcal{J}_8$	$\mathcal{J}_9$	$\mathcal{J}_{10}$	$\mathcal{J}_{11}$	Total
(i)	2	1	1	2	1	2	2	1	1	1	2	16
(ii)	3	2	1	2	1	3	3	2	1	2	3	23
(iii)	6	3	1	2	1	6	6	3	1	3	6	38
(iv)	9	6	1	2	1	9	9	6	1	6	9	59

Outline of proof is as follows.

Via the natural rational map $\pi: E\times F -\!-\!\!\rightarrow X$, we have 24 nodal curves on X, i.e., four branched nodal curves E_j $(j=1, \cdots, 4)$ which come from E, four branched nodal curves F_i $(i=1, \cdots, 4)$ which come from F, and 16 exceptional nodal curves C_{ij}.

First we prove the following Table C concerning the intersection numbers of nodal curves on X (Lemma (1.6) and (1.7) in § 1) by studying a certain involution on X which was first introduced by Nikulin [4].

Table C.

	E_j $(j=1,\cdots,4)$	F_i $(i=1,\cdots,4)$	other nodal curves
E_j	$E_j\cdot E_l=-2\delta_{jl}$	$E_j\cdot F_i=0$	there is unique j such that $D\cdot E_j=1$ and $D\cdot E_l=0$ $(l\neq j)$
F_i		$F_i\cdot F_k=-2\delta_{ik}$	there is unique i such that $D\cdot F_i=1$ and $D\cdot F_k=0$ $(k\neq i)$
other nodal curves			$D\cdot D'\equiv 0 \pmod 2$

By using Table C, we examine singular fibers and sections of Jacobian fibrations of X and we get Table A.

A divisor $\bigcup_i(E_i\cup F_i)\cup\bigcup_{i,j}C_{ij}$ on X is called the natural double Kummer pencil divisor, and a divisor on X which has the same configuration as the natural double Kummer pencil divisor is called a double Kummer pencil divisor. Let us put $\mathrm{Aut}_N(X):=\{f\in\mathrm{Aut}(X)\,;\, f^*|_{H^{2,0}(X)}=\mathrm{id}\}$.

Next we prove the following Lemma 1 (Lemma (1.8) and Corollary (1.13) in § 1) by using Torelli Theorem for complex tori of dimension 2.

LEMMA 1. *The group* $\mathrm{Aut}_N(X)$ *acts transitively on the set of all double Kummer pencil divisors on* X.

Using Table A and Lemma 1, we prove the following

LEMMA 2. *Let* φ *be a Jacobian fibration of* X. *Then there exist a singular fiber* Θ *of* φ *and* $g\in\mathrm{Aut}_N(X)$ *such that* $\mathrm{Supp}\,g(\Theta)$ *is contained in the natural double Kummer pencil divisor except for at most one component of* $g(\Theta)$.

By using Lemma 2 and by constructing certain automorphisms of X, we determine a minimal complete set of representatives of the orbit space $\mathcal{J}_m/\mathrm{Aut}_N(X)$ $(m=1, \cdots, 11)$. Finally by studying the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}_N(X)$ and the action of $\mathrm{Aut}(X)/\mathrm{Aut}_N(X)$ on $\mathcal{J}_m/\mathrm{Aut}_N(X)$, we determine a minimal complete set of representatives of the orbit space $\mathcal{J}_m/\mathrm{Aut}(X)$ $(m=1, \cdots, 11)$. As a corollary, we get Table B.

The contents of this paper are as follows.

In § 0, we fix some notation and recall some basic facts about Kummer surfaces and elliptic $K3$ surfaces. Main references of this section are Morrison

[**11**] and Shioda and Inose [8].

In § 1, we prove Table C and Lemma 1. We also study the quotient group $\mathrm{Aut}(X)/\mathrm{Aut}_N(X)$. In the course of proof, the condition that E and F are not mutually isogenous is essential. As for § 1, the author was very much inspired by works of Nikulin [**4**] and Shioda and Mitani [7].

In § 2, we classify all Jacobian fibrations of X according to the types of the singular fibers.

In § 3 and 4, we determine a minimal complete set of representatives of the orbit space $\mathcal{J}_m/\mathrm{Aut}(X)$ $(m=1, \cdots, 11)$.

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§ 0. Preliminaries.

Throughout this paper, we assume that the ground field is the complex number field $\boldsymbol{C}$. For a divisor we use a capital letter, and for its cohomology class the corresponding small letter, e.g., $d=c_1(\mathcal{O}(D))$. When a group G acts on a set S, by *a minimal complete set* (resp. *a non-minimal complete set*) *of representatives of the orbit space* S/G, we mean a subset of S which meets each orbit of S by G at exactly one (resp. at least one) point.

1. Kummer surfaces. Let A be an abelian surface. The Kummer surface $\mathrm{Km}(A)$ is the algebraic $K3$ surface obtained by the minimal resolution of the quotient surface $A/\langle -\mathrm{id}_A \rangle$. Then we have the natural rational map $\pi_A: A \dashrightarrow \mathrm{Km}\, A$ whose fundamental points are the 2-torsion points of A, say r_k $(k=1, \cdots, 16)$, and we let C_k denote the 16 nodal curves (i.e., nonsingular rational curves with self intersection number -2) on $\mathrm{Km}(A)$ corresponding to r_k. Via the morphism $\pi_A | A - \bigcup_k \{r_k\}$, we get a natural homomorphism $\pi_{A*}: H^2(A, \boldsymbol{Z}) \to (\bigoplus_k \boldsymbol{Z} c_k)^{\perp} \subset H^2(\mathrm{Km}(A), \boldsymbol{Z})$. The map π_{A*} satisfies the following properties:

$\pi_{A*}x \cdot \pi_{A*}y = 2x \cdot y$,

π_{A*} preserves the Hodge decompositions, and

π_{A*} is an isomorphism *onto* $(\bigoplus_k \boldsymbol{Z} c_k)^{\perp}$.

Especially, the induced map $\pi_{A*}: T_A \to T_{\mathrm{Km}(A)}$ is an isomorphism which preserves Hodge decomposition. Here, for an algebraic surface Y such that $H^2(Y, \boldsymbol{Z})$ is torsion free, we put:

$S_Y :=$ the Neron Severi group of Y (the algebraic lattice),

$T_Y := S_Y^{\perp}$ in $H^2(Y, \boldsymbol{Z})$ (the transcendental lattice).

For more detail, we refer the reader to Morrison [**11**], Shioda and Inose [8], and Pjateckiĭ-Šapiro and Šafarevič [**13**].

Let X be the Kummer surface $\mathrm{Km}(E \times F)$ where E and F are elliptic curves which are *not mutually isogenous*. The last condition on E and F is equivalent to the condition that the Picard number of $\mathrm{Km}(E\times F)$ is 18. *Throughout this paper we fix E, F and X arbitrarily.*

We use the following notation.

$\pi := \pi_{E\times F}\colon E\times F \dashrightarrow X$ (the natural rational map)

ω_X (resp. $\omega_{E\times F}$) := a nowhere vanishing holomorphic 2-form on X (resp. $E\times F$). (These are determined up to non-zero scalar multiples, and satisfy $\pi_* \boldsymbol{C}\omega_{E\times F} = \boldsymbol{C}\omega_X$.)

$\{P_i\}_{i=1,\cdots,4}$ (resp. $\{Q_i\}$) := the set of the 2-torsion points on E (resp. F).

$R_{ij} := (P_i, Q_j)$, $i, j=1, \cdots, 4$. (These are the 2-torsion points on $E\times F$.)

C_{ij} := the nodal curve on X corresponding to R_{ij}.

$E_j := \pi(E\times Q_j)$, $F_i := \pi(P_i\times F)$. (These are nodal curves on X.)

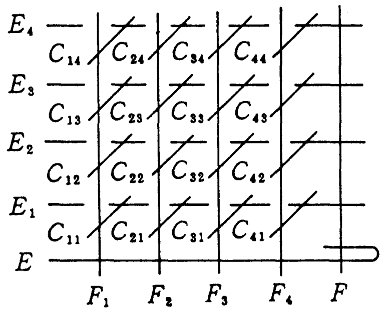
$B := \bigcup_{i=1}^{4}(E_i\cup F_i)$.

We call a nodal curve which is in B a *special* nodal curve, and a nodal curve which is not in B an *ordinary* nodal curve.

$K_{\mathrm{nat}} := B\cup(\bigcup_{i,j} C_{ij})$ (*the natural double Kummer pencil divisor*).

$E := \pi(E\times P)$, $F := \pi(Q\times F)$, for fixed $P\neq P_i$, $Q\neq Q_i$.

By definition, E_i, F_j, C_{ij}, E, F intersect as follows.



,

i.e.,

$$
\begin{aligned}
&C_{ij}\cdot C_{kl} = -2\delta_{ik}\delta_{jl}, \quad E^2 = F^2 = 0, \quad E_j\cdot E_l = -2\delta_{jl}, \quad E\cdot F = 2,\\
&F_i\cdot F_k = -2\delta_{ik}, \quad E\cdot E_l = F\cdot F_k = 0, \quad C_{ij}\cdot E_l = \delta_{jl},\\
&E\cdot F_k = F\cdot E_l = 1, \quad C_{ij}\cdot F_k = \delta_{ik}, \quad E\cdot C_{ij} = F\cdot C_{ij} = 0
\end{aligned}
\tag{0.1}
$$

(δ_{ij}=Kronecker's symbol).

We call a divisor consisting of 24 nodal curves which has the same type as K_{nat} *a double Kummer pencil divisor*.

As for $H^2(X, \boldsymbol{Z})$, $H^2(E\times F, \boldsymbol{Z})$, we get the following:

(0.2) (1) $H^2(E\times F, \boldsymbol{Z}) = S_{E\times F}\oplus T_{E\times F}$, $\quad S_{E\times F} = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$,

$$T_{E\times F} = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix},$$

(2) $\{e, f, c_{ij}\}$ is a basis of $S_X\otimes\boldsymbol{Q}$,

(3) $e_j = \frac{1}{2}\left(e - \sum_{i=1}^{4} c_{ij}\right)$, $\quad f_i = \frac{1}{2}\left(f - \sum_{j=1}^{4} c_{ij}\right)$ in S_X.

2. Elliptic $K3$ surfaces. Let Y be a $K3$ surface. We denote by $\mathcal{J}_Y$ the set of all *Jacobian fibrations* of Y, i.e., elliptic fibrations of Y with a global section. As is well-known, any elliptic fibration of Y is given by the morphism $\Phi_{|\Theta|}: Y\to\boldsymbol{P}^1$ defined by the complete linear system $|\Theta|$ which contains a divisor of the same type as a non-multiple singular fiber of an elliptic surface. (See Table 1.) By definition, an irreducible curve C is a section of $\Phi_{|\Theta|}$ if and only if C satisfies $C\cdot\Theta=1$. We note that every section of $\Phi_{|\Theta|}$ is a nodal curve. The biholomorphic automorphism group of Y, Aut(Y), acts on $\mathcal{J}_Y$ in an obvious manner; $f: \Phi_{|\Theta|}\mapsto\Phi_{|f(\Theta)|}$ for $f\in\mathrm{Aut}(X)$.

Let C_i ($i=1, 2$) be (not necessarily distinct) sections of $\varphi\in\mathcal{J}_Y$. Then there exists a unique symplectic automorphism f of Y (i.e., an automorphism whose action on $H^{2,0}(Y)=\boldsymbol{C}\omega_Y$ is trivial) such that $f(C_1)=C_2$ and $\varphi\circ f=\varphi$. On each non-singular fiber of φ, f acts as a translation. On a singular fiber, f acts by the rule in Table 1 (cf. Kodaira [**10**], p. 604). We call such f a translation automorphism of φ. We denote by $M_\varphi(Y)$ a subgroup of Aut(Y) consisting of all translation automorphisms of φ. $M_\varphi(Y)$ is naturally identified with the Mordell-Weil group of Y considered as an elliptic curve over $\boldsymbol{C}(\boldsymbol{P}^1)$ via φ.

LEMMA (0.3) (Shioda [**6**], p. 23 or Shioda and Inose [**8**], p. 120). *Let φ be a Jacobian fibration of a $K3$ surface Y. Let Θ_i ($i=1, \cdots, k$) be all the singular fibers of φ. Then,*

(1) $24 = \chi_{\mathrm{top}}(Y) = \sum_i \chi_{\mathrm{top}}(\Theta_i)$,

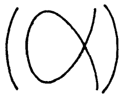
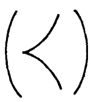
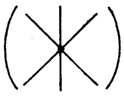
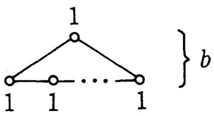
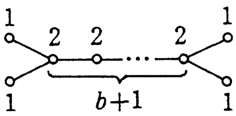
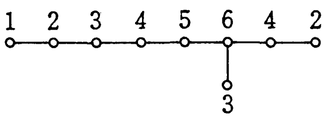
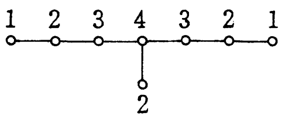
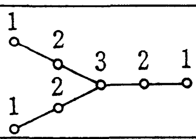
(2) *S_Y is generated by the classes of all irreducible components of Θ_i ($i=1, \cdots, k$) and all sections of φ. Hence, if one of Θ_i is neither of type* I_1 *nor of type* II, *then S_Y is generated by some classes of nodal curves.*

(3) *The Mordell-Weil group $M_\varphi(Y)$ is a finitely generated abelian group, which satisfies the equality,*

$$\operatorname{rank} M_\varphi(Y) = \operatorname{rank} S_Y - 2 - \sum_i (m(\Theta_i)-1),$$

where $m(\Theta_i)$ denotes the number of irreducible components of Θ_i.

Table 1. Non-multiple singular fibers of an elliptic surface.

Symbol	Structure (dual graph)	the number of components	the number of simple components	Euler number	Group structure
I_0	a non-singular elliptic curve	1	1	0	elliptic curve
I_1	a rational curve with one ordinary double point 	1	1	1	$\boldsymbol{C}^{\times}$
I_2		2	2	2	$\boldsymbol{C}^{\times}\times\boldsymbol{Z}/2\boldsymbol{Z}$
II	a rational curve with one ordinary cusp 	1	1	2	$\boldsymbol{C}$
III		2	2	3	$\boldsymbol{C}\times\boldsymbol{Z}/2\boldsymbol{Z}$
IV		3	3	4	$\boldsymbol{C}\times\boldsymbol{Z}/3\boldsymbol{Z}$
I_b $b\geqq 3$		b	b	b	$\boldsymbol{C}^{\times}\times\boldsymbol{Z}/b\boldsymbol{Z}$
I_b^* $b\geqq 0$		$b+5$	4	$b+6$	$\boldsymbol{C}\times(\boldsymbol{Z}/2\boldsymbol{Z})^2$ $b\equiv 0\,(2)$ $\boldsymbol{C}\times\boldsymbol{Z}/4\boldsymbol{Z}$ $b\equiv 1\,(2)$
II*		9	1	10	$\boldsymbol{C}$
III*		8	2	9	$\boldsymbol{C}\times\boldsymbol{Z}/2\boldsymbol{Z}$
IV*		7	3	8	$\boldsymbol{C}\times\boldsymbol{Z}/3\boldsymbol{Z}$

By a simple component, we mean a non-multiple irreducible component.

§1. Some properties on X.

First, we remark that the following natural exact sequence holds. Here for a subset $Z\subset Y$, we put $\mathrm{Aut}(Y; Z):=\{f\in\mathrm{Aut}(Y); f(Z)=Z\}$.

(1.1) $1 \longrightarrow \langle -\mathrm{id}_{E\times F}\rangle \longrightarrow \mathrm{Aut}(E\times F; \bigcup\{R_{ij}\}) \xrightarrow{\;-\;} \mathrm{Aut}(X; \bigcup C_{ij}) \longrightarrow 1.$

For $f\in\mathrm{Aut}(E\times F; \bigcup\{R_{ij}\})$, by $\bar{f}$, we denote a corresponding element of $\mathrm{Aut}(X; \bigcup C_{ij})$. If $f_*\omega_{E\times F}=\alpha\omega_{E\times F}$, we have $\bar{f}_*\omega_X=\alpha\omega_X$.

(1.2) For $\Theta=\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}\in\mathrm{Aut}(E\times F; \bigcup\{R_{ij}\})$, we put $\theta=\bar{\Theta}$.

We note that θ is an involution on X.

LEMMA (1.3). (1) $\theta_*|_{S_X}=\mathrm{id}, \quad \theta_*|_{T_X}=-\mathrm{id}$.
(2) X^θ (:=*the set of fixed points of* θ) $=B$.

PROOF. (1) is obvious by (0.2). By definition, we have,

$$(X-\bigcup C_{ij})^\theta=\pi(\{x\in E\times F-\bigcup C_{ij};\ \ \Theta x=x, \text{ or } -x\})=B-\bigcup C_{ij}.$$

On the other hand, since $\theta_*\omega_X=-\omega_X$, X^θ is a smooth closed submanifold of X. Then we have $X^\theta=B$. □

LEMMA (1.4). $\mathrm{Aut}(X)=\mathrm{Aut}(X; B)$, *i.e.*, $f(B)=B$ *for any* $f\in\mathrm{Aut}(X)$.

PROOF. (Following Nikulin [4], p. 1424.) By (1.3) and by the fact that $S_X\oplus T_X$ is of finite index in $H^2(X, \boldsymbol{Z})$, we have $(f\theta)_*=(\theta f)_*$ on $H^2(X, \boldsymbol{Z})$. Then by Torelli Theorem for $K3$ surfaces, we have $f\theta=\theta f$. Combining this with (1.3)(2), we get $f(B)=B$. □

Before proceeding, we remark the following.

(1.5) *For nodal curves* D_i $(i=1, 2)$ *on* X *and for* $f\in\mathrm{Aut}(X)$, *we have* $f(D_1)=D_2$ *if and only if* $f_*(d_1)=d_2$ *where* $d_i=c_1(\mathcal{O}_X(D_i))$. (Note that $h^0(\mathcal{O}_X(D_2))=1$.)

LEMMA (1.6). *Let* D_i $(i=1, 2)$ *be ordinary nodal curves on* X. *Then* $D_1\cdot D_2\equiv 0 \pmod 2$.

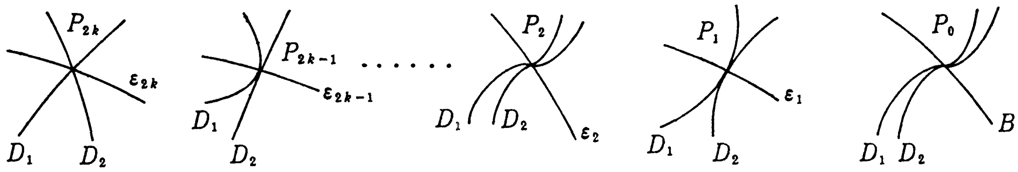
PROOF. If $D_1=D_2$, then we have $D_1\cdot D_2=-2$. Assume that $D_1\neq D_2$. By definition, we have

$$D_1\cdot D_2=\sum_{P\in D_1\cap D_2-B}\mathrm{mult}_P(D_1, D_2)+\sum_{P_0\in D_1\cap D_2\cap B}\mathrm{mult}_{P_0}(D_1, D_2).$$

By (1.3), (1.5), we have $\theta(D_i)=D_i$ $(i=1, 2)$ and θ acts on each D_i as an involution. Then the first sum above is even since $\mathrm{mult}_P(D_1, D_2)=\mathrm{mult}_{\theta(P)}(D_1, D_2)$ and $\theta(P)\neq P$ if $P\in D_1\cap D_2-B$. So, to prove (1.7) it is sufficient to show that

$\text{mult}_{P_0}(D_1, D_2)$ is even for each $P_0 \in D_1 \cap D_2 \cap B$. Assume that $\text{mult}_{P_0}(D_1, D_2) = 2k+1$ $(k=0, 1, 2, \cdots)$ for some $P_0 \in D_1 \cap D_2 \cap B$. By repeating blowing up, we get,

$$X_{2k} \xrightarrow[\text{at } P_{2k-1}]{\text{blowing up}} X_{2k-1} \longrightarrow \cdots\cdots \longrightarrow X_2 \xrightarrow[\text{at } P_1]{\text{blowing up}} X_1 \xrightarrow[\text{at } P_0]{\text{blowing up}} X_0 = X.$$



(Here $\varepsilon_i := \boldsymbol{P}(T_{P_{i-1}}(X))$ is the exceptional curve. For proper transforms of D_1 and D_2, we use the same letters on each X_i.) On X_{2k} we have $\text{mult}_{P_{2k}}(D_1, D_2) = 1$ by construction. On the other hand, by the property of blowing up, θ also acts on each X_i and preserves ε_i, D_1, D_2, and P_i. By construction, we see easily that on X_{2i}, $\theta | D_1$ and $\theta | D_2$ are involutions and $\theta |_{\varepsilon_{2i}}$ is an identity. Then on X_{2k}, we get $T_{P_{2k}}(D_1) = T_{P_{2k}}(D_2)$ and $\text{mult}_{P_{2k}}(D_1, D_2) \geqq 2$. This is contradiction. □

LEMMA (1.7). *Let D be an ordinary nodal curve on X. Then, there exist two special nodal curves E_j and F_i such that $D \cdot E_j = D \cdot F_i = 1$. Moreover D does not meet the other six special nodal curves.*

PROOF. Since θ acts on $D = \boldsymbol{P}^1$ as an involution, D and B meet at exactly two points transversely. (cf. Nikulin [4], p. 1434). So to prove (1.7), it is sufficient to show that the following 4 cases do not occur: (1) $D \cdot E_i = 2$ (for some i), (2) $D \cdot E_i = D \cdot E_j = 1$ (for some $i \neq j$), (3) $D \cdot F_i = 2$ (for some i), (4) $D \cdot F_i = D \cdot F_j = 1$ (for some $i \neq j$). For example, assume that (2) does occur. For simplicity of notation, we also assume $i=1$, $j=2$. In S_X we put,

$$d = ae + bf + \sum_{i,j} x_{ij} c_{ij}, \quad (a, b, x_{ij} \in \boldsymbol{Q}). \quad \text{(See (0.2).)}$$

Since we have $-2x_{ij} = D \cdot C_{ij} \equiv 0 \pmod 2$ by (1.6), we get $x_{ij} \in \boldsymbol{Z}$. By (0.1) and (0.2), we get

$$b + \sum_i x_{ij} = \begin{cases} 1 & (\text{if } j=1, 2) \\ 0 & (\text{if } j=3, 4) \end{cases}, \qquad a + \sum_j x_{ij} = 0 \quad (i=1, \cdots, 4).$$

Then, we get $b - a = 1/2$. On the other hand, since we have $x_{ij} \in \boldsymbol{Z}$, we get $b - a \in \boldsymbol{Z}$. Therefore (2) does not occur. Other cases also do not occur by a similar reason. □

LEMMA (1.8). *Let* D_k $(k=1, \cdots, 16)$ *be disjoint nodal curves on* X. *Then there exists* $f\in \mathrm{Aut}(X)$ *such that* $f(\bigcup_k D_k)=\bigcup_{i,j}C_{ij}$. *Hence, combining this with* (1.4), *we get* $f(\bigcup_k D_k\cup B)=K_{\mathrm{nat}}$. *Especially,* $K_D=\bigcup_k D_k\cup B$ *is a double Kummer pencil divisor.*

PROOF. By Nikulin [1], p. 262, we have $\sum_{k=1}^{16}d_k\in 2\cdot S_X$ and hence there exist an abelian surface A and a rational map $\pi_A: A \dashrightarrow X$ whose exceptional curves are D_k $(k=1, \cdots, 16)$. Hence via π_{A*} and π_*, we have a Hodge isometry $\phi_T: T_A \widetilde{\to} T_{E\times F}$. Then, by applying the theorem by Nikulin [3], p. 126, (or Morrison [11], p. 112), ϕ_T is extended to a Hodge isometry $\phi: H^2(A, \boldsymbol{Z}) \widetilde{\to} H^2(E\times F, \boldsymbol{Z})$. So we can apply the theorem of Shioda [6], p. 48 and we get $A\cong E\times F$. (Remark that $\mathrm{Pic}^0(E\times F)\cong E\times F$.) Therefore $f\in\mathrm{Aut}(X)$ induced from $F: A\cong E\times F$ which preserves the origins satisfies (1.8). □

Let M be either an abelian surface or a $K3$ surface. Since $H^{2,0}(M)=\boldsymbol{C}\omega_M$, we get the homomorphism $\alpha_M: \mathrm{Aut}(M)\to \boldsymbol{C}^\times$ characterized by $f_*\omega_M=\alpha_M(f)\omega_M$. Putting $\Gamma_M:=\mathrm{Im}(\alpha_M)$ and $\mathrm{Aut}_N(M):=\mathrm{Ker}(\alpha_M)$ (the symplectic automorphism group of M), we have the following exact sequence.

$$(1.9) \qquad 1 \longrightarrow \mathrm{Aut}_N(M) \longrightarrow \mathrm{Aut}(M) \xrightarrow{\alpha_M} \Gamma_M \longrightarrow 1 .$$

LEMMA (1.10). *Let* D_k $(k=1, \cdots, l)$ *be ordinary nodal curves on* X. *Let us put* $D:=D_1+\cdots+D_l$. *If* $D\cdot E_j\equiv D\cdot F_i\equiv 0 \pmod 2$ $(i, j=1, \cdots, 4)$ *then* $f_*(d)+d\in 2\cdot S_X$ *for any* $f\in\mathrm{Aut}_N(X)$.

PROOF. For $f\in \mathrm{Aut}_N(X)$, we have $f_*|T_X=\mathrm{id}$. (Because we have $f_*(x)\cdot\omega_X=f_*(x)\cdot f_*(\omega_X)=x\cdot\omega_X$ for $x\in T_X$ and then we get $f_*(x)-x\in S_X\cap T_X=\{0\}$.) Especially the induced map of f_* on T_X^*/T_X is identity. Here, for a non-degenerate lattice L, we set $L^*:=\{x\in L\otimes\boldsymbol{Q}\,;\, x\cdot L\in\boldsymbol{Z}\}=\mathrm{Hom}_{\boldsymbol{Z}}(L, \boldsymbol{Z})$. Then we see that the induced map of f_* on S_X^*/S_X is also identity by an easy lattice theoretic consideration. Hence we have $f_*(x)-x\in S_X$ for all $x\in S_X^*$. Let us consider $d/2$. Then $(d/2)\cdot C$ is an integer for every nodal curves on X by the assumption on D and (1.6). On the other hand, by considering a Jacobian fibration $\Phi_{|E|}$, we see that S_X is generated by some classes of nodal curves on X. (See (0.3) (2).) Hence we have $d/2\in S_X^*$. Therefore we have $f_*(d/2)-d/2\in S_X$ and $f_*(d)+d\in 2\cdot S_X$. □

LEMMA (1.11). $\mathrm{Aut}(X)=\mathrm{Aut}_N(X)\langle\bar{\xi}\rangle$ *(semi-direct product), where* $\bar{\xi}$ *is the element of* $\mathrm{Aut}(X; \bigcup_{i,j}C_{ij})$ *induced from the following* $\xi\in\mathrm{Aut}(E\times F; \bigcup_{i,j}\{R_{ij}\})$ *by* (1.1).

$E\times F$	$E_{\sqrt{-1}}\times E_{\omega}$	$E_{\rho}\times E_{\omega}$	$E_{\sqrt{-1}}\times E_{\rho}$	$E_{\rho}\times E_{\rho'}$
ξ	$\begin{pmatrix}\sqrt{-1} & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}1 & 0\\ 0 & \omega\end{pmatrix}$	$\begin{pmatrix}\sqrt{-1} & 1\\ 0 & 1\end{pmatrix}$	$\begin{pmatrix}-1 & 0\\ 0 & 1\end{pmatrix}$

(By E_{ξ} we denote the elliptic curve whose period is ξ in $H/SL_2(\boldsymbol{Z})$ where H is the upper half plane. And $\omega=(-1+\sqrt{-3})/2$, ρ, $\rho'\neq\sqrt{-1}$, ω in $H/SL_2(\boldsymbol{Z})$. Since E and F are not mutually isogenous, these cover all the cases.)

PROOF. By (1.9) it is sufficient to show that

$$\alpha_X|_{\langle\bar{\xi}\rangle}\colon \langle\bar{\xi}\rangle \xrightarrow{\sim} \Gamma_X .$$

Since E and F are not isogenous, we easily show that

$$\alpha_{E\times F}|_{\langle\xi\rangle}\colon \langle\xi\rangle \xrightarrow{\sim} \Gamma_{E\times F} .$$

So it is sufficient to show that if $\alpha\in\Gamma_X$, then $\alpha\in\Gamma_{E\times F}$. Let f be an automorphism of X such that $f_*\omega_X=\alpha\omega_X$. Put $\varphi=f_*|T_X$. Then $\tilde{\varphi}:=\pi_*^{-1}\circ\varphi\circ\pi_*$ is a Hodge isometry on $T_{E\times F}$, and satisfies $\tilde{\varphi}\omega_{E\times F}=\alpha\omega_{E\times F}$. So it is sufficient to show that there exists $g\in\mathrm{Aut}(E\times F)$ such that $g_*|T_X=\tilde{\varphi}$. To show this we use the following theorem by Shioda [**6**], p. 53.

THEOREM (1.12). *Let A be a two dimensional complex torus. Let ϕ be a Hodge isometry on $H^2(A, \boldsymbol{Z})$ such that* $\det\phi=1$. *Then there exists* $g\in\mathrm{Aut}(A)$ *satisfying either $g_*=\phi$ or $g_*=-\phi$.*

We put $\phi=\mathrm{id}_{S_{E\times F}}\oplus\tilde{\varphi}$. Then ϕ is a Hodge isometry on $H^2(E\times F, \boldsymbol{Z})$ and preserves effective classes on it. So if we can prove that $\det\phi=1$, i.e., $\det\tilde{\varphi}=1$, we get $g\in\mathrm{Aut}(E\times F)$ such that $g_*|T_X=\tilde{\varphi}$. Assume that $\det\tilde{\varphi}\neq1$. Then we have $\det\tilde{\varphi}=-1$ since $\tilde{\varphi}$ is an isometry on $T_{E\times F}$. Thus, putting $\phi'=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\oplus\tilde{\varphi}$, we see that ϕ' satisfies the condition of the above theorem. Hence there exists $g'\in\mathrm{Aut}(E\times F)$ such that $g'_*=\phi'$ or $-\phi'$. But this does not happen since E and F are not isogenous. Therefore we have $\det\tilde{\varphi}=1$. □

Combining (1.8) and (1.11), we get the following.

COROLLARY (1.13). *There exists $f\in\mathrm{Aut}_N(X)$ such that* $f(K_D)=K_{\mathrm{nat}}$ (Here K_D is same as in (1.8).)

Finally, we quote two theorems by Nikulin [1], [2] as lemmas.

LEMMA (1.14). *Let Y be a $K3$ surface. Let D_k $(k=1, \cdots, l)$ be disjoint nodal curves on Y. If $D:=\sum_{k=1}^{l}D_k\in 2\cdot S_Y$, then $l=0$, 8 or 16.*

LEMMA (1.15). *Let Y be a $K3$ surface. If $f\in\mathrm{Aut}_N(Y)$ is of finite order and* *not identity. Then the order of f and the number of the fixed points of f are as follows.*

order of f	2	3	4	5	6	7	8
number of fixed points of f	8	6	4	4	2	3	2

§2. Classification of $\mathcal{G}_X$ via types of the singular fibers.

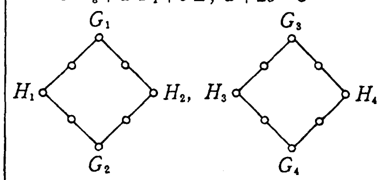
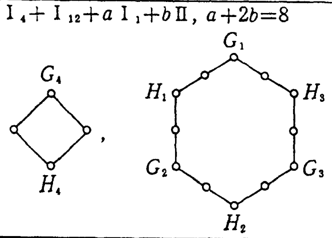
We use the following notation in §2, 3, and 4. By G_i, H_i $(i=1, \cdots, 4)$ we denote the 8 special nodal curves such that either $\{G_i\}=\{E_i\}$ and $\{H_i\}=\{F_i\}$ or $\{G_i\}=\{F_i\}$ and $\{H_i\}=\{E_i\}$ as a set. For fixed G_i, H_i $(i=1, \cdots, 4)$, we denote by C^{ij} the nodal curve in $\{C_{ij}\}$ meeting both G_j and H_i. By $\{D^{ij}\}$, where (i, j) moves some subsets of $\{1, \cdots, 4\}\times\{1, \cdots, 4\}$, we denote a collection of nodal curves such that D^{ij} meets G_j and H_i and D^{ij} do not meet one another. By R^{ij}, Q^{ij} etc., we denote a nodal curve which meets G_j and H_i.

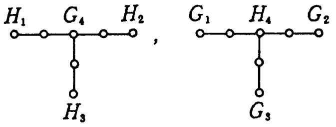
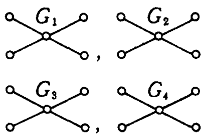
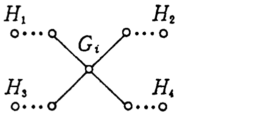
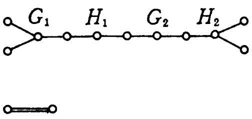
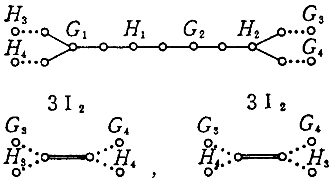
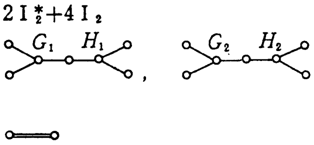
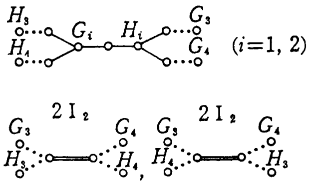
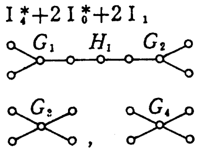
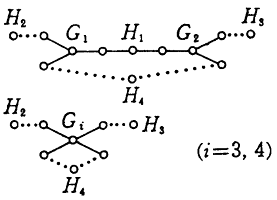
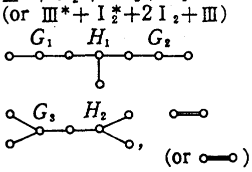
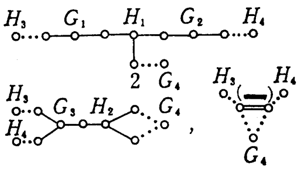
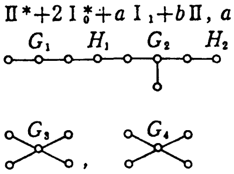
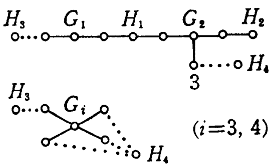
In this section we prove the following theorem.

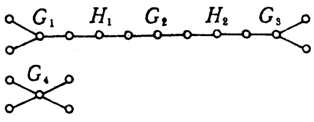
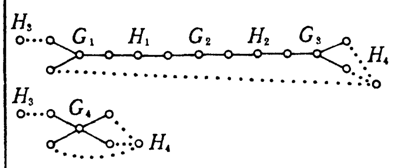
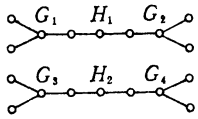
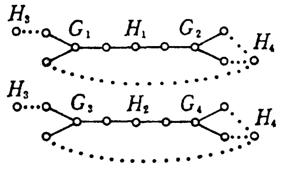
THEOREM (2.1). (1) *The set $\mathcal{G}_X$ is divided into eleven* Aut(X)*-stable subsets, $\mathcal{G}_1, \cdots, \mathcal{G}_{11}$ by the types of the singular fibers.*

(2) *For each $\mathcal{G}_m$ sections, Mordell-Weil groups, and configurations of sections and singular fibers of its members are described as in the following Table* 2.

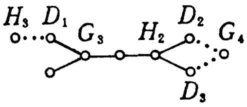
Table 2.

type	all the singular fibers (Figures of type I_1 and II are omitted.)	all the sections	Mordell-Weil group	configuration of singular fibers and sections
$\mathcal{G}_1$	$2\mathrm{I}_8+a\mathrm{I}_1+b\mathrm{II}$, $a+2b=8$ 		$\boldsymbol{Z}^2\oplus\boldsymbol{Z}/2\boldsymbol{Z}$	See figure in the remark (2.13)
$\mathcal{G}_2$	$\mathrm{I}_4+\mathrm{I}_{12}+a\mathrm{I}_1+b\mathrm{II}$, $a+2b=8$ 		$\boldsymbol{Z}^2\oplus\boldsymbol{Z}/2\boldsymbol{Z}$	〃

$\mathcal{J}_3$	$2\text{IV}^*+a\,\text{I}_1+b\,\text{II},\ a+2b=8$ 		$\boldsymbol{Z}^4$	〃
$\mathcal{J}_4$	$4\,\text{I}_0^*$ 	$H_1,\ H_2$ $H_3,\ H_4$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	 $(i=1,\ 2,\ 3,\ 4)$
$\mathcal{J}_5$	$\text{I}_6^*+6\,\text{I}_2$ 	$G_3,\ G_4$ $H_3,\ H_4$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	
$\mathcal{J}_6$	$2\,\text{I}_2^*+4\,\text{I}_2$ 	$G_3,\ G_4$ $H_3,\ H_4$	$(\boldsymbol{Z}/2\boldsymbol{Z})^2$	
$\mathcal{J}_7$	$\text{I}_4^*+2\,\text{I}_0^*+2\,\text{I}_1$ 	$H_2,\ H_3$ H_4 is a 2-section.	$\boldsymbol{Z}/2\boldsymbol{Z}$	 $(i=3,\ 4)$
$\mathcal{J}_8$	$\text{III}^*+\text{I}_2^*+3\,\text{I}_2+\text{I}_1$ (or $\text{III}^*+\text{I}_2^*+2\,\text{I}_2+\text{III}$) 	$H_3,\ H_4$ G_4 is a 2-section.	$\boldsymbol{Z}/2\boldsymbol{Z}$	
$\mathcal{J}_9$	$\text{II}^*+2\,\text{I}_0^*+a\,\text{I}_1+b\,\text{II},\ a+2b=2$ 	H_3 H_4 is a 3-section.	{id}	 $(i=3,\ 4)$

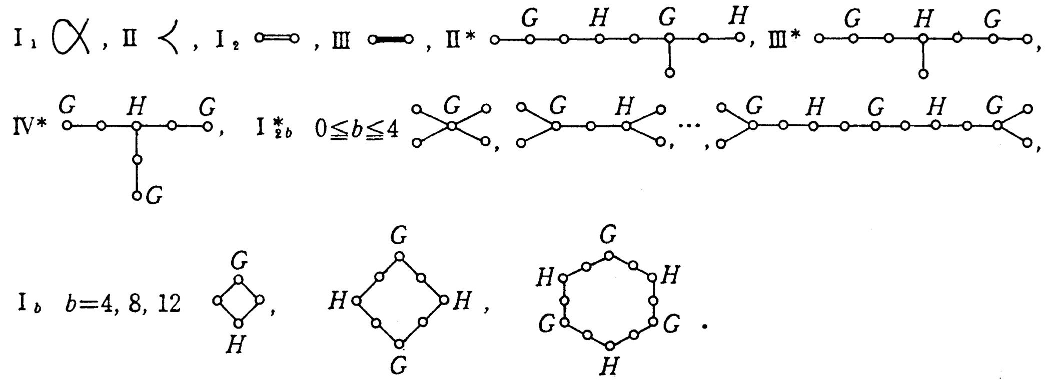
$\mathcal{J}_{10}$	$I_8^*+I_0^*+aI_1+bII,\ a+2b=4$ 	H_3 H_4 is a 3-section.	{id}	
$\mathcal{J}_{11}$	$2I_4^*+aI_1+bII,\ a+2b=4$ 	H_3 H_4 is a 3-section.	{id}	

By $\overset{G_i}{\circ}$ (resp. $\overset{H_i}{\circ}$, resp. $\circ$), we mean a nodal curve G_i (resp. H_i, resp. an ordinary nodal curve).

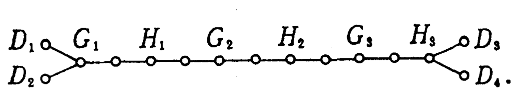
For example, by  , we mean that a section H_3 meets a singular fiber of type I_2^* in D_1 and 2-section G_4 meets this singular fiber in D_2 and D_3.

Let φ be a Jacobian fibration of X.

LEMMA (2.2). *Let Θ be a singular fiber of φ. Then Θ is one of the following form:*



PROOF. For example, we show that Θ is neither of type I_{10}^* nor of type I_{16}. If Θ is of type I_{10}^*, then by (1.6) and (1.7), Θ is as follows:



Then, by (1.6), a section of φ must be either H_4 or G_4. But this is impossible because, by (1.7), we have $D_i \cdot H_4=1$ for $i=1, 2$, and $D_j \cdot G_4=1$ for $j=3, 4$. If Θ is of type I_{16}, then Θ contains B. So for any ordinary nodal curve C, $C \cdot \Theta \geqq 2$ holds. Hence φ has no sections. □

LEMMA (2.3). *If all special curves are contained in some singular fibers of* φ, *then* $\varphi \in \mathcal{J}_1$ *or* $\mathcal{J}_2$ *or* $\mathcal{J}_3$. *Moreover* $\operatorname{rank} M_\varphi(X)=2, 2, 4$ *respectively.*

PROOF. Let C be a section of φ. Let $\Theta_1, \cdots, \Theta_k$ be the singular fibers of φ which are neither of type I_1 nor of type II. We note that C meets each Θ_i in a simple component. Since C is an ordinary nodal curve by the assumption, C meets each Θ_i in a special nodal curve. So we get $k=2$ because we have $C \cdot B=2$. Then types of Θ_1 and Θ_2 are either of (1) I_8, I_8 (2) I_4, I_{12} (3) IV*, IV* by (2.2). For each of three cases (1), (2), (3), by counting Euler number and $\operatorname{rank} M_\varphi(X)$ by (0.3) (1) and (3), we get the desired results. □

Until (2.12) we assume that at least one of special nodal curves is not in any singular fibers of φ.

LEMMA (2.4). (1) $\operatorname{rank} M_\varphi(X) = 0$.

Let $\Theta_1, \cdots, \Theta_k$ *be the singular fibers of* φ. *Then,*

(2) $24 = \sum_i \chi_{\mathrm{top}}(\Theta_i)$, $\quad 16 = \sum_i (m(\Theta)-1)$,

(3) φ *has at least one singular fiber which is neither of type* I_1 *nor of type* II.

PROOF. If (1) holds, then (2) holds by (0.3) (1), (3). Then (3) holds since $m(I_1)=m(\mathrm{II})=1$. Let us prove (1). Let $S_1, \cdots, S_l$ be all the special nodal curves not contained in any singular fibers of φ. Let C be an arbitrary smooth fiber of φ. We have $1 \leqq \#(C \cap (S_1 \cup \cdots \cup S_l)) \leqq C \cdot (S_1 + \cdots + S_l) = m$. Of course, m is independent of the choice of C. By (1.3), any $f \in M_\varphi(X)$ acts on the finite set $I_C = C \cap (S_1 \cup \cdots \cup S_l)$ as a permutation. So $f^{m!}$ fixes all the points of I_C for any C. Therefore, by definition of $M_\varphi(X)$, we get $f^{m!}=\mathrm{id}$ on X. Hence we have $\operatorname{rank} M_\varphi(X)=0$. □

Let Θ be a singular fiber of φ which is neither of type I_1 nor of type II.

LEMMA (2.5). (1) Θ *is one of the following form in* (2.2)*:*

$$I_2, \mathrm{III}, \mathrm{II}^*, \mathrm{III}^*, I_{2b}^*.$$

(2) *All sections of* φ *are special nodal curves.*

PROOF. If Θ is either I_b $(3 \leqq b)$ or IV* in (2.2), then Θ cannot meet any special nodal curves. Then (1) holds. Hence all the simple components of Θ are ordinary nodal curves. Then (2) holds by (1.7). □

We continue the proof of (2.1), and consider the following two cases separately:

Case (1). *At least one of singular fibers of* φ *is either of type* I_2 *or of type* III.

Case (2). *Otherwise.*

Case (1). We can see at once that either ($\sharp$) or ($\sharp\sharp$) holds:

($\sharp$) *All the sections of* φ *are* G_3, G_4, H_3, H_4 *and the remaining* G_1, G_2, H_1, H_2 *are in some fibers of* φ.

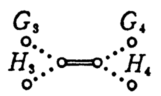
($\sharp\sharp$) *All the sections of* φ *are* H_3 *and* H_4. *The curve* G_4 *is a 2-section of* φ. *The remaining* G_1, G_2, G_3, H_1, H_2 *are in some fibers of* φ.

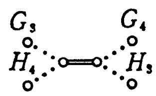
LEMMA (2.6). *Let* φ *be a Jacobian fibration satisfying* ($\sharp$). (We do not assume that one of the singular fibers of φ is of type I_2 or of type III.) *Then* $\varphi\in\mathcal{J}_5$ *or* $\varphi\in\mathcal{J}_6$ *holds, and* (2.1) (2) *holds for this* φ.

PROOF. By the condition ($\sharp$), any singular fiber of φ is one of the following types in (2.5); I_2, III, I_1, I_2^*, I_6^*. (Remark that φ has no singular fibers of type II because $M_\varphi(X)$ has a torsion element.) Then φ has either two singular fibers of type I_2^* or one singular fiber of type I_6^*. As for the latter case, putting $\alpha=\sharp\mathrm{I}_2$, $\beta=\sharp\mathrm{III}$, $\gamma=\sharp\mathrm{I}_1$, we get by (2.4):

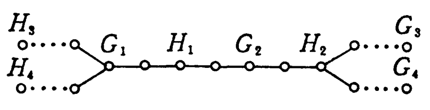
$$16=10+\alpha+\beta,\ 24=12+2\alpha+3\beta+\gamma,\ \text{and then,}\ \beta=\gamma=0,\ \alpha=6.$$

Hence we have $\varphi\in\mathcal{J}_5$. We show that (2.1) (2) holds for this φ. Since $\sharp M_\varphi(X)=4$, and the group structure of I_6^* is $\boldsymbol{C}\times(\boldsymbol{Z}/2\boldsymbol{Z})^2$, we have $M_\varphi(X)=(\boldsymbol{Z}/2\boldsymbol{Z})^2$. Each of six singular fibers of type I_2 meets four sections like either

(1) 

or (2) 

and a singular fiber of type I_6^* meets four sections like



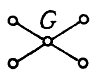
Put the number of singular fibers of type I_2 like (1) (resp. like (2)) m (resp. n). Let us take $f\in M_\varphi(X)$ such that $f(H_4)=G_4$. Then we have $f(H_3)=G_3$, and f has at least 2 fixed points on each of $m\mathrm{I}_2$, and on I_6^*. Then we get $2m+2\leqq 8$ by (1.15). Similarly, by taking $g\in M_\varphi(X)$ such that $g(H_3)=G_4$, we get $2n+2\leqq 8$. Hence we have $n=m=3$. (Remark that $m+n=6$.) For the former case, the proof is similar. □

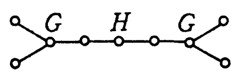
By a similar argument to (2.6), we get the following.

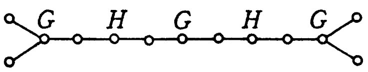
LEMMA (2.7). *Let* φ *be a Jacobian fibration satisfying* (##). *Then* $\varphi \in \mathcal{J}_8$ *holds and* (2.1) (2) *holds for this* φ.

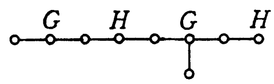
Case (2). Without loss of generality, we may assume that H_3 is a section of φ.

LEMMA (2.8). Θ *is one of the following form in* (2.2).

(1) 

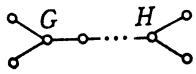
(2) 

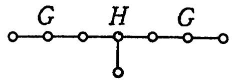
(3) 

(4) 

.

PROOF. If Θ is neither of (1), (2), (3), (4), Θ is either (5) or (6).

(5) 

(6) 

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If Θ is either (5) or (6), we easily show that φ satisfies either (#) or (##), and then φ has a singular fiber whose type is either I_2 or III. Hence (2.8) holds. □

LEMMA (2.9). *If* φ *has a singular fiber of type* (4) *in* (2.8), *then* $\varphi \in \mathcal{J}_9$ *holds and* (2.1) (2) *holds for this* φ.

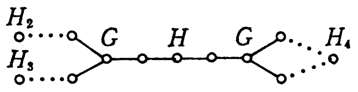
PROOF. Immediate. □

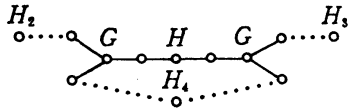
LEMMA (2.10). *If* φ *has a singular fiber of type* (3) *but not of type* (4) *in* (2.8), *then* $\varphi \in \mathcal{J}_{10}$ *holds and* (2.1) (2) *holds for this* φ.

PROOF. Immediate. □

LEMMA (2.11). *If* φ *has a singular fiber of type* (2) *but neither of type* (3) *nor of type* (4) *in* (2.8), *then either* $\varphi \in \mathcal{J}_7$ *or* $\varphi \in \mathcal{J}_{11}$ *holds and* (2.1) (2) *also holds for this* φ.

PROOF. We easily show that all the singular fibers of φ which are neither of type I_1 nor of type II are either (a) I_4^*, I_4^* or (b) I_4^*, I_0^*, I_0^*. When (a) holds, obviously we have $\varphi \in \mathcal{J}_{11}$ and (2.1) (2) holds. When (b) holds, we easily see that H_2, H_3 are sections of φ and H_4 is a 2-section of φ (by a suitable naming) and a configuration of a singular fiber of type I_4^* and H_2, H_3, and H_4 is either (c) or (d):

(c) 

(d) 

Assume that (c) holds. Take $f \in M_\varphi(X)$ such that $f(H_2)=H_3$. Then f has at least 10 fixed points on X. But this is impossible by (1.15). Hence (d) holds. Since $M_\varphi(X)=\boldsymbol{Z}/2\boldsymbol{Z}$, φ has no singular fibers of type II. Therefore the remaining singular fibers of φ are two singular fibers of type I_1. □

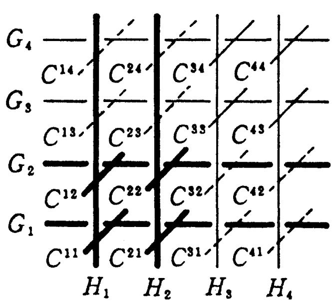
LEMMA (2.12). *If φ has a singular fiber of type* (1) *but neither of types* (2), (3), (4) *in* (2.8), *then $\varphi \in \mathcal{J}_4$ holds and* (2.1) (2) *holds for this φ.*

PROOF. Immediate. □

Hence (2.1) (1) is proved. And except for $\mathcal{J}_1$, $\mathcal{J}_2$ and $\mathcal{J}_3$, (2.1) (2) is also proved. We prove the rest in § 3. Q. E. D.

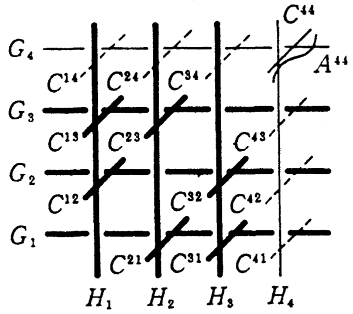
REMARK (2.13). Any $\mathcal{J}_m$ $(m=1, \cdots, 11)$ is non-empty. In fact we can construct elements $\Phi=\Phi_{|\Theta|}$ belonging to each $\mathcal{J}_m$ as follows. Here Θ is represented by bold-faced lines. Dotted lines (resp. dotted lines with index m) stand for sections (resp. m-sections).

$\mathcal{J}_1$



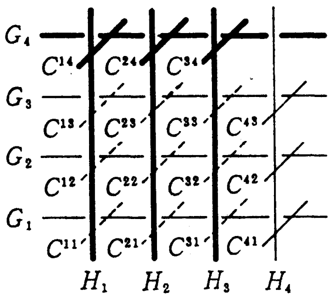
$H_3+C^{33}+G_3+C^{34}+G_4+C^{44}+H_4+C^{43}$ is another singular fiber of type I_8 of Φ. C^{13}, C^{14}, C^{23}, C^{24}, C^{31}, C^{32}, C^{41}, and C^{42} are sections of Φ which do not meet one another.

$\mathcal{J}_2$



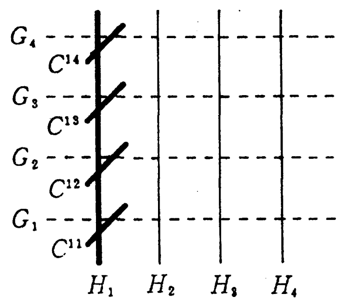
By (2.1) (1), there exists a nodal curve A^{44} such that $G_4+C^{44}+A^{44}+H_4$ is another singular fiber of type I_4 of Φ. C^{14}, C^{24}, C^{34}, C^{41}, C^{42}, C^{43} are sections of Φ which do not meet one another.

$\mathcal{J}_3$

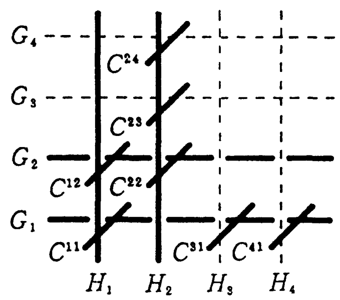


$G_1+G_2+G_3+2(C^{41}+C^{42}+C^{43})+3H_4$ is another singular fiber of type IV* of Φ. C^{ij} $(1\leqq i,j\leqq 3)$ are sections of Φ which do not meet one another.

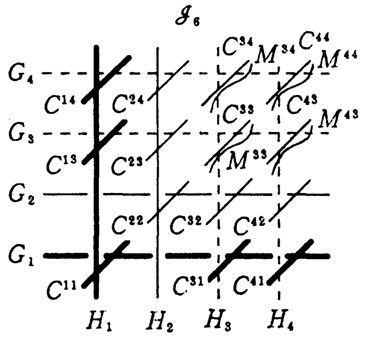
$\mathcal{J}_4$



$\mathcal{J}_5$

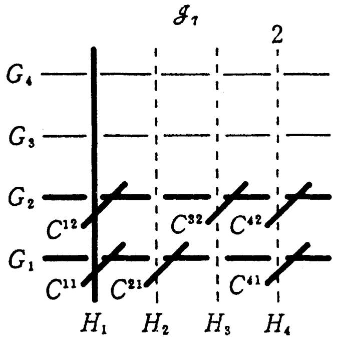


$\mathcal{J}_6$

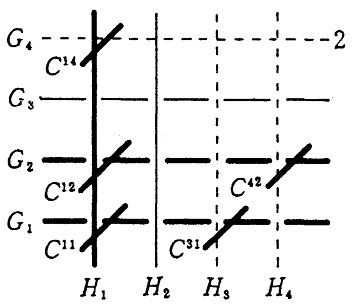


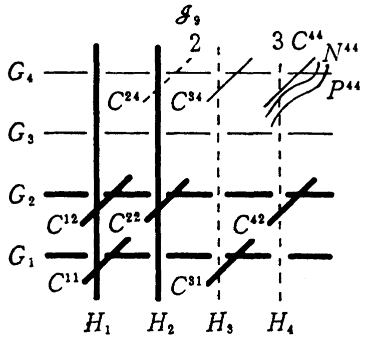
By (2.1), there exist four nodal curves M^{ij} $(3\leqq i,\ j\leqq 4)$ such that $C^{34}+M^{43}$, $C^{43}+M^{34}$, $C^{33}+M^{44}$, $C^{44}+M^{33}$ are other singular fibers of type I_2 of Φ. $C^{24}+C^{23}+C^{32}+C^{42}+2(H_2+C^{22}+G_2)$ is another singular fiber of type I_2^*. We note that M^{44} does not meet C^{ms} $(1\leqq m,\ s\leqq 4)$ except for C^{33}, C^{21} and C^{12}.

$\mathcal{J}_7$

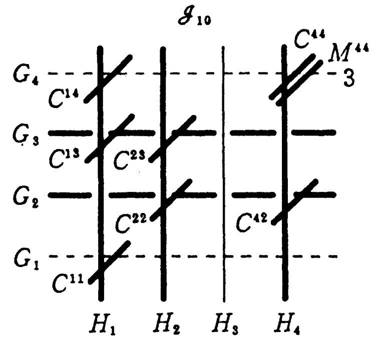


$\mathcal{J}_8$

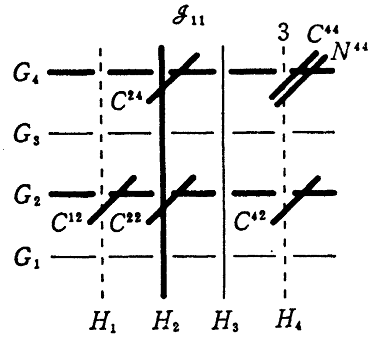




By (2.1), there exist nodal curves N^{44} and P^{44} such that $2G_4+C^{34}+C^{44}+N^{44}+P^{44}$ is another singular fiber of type I_0^* of Φ. We note that C^{24} is 2-section of Φ, and C^{24} does not meet N^{44}.



M^{44} is a nodal curve in the figure of $\mathcal{J}_6$ above.



N^{44} is a nodal curve in the figure of $\mathcal{J}_9$ above.

REMARK (2.14). We could not determine the value of a and b except for $\mathcal{J}_1$ and $\mathcal{J}_2$. As for $\mathcal{J}_8$, we could not determine which of $\mathrm{III}^*+\mathrm{I}_2^*+3\mathrm{I}_2+\mathrm{I}_1$ and $\mathrm{III}^*+\mathrm{I}_2^*+2\mathrm{I}_2+\mathrm{III}$ actually occurs.

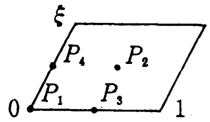
§ 3. A minimal complete set of representatives of $\mathcal{J}_m/\mathrm{Aut}(X)$ $(m=1, 2, 3)$.

In this section we find a minimal complete set of representatives (M. S. R.) of the orbit space $\mathcal{J}_m/\mathrm{Aut}(X)$ and prove (2.1) (2) for $m=1, 2, 3$. The cases for $m=4, \cdots, 11$ will be treated in the next section.

We use the following notation in § 3, 4.

$$\{i, j, k\} = \{p, q, r\} = \{2, 3, 4\}.$$

For E_ξ (see (1.11)), $P_1, \cdots, P_4$ stand for the following 2-torsion points of E_ξ.



We say X is of type (i), (ii), (iii) or (iv) if $E\times F$ is isomorphic to $E_{\sqrt{-1}}\times E_{\omega}$, $E_{\rho}\times E_{\omega}$, $E_{\sqrt{-1}}\times E_{\rho}$, or $E_{\rho}\times E_{\rho'}$. (See (1.11).)

We say an effective divisor D on X is *extendable* if there exists a double Kummer pencil divisor K_D such that $\operatorname{Supp} D\subset K_D$.

THEOREM (3.1). (I) *Put* $\varphi_{ip}^{(1)}=\Phi_{|\Theta_{ip}^{(1)}|}$ *where*

$$\Theta_{ip}^{(1)} = F_1+C_{11}+E_1+C_{i1}+F_i+C_{ip}+E_p+C_{1p} \qquad \text{and} \quad 2\leqq i,\ p\leqq 4.$$

(1) *The set* $\{\varphi_{ip}^{(1)}\}_{1\leq i,p\leq 4}$ *is an* M.S.R. *of* $\mathcal{J}_1/\mathrm{Aut}_N(X)$.

(2) *An* M.S.R. *of* $\mathcal{J}_1/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ip}:=\varphi_{ip}^{(1)}$.

Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_1/\mathrm{Aut}(X)$	φ_{22} φ_{32}	φ_{i2} $i=2,3,4$	φ_{ip} $i=2,3$ $p=2,3,4$	φ_{ip} $i=2,3,4$ $p=2,3,4$

(II) *Put* $\varphi_{ijk}^{(2)}=\Phi_{|\Theta_{ijk}^{(2)}|}$ *where*

$$\Theta_{ijk}^{(2)} = E_2+C_{i2}+F_i+C_{i3}+E_3+C_{j3}+F_j+C_{j4}+E_4+C_{k4}+F_k+C_{k2} \qquad \text{and} \quad \{i,\ j,\ k\} = \{2,\ 3,\ 4\}.$$

(1) *The set* $\{\varphi_{ijk}^{(2)}\}_{\{i,j,k\}=\{2,3,4\}}$ *is an* M.S.R. *of* $\mathcal{J}_2/\mathrm{Aut}_N(X)$.

(2) *An* M.S.R. *of* $\mathcal{J}_2/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ijk}:=\varphi_{ijk}^{(2)}$.

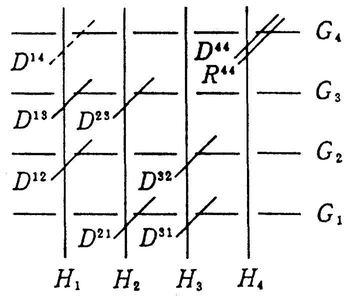
Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_2/\mathrm{Aut}(X)$	φ_{234}	φ_{234}, φ_{324}	φ_{234}, φ_{324}, φ_{342}	φ_{ijk} $\{i,j,k\}=\{2,3,4\}$

(III) *Put* $\varphi^{(3)}=\Phi_{|\Theta^{(3)}|}$ *where* $\Theta^{(3)}=F_1+F_2+F_3+2(C_{14}+C_{24}+C_{34})+3E_4$, *then* $\{\varphi^{(3)}\}$ *is an* M.S.R. *of both* $\mathcal{J}_3/\mathrm{Aut}_N(X)$ *and* $\mathcal{J}_3/\mathrm{Aut}(X)$.

PROOF. We give the proof only for (II), since the other cases are similar and easier. Assume $\varphi\in\mathcal{J}_2$. Then by a suitable G_i, H_i and D^{ms}, we have $\varphi=\Phi_{|\Theta|}$, where

$$\Theta = G_1+D^{21}+H_2+D^{23}+G_3+D^{13}+H_1+D^{12}+G_2+D^{32}+H_3+D^{31}.$$

The other singular fiber of type I_4 of φ can be written as follows: $\Theta'=G_4+D^{44}+H_4+R^{44}$. Since φ has at least one section, we put this section D^{14} without loss of generality. (As for D^{**} and R^{**}, see § 2.)



CLAIM (3.2). *Θ is extendable.*

PROOF of (3.2). We consider the elliptic fibration $\Phi_{|L|}$, where $L=D^{12}+D^{13}+2(H_1+D^{14}+G_4)+D^{44}+R^{44}$. Then, G_2 and G_3 become sections of $\Phi_{|L|}$, and H_4 becomes a 2-section. Hence we have $\Phi_{|L|}\in\mathcal{J}_8$. By the way, any component of a connected divisor $D=D^{23}+H_2+D^{21}+G_1+D^{31}+H_3+D^{32}$ does not meet L, and hence D is contained in one singular fiber L' of $\Phi_{|L|}$. By Theorem (2.1) L' must be of type III*, and then there exists a nodal curve D^{41}. Moreover, there exist at least two singular fibers of type I_2, say, $Q^{43}+D^{42}$, and $Q^{42}+D^{43}$. Then we have $\Phi_{|2H_4+D^{41}+D^{42}+D^{43}+D^{44}|}\in\mathcal{J}_4$. Hence, there exist nodal curves D^{11}, D^{22}, D^{33}, D^{34}, and $K_\Theta=\bigcup_{n,s=1}^4 D^{ns}\cup B$ becomes a double Kummer pencil containing $\operatorname{Supp}\Theta$. Therefore the claim is proved. □

Hence, by (1.13), there exists $h\in\mathrm{Aut}_N(X)$ such that $h(K_\Theta)=K_{\mathrm{nat}}$. Then, putting $\Theta'=h(\Theta)$ (as a divisor), we have $\operatorname{Supp}\Theta'\subset K_{\mathrm{nat}}$. So, if necessary, composing a suitable $g\in\mathrm{Aut}_N(X)$ induced by a translation on $E\times F$, we get $g(\Theta')=\Theta_{ijk}$ for some i, j, k. Therefore, to prove (1), it is sufficient to show that if φ_{ijk} and $\varphi_{i'j'k'}$ are in the same orbit, then $i=i'$, $j=j'$, and $k=k'$ hold. Under the above assumption, we have $f(\Theta_{i'j'k'})=\Theta_{ijk}$ by some $f\in\mathrm{Aut}_N(X)$. Since we have $f(B)=B$, we get the following:

$$f(C_{i'3}+C_{j'3}+C_{j'4}+C_{k'4}+C_{k'2}+C_{i'2})=C_{i3}+C_{j3}+C_{j4}+C_{k4}+C_{k2}+C_{i2}.$$

By the way, since $C_{i'3}+C_{j'3}+C_{j'4}+C_{k'4}+C_{k'2}+C_{i'2}$ satisfies the condition on (1.10), we have the following:

$$c_{i'3}+c_{j'3}+c_{j'4}+c_{k'4}+c_{k'2}+c_{i'2}+c_{i3}+c_{j3}+c_{j4}+c_{k4}+c_{k2}+c_{i2}\equiv 0 \pmod{2\cdot S_X}.$$

Since $\{i', j'\}\cap\{i, j\}\neq\emptyset$, $\{j', k'\}\cap\{j, k\}\neq\emptyset$, $\{k', i'\}\cap\{k, i\}\neq\emptyset$, we can put,

$$\{i', j'\}=\{x, y\},\qquad \{j', k'\}=\{u, v\},\qquad \{k', i'\}=\{\alpha, \beta\},$$
$$\{i, j\}=\{x, z\},\qquad \{j, k\}=\{u, w\},\qquad \{k, i\}=\{\alpha, \gamma\}.$$

Then we get, $c_{z3}+c_{y3}+c_{w4}+c_{v4}+c_{\gamma 2}+c_{\beta 2}\equiv 0 \pmod{2\cdot S_X}$. Therefore by (1.14), we get $C_{z3}=C_{y3}$, $C_{w4}=C_{v4}$, $C_{\gamma 2}=C_{\beta 2}$, i.e., $z=y$, $w=v$, $\gamma=\beta$. Hence $k=k'$, $i=i'$ and $j=j'$ hold. Next we prove (2). Since we have $\mathrm{Aut}(X)=\mathrm{Aut}_N(X)\cdot\langle\xi\rangle$

(cf. (1.11)), once an M.S.R. of $\mathcal{J}_i/\mathrm{Aut}_N(X)$ is found, we can find an M.S.R. of $\mathcal{J}_i/\mathrm{Aut}(X)$ by only examing how $\bar{\xi}$ acts on $\mathcal{J}_i/\mathrm{Aut}_N(X)$. The automorphism $\bar{\xi}$ acts on $\mathcal{J}_2/\mathrm{Aut}_N(X)$ as follows.

Type (i): $\varphi_{234} \xrightarrow[\bar{\xi}]{} \varphi_{324} \xrightarrow[\bar{\xi}]{} \varphi_{342} \xrightarrow[\bar{\xi}]{} \varphi_{243} \xrightarrow[\bar{\xi}]{} \varphi_{423} \xrightarrow[\bar{\xi}]{} \varphi_{432}$ (and $\varphi_{432} \xrightarrow{\bar{\xi}} \varphi_{234}$)

Type (ii): $\varphi_{234} \xrightarrow[\bar{\xi}]{} \varphi_{423} \xrightarrow[\bar{\xi}]{} \varphi_{342}$ (and $\varphi_{342} \xrightarrow{\bar{\xi}} \varphi_{234}$), $\quad \varphi_{243} \xrightarrow[\bar{\xi}]{} \varphi_{324} \xrightarrow[\bar{\xi}]{} \varphi_{432}$ (and $\varphi_{432} \xrightarrow{\bar{\xi}} \varphi_{243}$)

Type (iii): $\varphi_{234} \underset{\bar{\xi}}{\rightleftarrows} \varphi_{243}, \quad \varphi_{324} \underset{\bar{\xi}}{\rightleftarrows} \varphi_{423}, \quad \varphi_{342} \underset{\bar{\xi}}{\rightleftarrows} \varphi_{432}$

Type (iv): $\bar{\xi}=\theta$ acts as an identity on $\mathcal{J}_X$.

Q.E.D.

Finally we prove the rest of (2.1) (2) for $\mathcal{J}_2$ and $\mathcal{J}_3$. As for $\mathcal{J}_1$, the proof is similar for $\mathcal{J}_2$ and then omitted.

As for $\mathcal{J}_2$, by (3.1) and (2.3) it is enough to show that $\mathrm{Tor}\, M_\varphi(X)=\boldsymbol{Z}/2\boldsymbol{Z}$ for

$$\varphi = \Phi_{|H_1+C^{12}+G_2+C^{32}+H_3+C^{31}+G_1+C^{21}+H_2+C^{23}+G_3+C^{13}|}.$$

Note that φ has six sections C^{14}, C^{24}, C^{34}, C^{41}, C^{42}, C^{43}. By Lemma (1.15) in Cox and Zucker [**9**], p. 8, $f\in M_\varphi(X)$ defined by $f(C^{14})=C^{41}$ is a torsion element. Hence φ has no singular fibers of type II and then, by (2.3), φ has eight singular fibers of type I_1. Therefore any element of $M_\varphi(X)$ has at least 8 fixed points on X and then $\mathrm{Tor}\, M_\varphi(X)$ is 2-elementary. If f and g are 2-torsion elements in $M_\varphi(X)$, $f\circ g$ acts on singular fibers of type I_1 as an identity. Hence by (1.15), $f\circ g$ is an identity on X. Then we have $f=g$. Therefore $\mathrm{Tor}\, M_\varphi(X)$ $=\boldsymbol{Z}/2\boldsymbol{Z}$ holds.

As for $\mathcal{J}_3$, if $M_{\varphi^{(3)}}(X)$ has a torsion, we get $\mathrm{Tor}\, M_{\varphi^{(3)}}(X)=\boldsymbol{Z}/2\boldsymbol{Z}$ like as above. But this does not happen since the group structure of $\Theta^{(3)}$ is $\boldsymbol{C}\times\boldsymbol{Z}/3\boldsymbol{Z}$.

□

COROLLARY (3.3). *Let D^{ns} $(1\leqq n\neq s\leqq 4)$ be 12 disjoint nodal curves for arbitrarily fixed H_n, G_n $(n=1, 2, 3, 4)$.* (As for D^{**}, see § 2.) *Then there exists $\sigma\in\mathrm{Aut}_N(X)$ such that $\sigma(H_n)=G_n$, $\sigma(G_n)=H_n$ and $\sigma(D^{ns})=D^{sn}$ for all n, s with $1\leqq n\neq s\leqq 4$. Especially, there exists $\sigma'\in\mathrm{Aut}_N(X)$ such that $\sigma'(H_n)=G_n$, $\sigma'(G_n)$ $=H_n$ and $\sigma'(C^{ns})=C^{sn}$ for all n, s with $1\leqq n\neq s\leqq 4$.*

PROOF. We consider the Jacobian fibration $\varphi=\Phi_\Lambda$, where

$$\Lambda := |D^{23}+H_2+D^{24}+G_4+D^{34}+H_3+D^{32}+G_2+D^{42}+H_4+D^{43}+G_3|.$$

Then D^{12}, D^{13}, D^{14}, D^{21}, D^{31} and D^{41} are sections of φ and we have $\varphi \in \mathcal{J}_2$. Let us take three elements f_n $(n=2, 3, 4) \in M_\varphi(X)$ such that $f_n(D^{1n})=D^{n1}$. By Cox and Zucker (loc. cit.), f_2, f_3 and f_4 are torsion elements of $M_\varphi(X)$. Therefore we have $f_2=f_3=f_4$. Putting $\sigma=f_2=f_3=f_4$, we have $\sigma(H_n)=G_n$, $\sigma(G_n)=H_n$ and $\sigma(D^{ns})=D^{sn}$ for all n, s with $1 \leqq n \neq s \leqq 4$. □

Corollary (3.4). *Let A^{11}, B^{11}, D^{1s}, D^{s1} $(2 \leqq s \leqq 4)$ be 8 disjoint nodal curves on X for arbitrarily fixed H_n, G_n $(n=1, 2, 3, 4)$. Then,*

(1) $\Phi_1 := \Phi_{|A^{11}+\sum_{s=2}^4 D^{1s}+2H_1|}$ and $\Phi_2 := \Phi_{|B^{11}+\sum_{s=2}^4 D^{1s}+2H_1|}$ *are elements of* $\mathcal{J}_4$.

(2) *If any non-singular fiber of Φ_1 is isomorphic to E, then any non-singular fiber of Φ_2 is isomorphic to F.*

Proof. (1) is obvious. Let us consider the Jacobian fibration $\Phi_3 := \Phi_{|A^{11}+B^{11}+H_1+G_1|} \in \mathcal{J}_2$, and the involution $\sigma \in M_{\Phi_3}(X)$. Without loss of generality, we may assume that there exist 6 nodal curves D^{ns} $(2 \leqq n \neq s \leqq 4)$ and $\sum_{n=2}^4 (H_n+G_n)+\sum_{2 \leq n \neq s \leq 4} D^{ns}$ is another singular fiber of type I_{12} of Φ_3. By Cox and Zucker (loc. cit.), 6 sections D^{1s}, D^{s1} $(s=2, 3, 4)$ satisfy $\sigma(D^{1s})=D^{s1}$. Moreover we have $\sigma(B^{11})=A^{11}$ and $\sigma(H_1)=G_1$. Therefore σ translates a Jacobian fibration Φ_2 to a Jacobian fibration $\Phi_4 := \Phi_{|A^{11}+\sum_s D^{s1}+2G_1|}$. On the other hand, it is clear that if any non-singular fiber of Φ_1 is isomorphic to E, then any non-singular fiber of Φ_4 is isomorphic to F by (1.13) since $A^{11} \cup \bigcup_{s=2}^4 (D^{1s} \cup D^{s1})$ is extendable to a double Kummer pencil divisor. □

§ 4. A minimal complete set of representatives of $\mathcal{J}_m/\mathrm{Aut}(X)$ $(m=4, \cdots, 11)$.

Lemma (4.1). *For a fixed ordered pair (i, j, k, p, q, r) where $\{i, j, k\} = \{p, q, r\} = \{2, 3, 4\}$, there exists a unique nodal curve R_{ijkpqr} such that R_{ijkpqr} meets both E_1 and F_1 and does not meet any C_{ns} $(1 \leqq n, s \leqq 4)$ except for C_{ip}, C_{jq} and C_{kr}. Moreover R_{ijkpqr} is characterized in S_X by the following equality.*

$$r_{ijkpqr} = e+f-c_{ip}-c_{jq}-c_{kr}.$$

Proof. The curve M^{44} in (2.13) satisfies the condition on R_{ijkpqr} if we put $H_4=F_1$, $G_4=E_1$, $H_1=F_i$, $G_2=E_p$, $H_2=F_j$, $G_1=E_q$, $H_3=F_k$ and $G_3=E_r$. Let us show the uniqueness of R_{ijkpqr}. Put $r_{ijkpqr}=ae+bf+\sum_{n,s} x_{ns}c_{ns}$ where a, b, $x_{ns} \in \boldsymbol{Q}$. By the condition on R_{ijkpqr} and $R_{ijkpqr}^2=-2$, and (0.2) (3), we get $r_{ijkpqr}=\pm(e+f-c_{ip}-c_{jq}-c_{kr})$. Since $R_{ijkpqr} \cdot E \geqq 0$, we have $r_{ijkpqr}= e+f-c_{ip}-c_{jq}-c_{kr}$. Hence by (1.5), R_{ijkpqr} is unique. □

Theorem (4.2). (IV) *Put $\varphi_i^{(4)}=\Phi_{|\Theta_i^{(4)}|}$ $(i=1, 2)$ where $\Theta_1^{(4)}=2F_1+C_{11}+C_{12}+C_{13}+C_{14}$, $\Phi_2^{(4)}=2E_1+C_{11}+C_{21}+C_{31}+C_{41}$. Then $\{\varphi_1^{(4)}, \varphi_2^{(4)}\}$ is an* M.S.R. *of both $\mathcal{J}_4/\mathrm{Aut}_N(X)$ and $\mathcal{J}_4/\mathrm{Aut}(X)$.*

(V) *Put* $\varphi_{ip}^{(5)}=\Phi_{|\Theta_{ip}^{(5)}|}$ *where*

$$\Theta_{ip}^{(5)}=C_{k1}+C_{j1}+C_{1q}+C_{1r}+2(E_1+C_{i1}+F_i+C_{ip}+E_p+C_{1p}+F_1) \quad \text{and} \quad 2\leqq i,\ p\leqq 4.$$

(1) *The set* $\{\varphi_{ip}^{(5)}\}_{2\leqq i,p\leqq 4}$ *is an* S.R. (*a non-minimal set of representatives*) *of* $\mathcal{J}_5/\mathrm{Aut}_N(X)$.

(2) *The set* $\{\varphi_{22}^{(5)}\}$ *is an* M.S.R. *of both* $\mathcal{J}_5/\mathrm{Aut}_N(X)$ *and* $\mathcal{J}_5/\mathrm{Aut}(X)$.

(VI) *Put* $\varphi_{ip}^{(6)}=\Phi_{|\Theta_{ip}^{(6)}|}$ *where*

$$\Theta_{ip}^{(6)}=C_{k1}+C_{j1}+C_{1q}+C_{1r}+2(E_1+C_{11}+F_1) \quad \text{and} \quad 2\leqq i,\ p\leqq 4.$$

(1) *The set* $\{\varphi_{ip}^{(6)}\}_{2\leqq i,p\leqq 4}$ *is an* M.S.R. *of* $\mathcal{J}_6/\mathrm{Aut}_N(X)$.

(2) *An* M.S.R. *of* $\mathcal{J}_6/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ip}:=\varphi_{ip}^{(6)}$.

Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_6/\mathrm{Aut}(X)$	φ_{22} φ_{32}	φ_{i2} $i=2,3,4$	φ_{ip} $i=2,3$ $p=2,3,4$	φ_{ip} $i=2,3,4$ $p=2,3,4$

(VII) *Put* $\varphi_{ijp}^{(7)}=\Phi_{|\Theta_{ijp}^{(7)}|}$ *where*

$$\Theta_{ijp}^{(7)}=C_{ip}+C_{kp}+C_{j1}+C_{k1}+2(E_p+C_{1p}+F_1+C_{11}+E_1) \quad \text{and} \quad 2\leqq i\neq j\leqq 4,\ 2\leqq p\leqq 4.$$

(1) *The set* $\{\varphi_{ijp}^{(7)}\}_{2\leqq i\neq j\leqq 4,2\leqq p\leqq 4}$ *is an* S.R. *of* $\mathcal{J}_7/\mathrm{Aut}_N(X)$.

(2) *The set* $\{\varphi_{ijp}^{(7)}\}_{2\leqq i<j\leqq 4,2\leqq p\leqq 4}$ *is an* M.S.R. *of* $\mathcal{J}_7/\mathrm{Aut}_N(X)$.

(3) *An* M.S.R. *of* $\mathcal{J}_7/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ijp}:=\varphi_{ijp}^{(7)}$.

Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_7/\mathrm{Aut}(X)$	φ_{342} φ_{343}	φ_{34p} $p=2,3,4$	φ_{ij2} φ_{ij3} $2\leqq i<j\leqq 4$	φ_{ijp} $2\leqq i<j\leqq 4$ $p=2,3,4$

(VIII) *Put* $\varphi_{ijpq}^{(8)}=\Phi_{|\Theta_{ijpq}^{(8)}|}$ *where*

$$\Theta_{ijpq}^{(8)}=C_{jp}+2E_p+3C_{1p}+4F_1+3C_{11}+2E_1+C_{i1}+2C_{1q} \quad \text{and} \quad 2\leqq i\neq j\leqq 4,\ 2\leqq p\neq q\leqq 4.$$

(1) *The set* $\{\varphi_{ijpq}^{(8)}\}_{2\leqq i\neq j\leqq 4,2\leqq p\neq q\leqq 4}$ *is an* S.R *of* $\mathcal{J}_8/\mathrm{Aut}_N(X)$.

(2) *The set* $\{\varphi_{ij23}^{(8)}\}_{2\leqq i\neq j\leqq 4}$ *is an* M.S.R. *of* $\mathcal{J}_8/\mathrm{Aut}_N(X)$.

(3) *An* M.S.R. *of* $\mathcal{J}_8/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ij23}:=\varphi_{ij23}^{(8)}$.

Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_8/\mathrm{Aut}(X)$	φ_{2323}	φ_{2323} φ_{2423}	φ_{ij23} $2\leqq i<j\leqq 4$	φ_{ij23} $2\leqq i\neq j\leqq 4$

(IX) *Put* $\varphi_{ijp}^{(9)}=\Phi_{|\Theta_{ijp}^{(9)}|}$ *where*

$$\Theta_{ijp}^{(9)}=C_{jp}+2E_p+3C_{1p}+4F_1+5C_{11}+6F_1+3C_{k1}+4C_{i1}+2F_i \quad \text{and}$$
$$2\leqq i\neq j\leqq 4,\ 2\leqq p\leqq 4.$$

(1) *The set* $\{\varphi_{ijp}^{(9)}\}_{2\leq i\neq j\leq 4,\,2\leq p\leq 4}$ *is an* S.R. *of* $\mathcal{J}_9/\mathrm{Aut}_N(X)$.
(2) *The set* $\{\varphi_{223}^{(9)}\}$ *is an* M.S.R. *of both* $\mathcal{J}_9/\mathrm{Aut}_N(X)$ *and* $\mathcal{J}_9/\mathrm{Aut}(X)$.

(X) *Put* $\varphi_{ijkpqr}^{(10)}=\Phi_{|\Theta_{ijkpqr}^{(10)}|}$ *where*

$$\Theta_{ijkpqr}^{(10)}=C_{iq}+C_{1q}+C_{11}+R_{ijkpqr}+2(E_q+C_{kq}+F_k+C_{kp}+E_p+C_{jp}+F_j+C_{j1}+E_1)$$
$$\text{and } \{i,j,k\}=\{p,q,r\}=\{2,3,4\}.$$

(1) *The set* $\{\varphi_{ijkpqr}^{(10)}\}_{\{i,j,k\}=\{p,q,r\}}$ *is an* S.R. *of* $\mathcal{J}_{10}/\mathrm{Aut}_N(X)$.
(2) *The set* $\{\varphi_{ijk234}^{(10)}\}_{\{i,j,k\}=\{2,3,4\}}$ *is an* M.S.R. *of* $\mathcal{J}_{10}/\mathrm{Aut}_N(X)$.
(3) *An* M.S.R. *of* $\mathcal{J}_{10}/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ijk234}:=\varphi_{ijk234}^{(10)}$.

Type of X	(i)	(ii)	(iii)	(iv)
M.S.R. of $\mathcal{J}_{10}/\mathrm{Aut}(X)$	φ_{234234}	φ_{234234} φ_{324234}	φ_{234234} φ_{324234} φ_{423234}	φ_{ijk234} $\{i,j,k\}$ $=\{2,3,4\}$

(XI) *Put* $\varphi_{ijkpqr}^{(11)}=\Phi_{|\Theta_{ijkpqr}^{(11)}|}$ *where*

$$\Theta_{ijkpqr}^{(11)}=C_{i1}+C_{iq}+C_{11}+R_{ijkpqr}+2(F_i+C_{ir}+E_r+C_{1r}+F_1) \text{ and}$$
$$\{i,j,k\}=\{p,q,r\}=\{2,3,4\}.$$

(1) *The set* $\{\varphi_{ijkpqr}^{(11)}\}$ *is an* S.R. *of* $\mathcal{J}_{11}/\mathrm{Aut}_N(X)$.
(2) *The set* $\{\varphi_{ijkpqr}^{(11)}\}_{2\leq i<k\leq 4,\,2\leq p<r\leq 4}$ *is an* M.S.R. *of* $\mathcal{J}_{11}/\mathrm{Aut}_N(X)$.
(3) *An* M.S.R. *of* $\mathcal{J}_{11}/\mathrm{Aut}(X)$ *is given as follows where* $\varphi_{ijkpqr}:=\varphi_{ijkpqr}^{(11)}$.

Type of X	(i)	(ii)	(iii)		(iv)
M.S.R. of $\mathcal{J}_{11}/\mathrm{Aut}(X)$	φ_{234234} φ_{324324}	φ_{234234} φ_{324324} φ_{243243}	φ_{234234} φ_{324234} φ_{234324}	φ_{324324} φ_{234243} φ_{324243}	φ_{ijkpqr} $2\leqq i<k\leqq 4$ $2\leqq p<r\leqq 4$

COROLLARY (4.3). *For each* $\mathcal{J}_m$, $\#(\mathcal{J}_m/\mathrm{Aut}(X))$ (*the number of non-isomorphic elements*) *is as follows.*

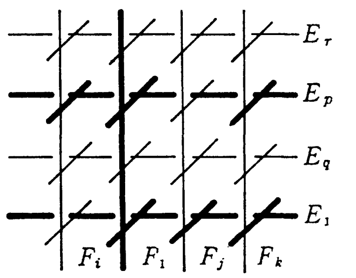
Type	$\mathcal{J}_1$	$\mathcal{J}_2$	$\mathcal{J}_3$	$\mathcal{J}_4$	$\mathcal{J}_5$	$\mathcal{J}_6$	$\mathcal{J}_7$	$\mathcal{J}_8$	$\mathcal{J}_9$	$\mathcal{J}_{10}$	$\mathcal{J}_{11}$	Total
(i)	2	1	1	2	1	2	2	1	1	1	2	16
(ii)	3	2	1	2	1	3	3	2	1	2	3	23
(iii)	6	3	1	2	1	6	6	3	1	3	6	38
(iv)	9	6	1	2	1	9	9	6	1	6	9	59

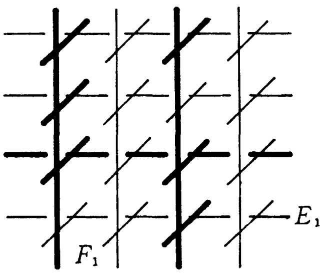
PROOF. We give a proof for (VII) and (X). For other cases, we only mention key claims because the verification of them is similar.

PROOF OF (VII). Obviously, we have $\varphi_{ijp}^{(7)} \in \mathcal{J}_7$. First we prove (1). Let $\varphi = \Phi_{|\Theta|}$ be an element of $\mathcal{J}_7$. We may assume that Θ is of type I_4^* and that Θ can be represented as follows:

$$\Theta = D^{13} + D^{43} + 2(G_3 + D^{23} + H_2 + D^{21} + G_1) + D^{31} + D^{41}.$$

Then H_1, H_3 are sections and H_4 is a 2-section of φ. By a similar method in the proof of Theorem (3.1), we see easily that Θ is extendable (to a double Kummer pencil divisor). Hence there exists $f \in \mathrm{Aut}_N(X)$ such that $\mathrm{Supp}\, f(\Theta) \subset K_{\mathrm{nat}}$. By the way, by (1.7), for any $h \in \mathrm{Aut}(X)$, either $h(\bigcup H_n) = \bigcup E_n$ and $h(\bigcup G_n) = \bigcup F_n$ or $h(\bigcup G_n) = \bigcup E_n$ and $h(\bigcup H_n) = \bigcup F_n$ hold. Then (if necessary, composing a suitable element of $\mathrm{Aut}_N(X; \bigcup_{n,s} C_{ns})$) we see that $f(\Theta)$ becomes either (a) or (b) for some $f \in \mathrm{Aut}_N(X)$:

(a)


(b)


Assume that $f(\Theta)$ is of type (b). Then, by composing a suitable automorphism g of X, constructed in the corollary (3.3), we see that $g \circ f(\Theta)$ is of type (a). Therefore (1) is proved.

Next we prove (2). It is sufficient to show the following.

CLAIM (4.4). *The fibrations* $\varphi_{ijp}^{(7)}$ *and* $\varphi_{i'j'p'}^{(7)}$ *are in the same orbit of* $\mathcal{J}_7/\mathrm{Aut}_N(X)$ *if and only if* $p = p'$, $\{i, j\} = \{i', j'\}$ *hold.*

PROOF OF (4.4). *If part:* Choose $g \in \mathrm{Aut}_N(X; \bigcup_{n,s} C_{ns})$ such that

$$E_p \longleftrightarrow E_1, \quad E_q \longleftrightarrow E_r, \quad \text{and} \quad F_1 \longleftrightarrow F_l \qquad (l = 1, \cdots, 4, \{p, q, r\} = \{2, 3, 4\}).$$

Then we have $g(\Theta^{(7)}_{ijp})=\Theta^{(7)}_{jip}$.

Only if part: Since $\Theta^{(7)}_{ijp}$ is a unique singular fiber of $\varphi^{(7)}_{ijp}$ of type I_4^*, $f(\Theta^{(7)}_{ijp})=\Theta^{(7)}_{i'j'p'}$ holds for some $f\in \mathrm{Aut}_N(X)$. Then we easily see that $f(C_{11}\cup C_{1p}\cup C_{k1}\cup C_{kp})=C_{11}\cup C_{1p'}\cup C_{k'1}\cup C_{k'p'}$. Hence, by (1.10), we have $C_{11}+C_{1p}+C_{k1}+C_{kp}+C_{11}+C_{1p'}+C_{k'1}+C_{k'p'}\equiv 0 \pmod{2\cdot S_X}$. Therefore, by (1.14), the claim holds. □

By the same method as in (3.1), we immediately see that (3) also holds.

Proof of (X). Obviously we have $\varphi^{(10)}_{ijkpqr}\in\mathcal{J}_{10}$. Let $\varphi=\Phi_{|\Theta|}$ be an element of $\mathcal{J}_{10}$. We may assume that Θ is of type I_8^* and represented as follows:

$$\Theta = D^{11}+Q^{11}+D^{13}+D^{23}+2(G_1+D^{31}+H_3+D^{32}+G_2+D^{42}+H_4+D^{43}+G_3).$$

Let us consider the Jacobian fibration $\varphi'=\Phi_{|D^{11}+Q^{11}+G_1+H_1|}\in\mathcal{J}_2$. Since D^{13} and D^{31} are sections, there exist nodal curves D^{24} and D^{34} such that another singular fiber of type I_{12} of φ' is $G_2+D^{32}+H_3+D^{34}+G_4+D^{24}+H_2+D^{23}+G_3+D^{43}+H_4+D^{42}$. By the way, since D^{13} is a section of φ' and $\varphi'\in\mathcal{J}_2$, there exist 6 disjoint sections D^{13}, D'^{12}, D'^{14}, D'^{21}, D'^{41} and D'^{31} as was seen in the proof (2.1) (2) for $\mathcal{J}_2$. Let us consider two elements σ and σ' of $M_{\varphi'}(X)$ such that $\sigma(D^{13})=D^{31}$, $\sigma'(D^{13})=D'^{31}$. By Cox and Zucker (loc. cit.), both σ and σ' are torsion elements of $M_{\varphi'}(X)$. Therefore $\sigma=\sigma'$ and $D^{31}=D'^{31}$ hold. So we can put $D'^{12}=D^{12}$, $D'^{14}=D^{14}$, $D'^{21}=D^{21}$, and $D'^{41}=D^{41}$. By (3.4), if any non-singular fiber of $\Phi_1:=\Phi_{|D^{11}+D^{12}+D^{13}+D^{14}+2H_1|}\in\mathcal{J}_4$ is isomorphic to E, any non-singular fiber of $\Phi_{|Q^{11}+D^{12}+D^{13}+D^{14}+2H_1|}$ is isomorphic to F. Thus, if necessary, changing the names of D^{11} and Q^{11}, we may assume that any non-singular fiber of Φ_1 is isomorphic to F. By Φ_1, $\Theta-Q^{11}$ is extended to a double Kummer pencil divisor $K_D=\bigcup_{1\le n\ne s\le 4}D^{ns}\cup D^{11}\cup D^{22}\cup D^{33}\cup D^{44}\cup B$. Then, by the assumption on Φ_1, there exists $f\in\mathrm{Aut}_N(X)$ such that $f(K_D)=K_{\mathrm{nat}}$, $f(D^{11})=C_{11}$, $f(\Theta-Q^{11})=\Theta^{(10)}_{ijkpqr}-R_{ijkpqr}$ for suitable (i, j, k, p, q, r) and $f(Q^{11})$ meets both E_1 and F_1 and does not meet any C_{ns} except for C_{ip}, C_{jq} and C_{kr}. Hence by (4.1), we have $f(Q^{11})=R_{ijkpqr}$ and (1) holds.

Next we prove (2). It is sufficient to show the following.

Claim (4.5). *Let $\mathfrak{S}_3$ be the permutation group of 3 letters 2, 3, 4. The fibrations $\varphi^{(10)}_{ijkpqr}$ and $\varphi^{(10)}_{i'j'k'p'q'r'}$ are in the same orbit of $\mathcal{J}_{10}/\mathrm{Aut}_N(X)$ if and only if $\begin{pmatrix} i & j & k \\ i' & j' & k'\end{pmatrix}=\begin{pmatrix} p & q & r \\ p' & q' & r'\end{pmatrix}$ holds as an element of $\mathfrak{S}_3$.*

Proof. *Only if part:* If $\varphi^{(10)}_{ijkpqr}$ and $\varphi^{(10)}_{i'j'k'p'q'r'}$ are in the same orbit of $\mathcal{J}_{10}/\mathrm{Aut}_N(X)$, $g(R_{ijkpqr}\cup C_{11})=R_{i'j'k'p'q'r'}\cup C_{11}$ holds for some $g\in\mathrm{Aut}_N(X)$. Then, by (1.10) and (1), we get $C_{ip}+C_{jq}+C_{kr}+C_{i'p'}+C_{j'q'}+C_{k'r'}\equiv 0 \pmod{2\cdot S_X}$. Hence only if part holds.

If part: It is sufficient to construct the following symplectic automorphisms: $\tau(\Theta_{ijkpqr}^{(10)})=\Theta_{jkiqrp}^{(10)}$, $\rho(\Theta_{ijkpqr}^{(10)})=\Theta_{kjirqp}^{(10)}$.

τ: Make a Jacobian fibration $\Phi_{|\Omega|}$ where $\Omega=C_{11}+R_{ijkpqr}+E_1+F_1$.

Then $\Phi_{|\Omega|}$ is in $\mathcal{G}_2$ and the other singular fiber of $\Phi_{|\Omega|}$ of type I_{12} is $\Omega'=F_i+C_{iq}+E_q+C_{kq}+F_k+C_{kp}+E_p+C_{jp}+F_j+C_{jr}+E_r+C_{ir}$ and C_{i1}, C_{j1}, C_{k1}, C_{1p}, C_{1q}, C_{1r} are sections. Take $\tau\in M_{\Phi_{|\Omega|}}$ such that $\tau(C_{j1})=C_{k1}$. Then by the group structures of Ω and Ω', we have

$$\tau(C_{11})=C_{11}, \quad \tau(C_{jp})=C_{kq}, \quad \tau(C_{kp})=C_{jq},$$
$$\tau(R_{ijkpqr})=R_{ijkpqr}=R_{jkiqpr}, \quad \tau(C_{kq})=C_{ir}, \quad \tau(C_{iq})=C_{jr}.$$

Since for the torsion element $\sigma\in M_{\Phi_{|\Omega|}}$ the equalities $\sigma(C_{1q})=C_{j1}$ and $\sigma(C_{1r})=C_{k1}$ hold, we have $\tau(C_{1q})=C_{1r}$. Hence $\tau(\Theta_{ijkpqr}^{(10)})=\Theta_{jkiqrp}^{(10)}$ holds for this τ.

ρ: Make a Jacobian fibration $\Phi_{|L|}$ where $L=C_{11}+R_{ijkpqr}+C_{iq}+C_{kq}+2(F_1+C_{1q}+E_q)$.

So we have $\Phi_{|L|}\in\mathcal{G}_8$, and F_i and F_k are sections of $\Phi_{|L|}$. Take $\rho\in M_{\Phi_{|L|}}(X)$ such that $\rho(F_i)=F_k$. Then, by a similar consideration as above, we see that $\rho(\Theta_{ijkpqr}^{(10)})=\Theta_{kjirqp}^{(10)}$ holds for this ρ. □

Since r_{ijkpqr} is explicitly represented in S_X, by the same method as in (3.1), we easily show (3).

Finally we mention key claims to find an M.S.R. of $\mathcal{G}_m/\mathrm{Aut}_N(X)$ from an S.R. of $\mathcal{G}_m/\mathrm{Aut}_N(X)$ for the other cases.

CLAIM (4.6). *All* $\varphi_{ip}^{(5)}$ *are in the same orbit of* $\mathcal{G}_5/\mathrm{Aut}_N(X)$.
(Make a Jacobian fibration in $\mathcal{G}_3$, and take suitable translation automorphisms of it.)

CLAIM (4.7). *If* $\varphi_{ip}^{(6)}$ *and* $\varphi_{i'p'}^{(6)}$ *are in the same orbit of* $\mathcal{G}_6/\mathrm{Aut}_N(X)$, *then* $i=i'$ *and* $p=p'$.

CLAIM (4.8). $\varphi_{ijpq}^{(8)}$ *and* $\varphi_{i'j'p'q'}^{(8)}$ *are in the same orbit of* $\mathcal{G}_8/\mathrm{Aut}_N(X)$ *if and only if the ordered pair* (i', j', p', q') *is one of the following six ordered pairs:*

$$(i, j, p, q), \quad (j, i, p, r), \quad (j, k, q, r), \quad (k, j, q, p), \quad (i, k, r, q), \quad (k, i, r, p).$$

(For if part, make a Jacobian fibration in $\mathcal{G}_1$, and take a suitable translation automorphism f of it. Then we have $f(\Theta_{ijpq}^{(8)})=\Theta_{ikrq}^{(8)}$. By taking a suitable $g\in\mathrm{Aut}_N(X; \bigcup_{n,s}C_{ns})$, we have $g(\Theta_{ijpq}^{(8)})=\Theta_{jipr}^{(8)}$.)

CLAIM (4.9). *All* $\varphi_{ijp}^{(9)}$ *are in the same orbit of* $\mathcal{G}_9/\mathrm{Aut}_N(X)$.
(Make a suitable Jacobian fibration in $\mathcal{G}_3$. Then $f(\Theta_{ijp}^{(9)})=\Theta_{ijq}^{(9)}$ holds for a suitable translation automorphism f of it. Make a suitable Jacobian fibration

in $\mathcal{J}_1$. Then the equalities $g(\Theta^{(9)}_{ij2})=\Theta^{(9)}_{ji2}$ and $h(\Theta^{(9)}_{ij2})=\Theta^{(9)}_{ik2}$ hold for suitable translation automorphisms g and h of it.)

CLAIM (4.10). *The fibrations* $\varphi^{(11)}_{ijkpqr}$ *and* $\varphi^{(11)}_{i'j'k'p'q'r'}$ *are in the same orbit of* $\mathcal{J}_{11}/\mathrm{Aut}_N(X)$ *if and only if* $j=j'$ *and* $q=q'$ *hold.*
(The other singular fiber of type I_4^* of $\varphi^{(11)}_{ijkpqr}$ is

$$\Gamma^{(11)}_{ijkpqr} = C_{j1}+S_{ijkpqr}+C_{k1}+C_{kq}+2(F_j+C_{jp}+E_p+C_{kp}+F_k).$$

Here S_{ijkpqr} is a nodal curve characterized by $s_{ijkpqr}=e+f-c_{1q}-c_{ip}-c_{kr}$.

If part: By taking a suitable element $\tau\in\mathrm{Aut}_N(X;\bigcup_{n,s}C_{ns})$, we have $\tau(\Theta^{(11)}_{ijkpqr})=\Gamma^{(11)}_{ijkrqp}$. By making a suitable Jacobian fibration in $\mathcal{J}_1$ and taking a suitable translation automorphism ρ of it, we have $\rho(\Theta^{(11)}_{ijkpqr})=\Theta^{(11)}_{kjirqp}$.)

As for $\mathcal{J}_4$, the statement is trivial since E and F are not mutually isogenous. This completes the proof. Q.E.D.

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