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Hardy-Sobolev spaces and maximal functions

Dedicated to Professor Hiroshi Fujita on the occasion of his sixtieth birthday

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§1. Introduction.

Throughout this paper, the letter Ω denotes an open subset of $\boldsymbol{R}^n$. No further restrictions are imposed on Ω unless the contrary is explicitly stated. Other notations used in this paper will be explained at the last part of this section.

Let f be a measurable function on Ω and let λ and q be positive numbers. Following A.P. Calderón [1], we define $N_q^\lambda(f)(x)$, $x\in\Omega$, as follows. Fix an $x\in\Omega$. For polynomial functions P on $\boldsymbol{R}^n$ of degree less than λ, we set

$$N_q^\lambda(f, P)(x) = \sup_{\substack{Q:\,\text{cube}\\ x\in Q\subset\Omega}} \frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_Q |f(y)-P(y)|^q dy\right)^{1/q}.$$

If there exists a polynomial P of degree less than λ for which $N_q^\lambda(f, P)(x)$ is finite, then such P is unique. If this is the case, we denote the unique P by P_x and set

$$N_q^\lambda(f)(x) = N_q^\lambda(f, P_x)(x).$$

If $N_q^\lambda(f, P)(x)=\infty$ for all polynomials P of degree less than λ, then we set $N_q^\lambda(f)(x)=\infty$.

The following theorem is due to Calderón (see [1; Theorem 4 and Lemma 7]).

THEOREM A. *Let k be a positive integer, $1<p\leqq\infty$, $1\leqq q\leqq p$ and $f\in L_{\mathrm{loc}}^q(\Omega)$. Then $N_q^k(f)$ belongs to $L^p(\Omega)$ if and only if all the weak derivatives $\partial^\alpha f$ of order $|\alpha|=k$ belong to $L^p(\Omega)$.*

As a matter of fact, Calderón's statement is slightly different from the one given above. He defined $N_q^\lambda(f)(x)$ by using balls in place of cubes and stated the result only for $1<q\leqq p<\infty$. Calderón's argument, however, actually covered the case $q=1$ or $p=\infty$, and still holds true if one replaces "balls" by "cubes".

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The purpose of the present paper is to generalize Theorem A to the case $p \leqq 1$. We shall explain a typical one of our results.

Take a function ϕ such that

$$\phi \in C_0^\infty(B(0, 1)) \quad \text{and} \quad \int \phi(x)dx = 1. \tag{1.1}$$

For $f \in \mathcal{D}'(\Omega)$, we define $f_{\phi,\Omega}^+(x)$, $x \in \Omega$, by

$$f_{\phi,\Omega}^+(x) = \sup\{|\langle f, (\phi)_t(x-\cdot)\rangle|\,;\ 0<t<\mathrm{dis}(x, \Omega^c)\}.$$

For p with $0<p\leqq 1$, we denote by $H^p(\Omega)$ the set of $f \in \mathcal{D}'(\Omega)$ for which $f_{\phi,\Omega}^+ \in L^p(\Omega)$. The set $H^p(\Omega)$ does not depend on the choice of ϕ (see [**9**]). (In Section 2, we shall summarize some properties of $H^p(\Omega)$ and other related spaces.)

One of our main results reads as follows.

THEOREM 1. *Let k be a positive integer and $0<p\leqq 1$, and suppose $1+k/n>1/p$. Then*

(i) *if f is a measurable function on Ω and $N_p^k(f) \in L^p(\Omega)$, then $f \in L_{\mathrm{loc}}^1(\Omega)$ and the weak derivatives $\partial^\alpha f$ of order $|\alpha|=k$ belong to $H^p(\Omega)$;*

(ii) *conversely, if $f \in \mathcal{D}'(\Omega)$ and $\partial^\alpha f \in H^p(\Omega)$ for $|\alpha|=k$, then $f \in L_{\mathrm{loc}}^1(\Omega)$ and $N_p^k(f) \in L^p(\Omega)$.*

We also give variants of Theorem 1, some of which hold in the case $1+k/n \leqq 1/p$ as well. The main results of the present paper will be found in Section 4.

DeVore and Sharpley [**3**] introduced a maximal function which is equivalent to $N_q^k(f)$ and, using it, studied the properties of the space

$$C_p^k(\Omega) = \{f \in L^p(\Omega)\,;\ N_p^k(f) \in L^p(\Omega)\}$$

for $k, p>0$. Our results will give a characterization of this space in the case that $\mathrm{dis}(x, \Omega^c)$, $x \in \Omega$, is bounded and that k and p satisfy the conditions of Theorem 1; see Remark h) in Section 4. We shall use some results of [**3**] in proving our results; the results of [**3**] which we need will be summarized in Section 3.

Calderón and Scott [**2**] used the maximal function $N_q^k(f)$ and the notion of Peano derivative to establish Sobolev type inequalities. Our results will give a relation between the distributional derivative and the Peano derivative; see Theorem 4 in Section 4.

NOTATIONS. The following notations are used throughout this paper. The letter C denotes a positive constant, which may be different in each occasion. The constant C depends only on the dimension n of the Euclidean space $\boldsymbol{R}^n$

and other explicitly indicated parameters. The letter $\boldsymbol{N}$ denotes the set of positive integers. A cube is a subset of $\boldsymbol{R}^n$ of the form $Q=\{(x_i)\in\boldsymbol{R}^n\,;\ a_i<x_i\leqq a_i+t\ (i=1,\cdots,n)\}$ with $(a_i)\in\boldsymbol{R}^n$ and $t>0$. We denote the sidelength of Q by $l(Q)=t$ and the center of Q by $x_Q=(a_i+t/2)$. If Q is a cube and A is a positive number, then AQ denotes the cube with the same center as Q and with $l(AQ)=Al(Q)$. If $a_i=2^jk_i\ (i=1,\cdots,n)$ and $t=2^j$ with $k_1,\cdots,k_n$ and j integers, then the corresponding cube Q is called a dyadic cube. If Q and Q' are dyadic cubes such that $Q\subset Q'$ and $l(Q')=2l(Q)$, then Q' is called the dyadic double of Q. For $x\in\boldsymbol{R}^n$ and $t>0$, the ball $B(x,t)$ is defined by $B(x,t)=\{y\in\boldsymbol{R}^n\,;\ |x-y|<t\}$. For $k\in\boldsymbol{N}\cup\{0\}$, we denote by $\mathcal{P}_k$ the set of polynomial functions on $\boldsymbol{R}^n$ of degree at most k. We use the multi-index notation in the customary way (see, for example, [**6**; page XV]). If E is a measurable subset of $\boldsymbol{R}^n$ and f is a measurable function defined on a set including E, then we shall abbreviate $\int_E f(x)dx$ to $\int_E f$ and define $\|f\|_{p,E}$, $0<p\leqq\infty$, by

$$\|f\|_{p,E}=\left(\int_E|f|^p\right)^{1/p}\qquad\text{for } 0<p<\infty$$

and by $\|f\|_{\infty,E}=\text{ess sup}\{|f(x)|\,;\ x\in E\}$. If $E=\boldsymbol{R}^n$, then we shall abbreviate $\int_E f$ and $\|f\|_{p,E}$ to $\int f$ and $\|f\|_p$ respectively. The Lebesgue measure of $E\subset\boldsymbol{R}^n$ is denoted by $|E|$. If p is a positive number and f is a measurable function on Ω, then $M_p(f)(x)$, $x\in\Omega$, is defined by

$$M_p(f)(x)=\sup_{t>0}\left(t^{-n}\int_{B(x,t)\cap\Omega}|f|^p\right)^{1/p}.$$

If $f\in\mathcal{D}'(\Omega)$ (i.e., f is a distribution on Ω) and $\psi\in C_0^\infty(\Omega)$, then $f(\psi)$ is denoted by $\langle f,\psi\rangle$. For a function f on $\boldsymbol{R}^n$ and for $t>0$, the function $(f)_t$ is defined by $(f)_t(x)=t^{-n}f(t^{-1}x)$, $x\in\boldsymbol{R}^n$.

§ 2. Hardy spaces over domains.

If ϕ is a function satisfying (1.1), $0<a\leqq\infty$, $0<b\leqq1$ and $f\in\mathcal{D}'(\Omega)$, then we define $f_{\phi,\Omega}^{+;a,b}(x)$, $x\in\Omega$, by

$$f_{\phi,\Omega}^{+;a,b}(x)=\sup\{|\langle f,(\phi)_t(x-\cdot)\rangle|\,;\ \ 0<t<\min\{a,b\operatorname{dis}(x,\Omega^c)\}\}.$$

Notice that $f_{\phi,\Omega}^{+;\infty,1}$ coincides with $f_{\phi,\Omega}^{+}$ of Section 1. For $x\in\boldsymbol{R}^n$, $t>0$ and $m\in\boldsymbol{N}$, we denote by $T_{m,\Omega}(x,t)$ the set of those $\psi\in C_0^\infty(B(x,t)\cap\Omega)$ such that $\|\partial^\alpha\psi\|_\infty\leqq t^{-n-|\alpha|}$ for $|\alpha|\leqq m$. If $0<a\leqq\infty$, $m\in\boldsymbol{N}$ and $f\in\mathcal{D}'(\Omega)$, then we define $f_{m,\Omega}^{*;a}(x)$, $x\in\Omega$, by

$$f_{m,\Omega}^{*;a}(x)=\sup\{|\langle f,\psi\rangle|\,;\ \ \psi\in\bigcup_{0<t<a}T_{m,\Omega}(x,t)\}.$$

Let ϕ be a function satisfying (1.1), $0<a,\ a'\leqq\infty$, $0<b\leqq 1$ and $m\in\boldsymbol{N}$. Suppose that both a and a' are finite or that both are equal to infinity. If $a=a'=\infty$, then we set $a'/a=1$. The inequality

$$f^{+;a,b}_{\phi;\Omega}(x)\leqq C_{\phi,m}f^{*;a}_{m,\Omega}(x) \tag{2.1}$$

is obvious. If $0<q<1$ and $m>n/q-n$, then

$$f^{*;a'}_{m,\Omega}(x)\leqq C_{\phi,a'/a,b,q}M_q(f^{+;a,b}_{\phi;\Omega})(x)\,; \tag{2.2}$$

this inequality can be easily deduced from the theorem of [8] (cf. in particular Proof of Corollary 1 at p. 58 of [8]). From (2.1) and (2.2), using the L^p-boundedness of the operator M_q for $q<p$, we can deduce the following results. If $0<p\leqq\infty$ and $m>n/p-n$, then

$$\|f^{*;a'}_{m,\Omega}\|_{p,\Omega}\leqq C_{\phi,a'/a,b,p}\|f^{+;a,b}_{\phi;\Omega}\|_{p,\Omega}\,. \tag{2.3}$$

If $0<p\leqq\infty$, $\phi'\in C_0^\infty(B(0,1))$, $\int\phi'=1$ and $0<b'\leqq 1$, then

$$\|f^{+;a',b'}_{\phi',\Omega}\|_{p,\Omega}\leqq C_{\phi,\phi',a'/a,b,p}\|f^{+;a,b}_{\phi;\Omega}\|_{p,\Omega}\,. \tag{2.4}$$

This latter result shows in particular that $H^p(\Omega)$ defined in Section 1 does not depend on the choice of the function ϕ.

Let $0<p\leqq 1$. We denote by $H^p_{\mathrm{loc}}(\Omega)$ the set of $f\in\mathcal{D}'(\Omega)$ which have the following property: For every $x\in\Omega$, there exists an open subset U of Ω such that $x\in U$ and $f|U\in H^p(U)$. We denote by $h^p(\Omega)$ the set of $f\in\mathcal{D}'(\Omega)$ for which $f^{+;1,1}_{\phi;\Omega}\in L^p(\Omega)$, where ϕ is a function satisfying (1.1). The inequality (2.4) shows that $h^p(\Omega)$ does not depend on the choice of ϕ. For $f\in\mathcal{D}'(\Omega)$, we set

$$\|f\|_{p,(\phi),\Omega}=\|f^{+;\infty,1}_{\phi;\Omega}\|_{p,\Omega}\quad\text{and}\quad\|f\|'_{p,(\phi),\Omega}=\|f^{+;1,1}_{\phi;\Omega}\|_{p,\Omega}\,.$$

For other equivalent definitions of $H^p(\Omega)$, $h^p(\Omega)$ and $H^p_{\mathrm{loc}}(\Omega)$, see the Remarks a)~c) given below.

For $f\in\mathcal{D}'(\Omega)$ and $x\in\Omega$, we say that $[f](x)$ exists if $\lim_{t\downarrow 0}\langle f,(\phi)_t(x-\cdot)\rangle$ exists for all ϕ satisfying (1.1) and if this limit does not depend on ϕ; if this is the case, we set

$$[f](x)=\lim_{t\downarrow 0}\langle f,(\phi)_t(x-\cdot)\rangle.$$

In [9], we showed that if $\Omega\neq\boldsymbol{R}^n$ then $C_0^\infty(\Omega)$ is dense in $H^p(\Omega)$. Hence, by the standard argument for the almost everywhere convergence (cf. for example [12; Chapt. II, Theorem 3.12]), we see the following fact: If $0<p\leqq 1$ and $f\in H^p_{\mathrm{loc}}(\Omega)$, then $[f](x)$ exists for almost every $x\in\Omega$ and moreover

$$\lim_{t\downarrow 0}\sup_{\phi\in T_{m,\Omega}(x,t)}\left|\langle f,\phi\rangle-[f](x)\int\phi\right|=0 \tag{2.5}$$

for $m>n/p-n$ and for almost every $x\in\Omega$. It is well known that if $f\in L_{\mathrm{loc}}^{1}(\Omega)$ then $[f](x)$ exists and is equal to $f(x)$ for almost every $x\in\Omega$.

REMARKS. a) If $\Omega=\boldsymbol{R}^{n}$, then our $H^{p}(\Omega)$ coincides with the usual $H^{p}(\boldsymbol{R}^{n})$ and our $h^{p}(\Omega)$ coincides with the space $h^{p}(\boldsymbol{R}^{n})$ of Goldberg [**4**]. Our $H_{\mathrm{loc}}^{p}(\boldsymbol{R}^{n})$ coincides with $\mathscr{H}_{p}(\boldsymbol{R}^{n})$ of Peetre [**10**]. Peetre [**10**] indicated the definition of $\mathscr{H}_{p}(M)$ for arbitrary smooth manifold M. If one considers Ω as an open submanifold of $\boldsymbol{R}^{n}$, then our $H_{\mathrm{loc}}^{p}(\Omega)$ coincides with Peetre's $\mathscr{H}_{p}(\Omega)$.

b) Denote by $H^{p}(\boldsymbol{R}^{n})|\Omega$ (resp. $h^{p}(\boldsymbol{R}^{n})|\Omega$) the set of the restrictions $f|\Omega$ of $f\in H^{p}(\boldsymbol{R}^{n})$ (resp. $f\in h^{p}(\boldsymbol{R}^{n})$). In [**9**], we showed that $H^{p}(\Omega)=H^{p}(\boldsymbol{R}^{n})|\Omega$ if Ω satisfies the following condition with some constant $A>1$: For each $x\in\Omega$ there exists an $x'\in\Omega^{c}$ such that $\mathrm{dis}(x, x')<A\,\mathrm{dis}(x, \Omega^{c})$ and $\mathrm{dis}(x', \Omega)>A^{-1}\,\mathrm{dis}(x, \Omega^{c})$. Proof of this result was based on the atomic decomposition for $H^{p}(\Omega)$. We can modify the argument of [**9**] to see that $h^{p}(\Omega)$ also admit certain atomic decomposition, and using it we see that $h^{p}(\Omega)=h^{p}(\boldsymbol{R}^{n})|\Omega$ if Ω satisfies the following condition: For $x\in\Omega$ with $\mathrm{dis}(x, \Omega^{c})<A^{-1}$, there exists an $x'\in\Omega^{c}$ such that $\mathrm{dis}(x, x')<A\,\mathrm{dis}(x, \Omega^{c})$ and $\mathrm{dis}(x', \Omega)>A^{-1}\,\mathrm{dis}(x, \Omega^{c})$ (here again A is a constant greater than 1).

c) Inequality (2.4) shows that $f\in h^{p}(\Omega)$ if and only if $f_{\phi;\Omega}^{+;a,b}\in L^{p}(\Omega)$ for some $a\in(0, \infty)$ and some $b\in(0,1]$. From this we see that $h^{p}(\Omega)=H^{p}(\Omega)$ if $\mathrm{dis}(x, \Omega^{c})$, $x\in\Omega$, is bounded.

§ 3. Some results of DeVore and Sharpley.

Let f be a measurable function on Ω and let λ and q be positive numbers. We denote by (λ) the largest integer less than λ. The maximal function $f_{\lambda,q}^{\flat}(x)$, $x\in\Omega$, is defined by

$$f_{\lambda,q}^{\flat}(x)=\sup_{\substack{Q:\ \mathrm{cube}\\ x\in Q\subset\Omega}}\ \inf_{P\in\mathscr{P}_{(\lambda)}}\frac{1}{|Q|^{\lambda/n}}\left(\frac{1}{|Q|}\int_{Q}|f-P|^{q}\right)^{1/q}$$

([**3**; p. 22]). We define $N_{q}^{\lambda,1}(f)(x)$ and $f_{\lambda;q}^{\flat;1}(x)$, $x\in\Omega$, in the following way: The definition of $N_{q}^{\lambda,1}(f)(x)$ (resp. $f_{\lambda;q}^{\flat;1}(x)$) is the same as that of $N_{q}^{\lambda}(f)(x)$ (resp. $f_{\lambda,q}^{\flat}(x)$) except that the sup is taken over the cubes Q such that $x\in Q\subset\Omega$ and $l(Q)<1$.

Theorems B, C and D below are due to DeVore and Sharpley [**3**]. Although they state the results only for $N_{q}^{\lambda}(f)$ and $f_{\lambda,q}^{\flat}$, their argument can be applied to $N_{q}^{\lambda,1}(f)$ and $f_{\lambda;q}^{\flat;1}$ without essential change.

THEOREM B. *Let p, q, r and λ be positive numbers such that $q<r$ and $1/r+\lambda/n>1/p$. Then*

$$\|f_{\lambda,q}^{\flat}\|_{p,\Omega}\leqq\|f_{\lambda,r}^{\flat}\|_{p,\Omega}\leqq C_{\lambda,p,q,r}\|f_{\lambda,q}^{\flat}\|_{p,\Omega}$$

and

$$\|f^{\flat;1}_{\lambda;q}\|_{p,\Omega} \leqq \|f^{\flat;1}_{\lambda;r}\|_{p,\Omega} \leqq C_{\lambda,p,q,r}\|f^{\flat;1}_{\lambda;q}\|_{p,\Omega}$$

for all measurable functions f on Ω.

This theorem follows from [**3**; Theorem 4.3].

THEOREM C ([**3**; Theorem 5.3]). *If λ and q are positive numbers, then*

$$f^{\flat}_{\lambda,q}(x) \leqq N^{\lambda}_{q}(f)(x) \leqq C_{\lambda,q} f^{\flat}_{\lambda,q}(x)$$

and

$$f^{\flat;1}_{\lambda;q}(x) \leqq N^{\lambda,1}_{q}(f)(x) \leqq C_{\lambda,q} f^{\flat;1}_{\lambda;q}(x)$$

for all measurable functions f on Ω and for all $x \in \Omega$.

We shall define the Peano derivative following DeVore-Sharpley [**3**; pp. 30-31]. Let f be a measurable function on Ω and $x \in \Omega$. Suppose there exist positive numbers q and λ, an open set $U \subset \Omega$ with $x \in U$, and a family of polynomials $\{P_Q;\ Q \text{ cube},\ x \in Q \subset U\}$ such that $\sup_Q \deg P_Q < \infty$ and

$$\sup_Q \frac{1}{|Q|^{\lambda/n}} \left(\frac{1}{|Q|} \int_Q |f - P_Q|^q \right)^{1/q} < \infty$$

(where the supremums are taken over the cubes Q satisfying $x \in Q \subset U$). Then, for multi-indices α with $|\alpha| < \lambda$, we set

$$D_\alpha f(x) = \lim_{\substack{l(Q) \to 0 \\ x \in Q \subset U}} \partial^\alpha_x P_Q(x).$$

This limit exists and $D_\alpha f(x)$ does not depend on λ, q, U and the family $\{P_Q\}$ (see [**3**; loc. cit.]). We call $D_\alpha f(x)$ the α-th Peano derivative of f at x.

THEOREM D ([**3**; Lemma 5.2 and Corollary 5.5]). *If f is a measurable function on Ω, q and λ are positive numbers, $x \in \Omega$ and $f^{\flat}_{\lambda,q}(x) < \infty$, then the α-th Peano derivatives of f at x exist for $|\alpha| < \lambda$ and the polynomial P_x that defines $N^{\lambda}_{q}(f)(x)$ is given by*

$$P_x(y) = \sum_{|\alpha| < \lambda} D_\alpha f(x) \frac{(y-x)^\alpha}{\alpha!}.$$

§4. Main results.

The following theorems are the main results of this paper.

THEOREM 2. *Let f be a measurable function on Ω, $0 < p \leqq 1$ and $k \in \boldsymbol{N}$, and suppose $1 + k/n > 1/p$ and $f^{\flat}_{k,p} \in L^p(\Omega)$. Then*

(i) *$f \in L^1_{\mathrm{loc}}(\Omega)$;*

(ii) *the weak derivatives $\partial^\alpha f$ with $|\alpha| = k$ belong to $H^p(\Omega)$ and*

$$\sum_{|\alpha|=k}\|\partial^\alpha f\|_{p,(\phi),\Omega} \leqq C_{k,p,\phi}\|f_{k,p}^{\flat}\|_{p,\Omega}.$$

THEOREM 3. *Let f be a measurable function on Ω, $0<p\leqq 1$ and $k\in\boldsymbol{N}$, and suppose $1+k/n>1/p$ and $f_{k,p}^{\flat,1}\in L^p(\Omega)$. Then*

(i) *$f\in L^1_{\mathrm{loc}}(\Omega)$;*

(ii) *the weak derivatives $\partial^\alpha f$ with $|\alpha|=k$ belong to $h^p(\Omega)$ and*

$$\sum_{|\alpha|=k}\|\partial^\alpha f\|'_{p,(\phi),\Omega} \leqq C_{k,p,\phi}\|f_{k,p}^{\flat,1}\|_{p,\Omega};$$

(iii) *if, in addition, $f\in L^p(\Omega)$, then $f\in h^p(\Omega)$ and*

$$\|f\|'_{p,(\phi),\Omega} \leqq C_{k,p,\phi}(\|f\|_{p,\Omega}+\|f_{k,p}^{\flat,1}\|_{p,\Omega}).$$

THEOREM 4. *Let $f\in\mathcal{D}'(\Omega)$, $0<p\leqq 1$ and $k\in\boldsymbol{N}$, and suppose $\partial^\alpha f\in H^p_{\mathrm{loc}}(\Omega)$ for $|\alpha|=k$. Then*

(i) *if $|\alpha|<k$, then $\partial^\alpha f\in H^p_{\mathrm{loc}}(\Omega)$ and $[\partial^\alpha f](x)$ exists almost everywhere on Ω;*

(ii) *we have*

$$\|[f]_{k,p}^{\flat}\|_{p,\Omega} \leqq C_{k,p,\phi}\sum_{|\alpha|=k}\|\partial^\alpha f\|_{p,(\phi),\Omega}$$

and

$$\|[f]_{k,p}^{\flat,1}\|_{p,\Omega} \leqq C_{k,p,\phi}\sum_{|\alpha|=k}\|\partial^\alpha f\|'_{p,(\phi),\Omega}$$

provided that the right-hand members are finite;

(iii) *if $|\alpha|<k$, then, for almost every $x\in\Omega$, the α-th Peano derivative $D_\alpha[f](x)$ exists and is equal to $[\partial^\alpha f](x)$;*

(iv) *if we set*

$$Q_x(y)=\sum_{|\alpha|<k}[\partial^\alpha f](x)\frac{(y-x)^\alpha}{\alpha!}$$

for almost every $x\in\Omega$ and if r is a positive number satisfying $1/r+k/n>1/p$, then

$$\limsup_{t\downarrow 0} t^{-k}\left(t^{-n}\int_{B(x,t)}|[f]-Q_x|^r\right)^{1/r}<\infty$$

for almost every $x\in\Omega$.

THEOREM 5. *Let $f\in\mathcal{D}'(\Omega)$, $0<p\leqq 1$ and $k\in N$, and suppose $1+k/n>1/p$ and $\partial^\alpha f\in H^p_{\mathrm{loc}}(\Omega)$ for $|\alpha|=k$. Then $f\in L^1_{\mathrm{loc}}(\Omega)$.*

Proofs of these theorems will be given in Section 5.

REMARKS. d) Theorem 1 of Section 1 follows from Theorems C, 2, 4 and 5.

e) The condition $1+k/n>1/p$ in Theorems 2 and 3 cannot be removed as we shall show in Section 5.

f) If $\mathrm{dis}(x, \Omega^c)$, $x\in\Omega$, is not bounded, then, in contrast with Theorem 3 (iii), the estimate

$$\|f\|_{p,(\phi),\Omega} \leqq C_{k,p,\phi,\Omega}(\|f\|_{p,\Omega}+\|f^{\flat}_{k,p}\|_{p,\Omega})$$

is false; see Section 5.

g) Theorem 4 is a generalization of the 1-dimensional result of Krotov [**5**].

h) Let $k\in\boldsymbol{N}$ and $0<p\leqq 1$. Let $c^k_p(\Omega)$ be the set of f in $L^p(\Omega)$ such that $f^{\flat}_{k;p}{}^{;1}\in L^p(\Omega)$, and let $w^k_p(\Omega)$ be the set of f in $h^p(\Omega)$ such that $\partial^\alpha f\in h^p(\Omega)$ for $|\alpha|=k$. Then, from Theorems 3, 4 and 5, we obtain the following result: If $1+k/n>1/p$, then both $c^k_p(\Omega)$ and $w^k_p(\Omega)$ are subsets of $L^1_{\mathrm{loc}}(\Omega)$ and $c^k_p(\Omega)=w^k_p(\Omega)$. Note that if $\mathrm{dis}(x, \Omega^c)$, $x\in\Omega$, is bounded, then $c^k_p(\Omega)$ coincides with the space $C^k_p(\Omega)$ of DeVore and Sharpley [**3**].

§ 5. Proofs of the main results.

We shall first prove Theorem 3. Before we proceed to the proof, we give a preliminary result concerning the decomposition of an open subset of $\boldsymbol{R}^n$.

For $a>0$ and $j\in\boldsymbol{N}\cup\{0\}$, we define $\mathcal{Q}^a_j=\mathcal{Q}^a_j(\Omega)$ as follows: $\mathcal{Q}^a_0$ is the set of maximal dyadic cubes Q such that $al(Q)<1$ and $aQ\subset\Omega$; if $j>0$, then $\mathcal{Q}^a_j$ is the set of the dyadic cubes whose dyadic doubles belong to $\mathcal{Q}^a_{j-1}$. For $x\in\Omega$, we set $d(x)=\min\{1, \mathrm{dis}(x, \Omega^c)\}$. Then we have the following lemma.

LEMMA 1. *If $a>6$, then $\mathcal{Q}^a_j$ have the following properties.*

(i) *For each j, $\mathcal{Q}^a_j$ is a disjoint family and the union of all the cubes in $\mathcal{Q}^a_j$ is equal to Ω.*

(ii) *If $Q\in\mathcal{Q}^a_j$ and $x\in(a/2)Q$, then $c_1\leqq 2^j al(Q)/d(x)\leqq c_2$.*

(iii) *If $Q\in\mathcal{Q}^a_k$, $R\in\mathcal{Q}^a_m$ and $3Q\cap 3R\neq\emptyset$, then $c_3\leqq 2^k l(Q)/2^m l(R)\leqq c_4$. If in addition $k\geqq m-1$, then $AQ\subset c_5 AR$ for all $A\geqq 1$.*

(iv) *If $0<A\leqq a/2$, then for each j the overlap of the family $\{AQ;\ Q\in\mathcal{Q}^a_j\}$ does not exceed $c_6(A+1)^n$ (i.e., no point of $\boldsymbol{R}^n$ belong to more than $c_6(A+1)^n$ of the cubes AQ, $Q\in\mathcal{Q}^a_j$).*

(v) *For each j, there exists a family of functions $\{\phi^j_Q;\ Q\in\mathcal{Q}^a_j\}$ such that $\phi^j_Q\in C^\infty_0(\boldsymbol{R}^n)$, $\mathrm{supp}\,\phi^j_Q\subset 3Q$, $0\leqq\phi^j_Q(x)\leqq 1$, $\sum_{Q\in\mathcal{Q}^a_j}\phi^j_Q(x)=1$ for all $x\in\Omega$, and $\|\partial^\alpha\phi^j_Q\|_\infty\leqq C_\alpha l(Q)^{-|\alpha|}$ for all α.*

Here $c_1, \cdots, c_6$ are positive constants depending only on the dimension n.

Proof of this lemma is left to the reader (cf., for example, [**11**; Chapt. VI, § 1]).

PROOF OF THEOREM 3. Take a positive number q such that $1+k/n>1/q>1/p$. We have $\|f^{\flat}_{k;q}{}^{;1}\|_{p,\Omega}\leqq\|f^{\flat}_{k;p}{}^{;1}\|_{p,\Omega}$ by Hölder's inequality.

For each cube $Q\subset\Omega$ with $l(Q)<1$, take a polynomial π_Q in $\mathcal{P}_{k-1}$ such that

$$\|f-\pi_Q\|_{q,Q}=\inf_{P\in\mathcal{P}_{k-1}}\|f-P\|_{q,Q}.$$

Then π_Q have the following estimates: First,

$$\|\pi_Q\|_{\infty,3Q}\leqq C_{k,q}\left(|Q|^{-1}\int_Q|f|^q\right)^{1/q};\tag{5.1}$$

secondly, if $b\geqq 1$, $Q\subset bQ'$, $Q'\subset bQ\subset\Omega$ and $bl(Q)<1$, then

$$\|\pi_Q-\pi_{Q'}\|_{\infty,3Q}\leqq C_{k,q,b}|Q|^{k/n-1/q}\left(\int_{bQ}(f_{k,q}^{\flat,1})^q\right)^{1/q}.\tag{5.2}$$

Proofs of these estimates are left to the reader (cf., for example, [**3**; §4, (4.5), (4.9)]).

Take a number a such that $a>\max\{6, c_1, 2c_5, c_5^2\}$. Let $\{\phi_Q^j\}$ be the partition of unity as mentioned in Lemma 1. For nonnegative integers j, set

$$f_j(x)=\sum_{Q\in\mathcal{G}_j^a}\pi_Q(x)\phi_Q^j(x),\qquad x\in\Omega.$$

We shall first prove the following: If $R\in\mathcal{G}_m^a$ and $j>m$, then

$$\int_{3R}|f_j-f_{j-1}|\leqq C_{k,q}(2^{(m-j)n}|R|)^{1+k/n-1/q}\left(\int_{c_5^2R}(f_{k,q}^{\flat,1})^q\right)^{1/q}.\tag{5.3}$$

In order to prove this, we write

$$f_j-f_{j-1}=\sum_{Q\in\mathcal{G}_j^a}\sum_{Q'\in\mathcal{G}_{j-1}^a}(\pi_Q-\pi_{Q'})\phi_Q^j\phi_{Q'}^{j-1}.$$

If $Q\in\mathcal{G}_j^a$, $Q'\in\mathcal{G}_{j-1}^a$ and $3Q\cap 3Q'\neq\emptyset$, then $Q\subset c_5Q'$ and $Q'\subset c_5Q$ (Lemma 1 (iii)), and hence, by (5.2), the inequality

$$|\pi_Q(x)-\pi_{Q'}(x)|\leqq C_{k,q}|Q|^{k/n-1/q}\left(\int_{c_5Q}(f_{k,q}^{\flat,1})^q\right)^{1/q}$$

holds for all $x\in 3Q$. In particular this inequality holds whenever $\phi_Q^j(x)\phi_{Q'}^{j-1}(x)\neq 0$. Hence

$$|f_j(x)-f_{j-1}(x)|\leqq C_{k,q}\sum_{Q\in\mathcal{G}_j^a}\phi_Q^j(x)|Q|^{k/n-1/q}\left(\int_{c_5Q}(f_{k,q}^{\flat,1})^q\right)^{1/q}.$$

Let $R\in\mathcal{G}_m^a$ and $j>m$. If $Q\in\mathcal{G}_j^a$ and $3Q\cap 3R\neq\emptyset$, then

$$|Q|\leqq C2^{(m-j)n}|R|\quad\text{and}\quad c_5Q\subset c_5^2R\qquad\text{(Lemma 1 (iii)).}$$

Using these estimates, we obtain

$$\begin{aligned}\int_{3R}|f_j-f_{j-1}|&\leqq C_{k,q}(2^{(m-j)n}|R|)^{1+k/n-1/q}\sum_{\substack{Q\in\mathcal{G}_j^a\\ 3Q\cap 3R\neq\emptyset}}\left(\int_{c_5Q}(f_{k,q}^{\flat,1})^q\right)^{1/q}\\&\leqq\text{(the right-hand member of (5.3)).}\end{aligned}$$

(To obtain the last inequality we used also the inequality $\sum a_n^{1/q}\leqq(\sum a_n)^{1/q}$ and Lemma 1 (iv).) Thus we proved (5.3).

Since $1+k/n-1/q>0$ and since $f_{k,q}^{\flat;1}\in L^{p}(\Omega)\subset L_{\mathrm{loc}}^{q}(\Omega)$, the estimate (5.3) implies that

$$\sum_{j>m}\int_{3R}|f_{j}-f_{j-1}|<\infty .$$

Hence $\lim_{j\to\infty}f_{j}$ exists in $L_{\mathrm{loc}}^{1}(\Omega)$. On the other hand $\lim_{j\to\infty}f_{j}(x)=f(x)$ for almost every $x\in\Omega$ (see [**3**; Lemma 4.1]). Combining these facts, we see that $f\in L_{\mathrm{loc}}^{1}(\Omega)$ and that $f_{j}\to f$ in $L_{\mathrm{loc}}^{1}(\Omega)$. Thus in particular the assertion (i) of the theorem is proved. (This proof of (i) is somewhat roundabout since (i) follows immediately from Theorem B. We need, however, the stronger results given above in the proof of (ii) and (iii).)

In order to prove (ii) and (iii) of the theorem, we shall prove the pointwise estimates

$$f_{\phi;\Omega}^{+;s,s}(x)\leqq C_{k,q,\phi}(M_{q}(f)(x)+M_{q}(f_{k,q}^{\flat;1})(x)) \tag{5.4}$$

and

$$\sum_{|\beta|=k}(\partial^{\beta}f)_{\phi;\Omega}^{+;s,s}(x)\leqq C_{k,q,\phi}M_{q}(f_{k,q}^{\flat;1})(x), \tag{5.5}$$

where ϕ is a function satisfying (1.1) and $s=c_{1}/a$. We can deduce the inequalities in (ii) and (iii) of the theorem from these estimates by using (2.4) and the L^{p}-boundedness of the operator M_{q}.

Let ϕ be a function satisfying (1.1), $x\in\Omega$ and $0<t<c_{1}d(x)/a$. Set $\psi=(\phi)_{t}(x-\cdot)$. Let m be the nonnegative integer such that $2^{-m-1}c_{1}d(x)/a\leqq t<2^{-m}c_{1}d(x)/a$. Take a cube R such that $R\in\mathcal{G}_{m}^{a}$ and $R\ni x$. Then $c_{1}\leqq 2^{m}al(R)/d(x)\leqq c_{2}$ (Lemma 1 (ii)) and, hence, $(c_{1}/2c_{2})l(R)\leqq t<l(R)$. Hence $\operatorname{supp}\psi\subset B(x,t)\subset 3R$ and $\|\partial^{\alpha}\psi\|_{\infty}\leqq C_{\phi,\alpha}t^{-n-|\alpha|}\leqq C_{\phi,\alpha}l(R)^{-n-|\alpha|}$.

First we shall estimate $\int f_{m}\psi$ and $\int f_{m}\partial^{\beta}\psi$ for $|\beta|=k$. Using (5.1) and Lemma 1 (iii), we have

$$\begin{aligned}\left|\int f_{m}\psi\right| &\leqq\left(\int|\psi|\right)\sup_{\substack{Q\in\mathcal{G}_{m}^{a}\\ 3Q\cap 3R\neq\emptyset}}\|\pi_{Q}\|_{\infty,3Q}\leqq C_{\phi,k,q}\sup_{\substack{Q\in\mathcal{G}_{m}^{a}\\ 3Q\cap 3R\neq\emptyset}}\left(|Q|^{-1}\int_{Q}|f|^{q}\right)^{1/q}\\ &\leqq C_{\phi,k,q}\left(|R|^{-1}\int_{c_{5}R}|f|^{q}\right)^{1/q}.\end{aligned}$$

Hence

$$\left|\int f_{m}\psi\right|\leqq C_{\phi,k,q}M_{q}(f)(x). \tag{5.6}$$

Let β be a multi-index with $|\beta|=k$. Since $\pi_{R}\in\mathcal{P}_{k-1}$, it holds that $\partial^{\beta}\pi_{R}=0$ and, hence,

$$\int f_{m}\partial^{\beta}\psi=(-1)^{|\beta|}\int\psi\partial^{\beta}(f_{m}-\pi_{R}).$$

Using Markov's inequality, (5.2) and Lemma 1 (iii), (iv), (v), we have

$$\|\partial^{\beta}(f_m-\pi_R)\|_{\infty,3R}=\sup_{y\in 3R}\Big|\sum_{\substack{Q\in\mathcal{Q}_m^a\\ 3Q\cap 3R\neq\emptyset}}\partial_y^{\beta}[(\pi_Q(y)-\pi_R(y))\phi_Q^m(y)]\Big|$$

$$\leqq C_{k,q}l(R)^{-|\beta|}|R|^{k/n-1/q}\Big(\int_{c_5R}(f_{k,q}^{\flat,1})^q\Big)^{1/q}\leqq C_{k,q}M_q(f_{k,q}^{\flat,1})(x).$$

Combining the estimate $\int|\phi|\leqq C_\phi$ with the last estimate, we obtain

$$\Big|\int f_m\partial^\beta\phi\Big|\leqq C_{k,q,\phi}M_q(f_{k,q}^{\flat,1})(x) \tag{5.7}$$

for $|\beta|=k$.

Next let $j>m$. Using (5.3) and the estimate $|R|\leqq 1$, we have

$$\Big|\int(f_j-f_{j-1})\phi\Big|\leqq\|\phi\|_\infty\int_{3R}|f_j-f_{j-1}|$$

$$\leqq C_{k,q,\phi}2^{(m-j)n(1+k/n-1/q)}|R|^{k/n-1/q}\Big(\int_{c_5^2R}(f_{k,q}^{\flat,1})^q\Big)^{1/q}$$

$$\leqq C_{k,q,\phi}2^{(m-j)n(1+k/n-1/q)}M_q(f_{k,q}^{\flat,1})(x).$$

Similarly we have, for $|\beta|=k$,

$$\Big|\int(f_j-f_{j-1})\partial^\beta\phi\Big|\leqq\|\partial^\beta\phi\|_\infty\int_{3R}|f_j-f_{j-1}|$$

$$\leqq C_{k,q,\phi}2^{(m-j)n(1+k/n-1/q)}\Big(|R|^{-1}\int_{c_5^2R}(f_{k,q}^{\flat,1})^q\Big)^{1/q}$$

$$\leqq C_{k,q,\phi}2^{(m-j)n(1+k/n-1/q)}M_q(f_{k,q}^{\flat,1})(x).$$

Since $1+k/n-1/q>0$, these estimates imply the following estimates:

$$\sum_{j>m}\Big|\int(f_j-f_{j-1})\phi\Big|\leqq C_{k,q,\phi}M_q(f_{k,q}^{\flat,1})(x), \tag{5.8}$$

$$\sum_{j>m}\Big|\int(f_j-f_{j-1})\partial^\beta\phi\Big|\leqq C_{k,q,\phi}M_q(f_{k,q}^{\flat,1})(x), \tag{5.9}$$

where $|\beta|=k$.

The estimates (5.4) and (5.5) now follow from (5.6), (5.7), (5.8), (5.9) and the fact that f_j converges to f in $L^1_{\mathrm{loc}}(\Omega)$. This completes the proof of Theorem 3.

PROOF OF THEOREM 2. We modify the definition of $\mathcal{Q}_j^a$ and $d(x)$ as follows: If $\Omega\neq\boldsymbol{R}^n$, then let $\mathcal{Q}_0^a$ be the set of maximal dyadic cubes Q such that $aQ\subset\Omega$, let $\mathcal{Q}_j^a$ for $j>0$ be defined inductively in the same way as before, and let $d(x)=\mathrm{dis}(x,\Omega^c)$; if $\Omega=\boldsymbol{R}^n$, then let $\mathcal{Q}_j^a$ be the set of the dyadic cubes with sidelength 2^{-j} and let $d(x)\equiv a$. Then Lemma 1 holds true for these modified $\mathcal{Q}_j^a$ and $d(x)$. Using the modified $\mathcal{Q}_j^a$ and $d(x)$, one can prove Theorem 2 in the same way as in the proof of Theorem 3. Details are left to the reader.

For the proofs of Theorems 4 and 5, we use the following lemmas.

LEMMA 2. *Let* Q *be a cube,* $\phi\in C_0^\infty(\boldsymbol{R}^n)$, $A>0$ *and* $k, m\in\boldsymbol{N}$. *Suppose* $\operatorname{supp}\phi\subset Q$, $\int\phi P=0$ *for all* $P\in\mathcal{P}_{k-1}$ *and* $\|\partial^\alpha\phi\|_\infty\leqq Al(Q)^{-n-|\alpha|}$ *for* $|\alpha|\leqq m$. *Then there exist functions* v_β, $|\beta|=k$, *such that* $v_\beta\in C_0^\infty(\boldsymbol{R}^n)$, $\operatorname{supp} v_\beta\subset Q$,

$$\|\partial^\alpha v_\beta\|_\infty\leqq C_{k,m}Al(Q)^{-n-|\alpha|} \qquad \text{for } |\alpha|\leqq m$$

and

$$\phi=l(Q)^k\sum_{|\beta|=k}\partial^\beta v_\beta .$$

PROOF. (The idea of the following proof can be found in [**7**; Lemma 3.5].) It is sufficient to consider the case $Q=(0, 1]^n$; the case for general Q can be reduced to this case by a change of variables. For $n\in\boldsymbol{N}$, let $D(n, 0)$ be the set of $\phi\in C_0^\infty(\boldsymbol{R}^n)$ such that $\operatorname{supp}\phi\subset(0, 1]^n$. For $n, k\in\boldsymbol{N}$, let $D(n, k)$ be the set of $\phi\in D(n, 0)$ such that $\int\phi P=0$ for all $P\in\mathcal{P}_{k-1}$. Take a function $g\in D(1, 0)$ such that $\int_{-\infty}^\infty g(x)dx=1$. We shall define the linear operators $T_j^{(n)}: D(n, 1)\to D(n, 0)$ $(j=1, \cdots, n)$ by induction. If $n=1$, then set

$$T_1^{(1)}\phi(x)=\int_{-\infty}^x\phi(t)dt, \qquad \phi\in D(n, 1).$$

Suppose $T_j^{(n-1)}$ $(j=1, \cdots, n-1)$ have been defined. Given $\phi\in D(n, 1)$, set

$$\phi_1(x')=\int_{-\infty}^\infty\phi(x', t)dt, \qquad x'\in\boldsymbol{R}^{n-1}.$$

Then $\phi_1\in D(n-1, 1)$. Define $T_j^{(n)}$ by

$$T_j^{(n)}\phi(x)=T_j^{(n-1)}\phi_1(x')g(x_n), \qquad j=1, \cdots, n-1,$$

and

$$T_n^{(n)}\phi(x)=\int_{-\infty}^{x_n}(\phi(x', t)-g(t)\phi_1(x'))dt,$$

where $x=(x', x_n)$ with $x'\in\boldsymbol{R}^{n-1}$ and $x_n\in\boldsymbol{R}$. Then it is easy to see that $T_j^{(n)}$ are well-defined linear operators from $D(n, 1)$ to $D(n, 0)$ and that

$$\phi(x)=\sum_{j=1}^n\frac{\partial}{\partial x_j}T_j^{(n)}\phi(x). \tag{5.10}$$

It is also easy to see that $T_j^{(n)}(D(n, k))\subset D(n, k-1)$ for all $k\in\boldsymbol{N}$ and that

$$\|\partial^\alpha T_j^{(n)}\phi\|_\infty\leqq A_\alpha\sum_{\beta\leqq\alpha}\|\partial^\beta\phi\|_\infty \tag{5.11}$$

for all α, where the constant A_α depends only on n, g and α. The assertion of the lemma for $Q=(0, 1]^n$ now follows from repeated application of (5.10) and (5.11). This completes the proof.

LEMMA 3. *Suppose Q, ψ, A, k and m satisfy the assumptions of Lemma 2. Also suppose $f\in\mathscr{D}'(\Omega)$ and $\Omega\supset Q$. Then*

$$|\langle f,\psi\rangle|\leqq C_{k,m}Al(Q)^k\sum_{|\beta|=k}\inf_{y\in Q}(\partial^\beta f)^{*,\sqrt{n}l(Q)}_{m,\Omega}(y).$$

This lemma can be easily derived from Lemma 2 once it is observed that $Q\subset B(y,\sqrt{n}\,l(Q))\cap\Omega$ for every $y\in Q$.

LEMMA 4. *Let $k\in\boldsymbol{N}$. Then for each cube $Q\subset\boldsymbol{R}^n$ there exists a function Φ_Q on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ which has the following properties.*

(i) *Φ_Q is smooth on $\boldsymbol{R}^n\times\boldsymbol{R}^n$ and* $\operatorname{supp}\Phi_Q(x,\cdot)\subset Q^i$ *for each $x\in\boldsymbol{R}^n$, where Q^i denotes the interior of Q.*

(ii) *For each $f\in\mathscr{D}'(Q^i)$, the function $\boldsymbol{R}^n\ni x\mapsto\langle f,\Phi_Q(x,\cdot)\rangle$ belongs to $\mathscr{P}_{k-1}$.*

(iii) $P(x)=\int P(y)\Phi_Q(x,y)dy$ *for all $P\in\mathscr{P}_{k-1}$.*

(iv) *If $A>0$ and $x\in AQ$, then $|\partial_x^\alpha\partial_y^\beta\Phi_Q(x,y)|\leqq C_{k,A,\alpha,\beta}l(Q)^{-n-|\alpha|-|\beta|}$ for all multi-indices α and β.*

PROOF. Take a function $\phi_0\in C_0^\infty((-1/2,1/2)^n)$ such that $\phi_0(x)\geqq 0$ and $\int\phi_0=1$. We regard $\mathscr{P}_{k-1}$ as a finite dimensional Hilbert space with the inner product $(P,Q)=\int P\bar{Q}\phi_0$. Let $\{\pi_i\}$ be an orthonormal basis of this Hilbert space. For each cube Q, set

$$\Phi_Q(x,y)=l(Q)^{-n}\sum_i\pi_i\Big(\frac{x-x_Q}{l(Q)}\Big)\pi_i\Big(\frac{y-x_Q}{l(Q)}\Big)\phi_0\Big(\frac{y-x_Q}{l(Q)}\Big).$$

This Φ_Q has all the properties of the lemma. This completes the proof.

We now proceed to the proofs of Theorems 4 and 5.

PROOF OF THEOREM 4. *Proof of* (i). As shown in Section 2, $[\partial^\alpha f](x)$ exists almost everywhere if $\partial^\alpha f\in H^p_{\mathrm{loc}}(\Omega)$. Thus it is sufficient to show that $\partial^\alpha f\in H^p_{\mathrm{loc}}(\Omega)$ for $|\alpha|<k$. Without loss of generality we may assume that $\partial^\alpha f\in H^p(\Omega)$ for $|\alpha|=k$. Fix a multi-index β with $|\beta|<k$ and fix an $x\in\Omega$. Take r such that $0<r<\operatorname{dis}(x,\Omega^c)/3\sqrt{n}$ and set $U=B(x,r)$. Take a function $\phi\in C_0^\infty((0,1)^n)$ such that $\int\phi=1$ and $\int\phi(y)y^\alpha dy=0$ for $0<|\alpha|\leqq k-1$. We shall show that $(\partial^\beta f)^{+;\infty,1}_{\phi;U}\in L^p(U)$, which will imply the desired result. Let $y\in U$, $0<t<r$ and let M be the nonnegative integer which satisfy $2^Mt<r\leqq 2^{M+1}t$. If $j\in\boldsymbol{N}$ and $j\leqq M$, then, applying Lemma 3 to

$$\psi=(-1)^{|\beta|}\partial^\beta((\phi)_{2^jt}(y-\cdot)-(\phi)_{2^{j-1}t}(y-\cdot))$$

and

$$Q=\{y+z \mid z\in(-2^j t, 0]^n\},$$

we obtain the estimate

(5.12) $$|\langle\partial^\beta f, (\phi)_{2^j t}(y-\cdot)-(\phi)_{2^{j-1}t}(y-\cdot)\rangle| \leqq C_{k,m,\phi,\beta}(2^j t)^{k-|\beta|}\sum_{|\alpha|=k}(\partial^\alpha f)^{*;\infty}_{m,\Omega}(y).$$

Similarly we have the estimate

(5.13) $$|\langle\partial^\beta f, (\phi)_r(y-\cdot)-(\phi)_{2^M t}(y-\cdot)\rangle| \leqq C_{k,m,\phi,\beta}r^{k-|\beta|}\sum_{|\alpha|=k}(\partial^\alpha f)^{*;\infty}_{m,\Omega}(y).$$

Hence

$$\begin{aligned}|\langle\partial^\beta f, (\phi)_t(y-\cdot)\rangle| &\leqq |\langle\partial^\beta f, (\phi)_r(y-\cdot)\rangle| \\ &\quad +(\text{the left-hand member of (5.13)}) \\ &\quad +\sum_{0<j\leqq M}(\text{the left-hand member of (5.12)}) \\ &\leqq |\langle\partial^\beta f, (\phi)_r(y-\cdot)\rangle|+C_{k,m,\phi,\beta}r^{k-|\beta|}\sum_{|\alpha|=k}(\partial^\alpha f)^{*;\infty}_{m,\Omega}(y) \\ &= I(y)+II(y), \qquad \text{say.}\end{aligned}$$

(Here we used the assumption $|\beta|<k$ to estimate the sum $\sum_{0<j\leqq M}$.) Thus we have

$$(\partial^\beta f)^{+;\infty,1}_{\phi;U}(y) \leqq \sup_{0<t<r}|\langle f, (\phi)_t(y-\cdot)\rangle| \leqq I(y)+II(y)$$

for all $y\in U$. The function $I(y)$ is the absolute value of a function which is smooth in a neighborhood of the closure of U; hence it belongs to $L^p(U)$. If we take m so that $m>n/p-n$, then the function $II(y)$ also belongs to $L^p(U)$ by virtue of (2.3). Hence $(\partial^\beta f)^{+;\infty,1}_{\phi;U}\in L^p(U)$. This proves (i).

Proofs of (ii), (iii) *and* (iv). Let Φ_Q be the function of Lemma 4. For cubes Q included in Ω, we set $P_Q=\langle f, \Phi_Q(x, \cdot)\rangle$. Then $P_Q\in\mathcal{P}_{k-1}$. We shall prove the following two claims.

The first claim: If $|\alpha|<k$, then

(5.14) $$\lim_{\substack{x\in Q\subset\Omega \\ l(Q)\to 0}}\partial^\alpha_x P_Q(x)=[\partial^\alpha f](x)$$

for almost every $x\in\Omega$.

The second claim: If Q is a cube included in Ω and $m\in\boldsymbol{N}$, then

(5.15) $$|[f](x)-P_Q(x)| \leqq C_{k,m}l(Q)^k\sum_{|\alpha|=k}(\partial^\alpha f)^{*,\sqrt{n}l(Q)}_{m,\Omega}(x)$$

for almost every $x\in Q$.

For the moment we assume these claims and prove (ii), (iii) and (iv). Take q such that $0<q<p$. Integrating the q-th power of (5.15), we obtain

$$|Q|^{-k/n}\Big(|Q|^{-1}\int_Q|[f]-P_Q|^q\Big)^{1/q} \leqq C_{k,m}\Big(|Q|^{-1}\int_Q\Big(\sum_{|\alpha|=k}(\partial^\alpha f)^{*,\sqrt{n}l(Q)}_{m,\Omega}\Big)^q\Big)^{1/q}.$$

Taking supremum over the cubes Q such that $x\in Q\subset\Omega$, with x fixed, we obtain

$$[f]^{\flat}_{k,q}(x) \leqq C_{k,m,q} M_q\Big(\sum_{|\alpha|=k}(\partial^\alpha f)^{*,\infty}_{m,\Omega}\Big)(x).$$

Hence, by the L^p-boundedness of the operator M_q, we have

$$\|[f]^{\flat}_{k,q}\|_{p,\Omega} \leqq C_{k,m,q,p}\Big\|\sum_{|\alpha|=k}(\partial^\alpha f)^{*,\infty}_{m,\Omega}\Big\|_{p,\Omega}.$$

We take m such that $m>n/p-n$. Then the above inequality, combined with Theorem B and (2.3), implies the first inequality of (ii). The second inequality of (ii) can be proved in a similar way (just use $(\partial^\alpha f)^{*,\sqrt{n}}_{m,\Omega}$ instead of $(\partial^\alpha f)^{*,\infty}_{m,\Omega}$). The assertion (iii) follows from (5.14), (5.15) and the definition of the Peano derivative. The assertion (iv) follows from (ii), (iii) and Theorems B, C and D.

Now we shall prove the first claim. Without loss of generality we may assume again that $\partial^\alpha f \in H^p(\Omega)$ for $|\alpha|=k$. Let $|\alpha|<k$ and let Q be a cube included in Ω. By Lemma 4 (iii) and by integration by parts, we see that

$$\int P(y)(\partial_x^\alpha \Phi_Q(x, y)-(-1)^{|\alpha|}\partial_y^\alpha \Phi_Q(x, y))dy = 0$$

for all $P\in\mathcal{P}_{k-1}$. From this fact and from Lemma 4 (i) and (iv), we see, using Lemma 3, that the estimate

$$\begin{aligned} &|\langle f(y), \partial_x^\alpha \Phi_Q(x, y)-(-1)^{|\alpha|}\partial_y^\alpha \Phi_Q(x, y)\rangle| \\ &\leqq C_{k,m,\alpha} l(Q)^{k-|\alpha|}\sum_{|\beta|=k}(\partial^\beta f)^{*,\sqrt{n}l(Q)}_{m,\Omega}(x) \end{aligned} \tag{5.16}$$

holds for all $x\in Q$. We have

$$\begin{aligned} \partial_x^\alpha P_Q(x) &= \langle f(y), \partial_x^\alpha \Phi_Q(x, y)\rangle \\ &= \langle f(y), (-1)^{|\alpha|}\partial_y^\alpha \Phi_Q(x, y)\rangle + \langle f(y), \partial_x^\alpha \Phi_Q(x, y)-(-1)^{|\alpha|}\partial_y^\alpha \Phi_Q(x, y)\rangle \\ &= I+II, \quad \text{say.} \end{aligned}$$

Using (2.5), we see that, for almost every $x\in\Omega$, the term $I=\langle\partial^\alpha f, \Phi_Q(x, \cdot)\rangle$ converges to $[\partial^\alpha f](x)$ as $l(Q)\to 0$ with $x\in Q\subset\Omega$. On the other hand, using (5.16) with $m>n/p-n$, we see that $II=O(l(Q)^{k-|\alpha|})$ almost everywhere. Hence II converges to zero almost everywhere. Thus (5.14) holds almost everywhere. This proves the first claim.

Finally we shall prove the second claim. Let Q be a cube included in Ω, and let $x\in Q$. Take a sequence of cubes Q_j such that $Q=Q_0\supset Q_1\supset\cdots$, $Q_j\ni x$ and $l(Q_j)=2^{-j}l(Q)$. Using Lemmas 3 and 4, we have

$$\begin{aligned} |P_{Q_j}(x)-P_{Q_{j-1}}(x)| &= |\langle f, \Phi_{Q_j}(x, \cdot)-\Phi_{Q_{j-1}}(x, \cdot)\rangle| \\ &\leqq C_{k,m}(2^{-j}l(Q))^k\sum_{|\alpha|=k}(\partial^\alpha f)^{*,\sqrt{n}l(Q)}_{m,\Omega}(x). \end{aligned}$$

Hence

$$\sum_{j=1}^{\infty} |P_{Q_j}(x)-P_{Q_{j-1}}(x)| \leqq C_{k,m} l(Q)^k \sum_{|\alpha|=k} (\partial^\alpha f)^{*,\sqrt{n}l(Q)}_{m,\Omega}(x).$$

This inequality and (5.14) with $\alpha=0$ imply (5.15). Thus the second claim is also proved. This completes the proof of Theorem 4.

PROOF OF THEOREM 5. Without loss of generality we may assume that $\partial^\alpha f \in H^p(\Omega)$ for $|\alpha|=k$. Take an integer m such that $m>n/p-n$. Then $(\partial^\alpha f)^{*,\infty}_{m,\Omega} \in L^p(\Omega)$ for $|\alpha|=k$ by virtue of (2.3). Let ϕ be the same function as in the proof of (i) of Theorem 4. For $x\in\Omega$ and $0<t<\mathrm{dis}(x, \Omega^c)$, set

$$f(x, t) = \langle f, (\phi)_t(x-\cdot)\rangle .$$

Let Q be a cube with $3Q\subset\Omega$, and let $j\in \boldsymbol{N}$. By repeatedly bisecting the sides of cubes, we can write Q as a union of 2^{jn} cubes each with sidelength $2^{-j}l(Q)$. Let R be one of these 2^{jn} cubes. Then using Lemma 3 we see that

$$\sup_{x\in R} |f(x, l(R))-f(x, l(R)/2)| \leqq C_{k,m,\phi} l(R)^k \sum_{|\alpha|=k} \inf_{y\in 3R} (\partial^\alpha f)^{*,\infty}_{m,\Omega}(y)$$

and, hence,

$$\int_R |f(x, l(R))-f(x, l(R)/2)|\,dx$$
$$\leqq C_{k,m,\phi} |R|^{1+k/n-1/p} \sum_{|\alpha|=k} \left(\int_R ((\partial^\alpha f)^{*,\infty}_{m,\Omega})^p\right)^{1/p}.$$

Taking the sum over all R's, we obtain

$$\int_Q |f(x, 2^{-j}l(Q))-f(x, 2^{-j-1}l(Q))|\,dx$$
$$\leqq C_{k,m,\phi} (2^{-jn}|Q|)^{1+k/n-1/p} \sum_{|\alpha|=k} \left(\int_Q ((\partial^\alpha f)^{*,\infty}_{m,\Omega})^p\right)^{1/p}.$$

Since $1+k/n-1/p>0$, the above estimate implies that

$$\sum_{j=1}^{\infty} \int_Q |f(x, 2^{-j}l(Q))-f(x, 2^{-j-1}l(Q))|\,dx < \infty .$$

Hence the function $f(\cdot, 2^{-j}l(Q))$ converges in $L^1(Q)$ as $j\to\infty$. On the other hand, $f(\cdot, 2^{-j}l(Q))$ converges to f in $\mathcal{D}'(\Omega)$. Hence, on Q^i, the distribution f coincides with an L^1-function. Since Q is an arbitrary cube satisfying $3Q\subset\Omega$, this means that $f\in L^1_{\mathrm{loc}}(\Omega)$. This completes the proof.

We shall prove the facts mentioned in Remarks e) and f), Section 4.

NOTE ON REMARK e). Let $f\in C_0^\infty(B(0, 1))$ and $\int f=1$, and let $B(x_0, r)\subset\Omega$. For $t>0$, set $f_t(x)=(f)_t(x-x_0)$. Let $0<p\leqq 1$ and $\lambda>0$. Then the following estimates hold for $0<t<r/10$:

(5.17) $$\|\partial^\alpha f_t\|'_{p,(\phi),\Omega} \geqq C_{\alpha,p,\phi,r} \quad \text{for each } \alpha,$$

$$\|f_t\|_{p,\Omega} = C_{f,p} t^{n/p-n}, \tag{5.18}$$

$$\|(f_t)^\natural_{\lambda,p}\|_{p,\Omega} \leqq C_{f,\lambda,p} t^{n/p-n-\lambda}. \tag{5.19}$$

(Proofs will be given below.) Hence, if $n/p-n-\lambda>0$, then, for every α, there exist no constant A independent of t such that

$$\|\partial^\alpha f_t\|'_{p,(\phi),\Omega} \leqq A(\|f_t\|_{p,\Omega}+\|(f_t)^\natural_{\lambda,p}\|_{p,\Omega}).$$

This fact implies that the condition $1+k/n>1/p$ in Theorems 2 and 3 cannot be removed.

We shall prove (5.17)~(5.19). In order to prove (5.17), take a function $\phi_\alpha \in C_0^\infty(B(0, 1))$ such that $\partial_x^\alpha \phi_\alpha(x)=1$ for $|x|<1/2$, and $\int \phi_\alpha=1$. Suppose $t<|x-x_0|<r/5$. Then for every $y \in \operatorname{supp} f_t$ it holds that $|x-y|<2|x-x_0|$ and, hence, that

$$(-1)^{|\alpha|}\partial_y^\alpha(\phi_\alpha)_{4|x-x_0|}(x-y) = (4|x-x_0|)^{-n-|\alpha|}.$$

It also holds that $4|x-x_0|<\min\{r, \operatorname{dis}(x, \Omega^c)\}$. Hence

$$(\partial^\alpha f_t)^{+;r,1}_{\phi_\alpha;\Omega}(x) \geqq \left|\int f_t(y)(-1)^{|\alpha|}\partial_y^\alpha(\phi_\alpha)_{4|x-x_0|}(x-y)dy\right| = (4|x-x_0|)^{-n-|\alpha|}.$$

From this estimate, we can deduce (5.17) by using (2.4). The equality (5.18) can be proved by a simple change of variables. Finally (5.19) follows from the estimate

$$(f_t)^\natural_{\lambda,p}(x) \leqq C_{f,p,\lambda} t^{-\lambda-n}(1+t^{-1}|x-x_0|)^{-\lambda-n/p}.$$

PROOF OF REMARK f). Let f_t, x_0, r, p and λ be the same as in Note on Remark e). Suppose $r>10$. Then, as shown above, the following estimates hold:

$$\|f_1\|_{p,\Omega} = C_{f,p}, \qquad \|(f_1)^\natural_{\lambda,p}\|_{p,\Omega} \leqq C_{f,\lambda,p},$$

$$(f_1)^{+;\infty,1}_{\phi;\Omega}(x) \geqq (4|x-x_0|)^{-n} \quad \text{if } 1<|x-x_0|<r/5,$$

where ϕ is a function satisfying $\phi(x)=1$ for $|x|<1/2$ as well as the condition (1.1). From the last estimate, it follows that

$$\|f_1\|_{p,(\phi),\Omega} \geqq C\left(\int_{1<|x-x_0|<r/5} |x-x_0|^{-np}dx\right)^{1/p}.$$

The right-hand member of this inequality tends to infinity as $r\to\infty$ since $p\leqq 1$. This implies the assertion of Remark f).

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