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# The Alder-Wainwright effect for stationary processes with reflection positivity

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## 1. Introduction.

The aim of this paper is to refine the results of Okabe [**13**] on the Alder-Wainwright effect for KMO-Langevin equations. Here by the *Alder-Wainwright effect*, we mean the long-time tail behaviors ($\propto t^{-(p+1)}$, $p>0$) of autocorrelation functions for stationary processes of non-Markovian type. This long-time tail was first observed by Alder and Wainwright in their computer experiment of molecular dynamics ([**1**] and [**2**]). Their numerical calculation suggested that the slowly developing viscous flow around a particle could explain the long-time tail behaviors.

The usual Langevin equation of Ornstein-Uhlenbeck type is not adequate for this Brownian particle because its autocorrelation function decays exponentially. This equation neglects the effect of the fluid flow around the particle which is generated by the accelerated motion of the particle. The appropriate hydrodynamic drag force on a spherical particle moving arbitrarily in $\boldsymbol{R}^3$ has been calculated by Stokes and Boussinesq by solving the linearlized Navier-Stokes equation ([**7**]). Then, the Langevin equation with this drag force becomes

$$m^*\frac{dX(t)}{dt} = -6\pi r\eta X(t)-6\pi r^2\left(\frac{\rho\eta}{\pi}\right)^{1/2}\int_{-\infty}^{t}\frac{1}{\sqrt{t-s}}\frac{dX(s)}{ds}ds+W(t), \tag{1.1}$$

where $m^*$ is an effective mass given by $m^*=m+(2/3)\pi r^3\rho$. Here we consider the motion of a sphere of radius $r$ and mass $m$ moving with an arbitrary velocity $X(t)$ at time $t$ in a fluid with viscosity $\eta$ and density $\rho$ subject to a random force $W(t)$ at time $t$. The second term of the right-hand side of (1.1) corresponds to the effect of the accelerated fluid flow around the particle. It has been shown (e.g. [**6**] and [**15**]) that the correlation function $R(t)$ of the stationary solution $X$ of (1.1) has a long-time tail $\propto t^{-3/2}$ as $t\to\infty$, which agrees with the above experiment. We remark that now the Alder-Wainwright effect has been observed by not only a computer experiment but also a physical experiment ([**14**]).

In their study of (1.1) ([**6**] and [**15**]), the noise $W$ is considered to be not a white noise but a colored noise such as

$$X(t)=\frac{1}{kT}\int_{-\infty}^{t}R(t-s)W(s)ds\,, \tag{1.2}$$

where $k$ is the Boltzman constant and $T$ is the temperature of the fluid. The requirement (1.2) to the noise $W$ comes from the Kubo's linear response theory ([**5**]) and therefore, Okabe ([8] and [**9**]) called it a *Kubo noise* and studied it rigorously.

The solution of (1.1) does not have a Markovian property but has a weaker property called the reflection positivity: the correlation function $R$ is of the form (2.9) below. Noticing this fact, Okabe [8] derived, as a generalization of (1.1), the first and second KMO-Langevin equations which describe the time evolution of real stationary Gaussian processes with reflection positivity. Here the *first* implies that the random noise is a white noise, while in the *second* KMO-Langevin equation the random noise is a Kubo noise. Furthermore, in [**13**], he showed, with a collaboration of Tomisaki, the Alder-Wainwright effect for both first and second KMO-Langevin equations as a correspondence between the decay $\propto t^{-(p+1)}$ of the correlation function $R$ and decay $\propto t^{-p}$ of the delay coefficient $\gamma$ $(0<p<1)$. For the Stokes-Boussinesq-Langevin equation (1.1), we have $p=1/2$ because $\gamma(t)=\text{const.}\,\chi_{(0,\infty)}(t)t^{-1/2}$.

In this paper, we refine the results of Okabe [**13**] on the Alder-Wainwright effect for KMO-Langevin equations in two points. First we generalize the region of the index $p$ of the decay of the delay coefficient $\gamma$ from $(0, 1)$ to $(0, \infty)$. Secondly we fill a gap of [**13**] in deriving the decay of the delay coefficient $\gamma$ from that of the correlation function $R$ for the first KMO-Langevin equations. In the first problem, we adopt a way which is different from the proof of [**13**] because their way can not be applied to the case $p\geqq 1$ straightway. Our proof is based on the *differentiation* compared to that of [**13**] being based on the *integration*. In our proof, we need to use a more general *Tauberian condition* than the monotonicity.

So far we consider only the time continuous case. In [**10**], [**11**] and [**12**], Okabe developed the theory of discrete KMO-Langevin equations which describe the time evolution of real, time-discrete and stationary Gaussian processes with reflection positivity. The Alder-Wainwright effect for the discrete KMO-Langevin equations also exist and will be discussed in a forthcoming paper ([**4**]).

## 2. Statements of Results.

We consider the first KMO-Langevin equation:

(2.1) $$\dot{X} = -\beta_1 X - \lim_{\varepsilon \downarrow 0} \gamma_{1,\varepsilon} * \dot{X} + \alpha_1 \dot{B}$$

and the second KMO-Langevin equation:

(2.2) $$\dot{X} = -\beta_2 X - \lim_{\varepsilon \downarrow 0} \gamma_{2,\varepsilon} * \dot{X} + \alpha_2 I .$$

Here for $j=1, 2$,

(2.3) $$\alpha_j > 0 \quad \text{and} \quad \beta_j > 0$$

and for $t \in \boldsymbol{R}$ and $\varepsilon > 0$,

(2.4) $$\gamma_{j,\varepsilon}(t) = \chi_{(0,\infty)}(t) \int_{\varepsilon}^{\infty} e^{-t\lambda} \rho_j(d\lambda)$$

with a Borel measure $\rho_j$ on $[0, \infty)$ satisfying

(2.5) $$\rho_i(\{0\}) = 0, \qquad \int_0^{\infty} (\lambda+1)^{-1} \rho_j(d\lambda) < \infty .$$

$\dot{B}$ is a stationary Gaussian random tempered distribution with a spectral measure

(2.6) $$\Delta_{\dot{B}}(d\xi) = d\xi$$

and $I$ is a stationary Gaussian random tempered distribution with a spectral measure

(2.7) $$\Delta_I(d\xi) = \sqrt{\frac{2}{\pi}} \frac{1}{\alpha_2} \left( \beta_2 + \xi^2 \int_0^{\infty} \frac{1}{\lambda^2 + \xi^2} \rho_2(d\lambda) \right) d\xi .$$

The random noise $I$ is called a *Kubo noise*, which is related to $X$ through the following relation (Theorem 8.3 in [8]):

(2.8) $$X(\phi) = \frac{1}{R(0)} \int_0^{\infty} R(t) \alpha_2 I(\phi(\cdot + t)) dt \qquad (\phi \in S(\boldsymbol{R})),$$

where $R$ is the correlation function of $X$. For the detailed theory such as the existence and uniqueness of solutions for KMO-Langevin equations, see [8].

Let $X_1 = (X_1(t); t \in \boldsymbol{R})$ (resp. $X_2 = (X_2(t); t \in \boldsymbol{R})$) be the unique real stationary Gaussian solution of (2.1) (resp. (2.2)) with mean 0 and covariance $R_1$ (resp. $R_2$) of the form:

(2.9) $$R_j(t) = \int_0^{\infty} e^{-|t|\lambda} \sigma_j(d\lambda) \qquad (t \in \boldsymbol{R}) \quad (j=1, 2),$$

where $\sigma_j$ $(j=1, 2)$ is a non-zero bounded Borel measure on $[0, \infty)$ satisfying

(2.10) $$\sigma_1(\{0\}) = 0, \quad \int_0^{\infty} (\lambda + \lambda^{-1}) \sigma_1(d\lambda) < \infty ,$$

(2.11) $$\sigma_2(\{0\}) = 0, \quad \int_0^{\infty} \lambda^{-1} \sigma_2(d\lambda) < \infty ,$$

respectively. Let us given a slowly varying function $L$ at infinity: $L$ is a positive measurable function, defined on some neighborhood $[X, \infty)$ of infinity, and satisfying

$$L(\lambda x)/L(x) \longrightarrow 1 \qquad (x\to\infty)\ \forall\lambda>0\,.$$

We put for $t\in\boldsymbol{R}$,

$$\gamma_j(t)=\chi_{(0,\infty)}(t)\int_0^\infty e^{-t\lambda}\rho_j(d\lambda) \qquad (j=1,2)\,. \tag{2.12}$$

We then have

THEOREM 2.1. *Let* $0<p<\infty$. *Then the following* (2.13) *and* (2.14) *are equivalent:*

$$\gamma_1(t)\sim t^{-p}L(t) \qquad as\ t\to\infty\,, \tag{2.13}$$

$$R_1(t)\sim\frac{\alpha_1{}^2 p}{\beta_1{}^3}t^{-(1+p)}L(t) \qquad as\ t\to\infty\,. \tag{2.14}$$

THEOREM 2.2. *Let* $0<p<\infty$. *Then the following* (2.15) *and* (2.16) *are equivalent:*

$$\gamma_2(t)\sim t^{-p}L(t) \qquad as\ t\to\infty\,, \tag{2.15}$$

$$R_2(t)\sim\frac{\sqrt{2\pi}\,\alpha_2 p}{\beta_2{}^2}t^{-(1+p)}L(t) \qquad as\ t\to\infty\,. \tag{2.16}$$

The Stokes-Boussinesq-Langevin equation (1.1) with a white noise or a Kubo noise as a random force is a special case of the above with $p=1/2$ (see § 4).

(2.13)⇒(2.14) and (2.15)⇔(2.16) both with $0<p<1$ were proved by Okabe [**13**]. He also proved (2.14)⇒(2.13) with $0<p<1$ under the condition

$$\int_0^\infty \lambda^{-1}\rho_1(d\lambda)<\infty \tag{2.17}$$

but according to Theorem 3.4 in [8] and (5.21) in [**9**], the Stokes-Boussinesq-Langevin equation (1.1) with a white noise as a random force does not satisfy (2.17) and in fact, it will be found that (2.13) and (2.17) never hold at the same time if $0<p<1$ (Lemma 3.1). Thus what are new in this paper are (2.13)⇔(2.14) with $p\geqq 1$, (2.15)⇒(2.16) with $p\geqq 1$ and (2.16)⇒(2.15) with $p>0$.

## 3. Preliminaries.

LEMMA 3.1. *Let* $\rho_1$ *be a Borel measure on* $[0,\infty)$ *which satisfies* (2.5) *and define a function* $\gamma_1$ *on* $\boldsymbol{R}$ *by* (2.12). *We assume* $0<p<1$ *and* (2.13). *Then*

$$\int_0^\infty \lambda^{-1}\rho_1(d\lambda)=\infty\,. \tag{3.1}$$

PROOF. We assume (2.13). Then, by the Karamata's Tauberian theorem (Theorem 1.7.1' in [3]),

$$U(a) \sim a^p L(1/a)/\Gamma(p+1) \qquad \text{as } a \downarrow 0, \tag{3.2}$$

where $U(a)=\rho_1([0, a])$. On the other hand, for any $a>0$,

$$\int_0^\infty \lambda^{-1}\rho_1(d\lambda) \geqq \int_0^a \lambda^{-1}\rho_1(d\lambda) \geqq a^{-1}\int_0^a \rho_1(d\lambda) = U(a)/a \tag{3.3}$$

and so, letting $a \downarrow 0$, we obtain (3.1). ■

The following two lemmata are easily proved by induction.

LEMMA 3.2. *Let $I$ be an open interval of $\boldsymbol{R}$, $n \in \boldsymbol{N}$ and $f \in C^\infty(I)$. Then*

$$\left(\frac{1}{f(\eta)}\right)^{(n)} = -\frac{f^{(n)}(\eta)}{f(\eta)^2} + \frac{F_n(\eta)}{f(\eta)^{n+1}}, \tag{3.4}$$

*where $F_n$ is a polynomial in $\{f^{(l)}; l=0, 1, \cdots, n-1\}$.*

LEMMA 3.3. *Let $I$ be an open interval of $\boldsymbol{R}$, $n \in \boldsymbol{N}$ and $f, g \in C^\infty(I)$. Then*

$$\left(\frac{g(\eta)}{f(\eta)}\right)^{(n)} = \frac{g^{(n)}(\eta)}{f(\eta)} + \frac{G_n(\eta)}{f(\eta)^{2^n}}, \tag{3.5}$$

*where $G_n$ is a polynomial in $\{g^{(l)}; l=0, 1, \cdots, n-1\}$ and $\{f^{(l)}; l=0, 1, \cdots, n\}$.*

LEMMA 3.4. *Let $\gamma$ be a locally integrable function on $[0, \infty)$. If $\lim_{t\to\infty}\gamma(t)=0$, then*

$$\lim_{\eta \downarrow 0} \eta \int_0^\infty e^{-\eta t}\gamma(t)dt = 0. \tag{3.6}$$

PROOF. Choose $M>0$ so large that $|\gamma(t)| \leqq \varepsilon$ for any $t \geqq M$. Then

$$\left|\eta\int_0^\infty e^{-\eta t}\gamma(t)dt\right| \leqq \eta\int_0^M e^{-\eta t}|\gamma(t)|dt + \varepsilon$$

and so

$$\limsup_{\eta \downarrow 0} \eta\int_0^\infty e^{-\eta t}\gamma(t)dt \leqq \varepsilon.$$

Since $\varepsilon$ is arbitrary, we obtain (3.6). ■

LEMMA 3.5. *Let $U \in L^1_{\mathrm{loc}}[0, \infty)$, $\rho \in \boldsymbol{R}$, $\varepsilon>0$ and $q \geqq 0$. We assume $q-\rho-\varepsilon<-1$ and*

$$U(t) \sim t^{-\rho} L(t) \qquad \text{as } t \to \infty. \tag{3.7}$$

*Then*

$$\lim_{\eta \downarrow 0} \eta^\varepsilon \int_0^\infty e^{-\eta t} t^q U(t)dt = 0. \tag{3.8}$$

PROOF. Choose $\delta>0$ and $M>0$ so that $t\geqq M$ implies $|U(t)|\leqq(1+\delta)t^{-\rho}L(t)$ and $t^{(q-\rho-\varepsilon+1)/2}L(t)\leqq 1$. Now

$$\left|\eta^{\varepsilon}\int_0^{\infty}e^{-\eta t}t^q U(t)dt\right| \leqq \eta^{\varepsilon}M^q\int_0^M |U(t)|\,dt+\int_M^{\infty}\eta^{\varepsilon}e^{-\eta t}t^q|U(t)|\,dt.$$

On the integrand of the second term, we have for some constant $c_1>0$,

$$\eta^{\varepsilon}e^{-\eta t}t^q|U(t)| \leqq (1+\delta)(\eta t)^{\varepsilon}e^{-\eta t}t^{(q-\rho-\varepsilon-1)/2}(t^{(q-\rho-\varepsilon+1)/2}L(t)) \leqq c_1 t^{(q-\rho-\varepsilon-1)/2}.$$

Since $t^{(q-\rho-\varepsilon-1)/2}$ is integrable over $[M, \infty)$, the integral tends to 0. ■

Let $X\in\boldsymbol{R}$. A function $f:[X, \infty)\to\boldsymbol{R}$ is called *slowly increasing* if

$$\lim_{\lambda\downarrow 1}\limsup_{x\to\infty}\sup_{t\in[1,\lambda]}\{f(tx)-f(x)\}\leqq 0 \qquad (\text{hence}=0) \tag{3.9}$$

(see [3]). We will use the following Tauberian condition:

(3.10) $U$ is eventually positive and $\log U$ is slowly increasing.

LEMMA 3.6. *Let $f$ be a positive non-increasing function on $[0, \infty)$ and $\rho>0$. Then $x^{\rho}f(x)$ satisfies the Tauberian condition* (3.10).

PROOF. For any $t\in[1, \lambda]$ and $x>0$, we have

$$\log(tx)^{\rho}f(tx)-\log x^{\rho}f(x) = \rho\log t+\log f(tx)-\log f(x)\leqq \rho\log t. \tag{3.11}$$

The lemma follows easily from this. ■

The following Karamata's Tauberian Theorem plays a crucial role in our proof of Theorems 2.1 and 2.2.

THEOREM 3.7. (Theorem 1.7.6 in [3]) *Assume $U(\cdot)\geqq 0$, $\rho>-1$, $L$ is slowly varying at infinity and*

$$\hat{U}(s):=s\int_0^{\infty}e^{-sx}U(x)dx, \tag{3.12}$$

*is convergent for $s>0$. Then*

$$U(x)\sim x^{\rho}L(x)/\Gamma(1+\rho) \qquad \textit{as } x\to\infty, \tag{3.13}$$

*implies*

$$\hat{U}(s)\sim s^{-\rho}L(1/s) \qquad \textit{as } s\downarrow 0. \tag{3.14}$$

*Conversely,* (3.14) *implies* (3.13) *if $U$ satisfies the Tauberian condition* (3.10).

LEMMA 3.8. *If $E$ is a non-negative non-increasing and integrable function on $[0, \infty)$, we put*

$$R(t)=\frac{1}{2\pi}\int_0^{\infty}E(t+s)E(s)ds \qquad (t\geqq 0), \tag{3.15}$$

(3.16)
$$c=\frac{1}{2\pi}\int_0^\infty E(s)ds\,.$$

*Then if* $p>0$ *and* $L$ *is slowly varying at infiinity,*

(3.17)
$$E(t)\sim t^{-p}L(t)\qquad as\ t\to\infty\,,$$

*if and only if*

(3.18)
$$R(t)\sim ct^{-p}L(t)\qquad as\ t\to\infty\,.$$

PROOF. Since $E$ is non-increasing, it follows that for any $t\geqq 0$ and $a\geqq 0$,

(3.19)
$$R(t)\leqq\frac{E(t)}{2\pi}\int_0^\infty E(s)ds=cE(t)\,,$$

and

(3.20)
$$R(t)\geqq\frac{E(t+a)}{2\pi}\int_0^a E(s)ds\,.$$

Now, we suppose (3.17). Then, using (3.19) and (3.20), we have

$$\frac{1}{2\pi}\int_0^a E(s)ds\leqq\liminf_{t\to\infty}\frac{R(t)}{t^{-p}L(t)}\leqq\limsup_{t\to\infty}\frac{R(t)}{t^{-p}L(t)}\leqq c$$

and so, letting $a\to\infty$, we obtain

$$\lim_{t\to\infty}\frac{R(t)}{t^{-p}L(t)}=c\,.$$

This shows (3.18).

Conversely, we assume (3.18). Then, again from (3.19) and (3.20) we have

$$1\leqq\liminf_{t\to\infty}\frac{E(t)}{t^{-p}L(t)}\leqq\limsup_{t\to\infty}\frac{E(t)}{t^{-p}L(t)}\leqq c\Big(\frac{1}{2\pi}\int_0^a E(s)ds\Big)^{-1}$$

and so, letting $a\to\infty$, we obtain

$$\lim_{t\to\infty}\frac{E(t)}{t^{-p}L(t)}=1\,.$$

This shows (3.17) and completes the proof of the lemma. ■

REMARK. It is in the proof of (3.18)⇒(3.17) that the assumption (2.17) was used in [**13**]. We remark that, in Lemma 3.8, (2.17) is not assumed.

## 4. Proof of Theorems 2.1 and 2.2.

THEOREM 4.1. *Let* $U\in L^1[0,\infty)$, $\gamma\in L^1_{\mathrm{loc}}[0,\infty)$, $\alpha>0$, $\beta>0$, $p>0$ *and* $L$ *be a slowly varying function at infinity. Assume that* $U$ *and* $\gamma$ *are positive, non-increasing, tend to zero as* $t\to\infty$ *and satisfy for any* $\eta>0$,

$$\int_0^\infty e^{-\eta t}U(t)dt = \sqrt{2\pi}\alpha \frac{1}{\beta+\eta+\eta\int_0^\infty e^{-\eta t}\gamma(t)dt}. \tag{4.1}$$

*Then*

$$\gamma(t) \sim t^{-p} L(t) \qquad \textit{as } t\to\infty \tag{4.2}$$

*if and only if*

$$U(t) \sim \frac{\sqrt{2\pi}\alpha p}{\beta^2} t^{-(1+p)} L(t) \qquad \textit{as } t\to\infty . \tag{4.3}$$

PROOF. It follows from Lemma 3.4 and (4.1) that

$$\lim_{\eta\downarrow 0}\int_0^\infty e^{-\eta t}U(t)dt = \int_0^\infty U(t)dt = \frac{\sqrt{2\pi}\alpha}{\beta}. \tag{4.4}$$

We put $n=[p]$, where $[p]$ is the greatest integer not greater than $p$.

We first assume (4.2). The idea of the proof is to differentiate both sides of (4.1) $n+1$ times with respect to $\eta$ so that we can apply Theorem 3.7 to our problem. By Lemma 3.2, we obtain

$$\int_0^\infty e^{-\eta t}t^{n+1}U(t)dt = (-1)^{n+1}\sqrt{2\pi}\alpha\left\{-\frac{f^{(n+1)}(\eta)}{f(\eta)^2}+\frac{F_{n+1}(\eta)}{f(\eta)^{n+2}}\right\}, \tag{4.5}$$

where

$$f(\eta) = \beta+\eta+\eta\int_0^\infty e^{-\eta t}\gamma(t)dt \tag{4.6}$$

and $F_{n+1}$ is a polynomial in $\{f^{(l)}; l=0, 1, \cdots, n\}$. From (4.6), we see that for any $l=1, 2, \cdots$,

$$f^{(l)}(\eta) = \delta_{l1}+(-1)^l\eta\int_0^\infty e^{-\eta t}t^l\gamma(t)dt+(-1)^{l-1}l\int_0^\infty e^{-\eta t}t^{l-1}\gamma(t)dt . \tag{4.7}$$

Since $n+1-p>0$ and

$$t^{n+1}\gamma(t) \sim t^{n+1-p} L(t), \qquad t^n\gamma(t) \sim t^{n-p} L(t) \qquad \text{as } t\to\infty ,$$

it follows from Theorem 3.7 that

$$\begin{aligned} f^{(n+1)}(\eta) &\sim (-1)^n\{(n+1)\Gamma(n+1-p)-\Gamma(n+2-p)\}\eta^{-(n+1-p)} L(1/\eta) \\ &= (-1)^n\Gamma(n+1-p)p\eta^{-(n+1-p)} L(1/\eta) \qquad \text{as } \eta\downarrow 0 . \end{aligned} \tag{4.8}$$

On the other hand, let $l=1, 2, \cdots, n$ and $\varepsilon>0$. Since $l-1-\varepsilon-p<-1$, it follows from Lemma 3.5 and (4.7) that

$$\lim_{\eta\downarrow 0}\eta^\varepsilon f^{(l)}(\eta) = 0 \tag{4.9}$$

and therefore

$$\lim_{\eta\downarrow 0}\eta^{\varepsilon}F_{n+1}(\eta)=0\,. \tag{4.10}$$

Furthermore, by Lemma 3.4 we have

$$\lim_{\eta\downarrow 0}f(\eta)=\beta\,. \tag{4.11}$$

Now we return to (4.5). From (4.8), (4.10) and (4.11) we obtain

$$\eta\int_0^\infty e^{-\eta t}t^{n+1}U(t)dt\sim\sqrt{2\pi}\alpha\eta^{p-n}L(1/\eta)\left\{\frac{\Gamma(n+1-p)p}{\beta^2}+\frac{\eta^{(n+1-p)/2}F_{n+1}(\eta)}{\eta^{-(n+1-p)/2}L(1/\eta)\beta^{n+2}}\right\}$$

$$\sim\frac{\sqrt{2\pi}\alpha\Gamma(n+1-p)p}{\beta^2}\eta^{p-n}L(1/\eta)\qquad\text{as }\eta\downarrow 0\,.$$

Since $n-p>-1$, we have from Lemma 3.6, Theorem 3.7 and the above,

$$t^{n+1}U(t)\sim\frac{\sqrt{2\pi}\alpha p}{\beta^2}t^{n-p}L(t)\qquad\text{as }t\to\infty\,.$$

Thus (4.3) follows.

Next, to prove the other part, let us assume (4.3). By (4.1), (4.4) and the integration by parts, we see that for any $\eta>0$,

$$\int_0^\infty e^{-\eta t}\gamma(t)dt=\frac{\sqrt{2\pi}\alpha}{\eta\int_0^\infty e^{-\eta t}U(t)dt}-\frac{\beta}{\eta}-1=\frac{\beta\int_0^\infty dte^{-\eta t}\int_t^\infty U(\tau)d\tau}{\int_0^\infty e^{-\eta t}U(t)dt}-1. \tag{4.12}$$

The idea of the proof is similar to that of the first half. This time we differentiate both sides of (4.12) $n$ times with respect to $\eta$. Then from Lemma 3.3 we obtain

$$\int_0^\infty e^{-\eta t}t^n\gamma(t)dt=(-1)^n\beta\frac{h^{(n)}(\eta)}{g(\eta)}+\frac{G_n(\eta)}{g(\eta)^{2^n}}-\delta_{0n}\,, \tag{4.13}$$

where

$$g(\eta)=\int_0^\infty e^{-\eta t}U(t)dt,\qquad h(\eta)=\int_0^\infty dte^{-\eta t}\int_t^\infty U(\tau)d\tau \tag{4.14}$$

and $G_n$ is a polynomial in $\{h^{(l)};l=0,1,\cdots,n-1\}$ and $\{g^{(l)};l=0,1,\cdots,n\}$ ($n\geqq 1$). When $n=0$, we put $G_n=0$. On the right hand side of (4.13), the first term will be found to be the main term.

First by (4.4),

$$\lim_{\eta\downarrow 0}g(\eta)=\sqrt{2\pi}\alpha/\beta\,. \tag{4.15}$$

For any $l=0,1,\cdots$ and $\eta>0$, we have

$$g^{(l)}(\eta)=(-1)^l\int_0^\infty e^{-\eta t}t^lU(t)dt,\qquad h^{(l)}(\eta)=(-1)^l\int_0^\infty dte^{-\eta t}t^l\int_t^\infty U(\tau)d\tau\,. \tag{4.16}$$

By the monotone density theorem (cf. Theorem 1.7.2 in [3] and its remark),

$$\int_t^\infty U(\tau)d\tau \sim \frac{\sqrt{2\pi}\alpha}{\beta^2} t^{-p} L(t) \qquad \text{as } t \to \infty .$$

Therefore, since $n-p>-1$, it follows from Theorem 3.7 that

$$h^{(n)}(\eta) \sim (-1)^n \frac{\sqrt{2\pi}\alpha \Gamma(n-p+1)}{\beta^2} \eta^{p-n-1} L(1/\eta) \qquad \text{as } \eta \downarrow 0 . \tag{4.17}$$

On the other hand, for any $\varepsilon>0$ and $l=0, 1, \cdots, n-1$, noting $l-\varepsilon-p<-1$, we have from Lemma 3.5

$$\lim_{\eta \downarrow 0} \eta^\varepsilon h^{(l)}(\eta) = 0 .$$

In the same way, we see that for any $l=0, 1, \cdots, n$ and $\varepsilon>0$,

$$\lim_{\eta \downarrow 0} \eta^\varepsilon g^{(l)}(\eta) = 0 .$$

Therefore it holds that for any $\varepsilon>0$,

$$\lim_{\eta \downarrow 0} \eta^\varepsilon G_n(\eta) = 0 . \tag{4.18}$$

Now we return to (4.13). By (4.15), (4.17) and (4.18), we see that

$$\eta\int_0^\infty e^{-\eta t} t^n \gamma(t) dt \sim \eta^{p-n} L(1/\eta)\left\{\Gamma(l-p+1)+\left(\frac{\beta}{\sqrt{2\pi}\alpha}\right)^{2^n} \frac{\eta^{(n+1-p)/2} G_n(\eta)}{\eta^{-(n+1-p)/2} L(1/\eta)}\right\}$$
$$\sim \Gamma(n-p+1)\eta^{p-n} L(1/\eta) \qquad \text{as } \eta \downarrow 0 .$$

By Lemma 3.6, we can apply Theorem 3.7 to the above and obtain

$$t^n \gamma(t) \sim t^{n-p} L(t) \qquad \text{as } t \to \infty .$$

This shows (4.2) and compeltes the proof of Theorem 4.1. ■

PROOF OF THEOREM 2.2. By (2.9), (2.11), (2.4), (2.5) and Lemma 2.8 in [8], we see that $R_2 \in L^1[0, \infty)$, $\gamma_2 \in L^1_{\text{loc}}[0, \infty)$ and both $R_2$ and $\gamma_2$ are positive, decreasing and tend to zero as $t \to \infty$. Furthermore, it follows from Theorem 8.5 in [8] that for any $\eta>0$,

$$\int_0^\infty e^{-\eta t} R_2(t) dt = \sqrt{2\pi}\alpha_2 \frac{1}{\beta_2+\eta+\eta\int_0^\infty e^{-\eta t}\gamma_2(t) dt} . \tag{4.19}$$

Then Theorem 2.2 follows immediately from Theorem 4.1. ■

PROOF OF THEOREM 2.1. First, as in the case of $\gamma_2$, we see that $\gamma_1$ is a positive, decreasing and locally integrable function on $[0, \infty)$. Let $E_1$ be the canonical representation kernel of $X_1$. By Theorems 2.1 and 2.2 in [8], $E_1$ is a positive, decreasing and integrable function on $[0, \infty)$ and satisfies for any $\eta>0$,

$$(4.20)\qquad \int_0^\infty e^{-\eta t}E_1(t)dt=\sqrt{2\pi}\alpha_1\frac{1}{\beta_1+\eta+\eta\int_0^\infty e^{-\eta t}\gamma_1(t)dt}.$$

Then it follows from Theorem 4.1 that (2.13) is equivalent to

$$(4.21)\qquad E_1(t)\sim\frac{\sqrt{2\pi}\alpha_1 p}{\beta_1^2}t^{-(1+p)}L(t)\qquad \text{as } t\to\infty.$$

On the other hand, by (E.4) in [8] and Lemma 3.4, we have

$$(4.22)\qquad R_1(t)=\frac{1}{2\pi}\int_0^\infty E_1(t+s)E_1(s)ds\qquad (t\geqq 0),$$

$$(4.23)\qquad \frac{1}{2\pi}\int_0^\infty E_1(s)ds=\frac{\alpha_1}{\sqrt{2\pi}\beta_1}.$$

Then by Lemma 3.8, we see that (4.21) is equivalent to (2.14). This completes the proof. ■

## 5. Examples.

Let $p$ be an arbitrary positive constant and $L$ a slowly varying function at infinity. In addition to the condition (2.5), we assume that $\rho_j$ $(j=1, 2)$ has a density $\rho_j(\lambda)$ $(j=1, 2)$ with

$$(5.1)\qquad \rho_j(\lambda)\sim\lambda^{p-1}L(1/\lambda)/\Gamma(p)\qquad \text{as } \lambda\downarrow 0.$$

Then, by the Karamata's Tauberian theorem (a version of Theorem 3.7 with $x\downarrow 0$, $s\to\infty$), the function $\gamma_1$ (resp. $\gamma_2$) defined by (2.12) satisfies (2.13) (resp. (2.15)) and hence (2.14) (resp. (2.16)) holds. Especially,

a) If we put $\rho_j(\lambda)=\lambda^{p-1}e^{-\lambda}/\Gamma(p)$, then we get $\gamma_j(t)=(t+1)^{-p}$ $(0<p<\infty)$.

b) If we put $\rho_j(\lambda)=\lambda^{p-1}/\Gamma(p)$, then we get $\gamma_j(t)=t^{-p}$. Here this time we assume $0<p<1$. The Stokes-Boussinesq-Langevin equation (1.1) is essentially this case with $p=1/2$.

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