J. Math. Soc. Japan
Vol. 41, No. 4, 1989

Resolvent estimates at low frequencies and limiting amplitude principle for acoustic propagators

By Hideo TAMURA

(Received June 22, 1988)

Introduction.

In the present paper we study the low frequency behavior of resolvents for perturbed acoustic operators with perturbations decreasing slowly at infinity and, as an application, we prove the principle of limiting amplitude for such operators.

We work in the 3-dimensional space R_x^3, $x=(x_1, x_2, x_3)$, and consider the following equation:

$$(0.1) \qquad (\partial/\partial t)^2 w = a(x)^2 \rho(x) \nabla\cdot(1/\rho(x))\nabla w\,.$$

As is well known, this equation governs the propagation of acoustic waves in an inhomogeneous medium with a local speed of sound $a(x)>0$ and an equilibrium density $\rho(x)>0$ which vary with $x\in R_x^3$. We deal with equation (0.1) under a Hilbert space formulation. First we assume that:

$$(a.0) \qquad 1/c < a(x) < c\,,$$

$$(\rho.0) \qquad 1/c < \rho(x) < c$$

for some $c>1$ and

(ρ.1) $\rho(x)$ is of C^1-class with bounded derivatives.

We now define the acoustic operator L as

$$(0.2) \qquad L = -a(x)^2 \rho(x) \nabla\cdot(1/\rho(x))\nabla\,.$$

Under the above assumptions, the operator L is symmetric in the Hilbert space $L^2(R_x^3\,; E(x)dx)$ with $E=a(x)^{-2}\rho(x)^{-1}$ and it admits a unique self-adjoint realization. We denote by the same notation L this realization and by $R(z\,; L)$ the resolvent of $L\,; R(z\,; L)=(L-z)^{-1}$, $\mathrm{Im}\, z\neq 0$. As is easily seen, L is positive (zero is not an eigenvalue) and the domain of L is given by $D(L)=H^2(R_x^3)$, $H^s(R_x^3)$ being the Sobolev space of order s. We further assume that the inhomogeneous medium under consideration is homogeneous at infinity. (This assumption will be made clear below.) Under suitable assumptions on the behavior as $|x|\to\infty$ of $a(x)$ and $\rho(x)$, we know that L has no eigenvalues and that the boundary values $R(\lambda\pm i0\,;L)$, $\lambda>0$, of $R(\lambda\pm i\kappa\,;L)$ as $\kappa\to 0$ exist in an appropriate weighted L^2 space topology (limiting absorption principle). The aim of the present work is to study the behavior of $R(\lambda\pm i0\,;L)$ at low frequencies ($\lambda\to 0$) and to prove, as an application, the validity of the limiting amplitude principle for equation (0.1).

We shall formulate the obtained result precisely. The formulation requires several assumptions and notations. To describe these assumptions, we follow the standard multi-index notations. We make the following assumptions on the behavior as $|x|\to\infty$ of $a(x)$ and $\rho(x)$:

(a.1) There exists $a_0>0$ for which $a(x)$ is decomposed as

$$a(x)=a_0+a_1(x)+a_2(x),$$

where $a_1=O(|x|^{-1-\theta})$ and

$$\partial_x^\alpha a_2=O(|x|^{-|\alpha|\ \theta}),\qquad |\alpha|\leqq 1,\quad \text{as } |x|\to\infty$$

for some $\theta>0$.

(ρ.2) There exists $\rho_0>0$ for which

$$\partial_x^\alpha(\rho(x)-\rho_0)=O(|x|^{-|\alpha|-\theta}),\qquad |\alpha|\leqq 1,\quad \text{as } |x|\to\infty$$

for the same θ as above.

Throughout the entire discussion, the constant θ is fixed with the meaning ascribed above and we assume, without loss of generality, that $0<\theta<1/2$.

We require several notations to describe the obtained result. Let $L_\beta^2=L_\beta^2(R_x^3)$ be the weighted L^2 space defined by

$$L_\beta^2=\{f(x):\langle x\rangle^\beta f(x)\in L^2\},\qquad \langle x\rangle=(1+|x|^2)^{1/2},$$

with the norm

$$|f|_\beta^2=\int\langle x\rangle^{2\beta}|f(x)|^2dx,$$

the integration with no domain attached being taken over the whole space. Let $A: L_\alpha^2\to L_\beta^2$ be a bounded operator. We denote by $\|A\|_{\alpha\to\beta}$ the operator norm when considered as an operator from L_α^2 into L_β^2. If, in particular, $A: L^2\to L^2$ is considered as an operator from L^2 into itself, then its norm is denoted by the simplified notation $\|A\|$.

THEOREM 0. *Assume* (a.0)$\sim$(a.1) *and* (ρ.0)$\sim$(ρ.2). *Then:*

(i) *L has no eigenvalues.*

(ii) *There exist limits $R(\lambda\pm i0\,;L)f$, $\lambda>0$, with $f\in L_\beta^2$, $\beta>1/2$, of $R(\lambda\pm i\kappa\,;L)f$ as $\kappa\to 0$ in the strong topology of $L_{-\beta}^2$.*

It is not the aim here to prove this theorem. The non-existence of eigenvalues embedded in continuous spectrum has been studied by many authors.

See, for example, [2], [8], [13] and the references there. Statement (i) may be proved by use of the method developed in these works, but for completeness, we will give a brief sketch of the proof of (i) in Appendix. The method is based on the idea of Froese and Herbst [4]. Once (i) is established, statement (ii) follows from the general theorem (Theorem 30.2.10) of Hörmander [5].

The aim here is to prove the following

THEOREM 1. *Assume* (a.0)~(a.1) *and* (ρ.0)~(ρ.2). *Let* $\alpha>1$ *and* $\beta>1/2$. *Then there exists* d, $0<d<1/2$, *such that*

$$\|R(\lambda\pm i0\,;\,L)\|_{\beta\to-\alpha}=O(\lambda^{-d}),\qquad \lambda\to 0.$$

REMARK. We can also prove that

$$\|R(\lambda\pm i0\,;\,L)\|_{\alpha\to-\alpha}=O(1),\qquad \lambda\to 0, \tag{0.3}$$

for $\alpha>1$, but we do not do this here. In proving the principle of limiting amplitude, the formulation as above is convenient and the fact $d<1/2$ is important.

The behavior of resolvents at low frequencies or at low energies plays a basic role in the study on the asymptotic behavior as t (time)$\to\pm\infty$ of solutions to the associated non-stationary problems. Thus, such a behavior has been studied by many authors. For example, Jensen-Kato [6] studied the Schrödinger operator $-\Delta+V$ with $V(x)$ having the decaying property $V=O(|x|^{-\gamma})$, $\gamma>2$, as $|x|\to\infty$. If we make the transformation $w\to v=\rho^{-1/2}w$, then equation (0.1) can be put into the form

$$(1/a(x))^2(\partial/\partial t)^2 v=\Delta v-V_\rho v,$$

where

$$V_\rho(x)=(3/4)\rho^{-2}|\nabla\rho|^2-(1/2)\rho^{-1}\Delta\rho.$$

If we assume the additional assumption $\partial_x^\alpha(\rho(x)-\rho_0)=O(|x|^{-2-\theta})$ for $|\alpha|=2$, then $V_\rho=O(|x|^{-2-\theta})$ as $|x|\to\infty$. Hence, the bound (0.3) can be proved by making use of the same argument as in [6], if $a(x)$ satisfies $a(x)-a_0=O(|x|^{-\gamma})$, $\gamma>2$, as $|x|\to\infty$. It should be noted here that the transformation as above is not used in the proof and hence the main theorem can be easily extended to general self-adjoint elliptic operators of the form

$$P=-a(x)^{-2}\sum_{1\leq j,\,k\leq 3}(\partial/\partial x_j)a_{jk}(x)(\partial/\partial x_k),$$

if, for example, the coefficients $a_{jk}(x)$ satisfy $\partial_x^\alpha(a_{jk}(x)-\delta_{jk})=O(|x|^{-|\alpha|-\theta})$, $|\alpha|\leq 1$, as $|x|\to\infty$, δ_{jk} being the Kronecker delta. Murata [10] also studied the low energy behavior of resolvents for general elliptic (not necessarily self-adjoint) operators, including the n-dimensional case, $n\geqq 1$. The results strongly depend on the space dimension. For an operator P of the above form, the bound (0.3) follows as a special case of the general results obtained by Murata [**10**], if, roughly speaking, the coefficients $a(x)$ and $a_{jk}(x)$ have the strong decaying property with rate $\gamma>2$. In general, the low energy behavior of resolvents depends heavily on the fact whether the operator under consideration has a zero energy resonance or bound state or not. The operator L which we consider here does not have such a resonance and bound state. This makes it possible for us to study the low frequency behavior of resolvents for a class of perturbations decreasing slowly at infinity.

The proof of Theorem 1 is done by a operator theoretical approach based on the commutator method. This method was first developed by Mourre [**9**] to prove the principle of limiting absorption for 3-body Schrödinger operators and its application has been extended by [**4**], [**7**] and [**11**], etc. to various spectral problems of N-body Schrödinger operators. In these works, it has been used to prove the principle of limiting absorption ([**11**]), to prove the non-existence of positive eigenvalues ([**4**]) and to study the resolvent smoothness as a function of energy ([**7**]). Through the present work, this remarkable method will be seen to be useful also to the low frequency analysis of resolvents.

As stated above, the resolvent behavior for low frequencies is important to the study on the time asymptotics of solutions of the associated non-stationary problems. As an application of the main theorem, we here study the asymptotic behavior as $t\to\infty$ of the solution to the following Cauchy problem:

$$(\partial/\partial t)^2 w+Lw=\exp(-it\sqrt{\omega})f, \qquad \omega>0,$$

with initial conditions $w=(\partial/\partial t)w=0$ at $t=0$. In Section 5 we will prove that for $f\in L^2_\beta$, $\beta>1/2$, the solution $w=w(t, x)$ behaves like

$$w=\exp(-it\sqrt{\omega})R(\omega+i0\,;L)f+o(1), \qquad t\to\infty,$$

in the strong topology of $L^2_{-\alpha}$, $\alpha>1$. This implies the validity of the limiting amplitude principle for L. This principle has been proved by many authors for various scattering problems, including the case of exterior boundary value problems. For related results, see, for example, [**1**], [**3**] and references there. The second aim of the present work is to show that such a principle holds true for a wide class of perturbations decreasing slowly at infinity.

§1. Reduction to main lemmas.

Throughout the entire discussion, all the assumptions $(a.0)\sim(a.1)$ and $(\rho.0)\sim(\rho.2)$ are always assumed to be satisfied. It is convenient to work in the L^2 space rather than in the original space $L^2(R^3_x\,;E(x)dx)$ with $E=a(x)^{-2}\rho(x)^{-1}$. We start by rewriting the statement of the main theorem in the form adapted to the L^2 space formalism.

Let a_0 and ρ_0 be as in assumptions (a.1) and (ρ.2), respectively. For brevity, these constants are assumed to be normalized as $a_0=\rho_0=1$. Set $E(x)=a(x)^{-2}\rho(x)^{-1}$ again and define the positive operator H acting on L^2 by

(1.1) $$H=-\nabla\cdot(1/\rho(x))\nabla.$$

Then we have $R(z;L)=Q(z;H)E$, $\operatorname{Im} z\neq 0$, where $Q(z;H)=(H-zE)^{-1}$. Therefore, the main theorem is obtained as an immediate consequence of the following

LEMMA 1.1. *Let the pair* (α, β) *be as in Theorem* 1. *Then*

$$\|Q(\lambda\pm i0;H)\|_{\beta\to-\alpha}=O(\lambda^{-d}),\qquad \lambda\to 0,$$

for some d, $0<d<1/2$.

Let $E(x)$ be as above. By assumption (a.1), we can decompose $E(x)$ as $E=E_0(x)+V_0(x)$ in such a way that:

(1.2) $$V_0=O(|x|^{-1-\theta}),\qquad |x|\to\infty;$$

(1.3) $$\sum_{0\le|\alpha|\le 1}|x|^{|\alpha|}|\partial_x^\alpha(E_0-1)|=O(|x|^{-\theta});$$

(1.4) $$\sum_{0\le|\alpha|\le 1}\langle x\rangle^{|\alpha|}|\partial_x^\alpha(E_0-1)|\leqq\delta_0,\qquad x\in R_x^3,$$

for $\delta_0>0$ small enough, δ_0 being fixed throughout.

LEMMA 1.2. *Let* $Q_0(z;H)=(H-zE_0)^{-1}$, $\operatorname{Im} z\neq 0$. *Then*

(1.5) $$\|Q_0(\lambda\pm i0;H)\|_{\alpha\to-\alpha}=O(1),\qquad \lambda\to 0,$$

for any $\alpha>1$.

We shall show that Lemma 1.1 follows from Lemma 1.2. Thus, the proof of the main theorem is reduced to that of Lemma 1.2.

PROOF OF LEMMA 1.1. We assume (1.5). Let $\alpha>1$ and $\beta>1/2$. We assert that:

(1.6) $$\|Q_0(\lambda\pm i0;H)\|_{\beta\to-\alpha}=O(\lambda^{-1/2}),$$

(1.7) $$\|Q_0(\lambda\pm i0;H)\|_{\beta\to-\beta}=O(\lambda^{-1}).$$

We first complete the proof of the lemma, accepting these assertions as proved.

Let σ, $1/2<\sigma<(1+\theta)/2$, be fixed arbitrarily. Take α and β close enough to 1 and 1/2, respectively. Then, by interpolation, $\|Q_0(\lambda\pm i0;H)\|_{\sigma\to-\sigma}=O(\lambda^{-\gamma})$ for any γ, $2-2\sigma<\gamma<1$. This, togther with (1.2), implies that $\mathrm{Id}-\lambda V_0Q_0(\lambda\pm i0;H)$: $L_\sigma^2\to L_\sigma^2$ is invertible for $\lambda>0$ small enough, Id being the identity operator, and

$$\|(\mathrm{Id}-\lambda V_0Q_0(\lambda\pm i0;H))^{-1}\|_{\sigma\to\sigma}=O(1),\qquad \lambda\to 0.$$

By interpolation again, we have

$$\|Q_0(\lambda\pm i0\,;H)\|_{\sigma\to-\alpha}=O(\lambda^{-\gamma})$$

for any γ, $1-\sigma<\gamma<1/2$. Since $Q(\lambda\pm i0\,;H)$ is represented as

$$Q(\lambda\pm i0\,;H)=Q_0(\lambda\pm i0\,;H)(\mathrm{Id}-\lambda V_0Q_0(\lambda\pm i0\,;H))^{-1},$$

the lemma follows immediately.

We now prove the assertions (1.6) and (1.7). We consider the "+" case only. Let $u=Q_0(\lambda-i\kappa\,;H)f$ with $f\in L^2$. Then u satisfies

$$Hu-\lambda E_0u+i\kappa E_0u=f. \tag{1.8}$$

Let $\langle\, ,\rangle$ denote the L^2 scalar product. Let $\phi(x)$ be a real-valued smooth function with bounded derivatives. We take the L^2 scalar product of ϕu with equation (1.8). Then we have

$$\langle\phi(1/\rho)\nabla u,\nabla u\rangle-(1/2)\langle(\nabla\cdot((1/\rho)\nabla\phi))u,u\rangle=\lambda\langle\phi E_0u,u\rangle+\mathrm{Re}\langle f,\phi u\rangle. \tag{1.9}$$

In the argument below, we use this identity with $\phi=\langle x\rangle^{-\gamma}$, $\gamma>0$.

We require another identity to prove (1.6) and (1.7). Let $\delta(<\theta)$, $0<\delta\ll1$, be fixed arbitrarily. Let $\chi(x)=\{\chi^j\}_{1\le j\le 3}$ be a real smooth vector field such that $\chi^j=(1-|x|^{-\delta})x_j/|x|$ for $|x|>R$, $R\gg1$. We write $\partial_j=\partial/\partial x_j$ and use the summation convention. By an easy calculation,

$$\mathrm{Re}\,\partial_k\chi^j\partial_ku\partial_j\bar u\geqq C_\delta\langle x\rangle^{-1-\delta}|\nabla u|^2,\qquad |x|>R. \tag{1.10}$$

We take the L^2 scalar product of $\chi^j\partial_ju+(1/2)(\partial_j\chi^j)u$ with equation (1.8). Then we have

$$\begin{aligned}&\mathrm{Re}\langle(1/\rho)(\partial_k\chi^j)\partial_ku,\partial_ju\rangle-(1/2)\langle\chi^j\partial_j(1/\rho)\partial_ku,\partial_ku\rangle\\&\quad-(1/4)\langle(\partial_k((1/\rho)\partial_k\partial_j\chi^j)u,u\rangle+(\lambda/2)\langle(\chi^j(\partial_jE_0))u,u\rangle\\&\quad=\mathrm{Re}\langle f,\chi^j\partial_ju+(1/2)(\partial_j\chi^j)u\rangle+\kappa\,\mathrm{Im}\langle E_0u,\chi^j\partial_ju\rangle.\end{aligned} \tag{1.11}$$

By elliptic estimate, the second term on the right side is dominated by $C\kappa\{|u|_0^2+|f|_0^2\}$, $|\cdot|_\beta$ being the norm of L^2_β.

We now set $\alpha=1+\delta$ and $\beta=(1+\delta)/2$. Assume that $f\in L^2_\alpha$. Then, by (1.5), it follows from (1.9) with $\phi=\langle x\rangle^{-2\alpha}$ that $|\nabla u|_{-\alpha}\leqq C|f|_\alpha$. Since

$$\kappa\langle E_0u,u\rangle=\mathrm{Im}\langle f,u\rangle\leqq|f|_\alpha|u|_{-\alpha}$$

and since $\beta<(1+\theta)/2$ by the choice of δ, we have by (1.10) and (1.11) that

$$|\nabla u|^2_{-\beta}\leqq C\{\lambda|u|^2_{-(1+\theta)/2}+|f|^2_\alpha\}.$$

We again use (1.9) with $\phi=\langle x\rangle^{-2\beta}$ to obtain

$$\lambda|u|^2_{-\beta}\leqq C\{|\nabla u|^2_{-\beta}+|f|^2_\alpha\}.$$

Thus we have $\lambda|u|^2_{-\beta}\leqq C|f|^2_\alpha$. This implies that

$$\|Q_0(\lambda-i\kappa\,;H)\|_{\alpha\to-\beta}=O(\lambda^{-1/2})$$

uniformly in $\kappa>0$ small enough and hence (1.6) is proved.

To prove (1.7), we repeat the same argument as above. Assume that $f\in L^2_\beta$, $\beta=(1+\delta)/2$. Then, by (1.6), it follows from (1.9) with $\phi=\langle x\rangle^{-2\alpha}$, $\alpha=1+\delta$, that $|\nabla u|^2_{-\alpha}\leqq C\lambda^{-1}|f|^2_\beta$ and also we have by (1.11) that

$$|\nabla u|^2_{-\beta}\leqq C\{\lambda|u|^2_{-(1+\theta)/2}+\lambda^{-1}|f|^2_\beta\}.$$

We use again (1.9) with $\phi=\langle x\rangle^{-2\beta}$ to obtain

$$\lambda|u|^2_{-\beta}\leqq C\{|\nabla u|^2_{-\beta}+\lambda^{-1}|f|^2_\beta\}.$$

This, together with the above estimate, proves (1.7) and the proof of the lemma is now complete. □

§ 2. Bounds at low frequencies.

The proof of Lemma 1.2 is based on the two lemmas below (Lemmas 2.2 and 2.3). In this section we prove Lemma 1.2, accepting these lemmas as proved.

PROOF OF LEMMA 1.2. We give the proof for the "+" case only. The proof is divided into several steps.

(1) Let H be defined by (1.1). We define

$$H(\lambda)=H-\lambda(E_0-1),\qquad \lambda>0, \tag{2.1}$$

and denote by $R(z\,;H(\lambda))$, $\operatorname{Im} z\neq 0$, the resolvent of $H(\lambda)$; $R(z\,;H(\lambda))=(H(\lambda)-z)^{-1}$. By Theorem 0, λ is not an eigenvalue of $H(\lambda)$ and hence by the general theorem (Theorem 30.2.10) due to Hörmander [**5**], there exist the boundary values $R(\lambda\pm i0\,;H(\lambda))$ of $R(\lambda\pm i\kappa\,;H(\lambda))$ as $\kappa\to 0$;

$$R(\lambda\pm i0\,;H(\lambda))f=s-\lim_{\kappa\downarrow 0}R(\lambda\pm i\kappa\,;H(\lambda))f,\qquad f\in L^2_\beta,$$

in the strong topology of $L^2_{-\beta}$, $\beta>1/2$.

LEMMA 2.1. *Let $Q_0(\lambda+i0\,;H)$ be as in Lemma* 1.2. *Then*

$$Q_0(\lambda+i0\,;H)=R(\lambda+i0\,;H(\lambda)).$$

REMARK. In the proof below, it is also proved that the existence of boundary values $Q_0(\lambda\pm i0\,;H)$ follows from that of $R(\lambda\pm i0\,;H(\lambda))$.

Before proving the lemma above, we introduce the new notation. X_β, $\beta\geqq 0$, denotes the multiplication operator by $\langle x\rangle^{-\beta}$;

$$X_\beta:\ \phi(x)\longrightarrow\langle x\rangle^{-\beta}\phi(x).$$

PROOF OF LEMMA 2.1. Write $Q_0(\kappa)$ and $R(\kappa)$ for $Q_0(\lambda+i\kappa\,;H)$ and $R(\lambda+i\kappa\,;H(\lambda))$, respectively. Then, for $\beta>1/2$, we have

$$\|R(\kappa)X_\beta\| = (2\kappa)^{-1/2}\|X_\beta(R(\kappa)-R(\kappa)^*)X_\beta\|^{1/2} = O(\kappa^{-1/2})$$

as $\kappa\to 0$. Similarly

$$\|X_\beta Q_0(\kappa)\| = \|X_\beta Q_0(\kappa)X_\beta\|^{1/2}O(\kappa^{-1/2}).$$

By interpolation, it follows that $\|X_\theta R(\kappa)X_\beta\|=O(\kappa^{-\gamma})$ for any γ, $1/2-\theta<\gamma<1/2$. We now use the resolvent identity

$$Q_0(\kappa)-R(\kappa) = i\kappa Q_0(\kappa)(E_0-1)R(\kappa).$$

Since $E_0-1=O(|x|^{-\theta})$ as $|x|\to\infty$, we obtain that

$$\|X_\beta(Q_0(\kappa)-R(\kappa))X_\beta\| = \|X_\beta Q_0(\kappa)X_\beta\|^{1/2}O(\kappa^\nu)$$

for $\nu=1/2-\gamma>0$. This implies that $\|X_\beta Q_0(\kappa)X_\beta\|=O(1)$ as $\kappa\to 0$ and hence we have

$$\|X_\beta(Q_0(\kappa)-R(\kappa))X_\beta\| = O(\kappa^\nu),$$

which completes the proof. □

(2) We again fix δ_0 sufficiently small and choose σ, $0<\sigma<\theta$, arbitrarily. Then, by assumption (ρ.2), we can decompose $\rho(x)$ as $\rho=\rho_1(x)+\rho_2(x)$ so that: ρ_2 has compact support and ρ_1 satisfies

$$\sum_{0\le|\alpha|\le 1}\langle x\rangle^{|\alpha|+\sigma}|\partial_x^\alpha(\rho_1-1)| \leqq \delta_0, \qquad x\in R_x^3. \tag{2.2}$$

We now define

$$H_1 = -\nabla\cdot(1/\rho_1(x))\nabla \tag{2.3}$$

and set

$$H_1(\lambda) = H_1-\lambda(E_0-1), \qquad \lambda>0.$$

LEMMA 2.2.

$$\|X_1R(\lambda+i0\,;H_1(\lambda))X_1\| = O(1), \qquad \lambda\to 0.$$

LEMMA 2.3. *For $\alpha>1$, there exists $\gamma=\gamma(\alpha)$, $0<\gamma<1$, such that*

$$\|X_\alpha(R(\lambda+i0\,;H_1(\lambda))-R(\mu+i0\,;H_1(\mu)))X_\alpha\| = O(\lambda^\gamma)+O(\mu^\gamma).$$

The proof of the above lemmas occupies the main body of the proof of Lemma 1.2 and hence of Theorem 1. We proceed with the argument, accepting these lemmas as proved. The proof of Lemmas 2.2 and 2.3 will be given in Sections 3 and 4, respectively.

(3) LEMMA 2.4. *One has the following statements:* (i) *The inverse H_1^{-1} exists as an operator from L_1^2 into L_{-1}^2;*

(ii) *As $\lambda\to 0$, $R(\lambda+i0\,;H_1(\lambda))$ is convergent to H_1^{-1} in the weak topology of L_{-1}^2.*

PROOF. (i) The uniqueness of solution $u\in L_{-1}^{2}$ to $H_{1}u=f$, $f\in L_{1}^{2}$, follows from the well known inequality

$$\int\langle x\rangle^{-2}|\phi(x)|^{2}dx\leqq 4\int|\nabla\phi(x)|^{2}dx. \tag{2.4}$$

Let $u_{\kappa}=R(i\kappa\,;H_{1})f$, $\kappa>0$, with $f\in L_{1}^{2}$. Then, by (2.4) again, $|u_{\kappa}|_{-1}=O(1)$, $\kappa\rightarrow 0$, and hence a subsequence of $\{u_{\kappa}\}$ is convergent to some $u_{0}\in L_{-1}^{2}$ as $\kappa\rightarrow 0$ in the weak topology of L_{-1}^{2}. The limit u_{0} satisfies $H_{1}u_{0}=f$. Thus the uniqueness of such a solution proves the statement (i).

(ii) Let $u_{\lambda}=R(\lambda+i0\,;H_{1}(\lambda))f$ with $f\in L_{1}^{2}$. Then, by Lemma 2.2, $|u_{\lambda}|_{-1}=O(1)$, $\lambda\rightarrow 0$. Hence, by the same argument as above, statement (ii) is proved. □

We now combine Lemmas 2.3 and 2.4 to obtain that for $\alpha>1$

$$\|X_{\alpha}(R(\lambda+i0\,;H_{1}(\lambda))-H_{1}^{-1})X_{\alpha}\|=O(\lambda^{\gamma}) \tag{2.5}$$

with some $\gamma>0$. Set

$$U=H-H_{1}=-\nabla\cdot(1/\rho(x)-1/\rho_{1}(x))\nabla.$$

Since the coefficient $1/\rho-1/\rho_{1}$ is of C^{1}-class and of compact support, we have by elliptic estimate that $|U\phi|_{\alpha}\leqq C_{\alpha}\{|H_{1}\phi|_{-\alpha}+|\phi|_{-\alpha}\}$ for ϕ such that $\langle x\rangle^{-\alpha}\phi\in H^{2}(R_{x}^{3})$. Therefore, it follows from (2.5) and Lemma 2.2 that

$$\|U(R(\lambda+i0\,;H_{1}(\lambda))-H_{1}^{-1})\|_{\alpha\rightarrow\alpha}=O(\lambda^{\gamma}) \tag{2.6}$$

for some $\gamma>0$.

(4) LEMMA 2.5. *One has the following statements:* (i) *The inverse* H^{-1} *exists as an operator from* L_{1}^{2} *into* L_{-1}^{2}.

(ii) *Let* U *be as above. Then* $\mathrm{Id}+UH_{1}^{-1}\colon L_{\alpha}^{2}\rightarrow L_{\alpha}^{2}$, $\alpha\geqq 1$, *is invertible and*

$$(\mathrm{Id}+UH_{1}^{-1})^{-1}=\mathrm{Id}-UH^{-1}. \tag{2.7}$$

PROOF. (i) This is proved in exactly the same way as in the proof of Lemma 2.4, (i).

(ii) This follows immediately from the relation $H_{1}^{-1}=H^{-1}(\mathrm{Id}+UH_{1}^{-1})$. □

We now write $R(\lambda)$ and $R_{1}(\lambda)$ for $R(\lambda+i0\,;H(\lambda))$ and $R(\lambda+i0\,;H_{1}(\lambda))$, respectively. Then, by the resolvent identity,

$$R_{1}(\lambda)=R(\lambda)(\mathrm{Id}+UR_{1}(\lambda)).$$

Making use of relation (2.7), we calculate

$$\mathrm{Id}+UR_{1}(\lambda)=(\mathrm{Id}+UH_{1}^{-1})[\mathrm{Id}+(\mathrm{Id}-UH^{-1})U(R_{1}(\lambda)-H_{1}^{-1})].$$

By (2.6), we see that $\mathrm{Id}+UR_{1}(\lambda)\colon L_{\alpha}^{2}\rightarrow L_{\alpha}^{2}$, $\alpha>1$, is invertible for $\lambda>0$ small enough. This, together with Lemmas 2.1 and 2.2, completes the proof of Lemma 1.2. □

§ 3. Commutator method.

In this section we prove Lemma 2.2 by making use of the commutator method developed by Mourre [**9**].

PROOF OF LEMMA 2.2. The proof of this lemma is also divided into several steps. The proof is done for the "+" case only.

(1) Let A be the generator of the dilation unitary group;

$$A=(1/2)\{x\cdot(1/i)\nabla+(1/i)\nabla\cdot x\}.$$

We calculate the commutator

$$B_1(\lambda)=i[H_1(\lambda),\ A]=2H_1(\lambda)+D_1(\lambda), \tag{3.1}$$

where

$$D_1(\lambda)=\nabla\cdot(x\cdot\nabla(1/\rho_1))\nabla+\lambda(x\cdot\nabla E_0+2(E_0-1)).$$

Let $f_\lambda(s)\in C_0^\infty(R^1)$, $0\leqq f_\lambda\leqq 1$, be a function such that f_λ has support in $(\lambda/3,\ 3\lambda)$ and $f_\lambda=1$ on $[\lambda/2,\ 2\lambda]$. We can take δ_0 in (1.4) and (2.2) so small that

$$f_\lambda(H_1(\lambda))B_1(\lambda)f_\lambda(H_1(\lambda))\geqq(\lambda/3)f_\lambda(H_1(\lambda))^2 \tag{3.2}$$

in the form sense.

Let $\chi\in C_0^\infty(R_x^3)$, $0\leqq\chi\leqq 1$, be a smooth cut-off function such that χ has support in $\{x:\ |x|<2\}$ and $\chi=1$ for $|x|\leqq 1$. For $\varepsilon>0$ small enough, we define

$$\rho_{1\varepsilon}(x)=1+\chi(\varepsilon x)(\rho_1(x)-1)$$

and

$$E_{0\varepsilon}(x)=1+\chi(\varepsilon x)(E_0(x)-1).$$

By definition, $\rho_{1\varepsilon}(x)=\rho_1(x)$ for $|x|\leqq\varepsilon^{-1}$ and $\rho_{1\varepsilon}(x)=1$ for $|x|\geqq 2\varepsilon^{-1}$; similarly for $E_{0\varepsilon}(x)$. We further define $H_1(\varepsilon\,;\lambda)$ by

$$H_1(\varepsilon\,;\lambda)=-\nabla\cdot(1/\rho_{1\varepsilon}(x))\nabla-\lambda(E_{0\varepsilon}-1). \tag{3.3}$$

LEMMA 3.1. *As $\lambda\to 0$, one has:*

(i) $\|(H_1+\lambda)^{-1/2}[(H_1(\lambda)-H_1(\varepsilon\,;\lambda)),\ A](H_1+\lambda)^{-1/2}\|=\varepsilon^\theta O(1)$,

(ii) $\|(H_1+\lambda)^{-1/2}[(d/d\varepsilon)H_1(\varepsilon\,;\lambda),\ A](H_1+\lambda)^{-1/2}\|=\varepsilon^{\theta-1}O(1)$,

(iii) $\|(H_1+\lambda)^{-1}[[H_1(\varepsilon\,;\lambda),\ A],\ A](H_1+\lambda)^{-1}\|=\varepsilon^{\theta-1}O(\lambda^{-1/2})+O(\lambda^{-1})$.

PROOF. Estimates (i) and (ii) are proved by a straightforward calculation. To prove (iii), we note that

$$\|(-\Delta+\lambda)(H_1+\lambda)^{-1}\|=O(1),\qquad\lambda\to 0. \tag{3.4}$$

Since $\nabla(1/\rho_1)=\delta_0 O(|x|^{-1})$ as $|x|\to\infty$, this follows from inequality (2.4). If we take account of (3.4), (iii) can be easily proved. □

REMARK. The original commutator method initiated by Mourre [9] requires the additional assumptions $\partial_x^\alpha(\rho_1(x)-1)=O(|x|^{-2})$ and $\partial_x^\alpha(E_0(x)-1)=O(|x|^{-2})$, $|\alpha|=2$, as $|x|\to\infty$, to guarantee that the double commutator

$$(H_1+1)^{-1}[[H_1(\lambda), A], A](H_1+1)^{-1}: \quad L^2 \longrightarrow L^2$$

is bounded. However, we dispense with such assumptions by applying the commutator method to $H_1(\varepsilon;\lambda)$ rather than to $H_1(\lambda)$ itself (see Tamura [12]).

Let $B_1(\varepsilon;\lambda)=i[H_1(\varepsilon;\lambda), A]$ and define

$$M(\varepsilon;\lambda) = f_\lambda(H_1(\lambda))B_1(\varepsilon;\lambda)f_\lambda(H_1(\lambda)).$$

Then, (3.2), together with Lemma 3.1, (i), implies that

$$M(\varepsilon;\lambda) \geqq (\lambda/4)f_\lambda(H_1(\lambda))^2 \tag{3.5}$$

for $\varepsilon>0$ small enough.

(2) It follows from (3.5) that $M(\varepsilon;\lambda)$ is non-negative and hence we can define $G_\kappa(\varepsilon;\lambda)$: $L^2\to L^2$ by

$$G_\kappa(\varepsilon;\lambda) = (H_1(\lambda)-\lambda-i\kappa-i\varepsilon M(\varepsilon;\lambda))^{-1}$$

for $\kappa>0$, $0<\kappa\leqq 1$, and $\varepsilon\geqq 0$ small enough.

LEMMA 3.2. *There exists ε_0, $0<\varepsilon_0\ll 1$, independent of λ such that for ε, $0<\varepsilon\leqq\varepsilon_0$,*

$$\|G_\kappa(\varepsilon;\lambda)\| = \varepsilon^{-1}O(\lambda^{-1}), \qquad \lambda\to 0,$$

uniformly in κ.

PROOF. The lemma is proved in exactly the same way as in the proof of Lemma 7.3 of [11], but for later reference, we here give a brief sketch of the proof, looking at the λ-dependence.

Let $\langle\ ,\ \rangle$ again denote the L^2 scalar product. Let $g_\lambda(s)=1-f_\lambda(s)$, f_λ being as above. We write f_λ and g_λ for $f_\lambda(H_1(\lambda))$ and $g_\lambda(H_1(\lambda))$, respectively. By (3.5), we have

$$|f_\lambda G_\kappa\phi|_0^2 \leqq (4/\lambda)(2\varepsilon)^{-1}\langle\phi,\ G_\kappa^*(2\varepsilon M(\varepsilon;\lambda))G_\kappa\phi\rangle.$$

Since

$$G_\kappa^*(2\varepsilon M(\varepsilon;\lambda))G_\kappa \leqq i(G_\kappa^*-G_\kappa)$$

in the form sense, we obtain

$$\|f_\lambda G_\kappa\| = \varepsilon^{-1/2}\|G_\kappa\|^{1/2}O(\lambda^{-1/2}). \tag{3.6}$$

We use the resolvent identity

$$G_\kappa(\varepsilon;\lambda) = G_\kappa(0;\lambda)[\mathrm{Id}+i\varepsilon M(\varepsilon;\lambda)G_\kappa(\varepsilon;\lambda)].$$

As is easily seen, $\|M(\varepsilon;\lambda)\|=O(\lambda)$ and hence

$$\|g_\lambda G_\kappa\| \leqq C\{\lambda^{-1}+\varepsilon\|G_\kappa\|\}.$$

Thus, we have

$$\|G_\kappa\| \leqq C\{\lambda^{-1}+\varepsilon^{-1/2}\lambda^{-1/2}\|G_\kappa\|^{1/2}\}$$

for $\varepsilon>0$ small enough. This proves the lemma. □

LEMMA 3.3. *As $\lambda\to 0$, one has:*

(i) $\|g_\lambda G_\kappa(\varepsilon\,;\lambda)\| = O(\lambda^{-1})$,

(ii) $\|g_\lambda G_\kappa(\varepsilon\,;\lambda)(H_1+\lambda)^{1/2}\| = O(\lambda^{-1/2})$,

(iii) $\|(H_1+\lambda)^{1/2}g_\lambda G_\kappa(\varepsilon\,;\lambda)(H_1+\lambda)^{1/2}\| = O(1)$,

where all the order relations are uniform in κ and ε.

REMARK. Similar estimates hold for $G_\kappa(\varepsilon\,;\lambda)^*$ and, as an immediate consequence, we obtain, for example, $\|G_\kappa(\varepsilon\,;\lambda)g_\lambda\|=O(\lambda^{-1})$. Such simple consequences of the lemma will be used without further comments throughout the proof.

PROOF. (i) This estimate has been already obtained in the proof of Lemma 3.2.

(ii) We denote by $m_\kappa(\varepsilon\,;\lambda)$ the norm under consideration. Then we have

$$m_\kappa(\varepsilon\,;\lambda)^2 = \|g_\lambda G_\kappa(H_1+\lambda)G_\kappa^* g_\lambda\|.$$

We calculate

$$(H_1+\lambda)G_\kappa^* = \mathrm{Id}+\lambda(E_0+1)G_\kappa^* - i(\kappa+\varepsilon M(\varepsilon\,;\lambda))G_\kappa^*.$$

Since

$$iG_\kappa(\kappa+\varepsilon M(\varepsilon\,;\lambda))G_\kappa^* = (1/2)(G_\kappa - G_\kappa^*),$$

it follows from (i) that $m_\kappa(\varepsilon\,;\lambda)^2=O(\lambda^{-1})$. This yields (ii).

(iii) This estimate is proved in the same way as above. Denote by $n_\kappa(\varepsilon\,;\lambda)$ the norm under consideration. Since

$$\|(H_1+\lambda)^{-1/2}g_\lambda(H_1+\lambda)^{1/2}\| = O(1), \qquad \lambda\to 0,$$

we have

$$n_\kappa(\varepsilon\,;\lambda)^2 \leqq C(1+n_\kappa(\varepsilon\,;\lambda)),$$

which proves (iii) at once. □

(3) Recall the notation X_β. We define

$$F_\kappa(\varepsilon\,;\lambda) = X_1 G_\kappa(\varepsilon\,;\lambda)X_1 \tag{3.7}$$

for κ, $0<\kappa\leqq 1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$, ε_0 being as in Lemma 3.2. We assert that:

$$\|(d/d\varepsilon)F_\kappa\| \leqq C\{1+\varepsilon^{-1/2}\|F_\kappa\|^{1/2}+\varepsilon^{\theta-1}\|F_\kappa\|\} \tag{3.8}$$

and

$$\|F_\kappa(\varepsilon_0\,;\lambda)\| = O(1), \qquad \lambda\to 0, \tag{3.9}$$

uniformly in κ. If the two assertions above are verified, then we have that $\|F_{\kappa}(0;\lambda)\|=O(1)$, $\lambda\to 0$, uniformly in κ and hence Lemma 2.2 follows immediately.

We again write f_{λ} and g_{λ} for $f_{\lambda}(H_1(\lambda))$ and $g_{\lambda}(H_1(\lambda))$, respectively. Differentiate $F_{\kappa}(\varepsilon;\lambda)$ in ε. Then we have

$$(d/d\varepsilon)F_{\kappa}(\varepsilon;\lambda)=\sum_{j=1}^{7}X_1Y_{\kappa}^{j}(\varepsilon;\lambda)X_1,$$

where

$$\begin{aligned}
Y_{\kappa}^{1}&=G_{\kappa}g_{\lambda}[H_1(\varepsilon;\lambda),A]g_{\lambda}G_{\kappa},\\
Y_{\kappa}^{2}&=G_{\kappa}f_{\lambda}[H_1(\varepsilon;\lambda),A]g_{\lambda}G_{\kappa},\\
Y_{\kappa}^{3}&=G_{\kappa}g_{\lambda}[H_1(\varepsilon;\lambda),A]f_{\lambda}G_{\kappa},\\
Y_{\kappa}^{4}&=-G_{\kappa}[H_1(\lambda)-\lambda-i\kappa-i\varepsilon M(\varepsilon;\lambda),A]G_{\kappa},\\
Y_{\kappa}^{5}&=i\varepsilon G_{\kappa}[(d/d\varepsilon)M(\varepsilon;\lambda)]G_{\kappa},\\
Y_{\kappa}^{6}&=G_{\kappa}[(H_1(\lambda)-H_1(\varepsilon;\lambda)),A]G_{\kappa},\\
Y_{\kappa}^{7}&=-i\varepsilon G_{\kappa}[M(\varepsilon;\lambda),A]G_{\kappa}.
\end{aligned}$$

(4) LEMMA 3.4. *As $\lambda\to 0$, one has:*

(i) $\|(H_1+\lambda)^{1/2}f_{\lambda}G_{\kappa}(\varepsilon;\lambda)X_1\|=\varepsilon^{-1/2}\|F_{\kappa}(\varepsilon;\lambda)\|^{1/2}O(1)$,

(ii) $\|(H_1+\lambda)^{1/2}g_{\lambda}G_{\kappa}(\varepsilon;\lambda)X_1\|=O(1)$.

PROOF. (i) This estimate is derived in the same way as used to derive (3.6).

(ii) Since $\|(H_1+\lambda)^{-1/2}X_1\|=O(1)$, $\lambda\to 0$, by (2.4), (ii) follows from Lemma 3.3, (iii). □

We now use the above lemma to evaluate the norm of $X_1Y_{\kappa}^{j}X_1$, $1\leqq j\leqq 6$. First, we have

$$\|X_1Y_{\kappa}^{j}X_1\|\leqq C\{1+\varepsilon^{-1/2}\|F_{\kappa}\|^{1/2}\}$$

for j, $1\leqq j\leqq 3$. Since $\|X_1A(H_1+\lambda)^{-1/2}\|=O(1)$, $\lambda\to 0$, it follows that

$$\|X_1Y_{\kappa}^{4}X_1\|=C\{1+\varepsilon^{-1/2}\|F_{\kappa}\|^{1/2}\}$$

and by Lemma 3.1, we have

$$\|X_1Y_{\kappa}^{j}X_1\|\leqq C\{1+\varepsilon^{\theta-1}\|F_{\kappa}\|\}$$

for j, $5\leqq j\leqq 6$.

(5) We require the new lemma below to evaluate the norm of $X_1Y_{\kappa}^{7}X_1$.

LEMMA 3.5. $\|[M(\varepsilon;\lambda),A]\|=\varepsilon^{\theta-1}O(\lambda)$.

By this lemma, we have

$$\|X_1Y_{\kappa}^{7}X_1\|\leqq C\{1+\varepsilon^{\theta-1}\|F_{\kappa}\|\}$$

and hence assertion (3.8) is proved.

The proof of Lemma 3.5 is done in almost the same way as in the proof of Lemmas 7.4 and 7.5 of [11], although we have to look at the dependence on ε and λ carefully.

LEMMA 3.6. *Let* $c_0 > \sup E_0(x)$. *Let* $f_{1\lambda}(s) \in C_0^\infty(R^1)$ *be a real function such that* $f_{1\lambda}$ *has support in* $(\lambda/3, 3\lambda)$ *and* $(d/ds)^k f_{1\lambda} = O(\lambda^{2-k})$. *Then*

$$\|(H_1(\lambda)+c_0\lambda)^{-1/2}[f_{1\lambda}(H_1(\lambda)), A](H_1(\lambda)+c_0\lambda)^{-1/2}\| = O(\lambda).$$

PROOF. It is easy to see that

$$\|(H_1(\lambda)+c_0\lambda)^{-1/2}[\exp(i\tau H_1(\lambda)), A](H_1(\lambda)+c_0\lambda)^{-1/2}\| \leqq C|\tau|.$$

The Fourier transform $\hat{f}_{1\lambda}(\tau)$ of $f_{1\lambda}(s)$ satisfies

$$|\hat{f}_{1\lambda}(\tau)| \leqq C\lambda^3(1+\lambda|\tau|)^{-3}.$$

Therefore, if we use the relation

$$[f_{1\lambda}(H_1(\lambda)), A] = (2\pi)^{-1/2}\int \hat{f}_{1\lambda}(\tau)[\exp(i\tau H_1(\lambda)), A]d\tau,$$

the lemma follows at once. □

PROOF OF LEMMA 3.5. We can write $f_\lambda = f_\lambda(H_1(\lambda))$ as

$$f_\lambda = (H_1(\lambda)+c_0\lambda)^{-1} f_{1\lambda}(H_1(\lambda))(H_1(\lambda)+c_0\lambda)^{-1}$$

with $f_{1\lambda}$ having the properties as in Lemma 3.6. Since

$$\|(H_1+\lambda)^{1/2}[(H_1(\lambda)+c_0\lambda)^{-1}, A]\| = O(\lambda^{-1/2}),$$

we have by Lemma 3.6 that

$$\|(H_1+\lambda)^{1/2}[f_\lambda, A]\| = O(\lambda^{1/2}). \tag{3.10}$$

We again set $B_1(\varepsilon;\lambda) = i[H_1(\varepsilon;\lambda), A]$ and calculate $[M(\varepsilon;\lambda), A]$ as

$$f_\lambda B_1(\varepsilon;\lambda)[f_\lambda, A] + f_\lambda[B_1(\varepsilon;\lambda), A]f_\lambda + [f_\lambda, A]B_1(\varepsilon;\lambda)f_\lambda.$$

Then, (3.10), together with Lemma 3.1, (iii), yields the estimate in the lemma. □

(6) We prove the other assertion (3.9).

LEMMA 3.7. *Let* $H_0 = -\Delta$. *Then*

$$\|X_1 R(\lambda+i\kappa; H_0)X_1\| = O(1), \qquad \lambda\to 0,$$

uniformly in κ, $0\leqq\kappa\leqq 1$.

PROOF. The proof uses the explicit integral kernel of $R(\lambda+i\kappa; H_0)$;

$$[R(\lambda+i\kappa; H_0)](x, y) = (4\pi)^{-1}|x-y|^{-1}\exp(i\sqrt{\lambda+i\kappa}\,|x-y|). \tag{3.11}$$

Let $u(x)=(R(\lambda+i\kappa\,;H_0)f)(x)$ with $f\in L_1^2$. Then, $|u(x)|\leqq(H_0^{-1}|f|)(x)$. By (2.4), $H_0^{-1}: L_1^2\to L_{-1}^2$ is bounded. This proves the lemma. □

Let $f_\lambda(s)$ be as before. Since $i[H_0, A]=2H_0$, we have

$$M_0(\lambda)=if_\lambda(H_0)[H_0, A]f_\lambda(H_0)\geqq(2\lambda/3)f_\lambda(H_0)^2.$$

This enables us to define $G_\kappa^0(\varepsilon\,;\lambda): L^2\to L^2$ by

$$G_\kappa^0(\varepsilon\,;\lambda)=(H_0-\lambda-i\kappa-i\varepsilon M_0(\lambda))^{-1}$$

for κ, $0<\kappa\leqq1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$. By Lemma 3.7,

$$\lim_{\varepsilon\downarrow0}\|X_1G_\kappa^0(\varepsilon\,;\lambda)X_1\|=O(1),\qquad \lambda\to0.$$

Therefore, the differential inequality (3.8) applied to $G_\kappa^0(\varepsilon\,;\lambda)$ gives

$$\|X_1G_\kappa^0(\varepsilon_0\,;\lambda)X_1\|=O(1),\qquad \lambda\to0, \tag{3.12}$$

and also we have

$$\|(H_0+\lambda)^{1/2}G_\kappa^0(\varepsilon_0\,;\lambda)X_1\|=O(1). \tag{3.13}$$

We look at the difference

$$G_\kappa-G_\kappa^0=G_\kappa^0\{H_0-H_1(\lambda)-i\varepsilon_0(M_0(\lambda)-M(\varepsilon_0\,;\lambda))\}G_\kappa$$

at $\varepsilon=\varepsilon_0$. By Lemma 3.4 and (3.13), we have

$$\|X_1(G_\kappa(\varepsilon_0\,;\lambda)-G_\kappa^0(\varepsilon_0\,;\lambda))X_1\|\leqq C\{1+\|F_\kappa(\varepsilon_0\,;\lambda)\|^{1/2}\}$$

for C dependent on ε_0. This implies

$$\|F_\kappa(\varepsilon_0\,;\lambda)\|\leqq C\{1+\|F_\kappa(\varepsilon_0\,;\lambda)\|^{1/2}\}$$

and hence assertion (3.9) follows immediately. Thus the proof of Lemma 2.2 is now complete. □

§4. Continuity at low frequencies.

In this section we prove Lemma 2.3 which had a basic role in proving the main theorem. By Lemma 2.2, $\|X_1R(\lambda+i0\,;H_1(\lambda))X_1\|=O(1)$ as $\lambda\to0$. Hence, by interpolation, it sufficies to prove the lemma for some $\alpha>1$. We do this for $\alpha=1+\sigma$, σ, $0<\sigma<\theta$, being fixed in (2.2), through a series of lemmas.

PROOF OF LEMMA 2.3. (0) The proof is long. First we shall explain briefly the strategy to prove the lemma.

We keep the same notations as in Section 3. Let $B_1(\varepsilon\,;\lambda)=i[H_1(\varepsilon\,;\lambda), A]$ again. We introduce the following auxiliary operator

$$\Gamma_\kappa(\varepsilon\,;\lambda)=(H_1(\lambda)-\lambda-i\kappa-i\varepsilon B_1(\varepsilon\,;\lambda))^{-1}$$

for κ, $0<\kappa\leqq1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$. The proof consists of the following three steps:

(a) To show that $\Gamma_\kappa(\varepsilon\,;\lambda): L^2\to L^2$ is well-defined as a bounded operator.

(b) To show that

$$\|X_1(\Gamma_\kappa(\varepsilon\,;\lambda)-R(\lambda+i\kappa\,;H_1(\lambda)))X_1\| = O(\varepsilon^\theta), \qquad \varepsilon\to 0,$$

uniformly in κ and $\lambda>0$ small enough.

(c) To show that

$$\|X_{1+\sigma}[(d/d\lambda)\Gamma_\kappa(\varepsilon\,;\lambda)]X_{1+\sigma}\| = \varepsilon^{-2}O(\lambda^{-1+\tau})$$

for any τ, $0<\tau<\sigma/2$.

If (a)~(c) are verified, then we have that

$$\|X_{1+\sigma}(R(\lambda+i\kappa\,;H_1(\lambda))-R(\mu+i\kappa\,;H_1(\mu)))X_{1+\sigma}\| = O(\varepsilon^\theta)+\varepsilon^{-2}(O(\lambda^\tau)+O(\mu^\tau)).$$

Take ε as $\varepsilon=(\lambda^\tau+\mu^\tau)^{1/(2+\theta)}$. Then we have

$$\|X_{1+\sigma}(R(\lambda+i0\,;H_1(\lambda))-R(\mu+i0\,;H_1(\mu)))X_{1+\sigma}\| = O(\lambda^\gamma)+O(\mu^\gamma)$$

with $\gamma=\tau\theta/(2+\theta)$. This yields the desired result.

(1) We define

$$M_1(\varepsilon\,;\lambda) = i\{[-\nabla\cdot(1/\rho_{1\varepsilon})\nabla,\ A]-\lambda f_\lambda[E_{0\varepsilon},\ A]f_\lambda\}.$$

Then $B_1(\varepsilon\,;\lambda)$ is represented as $B_1=M_1(\varepsilon\,;\lambda)-N_1(\varepsilon\,;\lambda)$, where

$$N_1 = i\lambda\{g_\lambda[E_{0\varepsilon},\ A]g_\lambda+f_\lambda[E_{0\varepsilon},\ A]g_\lambda+f_\lambda[E_{0\varepsilon},\ A]g_\lambda\}. \tag{4.1}$$

We can choose δ_0 in (1.4) and (2.2) so small that

$$M_1(\varepsilon\,;\lambda) \geqq (\lambda/3)f_\lambda(H_1(\lambda))^2. \tag{4.2}$$

This enables us to define $\Lambda_\kappa(\varepsilon\,;\lambda)\colon L^2\to L^2$ by

$$\Lambda_\kappa(\varepsilon\,;\lambda) = (H_1(\lambda)-\lambda-i\kappa-i\varepsilon M_1(\varepsilon\,;\lambda))^{-1}$$

for κ, $0<\kappa\leqq 1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$. We can also show that this operator can be extended to a bounded operator from $H^{-1}(R_x^3)$ into $H^1(R_x^3)$. We should note that it is not necessarily extended to a bounded operator from L^2 into $H^2(R_x^3)$.

LEMMA 4.1. *As $\lambda\to 0$ one has:*

(i) $\|f_\lambda\Lambda_\kappa(\varepsilon\,;\lambda)\| = \varepsilon^{-1}O(\lambda^{-1})$,

(ii) $\|g_\lambda\Lambda_\kappa(\varepsilon\,;\lambda)\| = O(\lambda^{-1})$,

(iii) $\|g_\lambda\Lambda_\kappa(\varepsilon\,;\lambda)(H_1+\lambda)^{1/2}\| = O(\lambda^{-1/2})$,

(iv) $\|(H_1+\lambda)^{1/2}g_\lambda\Lambda_\kappa(\varepsilon\,;\lambda)(H_1+\lambda)^{1/2}\| = O(1)$.

PROOF. We use the same argument as in the proof of Lemmas 3.2 and 3.3. By (4.2), we have

$$\|f_\lambda\Lambda_\kappa\| = \varepsilon^{-1/2}\|\Lambda_\kappa\|^{1/2}O(\lambda^{-1/2})$$

and by the resolvent identity, we have

$$\|(H_1+\lambda)^{1/2}g_\lambda\Lambda_\kappa\| \leqq C\{\lambda^{-1/2}+\varepsilon\|(H_1+\lambda)^{1/2}\Lambda_\kappa\|\}.$$

We combine the two estimates above to obtain that $\|(H_1+\lambda)^{1/2}\Lambda_\kappa\|=\varepsilon^{-1}O(\lambda^{-1/2})$. This proves (i) and (ii). Once (ii) is established, (iii) and (iv) are proved in the same way as in the proof of Lemma 3.3. □

(2) LEMMA 4.2. *One can define* $\Gamma_\kappa(\varepsilon;\lambda)\colon L^2\to L^2$ *by*

$$\Gamma_\kappa(\varepsilon;\lambda)=(H_1(\lambda)-\lambda-i\kappa-i\varepsilon B_1(\varepsilon;\lambda))^{-1}$$

for κ, $0<\kappa\leqq 1$, *and* ε, $0<\varepsilon\leqq\varepsilon_0$, *and one has*

$$\|\Gamma_\kappa(\varepsilon;\lambda)\|=\varepsilon^{-1}O(\lambda^{-1}).$$

REMARK. We can also show that $\Gamma_\kappa(\varepsilon;\lambda)$ is extended to a bounded operator from $H^{-1}(R_x^3)$ into $H^1(R_x^3)$.

PROOF. Let $N_1=N_1(\varepsilon;\lambda)$ be defined by (4.1). To prove the lemma, it suffices to show that $\mathrm{Id}+i\varepsilon N_1\Lambda_\kappa\colon L^2\to L^2$ is invertible. Then, $\Gamma_\kappa(\varepsilon;\lambda)$ is represented as

$$\Gamma_\kappa(\varepsilon;\lambda)=\Lambda_\kappa(\varepsilon;\lambda)(\mathrm{Id}+i\varepsilon N_1(\varepsilon;\lambda)\Lambda_\kappa(\varepsilon;\lambda))^{-1}. \tag{4.3}$$

By Lemma 4.1, we have

$$\varepsilon\|N_1(\varepsilon;\lambda)\Lambda_\kappa(\varepsilon;\lambda)\|=(\varepsilon+\delta_0)O(1),\qquad \lambda\to 0.$$

This proves the lemma. □

By Lemma 4.2, step (a) is completed.

(3) LEMMA 4.3. *As* $\lambda\to 0$, *one has:*

(i) $\|g_\lambda\Gamma_\kappa(\varepsilon;\lambda)\|=O(\lambda^{-1})$,

(ii) $\|g_\lambda\Gamma_\kappa(\varepsilon;\lambda)(H_1+\lambda)^{1/2}\|=O(\lambda^{-1/2})$,

(iii) $\|(H_1+\lambda)^{1/2}g_\lambda\Gamma_\kappa(\varepsilon;\lambda)(H_1+\lambda)^{1/2}\|=O(1)$.

PROOF. By Lemma 4.1, (ii), estimate (i) follows from (4.3), and (ii) and (iii) follow from (i). The proof is done in the same way as in the proof of Lemma 3.3. □

LEMMA 4.4.

$$\|f_\lambda\Gamma_\kappa(\varepsilon;\lambda)X_1\|\leqq C\{\varepsilon^{-1/2}\lambda^{-1/2}\|X_1\Lambda_\kappa(\varepsilon;\lambda)X_1\|^{1/2}+\lambda^{-1/2}\}.$$

PROOF. By (4.2), we have

$$\|f_\lambda\Lambda_\kappa X_1\|=\varepsilon^{-1/2}\|X_1\Lambda_\kappa X_1\|^{1/2}O(\lambda^{-1/2}). \tag{4.4}$$

We look at the difference

$$f_\lambda(\Gamma_\kappa-\Lambda_\kappa)X_1=-i\varepsilon f_\lambda\Lambda_\kappa N_1(\varepsilon;\lambda)\Gamma_\kappa X_1.$$

By Lemmas 4.1 and 4.3, it follows that

$$\|f_\lambda(\Gamma_\kappa-\Lambda_\kappa)X_1\| \leqq C\{\lambda^{-1/2}+\varepsilon\|f_\lambda\Gamma_\kappa X_1\|\}.$$

This, together with (4.4), proves the lemma. □

LEMMA 4.5.

$$\|X_1\Lambda_\kappa(\varepsilon;\lambda)X_1\| \leqq C\{\|X_1\Gamma_\kappa(\varepsilon;\lambda)X_1\|+\varepsilon\}.$$

PROOF. We evaluate the norm of the difference

$$X_1(\Gamma_\kappa-\Lambda_\kappa)X_1 = -i\varepsilon X_1\Lambda_\kappa N_1(\varepsilon;\lambda)\Gamma_\kappa X_1.$$

Making use of (4.4) and of Lemmas 4.1 and 4.3, we have

$$\|X_1(\Gamma_\kappa-\Lambda_\kappa)X_1\| \leqq C\{\varepsilon+\varepsilon^{1/2}\|X_1\Lambda_\kappa X_1\|^{1/2}+\varepsilon\lambda^{1/2}\|f_\lambda\Gamma_\kappa X_1\|\}$$

and hence, by Lemma 4.4,

$$\|X_1(\Gamma_\kappa-\Lambda_\kappa)X_1\| \leqq C\{\varepsilon+\varepsilon^{1/2}\|X_1\Lambda_\kappa X_1\|^{1/2}\},$$

from which the lemma follows at once. □

We now combine Lemmas 4.4 and 4.5 to obtain that

$$\|f_\lambda\Gamma_\kappa(\varepsilon;\lambda)X_1\| \leqq C\{\varepsilon^{-1/2}\lambda^{-1/2}\|X_1\Gamma_\kappa(\varepsilon;\lambda)X_1\|^{1/2}+\lambda^{-1/2}\}. \tag{4.5}$$

(4) The next task is to evaluate the norm of the difference $\Gamma_\kappa(\varepsilon;\lambda)-G_\kappa(\varepsilon;\lambda)$. Let $N(\varepsilon;\lambda)=B_1(\varepsilon;\lambda)-M(\varepsilon;\lambda)$. This is written as

$$N=g_\lambda B_1(\varepsilon;\lambda)g_\lambda+f_\lambda B_1(\varepsilon;\lambda)g_\lambda+g_\lambda B_1(\varepsilon;\lambda)f_\lambda.$$

We have shown in the proof of Lemma 2.2 that

$$\|f_\lambda G_\kappa X_1\| = \varepsilon^{-1/2}\|X_1G_\kappa X_1\|^{1/2}O(\lambda^{-1/2}) = \varepsilon^{-1/2}O(\lambda^{-1/2}).$$

Therefore, it follows from (4.5) that

$$\|X_1(\Gamma_\kappa-G_\kappa)X_1\| \leqq C\{\varepsilon^{1/2}+\varepsilon^{1/2}\|X_1\Gamma_\kappa X_1\|^{1/2}\}.$$

This implies that

$$\|X_1\Gamma_\kappa(\varepsilon;\lambda)X_1\| = O(1), \qquad \lambda\to 0,$$

uniformly in κ and ε. Thus we have

$$\|X_1(\Gamma_\kappa(\varepsilon;\lambda)-G_\kappa(\varepsilon;\lambda))X_1\| = O(\varepsilon^{1/2}), \qquad \varepsilon\to 0,$$

uniformly in κ and $\lambda>0$ small enough. On the other hand, by the differential inequality (3.8), we have

$$\|X_1(G_\kappa(\varepsilon;\lambda)-R(\lambda+i\kappa;H_1(\lambda)))X_1\| = O(\varepsilon^\theta).$$

(Recall that $0<\theta<1/2$.) Hence we obtain

$$\|X_1(\Gamma_\kappa(\varepsilon;\lambda)-R(\lambda+i\kappa;H_1(\lambda)))X_1\| = O(\varepsilon^\theta), \tag{4.6}$$

which completes the step (b).

For later reference, we summarize the basic estimates for $\Gamma_\kappa(\varepsilon;\lambda)$ obtained in steps (3) and (4).

LEMMA 4.6. *As $\lambda\to 0$, one has:* (i) $\|\Gamma_\kappa(\varepsilon;\lambda)\| = \varepsilon^{-1}O(\lambda^{-1})$, (ii) $\|\Gamma_\kappa(\varepsilon;\lambda)X_1\| = \varepsilon^{-1/2}O(\lambda^{-1/2})$, (iii) $\|X_1\Gamma_\kappa(\varepsilon;\lambda)X_1\| = O(1)$, (iv) $\|(H_1+\lambda)^{1/2}\Gamma_\kappa(\varepsilon;\lambda)X_1\| = \varepsilon^{-1/2}O(1)$.

(5) We differentiate $\Gamma_\kappa(\varepsilon;\lambda)$ in λ;

$$(d/d\lambda)\Gamma_\kappa = \Gamma_\kappa E_0\Gamma_\kappa + \varepsilon\Gamma_\kappa[E_{0\varepsilon}, A]\Gamma_\kappa.$$

LEMMA 4.7.

$$\|X_1\Gamma_\kappa(\varepsilon;\lambda)(E_0-1)\Gamma_\kappa(\varepsilon;\lambda)X_1\| = \varepsilon^{-1+\theta/2}O(\lambda^{-1+\theta/2}).$$

PROOF. Note that $E_0(x)-1=O(|x|^{-\theta})$ as $|x|\to\infty$. By interpolation, it follows from Lemma 4.6 that

$$\|X_1\Gamma_\kappa X_{\theta/2}\| = \varepsilon^{-1/2+\theta/4}O(\lambda^{-1/2+\theta/4}),$$

which completes the proof. □

LEMMA 4.8.

$$\|X_1\Gamma_\kappa(\varepsilon;\lambda)[E_{0\varepsilon}, A]\Gamma_\kappa(\varepsilon;\lambda)X_1\| = \varepsilon^{-1+\theta/2}O(\lambda^{-1+\theta/2}).$$

PROOF. Since $[E_{0\varepsilon}, A]=O(|x|^{-\theta})$ as $|x|\to\infty$, the same argument as in the proof of Lemma 4.7 proves the lemma. □

To prove (c), it suffices, by Lemmas 4.7 and 4.8, to show that

$$\|X_{1+\sigma}\Gamma_\kappa(\varepsilon;\lambda)^2X_{1+\sigma}\| = \varepsilon^{-2}O(\lambda^{-1+\tau}), \qquad \lambda\to 0, \tag{4.7}$$

for any τ, $0<\tau<\sigma/2$.

(6) Let $H_0=-\Delta$ again. We set $B_0=i[H_0, A]$ and define $\Gamma_\kappa^0(\varepsilon;\lambda)$: $L^2\to L^2$ by

$$\Gamma_\kappa^0(\varepsilon;\lambda) = (H_0-\lambda-i\kappa-i\varepsilon B_0)^{-1}$$

for κ, $0<\kappa\leq 1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$. Since $B_0=2H_0$, this operator is represented as

$$\Gamma_\kappa^0(\varepsilon;\lambda) = (1-2i\varepsilon)^{-1}R(z;H_0), \qquad z=(\lambda+i\kappa)/(1-2i\varepsilon),$$

and satisfies the same estimates as $\Gamma_\kappa(\varepsilon;\lambda)$ ((i)~(iv) of Lemma 4.6).

LEMMA 4.9. *For any γ, $1-\sigma<\gamma<1$, one has*

$$\|X_{1+\sigma}\Gamma_\kappa^0(\varepsilon;\lambda)^2X_{1+\sigma}\| = \varepsilon^{-1}O(\lambda^{-\gamma}).$$

PROOF. First we note that $(d/d\lambda)\Gamma_\kappa^0=(\Gamma_\kappa^0)^2$. Hence, by (3.11), the integral kernel of $\Gamma_\kappa^0(\varepsilon;\lambda)^2$ obeys the estimate $[\Gamma_\kappa^0(\varepsilon;\lambda)^2](x, y)=O(\lambda^{-1/2})$. This proves that for any $\alpha>3/2$, $\|X_\alpha(\Gamma_\kappa^0)^2X_\alpha\|=O(\lambda^{-1/2})$. On the other hand, we have

$$\|X_1(\Gamma_\kappa^0)^2X_1\| \leqq \|X_1\Gamma_\kappa^0\|\cdot\|\Gamma_\kappa^0X_1\| = \varepsilon^{-1}O(\lambda^{-1}).$$

Thus, the lemma follows by interpolation. □

(7) We now set

$$U(\varepsilon;\lambda)=H_1(\lambda)-H_0-i\varepsilon(B_1(\varepsilon;\lambda)-B_0).$$

Then, $U(\varepsilon;\lambda)$ is decomposed as $U=U_1(\varepsilon;\lambda)+U_2(\varepsilon)$, where $U_1=-\lambda(E_0-1)-i\varepsilon\lambda x\cdot\nabla E_{0\varepsilon}$ and $U_2=-\nabla\cdot a_\varepsilon(x)\nabla$ with

$$a_\varepsilon(x)=\{(1/\rho_1)-1\}-2i\varepsilon\{(1/\rho_{1\varepsilon})-1\}+i\varepsilon x\cdot\nabla(1/\rho_{1\varepsilon}).$$

By (2.2), the coefficient $a_\varepsilon(x)$ obeys the estimate

$$|a_\varepsilon(x)|\leqq C\delta_0\langle x\rangle^{-\sigma},\qquad 0<\sigma<\theta, \tag{4.8}$$

uniformly in ε.

We look at the difference

$$(\Gamma_\kappa^0)^2-(\Gamma_\kappa)^2=\sum_{j=1}^{2}\{(\Gamma_\kappa^0)^2U_j\Gamma_\kappa+\Gamma_\kappa^0U_j(\Gamma_\kappa)^2\}.$$

By Lemma 4.9, we have only to show that the above difference satisfies the estimate as in (4.7).

LEMMA 4.10.

(i) $\|X_{1+\sigma}(\Gamma_\kappa^0)^2U_1(\varepsilon;\lambda)\Gamma_\kappa X_{1+\sigma}\|=\varepsilon^{-2+\theta/2}O(\lambda^{-1+\theta/2})$,

(ii) $\|X_{1+\sigma}\Gamma_\kappa^0U_1(\varepsilon;\lambda)(\Gamma_\kappa)^2X_{1+\sigma}\|=\varepsilon^{-2+\theta/2}O(\lambda^{-1+\theta/2})$.

PROOF. We prove (i) only, because (ii) is proved in a similar way.

By definition, $U_2=\lambda O(|x|^{-\theta})$ as $|x|\to\infty$, and also we have by interpolation that

$$\|\Gamma_\kappa^0X_\theta\|\leqq\|\Gamma_\kappa^0\|^{1-\theta}\|\Gamma_\kappa^0X_1\|^\theta=\varepsilon^{-1+\theta/2}O(\lambda^{-1+\theta/2}).$$

Thus, the norm under consideration is dominated by

$$O(\lambda)\|X_1\Gamma_\kappa^0\|\cdot\|\Gamma_\kappa^0X_\theta\|\cdot\|\Gamma_\kappa X_1\|=\varepsilon^{-2+\theta/2}O(\lambda^{-1+\theta/2}).$$

This proves (i). □

By the lemma above, we have only to show that (4.7) is satisfied for the difference operator with $U_2(\varepsilon)$.

LEMMA 4.11.

(i) $\|X_\sigma\nabla(\Gamma_\kappa)^2X_{1+\sigma}\|\leqq C\{\|X_{1+\sigma}(\Gamma_\kappa)^2X_{1+\sigma}\|+\varepsilon^{-3/2+\sigma/2}\lambda^{-1+\sigma/2}\}$,

(ii) $\|X_\sigma\nabla(\Gamma_\kappa^0)^2X_{1+\sigma}\|=\varepsilon^{-3/2+\sigma/2}O(\lambda^{-1+\sigma/2})$.

PROOF. Let $f\in L^2_{1+\sigma}$. Set $u=(\Gamma_\kappa)^2f$. Then

$$[H_1(\lambda)-\lambda-i\kappa-i\varepsilon B_1(\varepsilon;\lambda)]u=\Gamma_\kappa f.$$

We take the L^2 scalar product of ϕu, $\phi=\langle x\rangle^{-2\sigma}$, with the above equation. Then we have

$$|X_\sigma \nabla u|_0^2 \leqq C\{|X_{1+\sigma}u|_0^2+\lambda|X_\sigma u|_0^2+|X_\sigma u|_0\cdot|X_\sigma \Gamma_\kappa f|_0\}.$$

By interpolation, it follows that

$$\|X_\sigma(\Gamma_\kappa)^2 X_{1+\sigma}\| = \varepsilon^{-3/2+\sigma/2}O(\lambda^{-3/2+\sigma/2})$$

and

$$\|X_\sigma \Gamma_\kappa X_{1+\sigma}\| = \varepsilon^{-1/2+\sigma/2}O(\lambda^{-1/2+\sigma/2}).$$

Combining these estimates proves (i).

(ii) By Lemma 4.9, $\|X_{1+\sigma}(\Gamma_\kappa^0)^2 X_{1+\sigma}\|=\varepsilon^{-1}O(\lambda^{-\gamma})$ for any γ, $1-\sigma<\gamma<1$. Hence, the same argument as above proves (ii). □

We have by Lemma 4.6, (iv), that

$$\|\nabla\Gamma_\kappa X_{1+\sigma}\| = \varepsilon^{-1/2}O(1), \qquad \lambda\to 0.$$

Hence, we have, by Lemma 4.11, (ii), and (4.8), the following

LEMMA 4.12.

$$\|X_{1+\sigma}(\Gamma_\kappa^0)^2 U_2(\varepsilon)\Gamma_\kappa X_{1+\sigma}\| = \varepsilon^{-2+\sigma/2}O(\lambda^{-1+\sigma/2}).$$

Thus, it remains to evaluate the norm of $\Gamma_\kappa^0 U_2(\Gamma_\kappa)^2$ only.

LEMMA 4.13. *For any* τ, $0<\tau<\sigma/2$,

$$\|X_{1+\sigma}(\Gamma_\kappa^0)\nabla\| = \varepsilon^{-1}O(\lambda^\tau)+O(1), \qquad \lambda\to 0.$$

PROOF. Let $f_\lambda(s)$ and $g_\lambda(s)$ be as before. We write

$$X_{1+\sigma}(\Gamma_\kappa^0)\nabla = X_{1+\sigma}f_\lambda(H_0)(\Gamma_\kappa^0)\nabla+X_{1+\sigma}g_\lambda(H_0)(\Gamma_\kappa^0)\nabla.$$

By the argument used in the proof of Lemma 4.9, $\|X_\alpha(H_0+\lambda)^{-1}\|=O(\lambda^{-1/4})$ for any $\alpha>3/2$. Take α close enough to 3/2. Then, by interpolation, it follows from (2.4) that $\|X_{1+\sigma}(H_0+\lambda)^{-1/2-\sigma}\|=O(\lambda^{-\nu})$ for any $\nu>\sigma/2$. Thus, we have

$$\|X_{1+\sigma}f_\lambda(H_0)(\Gamma_\kappa^0)\nabla\| = \varepsilon^{-1}O(\lambda^{\sigma-\nu}).$$

Since $\|X_{1+\sigma}g_\lambda(H_0)(\Gamma_\kappa^0)\nabla\|=O(1)$ as $\lambda\to 0$, this completes the proof. □

We can also estimate the norm of $X_{1+\sigma}(\Gamma_\kappa^0)\nabla$ as $\|X_{1+\sigma}(\Gamma_\kappa^0)\nabla\|=\varepsilon^{-1/2}O(1)$ as $\lambda\to 0$. Thus, we have by Lemmas 4.11, (i), and 4.13 and by (4.8) that

$$\|X_{1+\sigma}(\Gamma_\kappa^0)U_2(\varepsilon)(\Gamma_\kappa)^2 X_{1+\sigma}\| \leqq C\{\delta_0\|X_{1+\sigma}(\Gamma_\kappa)^2 X_{1+\sigma}\|+\varepsilon^{-2}\lambda^{-1+\tau}\}$$

for any τ, $0<\tau<\sigma/2$. Since δ_0 is small enough, this, together with Lemmas 4.9, 4.10 and 4.12, implies that

$$\|X_{1+\sigma}(\Gamma_\kappa)^2 X_{1+\sigma}\| = \varepsilon^{-2}O(\lambda^{-1+\tau}).$$

Thus, step (c) and hence the proof of Lemma 2.3 are now complete. □

§ 5. Principle of limiting amplitude.

In this section we study, as an application of the main theorem, the time asymptotics of solution to the following Cauchy problem:

$$(5.1)\qquad (\partial/\partial t)^2 w + Lw = \exp(-it\sqrt{\omega})f\,, \qquad \omega>0\,,$$

with initial conditions $w|_{t=0}=(\partial/\partial t)w|_{t=0}=0$, where L is defined by (0.2) and f is assumed to be in L^2_β, $\beta>1/2$.

THEOREM 5.1 (*principle of limiting amplitude*). *Assume* (a.0)~(a.1) *and* (ρ.0)~(ρ.2). *Let* $w=w(t, x)$ *be the solution to* (5.1) *with* $f\in L^2_\beta$, $\beta>1/2$. *Then* $w(t, x)$ *behaves like*

$$w = \exp(-it\sqrt{\omega})R(\omega+i0\,;\,L)f+o(1)\,, \qquad t\to\infty\,,$$

strongly in $L^2_{-\alpha}$, $\alpha>1$.

We may assume, without loss of generality, that $1/2<\beta<(1+\theta)/2$. The theorem above follows from the general theorem due to Eidus [**3**], Chapter 1, if the following two conditions (C.1) (low frequency behavior) and (C.2) (local Hölder continuity) are verified for the resolvent $R(\lambda\pm i0\,;\,L)$, $\lambda>0$:

(C.1) There exists d, $0<d<1/2$, such that

$$\|R(\lambda\pm i0\,;\,L)\|_{\beta\to-\alpha} = O(\lambda^{-d})\,, \qquad \lambda\to0\,;$$

(C.2) There exists γ, $0<\gamma<1$, such that

$$\|R(\lambda\pm i0\,;\,L)-R(\mu\pm i0\,;\,L)\|_{\beta\to-\alpha} \leqq C|\lambda-\mu|^\gamma$$

for λ, $\mu\in I$, $I\subset(0, \infty)$ being a compact interval fixed arbitrarily.

Condition (C.1) has been already verified and we will show in Appendix that (C.2) is really satisfied under the assumption of the theorem. Accepting this as proved, we shall give only a sketch of the proof. (For details, see [**3**].)

PROOF OF THEOREM 5.1. Let $\Theta(\lambda)$, $\lambda>0$, be the spectral resolution associated to L; $L=\int_0^\infty \lambda d\Theta(\lambda)$. Then

$$\Theta'(\lambda) = (d/d\lambda)\Theta(\lambda) = (2\pi i)^{-1}\{R(\lambda+i0\,;\,L)-R(\lambda-i0\,;\,L)\}\,, \qquad \lambda>0\,.$$

We extend $\Theta'(\lambda)$ to $\lambda\leqq0$ as $\Theta'(\lambda)=0$. The solution $w(t, x)$ to (5.1) is represented as $w=w_1(t, x)-iw_2(t, x)$, where

$$w_1 = \int[(\exp(-it\sqrt{\omega})-\exp(-it\sqrt{\lambda}))/(\lambda-\omega)]\Theta'(\lambda)fd\lambda\,,$$

$$w_2 = \int[(\sin t\sqrt{\omega})/(\lambda+\sqrt{\lambda\omega})]\Theta'(\lambda)fd\lambda\,.$$

By (C.1), the Riemann-Lebesgue theorem shows that

(5.2) $$w_2(t, x) = o(1), \qquad t\to\infty,$$

strongly in $L^2_{-\alpha}$. By (C.2), $R(\omega+i0\,;\,L)f$ is expressed as

$$R(\omega+i0\,;\,L)f = i\pi\Theta'(\omega)f + \text{p. v.}\int(\lambda-\omega)^{-1}\Theta'(\lambda)fd\lambda,$$

where the integral is taken in the principal value sense. We can also show, making use of (C.2) again, that as $t\to\infty$

$$\text{p. v.}\int[\exp(-it\sqrt{\lambda})/(\lambda-\omega)]\Theta'(\lambda)fd\lambda = -i\pi\exp(-it\sqrt{\omega})\Theta'(\omega)f + o(1)$$

strongly in $L^2_{-\alpha}$ and hence

$$w_1 = \exp(-it\sqrt{\omega})R(\omega+i0\,;\,L)f + o(1), \qquad t\to\infty,$$

strongly in $L^2_{-\alpha}$. This, together with (5.2), proves the theorem. □

Appendix 1. Absence of eigenvalues.

In this appendix we shall prove the statement (i) of Theorem 0. If this is verified, then statement (ii) (principle of limiting absorption) follows from the general theorem (Theorem 30.2.10) due to Hörmander [5] by making use of the same argument as in the proof of Lemma 2.1 (see also Remark after Lemma 2.1).

The proof is based on the idea of Froese and Herbst [4], where the Mourre commutator method has been efficiently used to prove the absence of positive eigenvalues for N-body Schrödinger operators.

Assume that $\phi\in H^2(R^3_x)$ is the eigenfunction associated with eigenvalue $\lambda>0\,;\,L\phi=\lambda\phi$. The proof consists of the following three steps:

(a) To show that $\langle x\rangle^\alpha\phi\in L^2$ for any $\alpha>0$;

(b) To show that $\exp(\alpha\langle x\rangle)\phi\in L^2$ for any $\alpha>0$;

(c) To show that $\phi=0$.

We prove (c) only. (a) and (b) are proved in the same way as in [4] and (c) is also proved by a slight modification.

The proof is done by contradiction. Assume that ϕ does not vanish identically. Let H be defined by (1.1) and let $V(x)=E(x)-1$, $E=a(x)^{-2}\rho(x)^{-1}$. Then we can rewrite $L\phi=\lambda\phi$ as $H\phi-\lambda V\phi=\lambda\phi$. By the unique continuation theorem (see, for example, Theorem 6.5.1, [2]), we may assume that ϕ is not of compact support, so that

(1) $$\int_{|x|\geq 3R}|\phi(x)|^2dx > c_R > 0$$

for $R\gg 1$ large enough. Let $\chi_R\in C^\infty(R_x^3)$, $0\leqq\chi_R\leqq 1$, be such that $\chi_R=0$ for $|x|<R$ and $\chi_R=1$ for $|x|\geqq 2R$. Set $\phi_R=\chi_R\phi$. Then ϕ_R obeys the equation

$$H\phi_R-\lambda V\phi_R=\lambda\phi_R+g_R,$$

where

$$g_R=[\chi_R, \nabla](1/\rho)\cdot\nabla\phi+\nabla\cdot((1/\rho)[\chi_R, \nabla])\phi.$$

The function g_R has support in $\{x: |x|<2R\}$ and satisfies $|g_R|_0\leqq C$ for C independent of R.

Let $F_\alpha(x)=\alpha|x|$, $\alpha>1$, so that $|\nabla F_\alpha|^2=\alpha^2$. We further define $\psi_{\alpha R}$ as

$$\psi_{\alpha R}=\exp(F_\alpha)\phi_R/|\exp(F_\alpha)\phi_R|_0,$$

so that $|\psi_{\alpha R}|_0=1$. This satisfies the equation

(2) $$H\psi_{\alpha R}-\alpha^2(1/\rho)\psi_{\alpha R}-\lambda V\psi_{\alpha R}+B_\alpha\psi_{\alpha R}=\lambda\psi_{\alpha R}+g_{\alpha R},$$

where $g_{\alpha R}=\exp(F_\alpha)g_R/|\exp(F_\alpha)\psi_R|_0$ and

$$B_\alpha=(1/\rho)\nabla F_\alpha\cdot\nabla+\nabla\cdot((1/\rho)\nabla F_\alpha).$$

The operator B_α satisfies the relation $B_\alpha^*+B_\alpha=0$ and takes the form

(3) $$B_\alpha=2i\alpha|x|^{-1}(1/\rho)A+\alpha x\cdot\nabla((1/\rho)|x|^{-1}),$$

where A is the generator of dilation unitary group. By (1), the function $g_{\alpha R}$ satisfies the estimate

(4) $$|\langle x\rangle g_{\alpha R}|_0\leqq C_R\exp(-(1/2)\alpha R).$$

LEMMA A.1. *Denote by* $\langle\ ,\ \rangle$ *the* L^2 *scalar product. Then:*

(0) $\langle\nabla\psi_{\alpha R}, \nabla\psi_{\alpha R}\rangle\leqq\gamma_0\alpha^2$, $\quad\gamma_0>0$,

(i) $\langle H\psi_{\alpha R}, \psi_{\alpha R}\rangle\geqq\gamma_1\alpha^2$, $\quad\gamma_1>0$,

(ii) $i\langle[H, A]\psi_{\alpha R}, \psi_{\alpha R}\rangle\geqq\gamma_2\alpha^2$, $\quad\gamma_2>0$,

(iii) $\mathrm{Im}\langle B_\alpha\psi_{\alpha R}, A\psi_{\alpha R}\rangle=2\alpha\langle|x|^{-1}(1/\rho)A\psi_{\alpha R}, A\psi_{\alpha R}\rangle+R^{-\theta}O(\alpha^2)$,

(iv) $\mathrm{Im}\langle\alpha^2(1/\rho)\psi_{\alpha R}, A\psi_{\alpha R}\rangle=R^{-\theta}O(\alpha^2)$,

(v) $\mathrm{Im}\langle V\psi_{\alpha R}, A\psi_{\alpha R}\rangle=R^{-\theta}O(\alpha)$,

for $\alpha>\alpha_R\gg 1$, *where* γ_j, $0\leqq j\leqq 2$, *are independent of* R.

PROOF. Since $B_\alpha+B_\alpha^*=0$, (0) and (i) follow from (2) and (4). Note that $\psi_{\alpha R}$ vanishes on $\{x: |x|<R\}$. For such a function $\psi\in H^2(R_x^3)$, we have $i\langle[H, A]\psi, \psi\rangle\geqq\gamma_3\langle H\psi, \psi\rangle$, $\gamma_3>0$. Hence, (ii) follows from (i) at once. We have, by assumption (ρ.2) and by (0), that

$$\alpha\,\mathrm{Im}\langle(x\cdot\nabla((1/\rho)|x|^{-1}))\psi_{\alpha R}, A\psi_{\alpha R}\rangle=R^{-\theta}O(\alpha^2).$$

This, together with (3), implies (iii). Estimate (iv) follows from assumption (ρ.2). By assumptions (a.1) and (ρ.2), we can decompose $V(x)$ as $V=V_1(x)+V_2(x)$, so that $V_1=O(|x|^{-(1+\theta)})$ and $\partial_x^\alpha V_2=O(|x|^{-(|\alpha|+\theta)})$, $0\leqq|\alpha|\leqq 1$, as $|x|\to\infty$.

Hence, (v) can be easily proved. □

We evaluate the quantity $i\langle[H, A]\psi_{\alpha R}, \psi_{\alpha R}\rangle$. We write

$$i\langle[H, A]\psi_{\alpha R}, \psi_{\alpha R}\rangle = i\{\langle A\psi_{\alpha R}, H\psi_{\alpha R}\rangle - \langle H\psi_{\alpha R}, A\psi_{\alpha R}\rangle\}.$$

By use of (2) and (4) and of Lemma A.1, (iii)$\sim$(v), we obtain

$$i\langle[H, A]\psi_{\alpha R}, \psi_{\alpha R}\rangle = -4\alpha\langle|x|^{-1}(1/\rho)A\psi_{\alpha R}, A\psi_{\alpha R}\rangle + R^{-\theta}O(\alpha^2).$$

On the other hand, by Lemma A.1, (ii),

$$i\langle[H, A]\psi_{\alpha R}, \psi_{\alpha R}\rangle \geqq \gamma_2\alpha^2.$$

This contradicts the fact that ϕ does not vanish identically and hence the absence of eigenvalues is now proved.

Appendix 2. Local Hölder continuity.

We begin by recalling the notations: $E(x) = a(x)^{-2}\rho(x)^{-1}$; $E = E_0(x) + V_0(x)$ with $V_0 = O(|x|^{-(1+\theta)})$, $|x| \to \infty$; $H(\lambda) = H - \lambda(E_0 - 1)$, H being defined by (1.1). Under these notations, we have $E(L-\lambda) = H(\lambda) - \lambda V_0 - \lambda$ and hence

$$R(\lambda \pm i0; L)E^{-1}\{\mathrm{Id} - \lambda V_0 R(\lambda \pm i0; H(\lambda))\} = R(\lambda \pm i0; H(\lambda)).$$

Thus, to prove the Hölder continuity (C.2), it suffices to show the following two facts: For β, $1/2 < \beta < (1+\theta)/2$,

(F.1) $R(\lambda \pm i0; H(\lambda)): L^2_\beta \to L^2_{-\beta}$ is locally Hölder continuous;

(F.2) $\mathrm{Id} - \lambda V_0 R(\lambda \pm i0; H(\lambda)): L^2_\beta \to L^2_\beta$ is invertible.

(F.2) follows from the principle of limiting absorption for L. In fact, we have

$$(\mathrm{Id} - \lambda V_0 R(\lambda \pm i0; H(\lambda))^{-1} = \mathrm{Id} + \lambda V_0 R(\lambda \pm i0; L)E^{-1}.$$

Of course, we can give a direct proof of (F.2) and, as a result, the existence of the boundary values $R(\lambda \pm i0; L)$ is obtained. However, we do not do this here, because it is not the aim here to prove the principle of limiting absorption for L.

(F.1) is proved in the same way as in the Schrödinger operators case ([**11**]). We give only a sketch for the "+" case.

We first note that $\|X_\beta R(\lambda + i0; H(\lambda))X_\beta\|$, $\beta > 1/2$, is locally bounded in $\lambda > 0$. Hence, to prove (F.1), it suffices, by interpolation, to show the following fact (F.1′):

(F.1′) $R(\lambda + i0; H(\lambda)): L^2_1 \to L^2_{-1}$ is locally Hölder continuous.

We have only to prove this only for λ in a small compact interval $I_0 = [\lambda_0 - \delta, \lambda_0 + \delta]$, $\lambda_0 > 0$ being fixed. We define

$$\rho_\varepsilon(x) = 1 + \chi(\varepsilon x)(\rho(x) - 1), \qquad 0 \leqq \varepsilon \ll 1,$$

in the same way as $\rho_{1\varepsilon}$ was defined in Section 3. We further define $H(\varepsilon; \lambda)$ as

$$H(\varepsilon; \lambda) = -\nabla\cdot(1/\rho_\varepsilon)\nabla - \lambda(E_{0\varepsilon}-1).$$

Let $f_0(s)\in C_0^\infty(R_s^1)$, $0\leqq f_0\leqq 1$, be a function such that f_0 has support in $(\lambda_0-3\delta, \lambda_0+3\delta)\subset(0, \infty)$ and $f_0=1$ on $[\lambda_0-2\delta, \lambda_0+2\delta]$. Then $f_0(H(\lambda))$ is continuous in $\lambda\in I_0$ in the $L^2\to L^2$ operator norm. By the same argument as in Appendix 1, we can show that $H(\lambda)$ has no positive eigenvalues and hence, for any compact operator K: $L^2\to L^2$,

$$\|f_0(H(\lambda))Kf_0(H(\lambda))\| = o(1), \qquad \delta\to 0,$$

uniformly in $\lambda\in I_0$. Thus we can take δ so small that

$$M(\varepsilon; \lambda) = if_0(H(\lambda))[H(\varepsilon; \lambda), A]f_0(H(\lambda)) \geqq \gamma f_0(H(\lambda))^2, \qquad \gamma>0,$$

in the form sense. This enables us to define $G_\kappa(\varepsilon; \lambda)$: $L^2\to L^2$ by

$$G_\kappa(\varepsilon; \lambda) = (H(\lambda)-\lambda-i\kappa-i\varepsilon M(\varepsilon; \lambda))^{-1}$$

for κ, $0<\kappa\leqq 1$, and ε, $0\leqq\varepsilon\leqq\varepsilon_0$. We set $F_\kappa(\varepsilon)=X_1G_\kappa(\varepsilon)X_1$. By an argument similar to that in Section 3, we see that $F_\kappa(\varepsilon)$ obeys the differential inequality as in (3.8) and hence it follows that

$$\|X_1(G_\kappa(\varepsilon; \lambda)-R(\lambda+i\kappa; H(\lambda)))X_1\| = O(\varepsilon^\theta)$$

uniformly in κ and $\lambda\in I_0$. As is easily seen, $\|M(\varepsilon; \lambda)-M(\varepsilon; \mu)\|=O(|\lambda-\mu|)$, $(\lambda, \mu)\in I_0\times I_0$, uniformly in ε. Since $\|X_1G_\kappa(\varepsilon; \lambda)\|=O(\varepsilon^{-1/2})$, we have

$$\|X_1(G_\kappa(\varepsilon; \lambda)-G_\kappa(\varepsilon; \mu))X_1\| = |\lambda-\mu|O(\varepsilon^{-1}).$$

Thus, if we take ε as $\varepsilon=|\lambda-\mu|^\nu$, $\nu=1/(1+\theta)$, it then follows that

$$\|X_1(R(\lambda+i\kappa; H(\lambda))-R(\mu+i\kappa; H(\mu)))X_1\| = O(|\lambda-\mu|^{\nu\theta})$$

uniformly in κ. This proves (F.1′).

References

[1] C. O. Bloom, A rate of approach to the steady state of solutions of second-order hyperbolic equations, J. Differential Equations, **19** (1975), 296-329.

[2] M. S. P. Eastham and H. Kalf, Schrödinger-type operators with continuous spectra, Reseach note in Math., **65**, Pitman, Boston · London · Melbourne, 1982.

[3] D. M. Eidus, The principle of limiting amplitude, Russian Math. Surveys, **24** (1969), 97-167.

[4] R. Froese and I. Herbst, Exponential bounds and absence of positive eigenvalues for N-body Schrödinger operators, Comm. Math. Phys., **87** (1982), 429-447.

[5] L. Hörmander, The analysis of linear partial differential operators IV, Springer, 1984.

[6] A. Jensen and T. Kato, Spectral properties of Schrödinger operators and time-decay of the wave functions, Duke Math. J., **46** (1979), 583-611.

[7] A. Jensen, E. Mourre and P. Perry, Multiple commutator estimates and resolvent smoothness in quantum scattering theory, Ann. Inst. H. Poincaré Phys. Théor.,

41 (1984), 207-225.

[8] K. Mochizuki, Growth properties of solutions of second order elliptic differential equations, J. Math. Kyoto Univ., **16** (1976), 351-373.

[9] E. Mourre, Absence of singular continuous spectrum for certain selfadjoint operators, Comm. Math. Phys., **78** (1981), 391-408.

[10] M. Murata, Asymptotic expansions in time for solutions of Schrödinger-type equations, J. Func. Anal. Appl., **49** (1982), 10-56.

[11] P. Perry, I. M. Sigal and B. Simon, Spectral analysis of N-body Schrödinger operators, Ann. of Math., **114** (1981), 519-567.

[12] H. Tamura, Principle of limiting absorption for N-body Schrödinger operators,—— a remark on the commutator method——, Lett. Math. Phys., **17** (1989), 31-36.

[13] J. Uchiyama, Polynomial growth or decay of eigenfunctions of second-order elliptic operators, Publ. RIMS. Kyoto Univ., **23** (1987), 975-1006.

Hideo TAMURA
Department of Mathematics
Ibaraki University
Mito, Ibaraki 310
Japan