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Necessary and sufficient conditions for optimality in nonlinear distributed parameter systems with variable initial state

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1. Introduction.

In this paper, we consider a nonlinear distributed control system, with time varying control constraints and an initial condition which is not determined by an a priori given function, but instead it is assumed to belong to a certain specified set (Lions [5] calls them "systems with insufficient data"). The cost criterion is a general convex integral functional.

Using the Dubovitski-Milyutin formalism, we are able to obtain a necessary and sufficient condition for the existence of an optimal solution. A very comprehensive presentation of the Dubovitski-Milyutin theory can be found in the monograph of Girsanov [3]. Our result extends Theorem 2.1 of Lions [5], since we allow for nonlinear dynamics and a nonquadratic cost criterion.

2. Preliminaries.

The mathematical setting is the following. Let $T=[0, b]\subseteq \boldsymbol{R}_+$ (a bounded time interval) and H a separable Hilbert space. Also let $X\subseteq H$ be a subspace of H carrying the structure of a separable reflexive Banach space, which imbeds continuously and densely into H. Identifying H with its dual (pivot space), we have $X\hookrightarrow H\hookrightarrow X^*$, with all embeddings being continuous and dense. Such a triple (X, H, X^*) of spaces is sometimes called "Gelfand triple" or "spaces in normal position". By $\|\cdot\|$ (resp. $|\cdot|$, $\|\cdot\|_*$) we will denote the norm of X (resp. of H, X^*). Also by $(\cdot, \cdot)$ we will denote the inner product in H and by $\langle\cdot, \cdot\rangle$ the duality brackets for the pair (X, X^*). The two are compatible in the sense that if $x\in X\subseteq H$ and $h\in H\subseteq X^*$, we have $(x, h)=\langle x, h\rangle$. Also let Y be another separable Banach space modelling the control space. By $P_{fc}(Y)$ we will denote the nonempty, closed, convex subsets of Y.

The optimal control problem under consideration is the following:

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$$\left\{\begin{array}{l} J(x, u) = \int_0^b L(t, x(t), u(t))dt \longrightarrow \inf \\ \text{s. t. } \dot{x}(t)+A(t, x(t)) = B(t, x(t))u(t) \text{ a. e.} \quad x(0)\in C,\ u(t)\in U(t) \text{ a. e.} \end{array}\right\} \tag{*}$$

We will need the following hypotheses concerning the data of (*).

H(A): $A: T\times X\to X^*$ is an operator s. t.

(1) $t\to A(t, x)$ is measurable,

(2) $x\to A(t, x)$ is continuously Frechet differetiable and strongly monotone uniformly in $t\in T$,

(3) $\|A(t, x)\|_* \leqq a(t)+b\|x\|$ a. e. with $a(\cdot)\in L_+^2$, $b>0$,

(4) $\langle A(t, x), x\rangle \geqq c\|x\|^2$, $c>0$.

H(B): $B: T\times H\to \mathcal{L}(Y, X^*)$ is an operator s. t.

(1) $t\to B(t, x)u$ is measurable for all $(x, u)\in H\times Y$,

(2) $x\to B(t, x)$ is continuous,

(3) $x\to B(t, x)u$ is continuously Frechet differentiable,

(4) $\|B(t, x)u\|_* \leqq \beta_1(t)+\beta_2(t)|x|+\beta_3(t)\|u\|$ a. e. with $\beta_1(\cdot)$, $\beta_2(\cdot)\in L_+^2$, $\beta_3(\cdot)\in L_+^\infty$.

H(L): $L: T\times H\times Y\to \boldsymbol{R}$ is an integrand s. t.

(1) $t\to L(t, x, u)$ is measurable,

(2) $(x, u)\to L(t, x, u)$ is convex and continuously Gateaux differentiable,

(3) for every $(x, u)\in L^\infty(H)\times L^2(Y)$, $J(x, u)$ is finite.

H(U): $U: T\to P_{fc}(Y)$ is a multifunction s. t. $\mathrm{Gr}\, U=\{(t, u)\in T\times Y: u\in U(t)\}\in B(T)\times B(Y)$ (where $B(T)$ is the Borel σ-field of T and $B(Y)$ the Borel σ-field of Y), $t\to|U(t)|=\sup\{\|u\|: u\in U(t)\}$ belongs in L_+^2 and if $S_U^2=\{u(\cdot)\in L^2(Y): u(t)\in U(t) \text{ a. e.}\}$, then $\mathrm{int}\, S_U^2\neq\emptyset$.

H(C): $C\subseteq H$ is a closed, convex set with a nonempty interior.

Following Lions [4], we define $W(T)=\{x(\cdot)\in L^2(X): \dot{x}\in L^2(X^*)\}$. This is a Banach space with norm $\|x\|_{W(T)}=\left[\int_0^b\|x(t)\|^2dt+\int_0^b\|\dot{x}(t)\|_*^2dt\right]^{1/2}$. It is well known that $W(T)\hookrightarrow C(T, H)$ i. e. the elements of $W(T)$ are continuous maps with values in H, eventually after changing each function on a set of measure zero.

Since our necessary and sufficient conditions, will involve the adjoint state, we need the following existence result. By $A_x(t, x(t))(\cdot)$, $B_x(t, x(t), u(t))$ and $L_x(t, x(t), u(t))$ we denote the derivatives with respect to x of the maps $A(t, x)$, $B(t, x, u)$ and $L(t, x, u)$ at the points $(t, x(t))$, $(t, x(t), u(t))$ and $(t, x(t), u(t))$ respectively. Also by $A_x^*(t, x(t))$, $B_x^*(t, x(t), u(t))$ we denote the adjoints of $A_x(t, x(t))$ and $B_x(t, x(t), u(t))$ respectively.

PROPOSITION 2.1. *If hypotheses* H(A), H(B), H(L) *hold,* $B_x(t, x(t), u(t))|_{X}(\cdot)$ *is dissipative and* $t\to L_x(t, x(t), u(t))$ *belongs in* $L^2(H)$, *then there exists* $p(\cdot)\in$

$W(T)$ *s.t.*

$$-\dot{p}(t)+A_x^*(t,\ x(t))p(t)=B_x^*(t,\ x(t),\ u(t))p(t)-L_x(t,\ x(t),\ u(t))\ a.e.\quad p(b)\in H.$$

PROOF. From the strong monotonicity of $A(t,\ \cdot)$, uniformly in $t\in T$, we have:

$$\langle A(t,\ x')-A(t,\ x(t)),\ x'-x(t)\rangle\geqq\theta\|x'-x(t)\|^2\qquad\theta>0$$

$$\Longrightarrow\langle A_x(t,\ x(t))(x'-x(t))+0(\|x'-x(t)\|),\ x'-x(t)\rangle\geqq\theta\|x'-x(t)\|^2.$$

Putting $x'-x(t)=\varepsilon p$, we see that

$$\langle A_x(t,\ x(t))\varepsilon p+0(\varepsilon\|p\|),\ \varepsilon p\rangle\geqq\theta\varepsilon^2\|p\|^2.$$

Divide by ε^2 and let $\varepsilon\to 0^+$. We get:

$$\langle A_x(t,\ x(t))p,\ p\rangle=\langle A_x^*(t,\ x(t))p,\ p\rangle\geqq\theta\|p\|^2.$$

Also by hypothesis $\langle -B_x^*(t,\ x(t),\ u(t))p,\ p\rangle\geqq 0$. Since $t\to L_x(t,\ x(t),\ u(t))$ belongs in $L^2(H)$, we can invoke Theorem 4.2, p. 167 of Barbu [**1**], and get that indeed there exists $p(\cdot)\in W(T)$ solving our problem. Q.E.D.

3. Necessary and sufficient conditions.

The next result gives us necessary and sufficient conditions for a triple $(x_0,\ x,\ u)\in H\times W(T)\times L^2(Y)$ to be a solution of $(*)$.

THEOREM 3.1. *If hypotheses* H(A), H(B), H(L), H(U), H(C) *hold, for the pair* $(x,\ u)\in W(T)\times L^2(Y)$ *we have* $\|A_x(t,\ x(t))\|_{\mathcal{L}(X,X^*)}\leqq\eta_1$, $\|B_x(t,\ x(t),\ u(t))\|_{\mathcal{L}(H,X^*)}\leqq\eta_2$, $B_x(t,\ x(t),\ u(t))|_X(\cdot)$ *is dissipative and* $t\to L_x(t,\ x(t),\ u(t))\in L^2(H)$, *then the triple* $(x(0)=x_0,\ x,\ u)\in H\times W(T)\times L^2(Y)$ *is a solution of* $(*)$ *if and only if*

$$\dot{x}(t)+A(t,\ x(t))=B(t,\ x(t))u(t)\ a.e.\qquad x(0)=x_0\in C,\ u(t)\in U(t),$$

there exists $p(\cdot)\in W(T)$ *satisfying the "adjoint equation"*

$$-\dot{p}(t)+A_x^*(t,\ x(t))p(t)=B_x^*(t,\ x(t),\ u(t))p(t)-L_x(t,\ x(t),\ u(t))\ a.e.,\quad p(b)=0$$

and the following "minimum principles" hold

$$(L_u(t,\ x(t),\ u(t))-B^*(t,\ x(t))p(t),\ v-u(t))_{Y,Y^*}\geqq 0\ \textit{for all}\ v\in U(t)\ \ a.e.\qquad\textit{and}$$

$$(-p(0),\ c-x_0)\geqq 0\quad\textit{for all}\ c\in C.$$

PROOF. As we already mentioned in the introduction, our approach is based on the Dubovitski-Milyutin formalism. So we need to analyze the cost criterion, the equality constraint (i.e. the evolution equation) and the initial data-control constraints (regarded here as an inequality constraint), by determining the cone of directions of decrease, the tangent cone and the cone of feasible directions respectively.

We will start with the cost criterion $J(\cdot,\ \cdot)$. Recalling that $J(\cdot,\ \cdot)$ is convex and using the monotone convergence theorem we get that

$$\nabla J(x, u)(h, v) = \int_0^b \nabla L(t, x(t), u(t))(h(t), v(t))dt,$$

where ∇ is the gradient operator.

But since by hypothesis H(L)(2), $L(t, \cdot, \cdot)$ is continuously Gateaux differentiable, from the total differential rule we have:

$$\nabla L(t, x(t), u(t))(h(t), v(t)) = L'_x(t, x(t), u(t))h(t) + L'_u(t, x(t), u(t))v(t).$$

Invoking Theorem 7.4 of Girsanov [3], we get that the cone of directions of decrease of the cost criterion $J(\cdot, \cdot)$ at (x, u) is given by

$$K_d = \{(h, v) \in W(T) \times L^2(Y) : J'(x, u)(h, v) < 0\}.$$

Assume $K_d \neq \emptyset$. Then we have:

$$\Longrightarrow K_d^* = \{-\lambda J(x, u) : \lambda \in \boldsymbol{R}_+\}.$$

Now we pass to the analysis of the equality constraint. This is determined by the dynamical equation of the system. So consider the map $P: H \times W(T) \times L^2(Y) \to L^2(X^*) \times H$ defined by

$$P(x'_0, x', u')(t) = (\dot{x}'(t) + A(t, x'(t)) - B(t, x'(t))u'(t), x'(0) - x'_0).$$

Observe that because of our hypotheses both $\hat{A}: W(T) \to X^*$ defined by $(\hat{A}x')(t) = A(t, x'(t))$ and $\hat{B}: W(T) \times L^2(Y) \to L^2(X^*)$ defined by $\hat{B}(x', u')(t) = B(t, x'(t))u'(t)$ are continuously Frechet differentiable at (x_0, x, u). So $P(\cdot, \cdot, \cdot)$ is continuously Frechet differentiable at (x_0, x, u) and furthermore

$$\begin{aligned} &P'(x_0, x, u)(h_0, h, v)(t) \\ &= (\dot{h}(t) + A_x(t, x(t))h(t) - B_x(t, x(t), u(t))h(t) - B(t, x(t))v(t), h(0) - h_0). \end{aligned}$$

We will show that $P'(x_0, x, u)$ is surjective. So let $(g, v, h_0, h_1) \in L^2(X^*) \times L^2(Y) \times H \times H$ be given and consider the following Cauchy problem:

$$\left\{ \begin{array}{l} \dot{h}(t) + A_x(t, x(t))h(t) = B_x(t, x(t), u(t))h(t) + B(t, x(t))v(t) + g(t) \quad \text{a.e.} \\ h(0) = h_0 + h_1. \end{array} \right\}$$

As in the proof of Proposition 2.1, we can check that all the hypotheses of Theorem 4.2 of Barbu [1] are satisfied. Hence the above Cauchy problem has a solution $h(\cdot) \in W(T)$. So for any $(g, h_1) \in L^2(X^*) \times H$, we can find $(h_0, h, v) \in H \times H \times L^2(Y)$ s.t. $P'(x_0, x, u)(h_0, h, v) = (g, h_1)$ i.e. $P'(x_0, x, u)$ is surjective. Hence we can apply Lyusternik's theorem (see Girsanov [3], Theorem 9.1) and deduce that if

$Q_1 = \{(x'_0, x', u') \in H \times W(T) \times L^2(Y) : P(x'_0, x', u') = 0\}$ (equality constraint set),

then the tangent space to Q_1 at (x_0, x, u) is given by

$$T(Q_1) = \{(h_0, h, v) \in H \times W(T) \times L^2(Y) : P'(x_0, x, u)(h_0, h, v) = 0\}$$
$$= \ker P'(x_0, x, u)$$
$$\Rightarrow (T(Q_1))^* = \{w^* \in H \times W(T)^* \times L^2(Y^*) : w^*(h_0, h, v) = 0$$
$$\text{for all } (h_0, h, v) \in T(Q_1)\}.$$

Finally we will analyze the initial data-control constraints. So set

$$Q_2 = C \times S_U^2 \subseteq H \times L^2(Y).$$

By hypothesis $\operatorname{int} C \neq \emptyset$ and $\operatorname{int} S_U^2 \neq \emptyset$, and $C \times S_U^2$ is convex. So Theorem 10.5 of Girsanov [**3**], tells us that the dual to the cone of feasible directions of Q_2 at (x_0, u) is given by

$$K(Q_2)_f^* = (C \times S_U^2)^* = C^* \times S_U^{2*}.$$

Hence $(c^*, u^*) \in K(Q_2)_f^*$ if and only if c^* supports C at x_0 and u^* support S_U^2 at u.

Now that we have in our disposal all the appropriate cones, we can apply the Dubovitski-Milyutin theorem [**2**] (see also Girsanov [**3**], Theorem 6.1) and get $y^* \in K_d^*$, $w^* \in T(Q_1)^*$ and $(c^*, u^*) \in K(Q_2)_f^*$, not all simultaneously zero s. t.

$$(0, y^*) + w^* + (c^*, 0, u^*) = 0$$
$$\Rightarrow y^*(h, v) + w^*(h_0, h, v) + (c^*, h_0) + u^*(v) = 0 \quad \text{for all } (h_0, h, v) \in H \times W(T) \times L^2(Y).$$

Recall from the analysis of the equality constraint that if $(h_0, h, v) \in T(Q_1)$ i.e. if $P'(x_0, x, u)(h_0, h, v) = 0$, then $w^*(h_0, h, v) = 0$. This means then that if for any $(h_0, v) \in H \times L^2(Y)$, we choose $h \in W(T)$, so that (h_0, h, v) solves the Cauchy problem

$$\dot{h}(t) + A_x(t, x(t))h(t) = B_x(t, x, (t), u(t))h(t) + B(t, x(t))v(t) \quad \text{a.e.,} \quad h(0) = h_0$$

(we already saw that such an $h \in W(T)$ always exists), then $w^*(h_0, h, v) = 0$ and in this case the Euler-Lagrange equation becomes

$$y^*(h, v) + (c^*, h_0) + u^*(v) = 0 \;\Rightarrow\; -\lambda J'(x, u)(h, v) + (c^*, h_0) + u^*(v) = 0.$$

Since $(h_0, v) \in H \times L^2(Y)$ is arbitrary, if $\lambda = 0$, then $c^* = 0$, $u^* = 0$ and so $w^* = 0$, a contradiction to the Dubovitski-Milyutin theorem. So $\lambda > 0$ and without any loss of generality, we can take $\lambda = 1$.

Consider the following adjoint Cauchy problem.

$$-\dot{p}(t) + A_x^*(t, x(t))p(t) = B_x^*(t, x(t), u(t))p(t) - L_x(t, x(t), u(t)) \quad \text{a.e.,} \quad p(b) = 0.$$

From Proposition 2.1 we know that the above Cauchy problem has a solution $p(\cdot) \in W(T)$. Using this adjoint state $p(\cdot)$, we get:

$$\int_0^b (L_x(t, x(t), u(t)), h(t))dt = \int_0^b \langle \dot{p}(t) - A_x^*(t, x(t))p(t) + B_x^*(t, x(t))p(t), h(t)\rangle dt$$
$$= \int_0^b \langle \dot{p}(t), h(t)\rangle dt - \int_0^b \langle A_x^*(t, x(t))p(t), h(t)\rangle dt + \int_0^b \langle B_x^*(t, x(t))p(t), h(t)\rangle dt.$$

From Lemma 5.5.1 of Tanabe [6], we know that:

$$\int_0^b \langle \dot{p}(t), h(t)\rangle dt = (p(t), h(t))\Big|_0^b - \int_0^b \langle p(t), \dot{h}(t)\rangle dt = -(p(0), h_0) - \int_0^b \langle p(t), \dot{h}(t)\rangle dt.$$

Also we have:

$$\int_0^b \langle A_x^*(t, x(t))p(t), h(t)\rangle dt = \int_0^b \langle p(t), A_x(t, x(t))h(t)\rangle dt \quad \text{and}$$

$$\int_0^b \langle B_x^*(t, x(t), u(t))p(t), h(t)\rangle dt = \int_0^b \langle p(t), B_x(t, x(t), u(t))h(t)\rangle dt.$$

Using these facts, we get:

$$\begin{aligned}&\int_0^b (L_x(t, x(t), u(t)), h(t))dt \\ &= \int_0^b \langle p(t), -\dot{h}(t) - A_x(t, x(t))h(t) + B_x(t, x(t), u(t))h(t)\rangle dt - (p(0), h_0).\end{aligned}$$

Recalling the choice of $h(\cdot)\in W(T)$, we get:

$$\int_0^b (L_x(t, x(t), u(t)), h(t))dt = \int_0^b \langle p(t), -B(t, x(t))v(t)\rangle dt - (p(0), h_0).$$

Use this back into the Euler-Lagrange equation, to get:

$$\begin{aligned}u^*(v) + (c^*, h_0) = &\int_0^b \langle p(t), -B(t, x(t))v(t)\rangle dt \\ &+ \int_0^b (L_u(t, x(t), u(t)), v(t))_{Y,Y*} dt - (p(0), h_0)\end{aligned}$$

for every $v\in L^2(Y)$ and every $h_0\in H$. Hence clearly

$$u^*(v) = \int_0^b (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v(t))_{Y,Y*} dt \quad \text{and} \quad c^*(h_0) = -(p(0), h_0).$$

Recall that u^* supports S_U^2 at u and c^* supports C at x_0. So we have:

$$\int_0^b (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v(t) - u(t))dt \geqq 0 \quad \text{for all } v\in S_U^2 \quad \text{and}$$

$$(-p(0), c - x_0) \geqq 0 \quad \text{for all } c\in C.$$

Suppose that for some $E\subseteqq T$ with $\lambda(E)>0$, we have:

$$\inf_{v\in U(t)} (L_u(t, x(t)), u(t)) - B^*(t, x(t))p(t), v - u(t))_{Y,Y*} < 0, \qquad t\in E.$$

Consider the multifunction $V: E\rightarrow 2^Y \setminus \{\emptyset\}$, defined by:

$$V(t) = \{v\in U(t): (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t))_{Y,Y*} < 0\}.$$

From our hypotheses H(B) and H(L), it is easy to see that

$$(t, v) \longrightarrow r(t, v) = (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t))_{Y,Y^*},$$

is measurable in t, continuous in v, hence jointly measurable. Thus

$$\operatorname{Gr} V = \{(t, v) \in E \times Y : r(t, v) < 0\} \cap \operatorname{Gr} U \in B(E) \times B(Y).$$

Apply Aumann's selection theorem (see Wagner [7]), to get $v_1 : E \to Y$ measurable s. t. $v_1(t) \in V(t)$ for $t \in E$. Let $v : T \to Y$ be defined by setting $v(t) = v_1(t)$ for $t \in E$ and $v(t) = u(t)$ for $t \in T \setminus E$. Clearly $v \in S_U^2$ and furthermore

$$\int_0^b (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v(t) - u(t))_{Y,Y^*} dt < 0$$

a contradiction. So we have:

$$\inf_{v \in U(t)} (L_u(t, x(t), u(t)) - B^*(t, x(t))p(t), v - u(t)) \geqq 0 \quad \text{a.e.}$$

while $\inf_{c \in C}(-p(0), c) = (-p(0), x_0)$.

Finally we remove the hypothesis $K_a \neq \emptyset$. If $K_a = \emptyset$, then

$$\int_0^b (L_x(t, x(t), u(t)), h(t)) dt + \int_0^b (L_u(t, x(t), u(t)), v(t))_{Y,Y^*} dt = 0$$

for all $(h, v) \in W(T) \times L^2(Y)$. Hence we have:

$$L_x(t, x(t), u(t)) = 0, \qquad L_u(t, x(t), u(t)) = 0.$$

The solution of the adjoint equation is

$$p(t) = 0$$

and so the minimum principle becomes obvious. This completes the necessity part of the proof.

For the sufficiency part, we apply Theorem 15.2 of Girsanov [3]. Note that $J(\cdot, \cdot)$ is a convex function, which is finite everywhere. Also through a simple application of Fatou's lemma, we can check that $J(\cdot, \cdot)$ is l. s. c. A convex, l. s. c. function which is finite everywhere, is continuous. So $J(\cdot, \cdot)$ is continuous, convex. The Slater type requirement of Theorem 15.2 of Girsanov [3], is automatically satisfied, since by hypothesis $\operatorname{int} S_U^2 \neq \emptyset$ and $\operatorname{int} C \neq \emptyset$. Thus an application of Theorem 15.2 of Girsanov [3], gives us the sufficiency part. Q.E.D.

REMARK. If $U(\cdot)$ is not L^2-bounded (i. e. $t \to |U(t)|$ is not in L_+^2), then the minimum principle has integral form i. e. $\int_0^b (L_u(t, x(t)), u(t)) - B^*(t, x(t))p(t), v(t) - u(t))_{Y,Y^*} \geqq 0$ for all $v \in S_U^2$.

4. An example.

In this section we work out a concrete example of a parabolic distributed parameter control system, to which our result applies.

So let $T = [0, b]$ and let V be a bounded domain in $\boldsymbol{R}^n$, with a smooth boundary $\partial V = \Gamma$. We consider the following distributed parameter optimal control problem defined on $T\times V$

$$\left\{\begin{array}{l} \hat{J}(x, u)=\int_0^b\int_V \hat{L}(t, z, x(t, z), u(t, z))dzdt \longrightarrow \inf \\ \text{s.t. } \dfrac{\partial x(t, z)}{\partial t}+\hat{A}(t)x(t, z)=u(t, z) \text{ on } (0, b)\times V \\ x(t, z)=0 \text{ on } T\times\Gamma \\ x(0, \cdot)\in C\subseteq L^2(V), \ \left(\int_V \|u(t, v)\|^2 dv\right)^{1/2}\leqq r(t). \end{array}\right\} \tag{**}$$

Here $\hat{A}(t)$ is the formal second order elliptic partial differential operator in divergence form, defined by $\hat{A}(t)y=-\sum_{i,j=1}^n(\partial/\partial z_i)(a_{ij}(t, z)(\partial y(z)/\partial z_j))$. We assume that $a_{ij}(\cdot, \cdot)\in L^\infty(T\times V)$ and that they satisfy the following strong ellipticity condition:

$$\sum_{i,j=1}^n a_{ij}(t, z)\eta_i\eta_j \geqq \theta\sum_{i=1}^n \eta_i^2$$

for all $(t, z)\in T\times V$, $\eta=(\eta_i)_{i=1}^n\in \boldsymbol{R}^n$ and with $\theta>0$.

For this example $X=H_0^1(V)$, $H=L^2(V)$ and $X^*=H^{-1}(V)$. Clearly (X, H, X^*) is a Gelfand triple. On $X\times X$ we consider the following bilinear Dirichlet form:

$$a(t, x, y)=\int_V \sum_{i,j=1}^n a_{ij}(t, z)\frac{\partial x(z)}{\partial z_i}\frac{\partial y(z)}{\partial z_j}dz.$$

Since $a_{ij}(\cdot, \cdot)\in L^\infty(T\times V)$ and using Poincaré's inequality, we have:

$$|a(t, x, y)|\leqq c\|x\|_{H_0^1(V)}\|y\|_{H_0^1(V)}.$$

Let $A(t): H_0^1(V)=X\to H^{-1}(V)=X^*$ be the continuous, linear operator defined by

$$a(t, x, y)=\langle A(t)x, y\rangle \qquad x, y\in H_0^1(V).$$

Making use of the strong ellipticity condition, we can show that

$$\langle A(t)x, x\rangle \geqq \hat{c}\|x\|_{H_0^1(V)}^2.$$

We set $Y=L^2(V)$ (the control space) and set $U(t)=\{u\in L^2(V): \|u\|_2\leqq r(t)\}$. Assume $r(\cdot)\in L_+^2$ and $0<\delta\leqq r(t)$. Let $\mathring{B}(\delta/(\max(b, 1)))=\mathring{B}=\{u\in L^2(Y): \|u\|_{L^2(Y)}<\delta/(\max(b, 1))\}$. Then $\mathring{B}\subseteq S_U^2$ and so $\operatorname{int} S_U^2\neq\emptyset$. Also we assume that $C\subseteq L^2(V)$ is nonempty, closed, convex, solid (i.e. $\operatorname{int} C\neq\emptyset$).

Finally let $\hat{L}: T\times V\times\boldsymbol{R}\times\boldsymbol{R}\to\boldsymbol{R}$ be a integrand s.t.

(i) $(t, z)\to\hat{L}(t, z, x, u)$ is measurable,

(ii) $(x, u)\to\hat{L}(t, z, x, u)$ is convex and continuously differentiable,

(iii) for every $x\in L^\infty(T, L^2(V))$ and every $u\in L^2(T, L^2(V))=L^2(T\times V)$, $\hat{J}(x, u)$ is finite.

Define $L: T\times L^2(V)\times L^2(V)\to\boldsymbol{R}$ by $L(t, x, u)=\int_V \hat{L}(t, z, x(z), u(z))dz$. Using the above hypotheses (i)→(iii) about $\hat{L}$, we have that $L(t, x, u)$ satisfies $H(L)$.

Furthermore $L_x(t, x, u)(h)=\int_V \hat{L}_x(t, z, x(z), u(z))h(z)dz$ and $L_u(t, x, u)(v)=\int_V \hat{L}_u(t, z, x(z), u(z))v(z)dz$.

Now rewrite optimal control problem (**) in the following abstract form:

$$\left\{\begin{array}{l}\int_0^b L(t, x(t), u(t))dt \longrightarrow \inf \\ \text{s.t. } \dot{x}(t)+A(t)x(t)=u(t) \text{ a.e. } \quad x(0)\in C, \quad u(t)\in U(t) \text{ a.e.}\end{array}\right\} \qquad (**)'$$

This is a particular case of the more general problem studied in Section 3. So we can apply Theorem 3.1 and get the following necessary and sufficient condition for a triple $(x_0, x, u)\in L^2(V)\times W(T)\times L^2(T\times V)$, to be a solution of (**). Recall that $W(T)=\{x\in L^2(T, H_0^1(V)): \dot{x}\in L^2(T, H^{-1}(V))\}$. Also $A^*(t)$ is the formal adjoint of the operator A(t), and $(t, z)\to \hat{L}_x(t, z, x(t, z), u(t, z))\in L^1(T\times z)$.

THEOREM 4.1. *If the above hypotheses hold, then* $(x_0, x, u)\in L^2(V)\times W(T)\times L^2(T\times V)$ *solves* (**) *if and only if*

(i) $$\frac{\partial x(t, z)}{\partial t}+A(t)x(t, z)=u(t, z) \quad on\ T\times V$$

$$x|_{T\times\Gamma}(t, z)=0, \ x(0, \cdot)=x_0(\cdot)\in C, \ \left(\int_V |u(t, z)|^2 dz\right)^{1/2}\leqq r(t),$$

(ii) *there exists* $p(\cdot)\in W(T)$ *satisfying the "adjoint equation"*

$$-\frac{\partial p(t, z)}{\partial t}+A^*(t)p(t, z)=\hat{L}_x(t, z, x(t, z), u(t, z)) \quad on\ T\times V$$

$$p(t, z)=0 \quad on\ T\times\Gamma, \ p(b, z)=0, \ z\in V,$$

(iii) *the following "minimum principles" hold*

$$\int_V (-p(t, z)+\hat{L}_u(t, z, x(t, z), u(t, z)))(v(z)-u(t, z))dz \geqq 0 \quad a.e.$$

for all $v\in L^2(V)$ *s.t.* $\|v\|_2\leqq r(t)$ *and*

$$\int_V -p(0, z)(c(z)-x(0, z))dz \geqq 0 \qquad for\ all\ c(\cdot)\in C.$$

REMARK. If $r(\cdot)$ is not in L_+^2, but simply measurable, then the minimum principle has an integral form $\int_0^b\int_V (-p(t, z)+\hat{L}_u(t, z, x(t, z), u(t, z)))(v(t, z)-u(t, z)) \geqq 0$ for all $v\in L^2(T\times V)$ s.t. $\|v(t, \cdot)\|_{L^2(V)}\leqq r(t)$ a.e.

Finally we will conclude with some special cases of the problem studied in this paper

(1) $C=H$, $S_U^2=L^2(Y)$: Then from the maximum principles we get

$$B^*(t, x(t))p(t)=L_u(t, x(t), u(t)) \text{ a.e. } \quad \text{and} \quad p(0)=0.$$

(2) $C \subseteq L^2(Y)$ with $\operatorname{int} C \neq \emptyset$, $S_U^2 = L^2(Y)$: The maximum principles give us

$$B^*(t, x(t))p(t) = L_u(t, x(t), u(t)) \quad \text{a.e.} \qquad \text{and} \qquad (-p(0), c - x(0)) \geqq 0$$

for all $x \in C$.

Finally if $C = \{0\}$, then although $\operatorname{int} C = \emptyset$, it can be easily seen looking at the proof of Theorem 3.1 that the second minimum principle disappears and we have:

(3) $C = \{0\}$, $S_U^2 = L^2(U)$: The first minimum principle tells us that

$$B^*(t, x(t))p(t) = L_u(t, x(t), u(t)) \quad \text{a.e.}$$

In the particular case of our example we have in all cases that the adjoint state is $p(t, z) = \hat{L}_u(t, z, x(t, z), u(t, z))$ a.e.

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