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Reflective elastic waves at the boundary as a propagation of singularities phenomenon

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1. Introduction.

Let us consider an elastic body Ω in R^3 with a smooth boundary $\partial\Omega$. If the medium is isotropic, it is well known that the displacement $u(x, t)={}^t(u_1, u_2, u_3)$ of Ω satisfies the following boundary value problem

$$\rho\partial^2 u/\partial t^2=(\lambda+\mu)\,\mathrm{grad}(\mathrm{div}\, u)+\mu\Delta u \quad \text{in } \Omega\times R, \tag{1.1}$$

$$\sum_{i=1}^{3} n_i(x)\Big\{\lambda(\mathrm{div}\, u)\delta_{ij}+\mu\Big(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial u_i}\Big)\Big\}=0 \quad (j=1, 2, 3) \text{ on } \partial\Omega\times R, \tag{1.2}$$

where ρ is the density of the medium, λ and μ are the "Lamé constants" which are positive, and $n(x)=(n_1(x), n_2(x), n_3(x))$ is the outer unit normal vector to the boundary $\partial\Omega$.

Solutions $u(t, x)$ of the boundary value problem (1.1) and (1.2) in the half space $\Omega=\{x\in R^3: x_3>0\}$ are theoretical models of seismic waves. It is not difficult to show that the rotation of u and the divergence of u are solutions of usual wave equations $\partial_t^2-(\mu/\rho)\Delta$ and $\partial_t^2-((\lambda+2\mu)/\rho)\Delta$, respectively. By this observation and one from a seismogram it is said that there are two kind of seismic waves with speeds $(\mu/\rho)^{1/2}$ and $((\lambda+2\mu)/\rho)^{1/2}$, which are called an S wave and a P wave, respectively. By constructing a special solution of (1.1) and (1.2) in the half space seismology insists the following two phenomena:

(1.3) If a P wave only makes incidence to the boundary, then both a P wave and an S wave reflect from the boundary.

(1.4) There are two kinds of S waves. One of them is called an SV wave, if a reflective phenomenon for its incident wave is similar to one for an incident P wave. The other is called an SH wave, if only an S wave reflects for its incident wave.

In this paper we shall prove (1.3) and (1.4) from the viewpoint of a propagation of singularities of every solution to the boundary value problem (1.1) and (1.2) with an arbitrary domain Ω in R^3. Namely, we shall show the following facts: Let $C_{\text{in}}^{(p)}$ be the incoming characteristic curve of $\partial_t^2-((\lambda+2\mu)/\rho)\Delta$ hitting on $x_0\in\partial\Omega$ at time t_0 with a direction $\omega\in S^2$, and $C_{\text{in}}^{(s)}(\omega)$ be the incoming characteristic curve of $\partial_t^2-(\mu/\rho)\Delta$ hitting on $x_0\in\partial\Omega$ at time t_0 with a direction ω_+ decided from ω (see i) of Definition 2.2). We denote by $C_{\text{out}}^{(p)}$ and $C_{\text{out}}^{(s)}$ the outgoing characteristic curves of $\partial_t^2-((\lambda+2\mu)/\rho)\Delta$ and $\partial_t^2-(\mu/\rho)\Delta$ decided by the rule of geometrical optics from $C_{\text{in}}^{(p)}$ and $C_{\text{in}}^{(s)}$, respectively. If a solution $u(x, t)$ of (1.1) and (1.2) is not C^∞ on $C_{\text{in}}^{(p)}$ and is C^∞ near $C_{\text{in}}^{(s)}$, then $u(x, t)$ is not C^∞ on $C_{\text{out}}^{(p)}\cup C_{\text{out}}^{(s)}$. This is a corollary of Theorem 4.1. On the other hand if a solution of (1.1) and (1.2) is not C^∞ on $C_{\text{in}}^{(s)}$ and is C^∞ near $C_{\text{in}}^{(p)}$, then we have one of the following two phenomena; i) $u(x, t)$ is not C^∞ on $C_{\text{out}}^{(p)}\cup C_{\text{out}}^{(s)}$. ii) $u(x, t)$ is not C^∞ on $C_{\text{out}}^{(s)}$ and is C^∞ near $C_{\text{out}}^{(p)}$. This is a corollary of Theorem 4.4. Thus we can give a proof of (1.3) and (1.4) from the viewpoint of a propagation of singularities.

2. Definitions of rays.

In this section we shall define several rays which are used to explain reflective phenomena. First we remark that the principal symbol of the elastic wave equation (1.1) has the following property;

$$\det(\rho\tau^2 I_3-(\lambda+\mu)\xi^t\xi-\mu|\xi|^2 I_3)=\rho^6(\tau^2-\alpha^2|\xi|^2)^2(\tau^2-\beta^2|\xi|^2), \tag{2.1}$$

where I_3 is the 3×3 identity matrix, ξ is a column vector of R^3 and $\alpha^2=\mu/\rho$, $\beta^2=(\lambda+2\mu)/\rho$. This fact also suggests that there are two kind of seismic waves with speeds α and β.

We say that a ray $\gamma(t)=(x(t), t, \xi(t), \tau(t))$ in $T^*(\Omega\times R)$ parametrized by time t is an incoming (outgoing) ray to the boundary $\partial\Omega$, if $x(t_0)=x_0\in\partial\Omega$ and

$$n(x_0)\cdot\frac{dx}{dt}(t_0)>0 \qquad \left(n(x_0)\cdot\frac{dx}{dt}(t_0)<0\right). \tag{2.2}$$

First we shall define an incident P (S) ray and its reflected P (S) ray.

DEFINITION 2.1. Let x_0 be an arbitrary point of $\partial\Omega$ and ω be an element of S^2 such that $n(x_0)\cdot\omega>0$.

i) The incoming half null bicharacteristic $\{(\beta\omega(t-t_0)+x_0, t, -\varepsilon\omega, \varepsilon\beta)\in T^*(\Omega\times R): t<t_0\}$ of $\tau^2-\beta^2|\xi|^2$ is called an incident P ray with a direction ω and is denoted by $\gamma_i^{(p)}(\omega)$, where $\varepsilon^2=1$. By the geometrical optics the reflected direction of ω at $x_0\in\partial\Omega$ is $\omega_r=\omega-2(n(x_0)\cdot\omega)n(x_0)$. Thus the outgoing half null bicharacteristic $\gamma_r^{(p)}(\omega)=\{(\beta\omega_r(t-t_0)+x_0, t, -\varepsilon\omega_r, \varepsilon\beta)\in T^*(\Omega\times R): t_0<t\}$ of $\tau^2-\beta^2|\xi|^2$ is called a reflected P ray of $\gamma_i^{(p)}(\omega)$.

(ii) Similarly $\gamma_i^{(s)}(\omega)=\{(\alpha\omega(t-t_0)+x_0, t, -\varepsilon\omega, \varepsilon\alpha)\in T^*(\Omega\times R): t<t_0\}$ and $\gamma_r^{(s)}(\omega)=\{(\alpha\omega_r(t-t_0)+x_0, t, -\varepsilon\omega_r, \varepsilon\alpha)\in T^*(\Omega\times R): t_0<t\}$ are called an incident S

ray with a direction ω and its reflected S ray, respectively.

Let π be the projection from $T^*(R^3\times R)$ to $T^*(\Omega\times R)$. Then we remark that $\pi((x_0, t_0, -\varepsilon\omega, \varepsilon\beta))=\pi((x_0, t_0, -\varepsilon\omega_r, \varepsilon\beta))=(x_0, t_0, -\varepsilon(\omega-(n(x_0)\cdot\omega)n(x_0)), \varepsilon\beta)$. We shall define a transfered reflected ray. We consider a point $(x_0, t_0, -\varepsilon(\omega-(n(x_0)\cdot\omega)n(x_0))+An(x_0), \varepsilon\beta)$ such that $\pi((x_0, t_0, -\varepsilon\omega, \varepsilon\beta))=\pi((x_0, t_0, -\varepsilon(\omega-(n(x_0)\cdot\omega)n(x_0))+An(x_0), \varepsilon\beta))$ and $(\varepsilon\beta)^2-\sigma^2|\varepsilon(\omega-(n(x_0)\cdot\omega)n(x_0))+An(x_0)|^2=0$. Thus

$$A=\pm a(\omega)=\pm[\beta^2/\alpha^2-1+(n(x_0)\cdot\omega)^2]^{1/2}. \tag{2.3}$$

Making use of $a(\omega)$, we have the following

DEFINITION 2.2. Let $x_0\in\partial\Omega$ and $\omega\in S^2$ such that $n(x_0)\cdot\omega>0$.

i) The outgoing half null bicharacteristic $\gamma_{\mathrm{tr}}^{(s)}(\omega)=\{(\alpha^2\omega_-(t-t_0)/\beta+x_0, t, -\varepsilon\omega_-, \varepsilon\beta)\in T^*(\Omega\times R): t_0<t\}$ of $\tau^2-\alpha^2|\xi|^2$, where $\omega_\pm=\omega-(n(x_0)\cdot\omega)n(x_0)\pm a(\omega)n(x_0)$ with $|\omega_\pm|=\beta/\alpha$, is called the transfered reflected S ray of the incident P ray $\gamma_i^{(p)}(\omega)$. The incoming half null bicharacteristic $\{(\alpha^2\omega_+(t-t_0)/\beta+x_0, t, -\varepsilon\omega_+, \varepsilon\beta)\in T^*(\Omega\times R): t<t_0\}$ is denoted by $\gamma_{\mathrm{in}}^{(s)}(\omega)$.

ii) If $\alpha^2>\beta^2(1-(n(x_0)\cdot\omega)^2)$, then similarly we can define the transfered reflected P ray $\gamma_{\mathrm{tr}}^{(p)}(\omega)=\{(\beta^2\tilde{\omega}_-(t-t_0)/\alpha+x_0, t, -\varepsilon\omega_-, \varepsilon\alpha)\in T^*(\Omega\times R): t_0<t\}$ and the incoming P ray $\gamma_{\mathrm{in}}^{(p)}(\omega)=\{(\beta^2\tilde{\omega}_+(t-t_0)/\alpha+x_0, t, -\varepsilon\omega_+, \varepsilon\alpha)\in T^*(\Omega\times R): t<t_0\}$ for the incident S ray $\gamma_i^{(s)}(\omega)$, where $\tilde{\omega}_\pm=\omega-(n(x_0)\cdot\omega)n(x_0)\pm[\alpha^2/\beta^2-1+(n(x_0)\cdot\omega)^2]^{1/2}\cdot n(x_0)$ with $|\tilde{\omega}_\pm|=\alpha/\beta$.

In Remark 3.1 we shall give precise meanings of these rays.

3. Reduction to the first order systems.

Throughout this paper we assume that a considered solution $u(x)$ of (1.1) and (1.2) is an extensible distribution, i.e., there exists a distribution $U(x, t)$ in $R^3\times R$ such that $U=u$ as an element of $\mathcal{D}'(\Omega\times R)$.

Since (1.1) and (1.2) are rotation free, we may assume that $x_0=0\in\partial\Omega$ and Ω is defined by $\{x\in R^3: x_3>g(x')\}$ in a neighbourhood U_0 of 0, where $x'=(x_1, x_2)$ and $(\nabla g)(0)=(\partial g/\partial x_1(0), \partial g/\partial x_2(0))=0$. Thus $n(0)=(0, 0, -1)$. By making use of the coordinate transformation $y'=x'$, $y_3=x_3-g(x')$ such that $\Omega\cap U_0$ is transformed into $\{y: y_3>0\}$ and putting $U(y, t)={}^t(\Lambda(D_{y'}, D_t)^t u, D_{y_3}{}^t u)$, where $\Lambda(D_{y'}, D_t)$ is a scalar pseudodifferential operator with the symbol $\Lambda_1(\eta', \tau)=(\tau^2+|\eta'|^2+1)^{1/2}$, the boundary value problem (1.1) and (1.2) is reduced to the following one (see Section 1.1 of [6]).

$$\begin{cases} D_{y_3}U=M(y', D_{y'}, D_t)U & \text{in } y_3>0, \\ B(y', D_{y'}, D_t)U=0 & \text{on } y_3=0, \end{cases} \tag{3.1}$$

where $M(y', D_{y'}, D_t)$ is a pseudodifferential operator of order 1 with a form of a 6×6 matrix and $B(y', D_{y'}, D_t)$ is a pseudodifferential operator of order 0 with a form of a 3×6 matrix. Here the principal symbol $M_1(y', \eta', \tau)$ of $M(y', D_{y'}, D_t)$ satisfies that $\det(\eta_3 I_6 - M_1(y', \eta', \tau)) = \{(\eta_3 - a(y', \eta'))^2 + s(y', \eta', \tau)\}^2\{(\eta_3 - a(\eta', \tau))^2 + p(y', \eta', \tau)\}$, where

$$a(y', \eta') = \langle \eta', \nabla g\rangle / |G|^2, \tag{3.2}$$

$$s(y', \eta', \tau) = \{|\eta'|^2 - \tau^2/\alpha^2 - \langle \eta', \nabla g\rangle^2/|G|^2\}/|G|^2, \tag{3.3}$$

$$p(y', \eta', \tau) = \{|\eta'|^2 - \tau^2/\beta^2 - \langle \eta', \nabla g\rangle^2/|G|^2\}/|G|^2 \tag{3.4}$$

with $G = {}^t(-\nabla g, 1)$. Furthermore the principal symbol $(B_1, B_2)(y', \eta', \tau)$ of $B(y', D_{y'}, D_t) = (B_1, B_2)(y', D_{y'}, D_t)$ is

$$\begin{cases} B_1(y', \eta', \tau) = (\lambda G {}^t\bar{\eta} + \mu \bar{\eta} {}^t G + \mu G \cdot \bar{\eta}) \Lambda_1^{-1}, \\ B_2(y', \eta', \tau) = (\lambda + \mu) G {}^t G + \mu |G|^2 I_3, \end{cases} \tag{3.5}$$

where $\bar{\eta} = (\eta_1, \eta_2, 0)$.

Since $u(x, t)$ is an extensible distribution, from Theorem 4.3.1 of [**2**] we can regard $U(x, t)$ as an element of $C^\infty(\bar{R}_+; \mathcal{D}'(R^2_{y'} \times R))$. In order to state theorems on a propagation of singularities we use the notation $WF(G)$ for a distribution G, which is the wave front set of Hörmander (see (2.5.2) and Proposition 2.5.5 in [**1**]). For $F \in C^\infty(\bar{R}_+: \mathcal{D}'(R^2_{y'} \times R_t))$ we say that F is micro-locally smooth at $\rho \in T^*(R^2_{y'} \times R_t)\setminus 0$, if there exists a properly supported pseudodifferential operator $A(y', t, D_{y'}, D_t)$ of order 0 such that the principal symbol of A is not zero at ρ and $(AF)(y, t) \in C^\infty([0, \varepsilon] \times R^3_{y', t})$ for some $\varepsilon > 0$.

We denote by ρ_0 $(0, t_0, -\varepsilon\eta'_0, \varepsilon\tau_0) \in T^*(\{y \in R^3 : y_3 = 0\})$ such that $\varepsilon^2 = 1$ and $\tau_0 |\eta'_0| (ps)(0, t_0, -\varepsilon\eta'_0, \varepsilon\tau_0) \neq 0$. In a conic neighbourhood of Γ_0 of ρ_0 we define functions

$$a^\pm(y', \eta', \tau) = \begin{cases} a \pm i s^{1/2} & \text{if } s > 0 \text{ at } \rho_0, \\ a \pm \varepsilon(-s)^{1/2} & \text{if } s < 0 \text{ at } \rho_0, \end{cases} \tag{3.6}$$

$$b^\pm(y', \eta', \tau) = \begin{cases} a \pm i p^{1/2} & \text{if } p > 0 \text{ at } \rho_0, \\ a \pm \varepsilon(-p)^{1/2} & \text{if } p < 0 \text{ at } \rho_0. \end{cases} \tag{3.7}$$

Making use of these functions, we define vector valued functions

$${}^t s_j^\pm(y', \eta', \tau) = ({}^t w_j^\pm, a^{\pm\, t} w_j^\pm) \qquad (j = 1, 2), \tag{3.8}$$

$${}^t s_3^\pm(y', \eta', \tau) = ({}^t w_3^\pm, b^{\pm\, t} w_3), \tag{3.9}$$

where

$$w_1^\pm = {}^t(a^\pm \eta_1, a^\pm \eta_2, -|\eta'|^2) \Lambda_1^{-2}, \tag{3.10}$$

$$w_2^\pm = {}^t(-\eta_2, \eta_1, 0) \Lambda_1^{-1}, \qquad w_3^\pm = {}^t(\eta_1, \eta_2, b^\pm) \Lambda_1^{-1}. \tag{3.11}$$

Then the 6×6 matrix $S_0(y', \eta', \tau)=(s_1^+, s_2^+, s_3^+, s_1^-, s_2^-, s_3^-)$ is positively homogeneous of degree 0 and is non-singular in Γ_0.

From the argument of Section 1.3 of [**6**] (see also Section 2 in [**5**]) there exists an elliptic pseudodifferential operator $S(y', D_{y'}, D_t)$ of order 0 whose principal symbol is equal to S_0 in Γ_0 such that the boundary value problem (3.1) is micro-locally reduced to the following

$$\left\{\begin{array}{ll} D_{y_3}V-\begin{pmatrix} H^+ & & & \\ & h^+ & & 0 \\ 0 & & H^- & \\ & & & h^- \end{pmatrix}V = F & \text{in } y_3>0\,, \\ C(y', D_{y'}, D_t)V = G & \text{on } y_3=0\,, \end{array}\right. \tag{3.12}$$

where $V=S^{-1}U$, $C=BS$, $\rho_0\notin WF(G)$ and F is smooth at ρ_0. Moreover the principal symbol of $H^{\pm}(y', D_{y'}, D_t)$ is a diagonal matrix $a^{\pm}(y', \eta', \tau)I_2$ and the principal symbol of $h^{\pm}(y', D_{y'}, D_t)$ is $b^{\pm}(y', \eta', \tau)$, where I_2 is the 2×2 identity matrix.

REMARK 3.1. We shall explain relations between rays defined in Section 2 and the Hamilton vector fields of $\eta_3-a^{\pm}(y', \eta', \tau)$ and $\eta_3-b^{\pm}(y', \eta', \tau)$. Let χ be a coordinate transform of $T^*(R^3\times R)$ induced from the coordinate transform $y'=x'$, $y_3=x_3-g(x')$, that is, $\chi(x, t, \xi, \tau)=(x', x_3-g(x'), t, \xi'+(\nabla g)(x')\xi_3, \xi_3, \tau)$. Then it is not difficult to show that the image $\chi(\gamma_i^{(p)}(\omega))$ $(\chi(\gamma_r^{(p)}(\omega)))$ of $\gamma_i^{(p)}(\omega)$ $(\gamma_r^{(p)}(\omega))$ by χ is the half Hamilton vector field of $\eta_3-b^+(y', \eta', \tau)$ $(\eta_3-b^-(y', \eta', \tau))$ starting at $(0, t_0, -\varepsilon\omega, \varepsilon\beta)$ $((0, t_0, -\varepsilon\omega_r, \varepsilon\beta))$ and belonging to $T^*\{(y, t); y_3>0\}$. Similarly $\chi(\gamma_{\text{in}}^{(s)}(\omega))$ $(\chi(\gamma_{\text{tr}}^{(s)}(\omega)))$ is the half Hamilton vector field of $\eta_3-a^+(y', \eta', \tau)$ $(\eta_3-a^-(y', \eta', \tau))$ starting at $(0, t_0, -\varepsilon\omega_+, \varepsilon\beta)$ $((0, t_0, -\varepsilon\omega_-, \varepsilon\beta))$ and belonging to $T^*\{(y, t); y_3>0\}$. The same facts hold for the incident S ray $\gamma_i^{(s)}(\omega)$, if $\alpha^2>\beta^2(1-(n(0)\cdot\omega)^2)$.

Next we shall get a simple form of the boundary operator. Let ρ_0 be $(0, t_0, -\varepsilon\eta_0, \varepsilon\tau_0)$ with $\tau_0|\eta_0'|(ps)(0, t_0, -\varepsilon\eta_0', \varepsilon\tau_0)\neq 0$ and $E_1(y', D_{y'}, D_t)$ be a pseudodifferential operator of order 0 with a form of a 3×3 matrix whose principal symbol of E_1 is non-singular at ρ_0. Put $(E_1C)(y', D_{y'}, D_t)=(F^+, F^-)$ $(y', D_{y'}, D_t)$, where $F^{\pm}$ has a form of a 3×3 matrix. We can choose E_1 as follows:

LEMMA 3.2. *Then there exists a pseudodifferential operator $E_1(y', D_{y'}, D_t)$ satisfying the above conditions such that in $\{y'=0\}\cap\Gamma_0$ the principal symbol of $F^{\pm}$ is*

$$\begin{pmatrix} (\rho\tau^2-2\mu|\eta'|^2)|\eta'|^2\Lambda_1^{-4} & 0 & 2\mu b^{\pm}|\eta'|^2\Lambda_1^{-3} \\ 0 & \mu a^{\pm}|\eta'|^2\Lambda_1^{-3} & 0 \\ -2\mu a^{\pm}|\eta'|^2\Lambda_1^{-3} & 0 & (\rho\tau^2-2\mu|\eta'|^2)\Lambda_1^{-2} \end{pmatrix} \tag{3.13}$$

PROOF. Define a pseudodifferential operator $E_1(y', D_{y'}, D_t)$ of order 0 such that in $\{y'=0\}\cap\Gamma_0$ the j-th column vectors $e_j(y', \eta', \tau)$ $(j=1, 2, 3)$ of the principal symbol of E_1 are equal to $e_1={}^t(\eta_1, -\eta_2, 0)\Lambda_1^{-1}$, $e_2={}^t(\eta_2, \eta_1, 0)\Lambda_1^{-1}$ and $e_3={}^t(0, 0, 1)$. Making use of (3.6) to (3.11) we can show that the principal symbol of E_1C is given by (3.13) at $y'=0$. The proof is completed.

Let $T(y', D_{y'}, D_t)$ be a pseudodifferential operator of order 0 with a form of a 6×6 matrix whose principal symbol is the identity matrix in $\{y'=0\}\cap\Gamma_0$, where Γ_0 is a conic neighbourhood of ρ_0. Moreover we assume that T has the form $\begin{pmatrix}\tilde{A} & 0\\ 0 & \tilde{B}\end{pmatrix}$ with $\tilde{A}=\begin{pmatrix}A & 0\\ 0 & 1\end{pmatrix}$ and $\tilde{B}=\begin{pmatrix}B & 0\\ 0 & 1\end{pmatrix}$, where $A(y', D_{y'}, D_t)$ and $B(y', D_{y'}, D_t)$ are pseudodifferential operators with a form of 2×2 matrices. We can specify T satisfying the following condition.

LEMMA 3.3. *We assume that* $\tau_0-2\alpha^2|\eta_0'|^2\neq 0$. *Then there exist a pseudodifferential operator* $T(y', D_{y'}, D_t)$ *of order* 0 *satisfying the above conditions such that the symbol of the sum of the second and fifth column vector of* $(E_1CT)\cdot(y', D_{y'}, D_t)$ *belongs to* $S^{-\infty}(\Gamma_0)$.

PROOF. Let $c_j(y', D_{y'}, D_t)$ be the j-th column vector of E_1C and $a_{ij}(y', D_{y'}, D_t)$ and $b_{ij}(y', D_{y'}, D_t)$ be the (i, j) components of A and B, respectively. Then the required condition is that the symbol of $c_1a_{12}+c_2a_{22}+c_4b_{12}+c_5b_{22}$ belongs to $S^{-\infty}(\Gamma_0)$. We assume $b_{22}(y', D_{y'}, D_t)=1$ and fine the symbol of ${}^t(a_{12}, a_{22}, b_{12})$ with a form $\sum_{j=0}^{\infty}\beta_j(y', \eta', \tau)$, where column vector β_j is a positively homogeneous function of (η', τ) of order $-j$. From the assumption on ρ_0 it follows that a 3×3 square matrix (c_{10}, c_{20}, c_{40}), where $c_{j0}(y', \eta', \tau)$ is the principal symbol of $c_j(y', D_{y'}, D_t)$, is non-singular in a small conic neighbourhood Γ_0 of ρ_0. Thus by calculuses of symbols of pseudodifferential operators we can inductively decide $\beta_j(y', \eta', \tau)$. In the case $j=0$ making use of (3.13) and $a^+=-a^-$ at $y'=0$, we can derive $\beta_0={}^t(0, 1, 0)$ at $y'=0$ by Cramer's formula. If we put $a_{11}=b_{11}=1$ and $a_{21}=b_{21}=0$, we have all desired properties on T. The proof is completed.

Next we put $E_2(y', D_{y'}, D_t)$ to be a pseudodifferential operator with a form of a 3×3 matrix whose principal symbol of E_2 is the identity matrix in $\{y'=0\}\cap\Gamma_0$. The desired E_2 satisfies the following

LEMMA 3.4. *We assume that* $\tau_0-\alpha^2|\eta_0'|^2>0$ *and* $\tau_0-2\alpha^2|\eta_0'|^2\neq 0$. *Then there exists a pseudodifferential operator* $E_2(y', D_{y'}, D_t)$ *of order* 0 *satisfying the above conditions such that the symbols of* (1, 2), (1, 5), (2, 4), (2, 6), (3, 2) *and* (3, 5) *components of* $(E_2E_1CT)(y', D_{y'}, D_t)$ *belong to* $S^{-\infty}(\Gamma_0)$.

PROOF. Let $b_j(y', D_{y'}, D_t)$ $(j=1, \cdots, 6)$ be the j-th column vector of E_1CT

and $e_j(y', D_{y'}, D_t)$ $(j=1, 2, 3)$ be the j-th line vector of E_2. By Lemma 3.3 the required conditions are that the symbols of ${}^t e_1 \cdot b_2 = {}^t e_3 \cdot b_2 = {}^t e_2 \cdot b_4 = {}^t e_2 \cdot b_6$ belong to $S^{-\infty}(\Gamma_0)$. We shall fine the symbol of e_j with a form $\sum_{k=0}^{\infty} e_{jk}(y', \eta', \tau)$, where e_{jk} is a positively homogeneous function of (η', τ) of order $-k$. From (3.13) the principal symbol of b_2 is not zero vector. Thus we can put $e_{10} = (1, 0, 0) - ({}^t b_{20} \cdot (1, 0, 0)){}^t b_{20}/|b_{20}|^2$ and $e_{30} = (0, 0, 1) - ({}^t b_{20} \cdot (0, 0, 1)){}^t b_{20}/|b_{20}|^2$, where b_{j0} is the principal symbol of b_j. Making use of calculuses of symbols of pseudo-differential operators, we can inductively decide e_{jk} for $j=1, 3$, $k \geqq 1$. In order to decide e_2 we first remark that in a conic neighbourhood Γ_0 of ρ_0 b_{40} and b_{60} are linearly independent by (3.13) and the assumption on ρ_0. Thus a linear equation $({}^t e_{2k} \cdot b_{40})(y', \eta', \tau) = f_1(y', \eta', \tau)$, $({}^t e_{2k} \cdot b_{60})(y', \eta', \tau) = f_2(y', \eta', \tau)$ is solvable for any C^∞ homogeneous function (f_1, f_2) of order $-k$. In the case $j=0$ we look for $e_{20}(y', \eta', \tau)$ with a form $(0, 1, 0) + X{}^t b_{40} + Y{}^t b_{60}$. Then an equivalent condition of ${}^t e_{20} \cdot b_{40} = {}^t e_{20} \cdot b_{60} = 0$ is that scalar functions $X(y', \eta', \tau)$ and $Y(y', \eta', \tau)$ satisfy a linear equation $(b_{40} \cdot b_{40})X + (b_{40} \cdot b_{60})Y = -(0, 1, 0) \cdot {}^t b_{40}$, $(b_{40} \cdot b_{60})X + (b_{60} \cdot b_{60})Y = -(0, 1, 0) \cdot {}^t b_{60}$. Since by (3.13) ${}^t b_{20} \cdot (1, 0, 0) = {}^t b_{20} \cdot (0, 0, 1) = {}^t b_{40} \cdot (0, 1, 0) = {}^t b_{60} \cdot (0, 1, 0) = 0$ in $\{y'=0\} \cap \Gamma_0$, the principal symbol of E_2 is the identity matrix in $\{y'=0\} \cap \Gamma_0$. The proof is completed.

4. Theorems on reflective phenomena.

In this section we shall prove theorems on reflective phenomena of singularities corresponding to (1.3) and (1.4) of Introduction. Let $\rho_0 = (0, t_0, -\varepsilon\eta_0', \varepsilon\tau)$ satisfy the conditions of Lemmas 3.2, 3.3 and 3.4. Denote $T^{-1}(y', D_{y'}, D_t)V(y, t)$ by $W(y, t) = {}^t(w_1, \cdots, w_6)$, where T is a pseudodifferential operator of Lemma 3.3. Then $W(y, t)$ satisfies the following boundary value problem

$$D_{y_3}W - \begin{pmatrix} \tilde{H}^+ & & & 0 \\ & \tilde{h}^+ & & \\ & & \tilde{H}^- & \\ 0 & & & \tilde{h}^- \end{pmatrix} W = F_1 \qquad \text{in } y_3 > 0\,, \tag{4.1}$$

$$C_1(y', D_{y'}, D_t)W = G_1 \qquad \text{on } y_3 = 0\,, \tag{4.2}$$

where $C_1 = E_1 E_2 BST$, $\rho_0 \notin WF(G_1)$ and F_1 is smooth at ρ_0. Moreover the principal symbol of $\tilde{H}^\pm(y', D_{y'}, D_t)$ is a diagonal matrix $a^\pm(y', \eta', \tau)I_2$ and the principal symbol of $\tilde{h}^\pm(y', D_{y'}, D_t)$ is $b^\pm(y', \eta', \tau)$. First we shall consider an incident P ray of singularities.

THEOREM 3.1. *Let $u(x, t)$ be a solution of* (1.1) *and* (1.2). *We assume that $\omega \in S^2$ satisfies that $0 < n(0) \cdot \omega < 1$ and $h(\beta^2/(1-(n(0)\cdot\omega)^2)) \neq 0$, where $h(s) = s^3 - 8\alpha^2 s^2 + (24\alpha^4 - 16\alpha^6/\beta^2)s - 16\alpha^6 + 16\alpha^8/\beta^2$. If $\gamma_i^{(p)}(\omega) \subset WF(u)$ and $\gamma_{\text{in}}^{(s)}(\omega) \cap WF(u) = \emptyset$,* *then* $\gamma_r^{(p)}(\omega) \cup \gamma_{tr}^{(s)}(\omega) \subset WF(u)$.

PROOF. Let us consider the problem in a conic neighbourhood Γ_1 of $\rho_1 = (0, t_0, -\varepsilon\omega', \varepsilon\beta)$, which satisfies the conditions of Lemmas 3.2, 3.3 and 3.4 by $n(0)\cdot\omega = -\omega_3$. From the hypothesises on $WF(u)$ of Theorem 4.1 and Theorem 2.5.11′ of [1] it follows that $WF(U) \cap \chi(\gamma_{in}^{(s)}(\omega)) = \emptyset$ and $\chi(\gamma_i^{(p)}(\omega)) \subset WF(U)$, where U is a distribution appeared in (3.1) and χ is the coordinate transform of $T^*(R^3 \times R)$ defined in Remark 3.1. Let $A(y, t, D_y, D_t)$ be a pseudodifferential operator of order 0 such that the principal symbol of A is not zero in a conic neighbourhood Γ of some point belonging to $\chi(\gamma_i^{(p)}(\omega)) \cup \chi(\gamma_{in}^{(s)}(\omega))$ and the symbol of A vanishes in the complement of a conic neighbourhood of $\tilde{\Gamma}$ with $\Gamma \Subset \tilde{\Gamma}$. Then by the relation $AW = A(T^{-1}S^{-1})U$ and Proposition A.1 of [4] it follows that $WF(W) \subset \chi(\gamma_i^{(p)}(\omega))$ and $WF(W) \cap \chi(\gamma_{in}^{(s)}(\omega)) = \emptyset$. Since W satisfies the hyperbolic equation (4.1) in $y_3 > 0$, from Remark 3.1 we have that $(WF(w_1) \cup WF(w_2)) \cap \chi(\gamma_{in}^{(s)}(\omega)) = \emptyset$ and $WF(w_3) \subset \chi(\gamma_i^{(p)}(\omega))$. Thus making use of properties of fundamental solutions of first order hyperbolic equations which are satisfied by ${}^t(w_1, w_2)$ and w_3, we have that

$$(4.3) \qquad \rho_1 \notin WF(w_{1|y_3=0}) \cup WF(w_{2|y_3=0}), \qquad \rho_1 \in WF(w_{3|y_3=0}).$$

Let $B_3(y', D_{y'}, D_t)$ be $\begin{pmatrix} b_{14} & b_{16} \\ b_{34} & b_{36} \end{pmatrix}$, where $b_{ij}(y', D_{y'}, D_t)$ is the (i, j) component of C_1. Then from Lemma 3.4, (4.2) and (4.3) on $y_3 = 0$ w_3, w_4 and w_6 satisfy the following condition;

$$B_3\begin{pmatrix} w_4 \\ w_6 \end{pmatrix} = -\begin{pmatrix} b_{13} \\ b_{33} \end{pmatrix} w_3 + G_2,$$

where $\rho_0 \in WF(G_2)$. By (3.13) the principal symbol of B_3 is non-singular at ρ_1. Let us check the condition that the principal symbols of components of $B_3^{-1}{}^t(b_{13}, b_{33})$ are not zero at ρ_1. By Cramer's formula one of equivalent conditions that the principal symbols of the first and second components of $B_3^{-1}{}^t(b_{13}, b_{33})$ are not zero at ρ_0 is one that the principal symbols of $b_{13}b_{36} - b_{33}b_{16}$ and $b_{14}b_{33} - b_{34}b_{13}$ are not zero at ρ_0. From (3.13) the principal symbol of $b_{13}b_{36} - b_{33}b_{16}$ is clearly not zero at ρ_0. In $\{y' = 0\} \cap \Gamma_0$ the principal symbol of $b_{14}b_{33} - b_{34}b_{13}$ is equal to $(\tau^2 - 2\alpha^2|\eta'|^2)^2 - 4\alpha^4(\tau^2 - \beta^2|\eta'|^2)^{1/2}(\tau^2 - \alpha^2|\eta'|^2)^{1/2}/(\alpha\beta)$. Thus the condition that the principal symbol of $b_{14}b_{33} - b_{34}b_{13}$ is not zero at ρ_1 is equivalent to one that $\tau^2 h(\tau^2/|\eta'|^2)/|\eta'|^2$ is not zero at ρ_1. This condition is our assumption $h(\beta^2/(1-(n(0)\cdot\omega)^2)) \neq 0$. Thus we can conclude $\rho_1 \in WF(w_{4|y_3=0}) \cap WF(w_{6|y_3=0})$. The argument of deriving (4.3) from the assumptions on $WF(u)$ is invertible. Thus we can show that $\gamma_r^{(p)}(\omega) \supset \gamma_{tr}^{(s)}(\omega) \subset WF(u)$. The proof is completed.

On the polynomial $h(s)$ we have the following

REMARK 4.2. $h(s)$ is a polynomial which appears in the analysis of Rayleigh waves in the half space case (see [2]). It is easy to show that $h(s)=0$ has only one simple root in $0<s<\alpha^2$. However we want to look for roots which are greater than β^2. Since $h(\beta^2)=\lambda^4/(\lambda+2\mu)$ and $h'(s)\geqq 0$ for $\lambda\geqq 4\mu$, if $n(0)\cdot\omega$ is closed to 0 or 1, $h(\beta^2/(1-(n(0)\cdot\omega)^2))$ is positive, and $h(s)$ is also positive for $s\geqq\beta^2$ if $\lambda\geqq 4\mu$.

Next we shall consider an incident S ray of singularities passing through $(0, t_0, -\varepsilon\omega, \varepsilon\alpha)$. First we show a simple case.

THEOREM 4.3. *Let $u(x, t)$ be a solution of* (1.1) *and* (1.2). *We assume that $0<n(0)\cdot\omega<1$ and $0<\alpha^2/(1-(n(0)\cdot\omega)^2)<\beta^2$. In this case the incoming ray $\gamma_{\mathrm{in}}^{(p)}(\omega)$ and the transfered reflective ray $\gamma_{\mathrm{tr}}^{(p)}(\omega)$ for the incident S ray $\gamma_i^{(s)}(\omega)$ do not exist. If $\gamma_i^{(s)}(\omega)$ is contained in $WF(u)$, then $\gamma_r^{(s)}(\omega)$ is also contained in $WF(u)$.*

PROOF. By the argument of deriving (4.3) we only show that $\rho_2=(0, t_0, -\varepsilon\omega', \varepsilon\alpha)$ belongs to $WF(w_{4|y_3=0})\cup WF(w_{5|y_3=0})$. From the assumption $\alpha^2/(1-(n(0)\cdot\omega)^2)<\beta^2$ a point $(0, t_0, -\varepsilon\omega, \varepsilon\alpha)$ is not a zero point of $\tau^2-\beta^2|\xi|^2$, that is, $p(0, -\varepsilon\omega', \varepsilon\alpha)>0$. Thus $w_6(y, t)$ is a solution of a backward parabolic equation $(D_{y_3}-h^-)w_6=0$ in $y_3>0$. It follows that ρ_2 does not belong to $WF(w_{6|y_3=0})$. We assume that ρ_2 does not belong to $WF(w_{4|y_3=0})\cup WF(w_{5|y_3=0})$. Let $\tilde{B}(y', D_{y'}, D_t)$ be a 3×3 square matrix whose first, second and third column vector are equal to these of C_1. Then by Lemma 3.4 the boundary condition $C_1W=G_1$ is reduced to $\tilde{B}{}^t(w_1, w_2, w_3)=\tilde{G}_1$, where ρ_2 does not to $WF(\tilde{G}_1)$. By (3.13) the principal symbol of $\tilde{B}$ is non-singular at ρ_0. Thus we can conclude that ρ_2 does not belong to $WF(w_{1|y_3=0})\cup WF(w_{2|y_3=0})$. This is a contradiction to $\gamma_i^{(s)}(\omega)\subset WF(u)$. The proof is completed.

Second we shall consider a case $\beta^2<\alpha^2/(1-(n(0)\cdot\omega)^2)$. In this case there are two reflective ray of singularities of $\gamma_i^{(s)}(\omega)$ and reflective phenomena are more interested than P singularities case.

THEOREM 4.4. *Let $u(x, t)$ be a solution of* (1.1) *and* (1.2). *We assume that $0<n(0)\cdot\omega<1$, $\beta^2<\alpha^2/(1-(n(0)\cdot\omega)^2)$ and $h(\alpha^2/(1-(n(0)\cdot\omega)^2))\neq 0$. If $\gamma_i^{(s)}(\omega)\subset WF(u)$ and $\gamma_{\mathrm{in}}^{(s)}(\omega)\cap WF(u)=\emptyset$, then one of the following two reflective phenomena occurs:* a) $\gamma_r^{(s)}(\omega)\cup\gamma_{\mathrm{tr}}^{(p)}(\omega)\subset WF(u)$, b) $\gamma_r^{(s)}(\omega)\subset WF(u)$ *and* $\gamma_{\mathrm{tr}}^{(p)}(\omega)\cap WF(u)=\emptyset$.

PROOF. From the assumption on $WF(u)$ and the argument of deriving (4.3) it follows that $\rho_2\in WF(w_{1|y_3=0})\cup WF(w_{2|y_3=0})$ and $\rho_2\notin WF(w_{3|y_3=0})$. First we assume $\rho_2\notin WF(w_{1|y_3=0})$. Then by Lemma 3.4 (4.2) is reduced to the following equation on $y_3=0$:

$$\begin{pmatrix} b_{14} & b_{16} \\ b_{34} & b_{35} \end{pmatrix}\begin{pmatrix} w_4 \\ w_6 \end{pmatrix} = G_2\,, \qquad b_{22}w_2+b_{25}w_5 = G_3\,,$$

where $b_{ij}(y', D_{y'}, D_t)$ is the (i, j) component of $C_1(y', D_{y'}, D_t)$ and ρ_2 does not belong to $WF(G_2)\cup WF(G_3)$. By (3.13) and the condition $\rho_2\in WF(w_{2|y_3=0})$ we can conclude that $\rho_2\in WF(w_{5|y_3=0})$ and $\rho_2\notin WF(w_{4|y_3=0})\cap WF(w_{6|y_3=0})$. By the argument of deriving (4.3) it follows that $\gamma_r^{(s)}(\omega)\subset WF(u)$ and $\gamma_{tr}^{(p)}(\omega)\cap WF(u)=\emptyset$.

Second we assume that ρ_2 belongs to $WF(w_{1|y_3=0})$. Then making use of Lemma 3.4 and $C_1W=G_1$ on $y_3=0$, we have

$$\begin{pmatrix} b_{14} & b_{16} \\ b_{34} & b_{36} \end{pmatrix}\begin{pmatrix} w_4 \\ w_6 \end{pmatrix} = -\begin{pmatrix} b_{11} \\ b_{31} \end{pmatrix} w_1+G_4 \qquad \text{on } y_3=0\,,$$

where $\rho_2\notin WF(G_4)$. Thus by the same way of proving Theorem 4.1 we get $\rho_2\in WF(w_{4|y_3=0})\cup WF(w_{6|y_3=0})$. It implies that $\gamma_r^{(s)}(\omega)\cup\gamma_{tr}^{(p)}(\omega)\subset WF(u)$. The proof is completed.

REMARK 4.4. In the case $\beta^2=\alpha^2/(1-(n(0)\cdot\omega)^2)$ ρ_2 is a glancing point of $\tau^2-\beta^2|\xi|^2$, that is, $p(0, t_0, -\varepsilon\omega', \varepsilon\alpha)=0$. Thus if Ω is concave, it seems that there exist singularities of $u(x, t)$ propagating on $\partial\Omega$ instead of a transfered P ray of singularities. This example is shown in [8].

5. Half space case.

We assume $\partial\Omega$ is a hyperplane in R^3. Then in a conic neighbourhood of $(0, t, \xi', \tau)$ such that $\tau|\xi'|^2(\tau^2-\alpha^2|\xi'|^2)(\tau^2-\beta^2|\xi'|^2)\neq 0$, the problem (1.1) and (1.2) is micro-locally reduced to the problem (3.12), where the full symbols of $H^{\pm}(D_{y'}, D_t)$ and $h^{\pm}(D_{y'}, D_t)$ are $\pm aI_2=\pm(\tau^2/\alpha^2-|\xi'|^2)^{1/2}I_2$ and $\pm b=\pm(\tau^2/\beta^2-|\xi'|^2)^{1/2}$, respectively and the full symbol of C is given by (3.13). Let $V=S^{-1\,t}(\Lambda^t u, D_{x_3}{}^t u)={}^t(v_1, \cdots, v_6)$, where the full symbol of $S=(s_1^+, s_2^+, s_3^+, s_1^-, s_2^-, s_3^-)$ is given by $s_j^{\pm}={}^t({}^tw_j^{\pm}, \pm a\,{}^tw_j^{\pm})$ $(j=1, 2)$ and $s_3^{\pm}={}^t({}^tw_3, \pm bw_3^{\pm})$ with $w_1^{\pm}={}^t(\pm a\xi_1, \pm a\xi_2, -|\xi'|^2)\Lambda_1^{-2}$, $w_2^{\pm}={}^t(-\xi_2, \xi_1, 0)\Lambda_1^{-1}$ and ${}^tw_3^{\pm}={}^t(\xi_1, \xi_2, \pm b)\Lambda_1^{-1}$. By the form of the symbol of $S(D_{y'}, D_t)$ it follows that

$$\begin{aligned}(5.1)\qquad \hat{v}_2(\xi', x_3, \tau) &= \Lambda_1(-a\xi_2, a\xi_1, 0, -\xi_2\Lambda_1, \xi_1\Lambda_1, 0)\begin{pmatrix} \Lambda_1\hat{u} \\ D_{x_3}\hat{u} \end{pmatrix}/(2a|\xi'|^2) \\ &= \Lambda_1^2(D_{x_3}+a)(\widehat{\operatorname{rot} u}(\xi', x_3, \tau))_3/(2a|\xi'|^2)\,,\end{aligned}$$

where $\hat{u}(\xi', x_3, \tau)$ and $\hat{v}_2(\xi', x_3, \tau)$ is the partial Fourier transforms with respect to (x', t) of $u(x, t)$ and $v_2(x, t)$, respectively, and $(\widehat{\operatorname{rot} u}(\xi', x_3, \tau))_3$ is the third component of the partial Fourier transform of the rotation of $u(x, t)$ with respect to (x', t). Similarly we have

$$(5.2)\qquad v_5(\xi', x_3, \tau) = -\Lambda_1^2(D_{x_3}-a)(\widehat{\operatorname{rot} u})/(2a|\xi'|^2)\,.$$

Making use of (5.1) and (5.2), we have that $S^t(0, v_2, 0, 0, v_5, 0)={}^t(\Lambda u_H, D_{x_3}u_H)$, where $\hat{u}_H={}^t(-\xi_2|\xi'|^{-2}\widehat{(\mathrm{rot}\,u)}_3,\ \xi_1|\xi'|^2\widehat{(\mathrm{rot}\,u)}_3,\ 0)$. Since v and $(0, v_2, 0, 0, v_5, 0)$ micro-locally satisfies the same boundary value problem, we can conclude that $u-u_H$ micro-locally satisfies the boundary value problem (1.1) and (1.2) and u_H micro-locally satisfies the following boundary value problem:

$$\begin{cases} (\partial_t^2-\alpha^2\Delta)u_H=0 & \text{in } x_3>0\,, \\ \partial u_H/\partial x_3=0 & \text{on } x_3=0\,. \end{cases}$$

Here we used that $(\partial_t^2-\alpha^2\Delta)(\mathrm{rot}\,u)=0$ in $x_3>0$, $D_{x_3}(\mathrm{rot}\,u)_3=a|\xi'|^2\Lambda_1^2(v_2-v_4)$ and the boundary condition of ${}^t(0, v_2, 0, 0, v_5, 0)$ is $\alpha^2|\xi'|^2\Lambda_1^{-3}a(\hat{v}_2-\hat{v}_5)(\xi', 0, \tau)=0$. This is a reason why in Theorem 3.4 there are two ways of reflective phenomena.

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