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The maximum Markovian self-adjoint extensions of generalized Schrödinger operators

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0. Introduction.

Let G be an open set in $\boldsymbol{R}^d$ and let m be a Radon measure on G. Let S be a symmetric linear operator on $L^2(G, m)$ with the domain $\mathcal{D}[S]$ being dense in $L^2(G, m)$. Let us define a symmetric form by $\mathcal{E}_{(S)}(u, v)=(-Su, v)_m$, $u, v\in \mathcal{D}[S]$ and assume that the symmetric form $\mathcal{E}_{(S)}$ is Markovian in the sense of [8]. Then, the Friedrichs extension of S, the self-adjoint operator associated with the smallest closed extension of $\mathcal{E}_{(S)}$, generates Markovian semigroup ([8; Theorem 2.11]). Let us denote by $\mathcal{A}_{\mathcal{M}}(S)$ the family of all self-adjoint extensions which generate Markovian semigroups, and let us call an element of $\mathcal{A}_{\mathcal{M}}(S)$ a *Markovian extension* of S. Recall that semi-order "$<$" on $\mathcal{A}_{\mathcal{M}}(S)$ is defined by

$$A_1 < A_2 \quad \text{if } \mathcal{D}[A_1] \subset \mathcal{D}[A_2] \text{ and}$$
$$(\sqrt{-A_1}u, \sqrt{-A_1}u)_m \geqq (\sqrt{-A_2}u, \sqrt{-A_2}u)_m \qquad \text{for } u\in\mathcal{D}[\sqrt{-A_1}].$$

Then, the Friedrichs extension of S is the minimum one of $\mathcal{A}_{\mathcal{M}}(S)$ with this semi-order. Now, it is natural to ask whether the maximum element of $\mathcal{A}_{\mathcal{M}}(S)$ exists and what is the maximum one if it exists.

In the case that m is the Lebesgue measure and S is the Laplacian Δ defined on $C_0^\infty(G)$ (in notation $\Delta \upharpoonright C_0^\infty(G)$), the maximum element of $\mathcal{A}_{\mathcal{M}}(\Delta)$ is the self-adjoint operator associated with the Sobolev space $W^{1,2}(G)$ ([8; Theorem 2.3.1]). Here $C_0^\infty(G)$ is the space of infinitely differentiable functions with compact support in G. In this paper, we shall extend this result to "*generalized Schrödinger operators*". More precisely, let ρ be a measurable function on G which is strictly positive almost everywhere and locally square integrable with respect to the Lebesgue measure λ^d. Let us assume that ρ is differentiable in the sense of the Schwartz distribution and its derivatives $\nabla_i\rho$ are also locally square integrable. Then, we define a generalized Schrödinger operator by

$$(0.1)\qquad L_\rho\varphi = \Delta\varphi+2\sum_{i=1}^{d}\nabla_i\rho/\rho\cdot\nabla_i\varphi, \qquad \varphi\in C_0^\infty(G),$$

which is a symmetric operator on $L^2(G, \rho^2\lambda^d)$. We shall pay attention to the maximum element of $\mathcal{A}_{\mathfrak{M}}(L_\rho)$, and obtain the following theorem.

THEOREM. *Let us define*

$$\mathcal{F}^+ = \left\{ u \in L^2(G, \rho^2\lambda^d);\ \begin{array}{c} \text{there exist } g_i \in L^2(G, \rho^2\lambda^d) \text{ such that} \\ (u, -\nabla_i\varphi - 2\nabla_i\rho/\rho\cdot\varphi)_{\rho^2\lambda^d} = (g_i, \varphi)_{\rho^2\lambda^d}, \\ \text{for any } \varphi \in C_0^\infty(G)\ (i=1, \cdots, d) \end{array} \right\}.$$

Let us denote g_i by $D_i u$ and define the symmetric form by

$$\mathcal{E}^+(u, v) = \sum_{i=1}^d \int_G D_i u \cdot D_i v \rho^2 d\lambda^d, \qquad u, v \in \mathcal{F}^+.$$

Then, the self-adjoint operator associated with $(\mathcal{E}^+, \mathcal{F}^+)$ is the maximum element of $\mathcal{A}_{\mathfrak{M}}(L_\rho)$.

In recent years, the operators of the type (0. 1) have been investigated in physical literatures ([**2**], [**3**], [**4**], [**16**], [**18**]), and in relation to the quantum field theory the infinite dimensional versions of (0.1) also have been done ([**5**], [**6**]). In particular, in Albeverio and Kusuoka [**5**], they characterized the maximum element when the symmetric operator S is one associated with a "classical Dirichlet space" on an infinite dimensional vector space. Their method is very probabilistic and without using the hypoellipticity of the symmetric operator S, and the proof of our theorem depends heavily on it. But since in our case the basic measure m is not supposed to be a finite measure, we can not follow the argument in [**5**, §2]. Hence, we need a different idea, that is, the regular representation of Dirichlet forms (cf. [**10**]).

As an application of the above theorem, we obtain the necessary and sufficient condition for the symmetric operator $L_\rho \uparrow C_0^\infty(G)$ having a unique Markovian extension, i.e. $\#(\mathcal{A}_{\mathfrak{M}}(L_\rho))=1$. Indeed, let $(\mathcal{E}^\circ, \mathcal{F}^\circ)$ be the smallest closed extension of $\mathcal{E}_{(L_\rho)}$. Then we can say that the operator $L_\rho \uparrow C_0^\infty(G)$ has a unique Markovian extension if and only if the space $\mathcal{F}^\circ$ is identified with $\mathcal{F}^+$.

In the case that $\rho=1$, the spaces $\mathcal{F}^+$ and $\mathcal{F}^\circ$ are nothing but the Sobolev spaces $W^{1,2}(G)$ and $W_0^{1,2}(G)$ respectively. On the other hand, we see from Theorem 3.31 in [**1**] that the space $W^{1,2}(G)$ is identified with $W_0^{1,2}(G)$ if and only if $\boldsymbol{R}^d \setminus G$, the complement of G, is (1, 2)-polar. Combining these facts, we see that if $\boldsymbol{R}^d \setminus G$ is (1, 2)-polar, the symmetric operator $\Delta \uparrow C_0^\infty(G)$ has a unique Markovian extension which is nothing but the Friedrichs extension. In particular, the operator $\Delta \uparrow C_0^\infty(\boldsymbol{R}^d \setminus \{0\})$ has a unique Markovian extension if and only if $d \geqq 2$. But, it is known that the operator $\Delta \uparrow C_0^\infty(\boldsymbol{R}^d \setminus \{0\})$ is essentially self-adjoint if and only if $d \geqq 4$ ([**15**; Theorem X.111]). Thus, we see that the uniqueness of Markovian extension is really a weaker notion than the essential self-adjointness.

In § 3, we shall give another three examples of symmetric operators which have a unique Markovian extension but are not essentially self-adjoint.

In Example 1, we consider the case that G is an interval (r_0, r_1) in $\boldsymbol{R}^1$. In Wielens [**18**], he studied the relation between the boundary classification of Feller and the essential self-adjointness. In particular, he showed that the operator L_ρ is essentially self-adjoint if both r_0 and r_1 are strong entrance boundaries but is not if r_0 or r_1 is a weak entrance boundary (cf. [**16**; pp. 111]). On the other hand, we shall show that L_ρ has a unique Markovian extension if and only if both r_0 and r_1 are not regular boundaries. Thus, we can say that the difference between the uniqueness of Markovian extension and the essential self-adjointness appears if either r_0 or r_1 is a weak entrance boundary.

In Example 2, we deal with the case that G is the complement of the origin $\{0\}$ and $\rho(x)$ is the function $|x|^\gamma$. Since the operator $L_{|x|^\gamma}$ is spherically symmetric, the uniqueness problem for $L_{|x|^\gamma}$ is reduced to Example 1. But we prove the uniqueness of Markovian extension in relation to the capacity of $\{0\}$, and which enable us to extend to more general cases (Remark 2).

We can regard the Dirichlet space $(\mathcal{E}^+, \mathcal{F}^+)$ as the Sobolev space with the weight function ρ^2. On the other hand, Sobolev spaces with the weight function $\operatorname{dist}(x, \partial G)^\mu$ were investigated in detail (cf. [**13**]). In Example 3, we shall deal with the case that G is a bounded Lipschitz domain and $\rho(x)$ is a function such that $0<c_1 \operatorname{dist}(x, \partial G)^\mu \leqq \rho(x) \leqq c_2 \operatorname{dist}(x, \partial G)^\mu$, and state that the operator L_ρ has a unique Markovian extension if $\mu \leqq -1/2$ or $\mu > 1/2$.

1. Preliminaries.

Let G be an open set of $\boldsymbol{R}^d$ and ρ be a measurable function on G such that

$$(1.1)\qquad \begin{aligned} &\text{i)}\quad \rho > 0, \quad \lambda^d\text{-a.e.} \quad \text{on } G \\ &\text{ii)}\quad \rho \in L^2_{\mathrm{loc}}(G, \lambda^d) \quad \text{and} \quad \nabla_i \rho \in L^2_{\mathrm{loc}}(G, \lambda^d)\ (i=1, \cdots, d). \end{aligned}$$

Let us denote $\rho^2\lambda^d$ by m and $2\nabla_i\rho/\rho$ by β_i simply. Then, we define a symmetric operator on $L^2(G, m)$ by

$$(1.2)\qquad L_\rho \varphi = \Delta\varphi + \sum_{i=1}^d \beta_i \cdot \nabla_i \varphi, \qquad \varphi \in C_0^\infty(G).$$

Denote by $\mathcal{A}_{\mathcal{M}}(L_\rho)$ the totality of Markovian extensions of L_ρ. For $A \in \mathcal{A}_{\mathcal{M}}(L_\rho)$ let $\mathcal{F}_A = \mathcal{D}[\sqrt{-A}]$ and $\mathcal{E}_A(u, v) = (\sqrt{-A}u, \sqrt{-A}u)_m$. Then, $(\mathcal{E}_A, \mathcal{F}_A)$ becomes a Dirichlet form on $L^2(G, m)$. We set $\mathcal{A}^*_{\mathcal{M}}(L_\rho) = \{(\mathcal{E}_A, \mathcal{F}_A);\ A \in \mathcal{A}_{\mathcal{M}}(L_\rho)\}$. It was shown in Fukushima [**10**] that each $(\mathcal{E}, \mathcal{F}) \in \mathcal{A}^*_{\mathcal{M}}(L_\rho)$ has a regular representation. In this section, we fix $(\mathcal{E}, \mathcal{F}) \in \mathcal{A}^*_{\mathcal{M}}(L_\rho)$ and its regular representation $(\tilde{G}, \tilde{m}, \tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \Phi)$, i.e. $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ is a regular Dirichlet form on $L^2(\tilde{G}, \tilde{m})$ and Φ is an isometrically isomorphic map between two Dirichlet rings (cf. [**9**]). The map Φ is constructed through the Gel'fand representation of a certain Banach algebra $(R, \|\ \|_\infty)$ included in $\overline{\mathcal{F}_b}^{\|\ \|_\infty}$, where $\mathcal{F}_b=\mathcal{F}\cap L^\infty(G, m)$ (see [**9**], [**10**] for detail). In our case, we can assume that the Banach algebra R includes $C_0^\infty(G)$. Theorem 2 and (5.7) in [**9**] say that Φ can be extended to a unitary map from $L^2(G, m)$ to $L^2(\tilde{G}, \tilde{m})$ and the Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ has the following relation with $(\mathcal{E}, \mathcal{F})$:

$$
\begin{aligned}
&\tilde{\mathcal{F}} = \Phi(\mathcal{F}) \\
&\tilde{\mathcal{E}}(u, v) = \mathcal{E}(\Phi^{-1}u, \Phi^{-1}v) \qquad \text{for } u, v \in \tilde{\mathcal{F}}\,.
\end{aligned} \tag{1.3}
$$

Moreover, denoting by L and $\tilde{L}$ the self-adjoint operators corresponding to $(\mathcal{E}, \mathcal{F})$ and $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ respectively, we see from the relation (6.8) in [**10**] that for any $f \in \mathcal{D}[L]$, $\Phi(f)$ belongs to $\mathcal{D}[\tilde{L}]$ and

$$
\tilde{L}\Phi(f) = \Phi(Lf)\,. \tag{1.4}
$$

Let Γ be so called Carré du Champ operator, i.e. $\Gamma(f, g)=L(fg)-fLg-gLf$ and let $\tilde{\Gamma}$ be the Carré du Champ operator associated with the form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Note that for $f, g \in C_0^\infty(G)$

$$
\Gamma(f, g) = 2\sum_{i=1}^{d} \nabla_i f \cdot \nabla_i g\,. \tag{1.5}
$$

We simply denote $\Gamma(f, f)$ by $\Gamma(f)$.

LEMMA 1. *For* $f \in C_0^\infty(G)$

$$
\Phi(\Gamma(f)) = \tilde{\Gamma}(\Phi(f))\,. \tag{1.6}
$$

PROOF. By (1.4), we have for $f \in C_0^\infty(G)$

$$
\begin{aligned}
\Phi(\Gamma(f)) &= \Phi(Lf^2 - 2fLf) \\
&= \tilde{L}\Phi(f)^2 - 2\Phi(f)\tilde{L}\Phi(f) \\
&= \tilde{L}\Phi(f)^2 - 2\Phi(f)\tilde{L}\Phi(f) \\
&= \tilde{\Gamma}(\Phi(f))\,. \qquad \text{q.e.d.}
\end{aligned}
$$

Let $C_0(G)$ be the family of all continuous functions with compact support in G. Let $u, v \in C_0(G)$ with $\operatorname{supp}[u] \cap \operatorname{supp}[v] = \emptyset$. Take relatively compact open sets U, U', V, V' so that i) $\operatorname{supp}[u] \subset U' \subset \overline{U}' \subset U$ and $\operatorname{supp}[v] \subset V' \subset \overline{V}' \subset V$, ii) $U \cap V = \emptyset$, and choose $f, g \in C_0^\infty(G)$ such that

$$
f = \begin{cases} 1 & \text{on } U' \\ 0 & \text{on } G-U \end{cases}, \qquad g = \begin{cases} 1 & \text{on } V' \\ 0 & \text{on } G-V\,. \end{cases}
$$

Then, noting that $\Phi(u)=\Phi(fu)=\Phi(f)\Phi(u)$, we have $\Phi(f)=1$ on $\{\Phi(u)\neq 0\}$. By the same reason, $\Phi(g)=1$ on $\{\Phi(v)\neq 0\}$. Note that for $f\in C_0^\infty(G)\subset R$, $\Phi(f)$ is a continuous function on $\tilde{G}$ by the definition of the Gel'fand representation. Then we have

$$\text{(1.7)}\qquad \begin{aligned} &\operatorname{supp}[\Phi(u)]\subset\{\Phi(f)>0\}\\ &\operatorname{supp}[\Phi(v)]\subset\{\Phi(g)>0\}. \end{aligned}$$

On the other hand, $\{\Phi(f)>0\}\cap\{\Phi(g)>0\}=\emptyset$ because $\Phi(f)\Phi(g)=\Phi(fg)=0$, which leads to $\operatorname{supp}[\Phi(u)]\cap\operatorname{supp}[\Phi(v)]=\emptyset$. Next take $w\in C_0(G)$ such that $w=k$ (constant) on U. Then, since $\Phi(w)\Phi(f)=\Phi(wf)=\Phi(kf)=k\Phi(f)$, $\Phi(w)=k$ on $\{\Phi(f)>0\}\supset\operatorname{supp}[\Phi(u)]$. Thus, the above observation leads us to the next lemma.

LEMMA 2. *For $u, v, w\in C_0(G)$ such that* $\operatorname{supp}[u]\cap\operatorname{supp}[v]=\emptyset$ *and $w=k$ (constant) on neighbourhood of* $\operatorname{supp}[u]$,

$$\text{(1.8)}\qquad \begin{aligned} &\text{i)}\ \operatorname{supp}[\Phi(u)]\cap\operatorname{supp}[\Phi(v)]=\emptyset\\ &\text{ii)}\ \Phi(w)=k \qquad \textit{on some neighbourhood of } \operatorname{supp}[\Phi(u)]. \end{aligned}$$

Now, according to the Beuling-Deny formula, the regular Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$ can be decomposed as

$$\tilde{\mathcal{E}}(u, v)=\tilde{\mathcal{E}}^c(u, v)+\int_{\tilde{G}\times\tilde{G}-d}(\tilde{u}(x)-\tilde{u}(y))(\tilde{v}(x)-\tilde{v}(y))\tilde{J}(dxdy)$$
$$+\int_{\tilde{G}}\tilde{u}(x)\tilde{v}(x)\tilde{k}(dx), \qquad \text{for } u, v\in\tilde{\mathcal{F}}$$

where $\tilde{J}$ is a symmetric positive Radon measure on the product space $\tilde{G}\times\tilde{G}$ off the diagonal d and $\tilde{k}$ is a positive Radon measure on $\tilde{G}$. (see Lemma 4.5.4 in [8]). Here, $\tilde{u}$ and $\tilde{v}$ mean quasi-continuous versions of u and v. Let us define Radon measures J on $G\times G-d$ and k on G as ones corresponding to the following operators respectively: for $f, g\in C_0(G)$ with $\operatorname{supp}[f]\cap\operatorname{supp}[g]=\emptyset$

$$\text{(1.9)}\qquad \int_{\tilde{G}\times\tilde{G}-d}\Phi(f)(x)\Phi(g)(y)\tilde{J}(dxdy)$$

and for $f\in C_0(G)$

$$\text{(1.10)}\qquad \int_{\tilde{G}}\Phi(f)(x)\tilde{k}(dx).$$

Note that (1.9) is well defined by Lemma 2. Moreover, let us define the symmetric form $\mathcal{E}^c(u, v)$ by

$$\text{(1.11)}\qquad \mathcal{E}^c(u, v)=\tilde{\mathcal{E}}^c(\Phi(u), \Phi(v)), \qquad u, v\in\mathcal{F}\cap C_0(G).$$

Then, the Dirichlet form $\mathcal{E}$ can be decomposed as

$$\begin{aligned}\mathcal{E}(u, v) &= \tilde{\mathcal{E}}(\Phi(u), \Phi(v)) \\ &= \tilde{\mathcal{E}}^c(\Phi(u), \Phi(v))+\int_{\tilde{G}\times\tilde{G}-d}(\Phi(u)(x)-\Phi(u)(y))(\Phi(u)(x)-\Phi(y))J(dxdy) \\ &\qquad +\int_{\tilde{G}}\Phi(u)(x)\Phi(v)(x)\tilde{k}(dx) \\ &= \mathcal{E}^c(u, v)+\int_{G\times G-d}(u(x)-u(y))(v(x)-v(y))J(dxdy)+\int_G u(x)v(x)k(dx), \\ &\qquad\qquad \text{for } u, v\in\mathcal{F}\cap C_0(G).\end{aligned} \tag{1.12}$$

This decomposition is nothing but the Beuling-Deny formula for the Dirichlet form $(\mathcal{E}, \mathcal{F})$ because the strong local property of $\tilde{\mathcal{E}}^c$ leads to that of $\mathcal{E}^c$ by Lemma 2 (cf. [7]). Thus, the equality (1.5) gives us

$$\begin{aligned}\sum_{i=1}^d\int_G \nabla_i f\cdot\nabla_i g dm &= \mathcal{E}^c(f, g)+\int_{G\times G-d}(f(x)-f(y))(g(x)-g(y))J(dxdy) \\ &\qquad +\int_G f(x)g(x)k(dx), \qquad f, g\in C_0^\infty(G).\end{aligned} \tag{1.13}$$

Hence, $\int_G f(x)g(y)J(dxdy)=0$ for $f, g\in C_0^\infty(G)$ with $\operatorname{supp}[f]\cap\operatorname{supp}[g]=\emptyset$, and which implies that $J=0$. Next by applying (1.13) for $f, g\in C_0^\infty(G)$ such that $g=1$ on some neighbourhood of $\operatorname{supp}[f]$, we have $k=0$. Thus, for $f, g\in\mathcal{F}\cap C_0(G)$

$$\mathcal{E}(f, g) = \mathcal{E}^c(f, g)$$

and consequently

$$\tilde{\mathcal{E}}(\Phi(f), \Phi(g)) = \mathcal{E}(f, g) = \mathcal{E}^c(f, g) = \tilde{\mathcal{E}}^c(\Phi(f), \Phi(g)). \tag{1.14}$$

Let $\tilde{\boldsymbol{M}}=(\tilde{P}_x, \tilde{X}_t)$ be the Hunt process corresponding to the regular Dirichlet form $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}})$. Then it was proved in [8] that the additive functional $\tilde{A}_t^{[u]}=\tilde{u}(X_t)-\tilde{u}(X_0)$, $u\in\tilde{\mathcal{F}}$ can be decomposed as

$$\tilde{u}(X_t)-\tilde{u}(X_0) = \tilde{M}_t^{[u]}+\tilde{N}_t^{[u]},$$

where $\tilde{M}^{[u]}$ is a martingale additive functional of finite energy and $\tilde{N}^{[u]}$ is a continuous additive functional of zero energy. Denote by $\tilde{M}^{c[u]}$ (resp. $\tilde{M}^{d[u]}$) a continuous (resp. purely discontinuous) part of $\tilde{M}^{[u]}$. Let us denote by $\tilde{\mu}^c_{\langle\Phi(f)\rangle}$ (resp. $\tilde{\mu}^d_{\langle\Phi(f)\rangle}$) the Revuz measure corresponding to the continuous additive functional $\langle\tilde{M}^{c[\Phi(f)]}\rangle$ (resp. $\langle\tilde{M}^{d[\Phi(f)]}\rangle$). Then, by Theorem 2.1 in [**11**] the equality (1.14) implies that $\tilde{\mu}^d_{\langle\Phi(f)\rangle}=0$ for $f\in\mathcal{F}\cap C_0(G)$. Thus we get

LEMMA 3. i) *For* $f\in\mathfrak{F}\cap C_0(G)$

$$\tilde{M}^{[\Phi(f)]}=\tilde{M}^{c[\Phi(f)]}. \tag{1.15}$$

ii) *For* $f\in\mathfrak{F}\cap C_0(G)$ *and* $u\in\tilde{\mathfrak{F}}$

$$\tilde{\mathcal{E}}(u,\Phi(f))=\tilde{\mathcal{E}}^c(u,\Phi(f)). \tag{1.16}$$

Here, we mention a lemma concerned with the measure $\tilde{\mu}^c_{\langle u\rangle}$.

LEMMA 4. *It holds that for* $f\in C_0^\infty(G)$

$$d\tilde{\mu}^c_{\langle\Phi(f)\rangle}=\tilde{\Gamma}(\Phi(f))d\tilde{m}. \tag{1.17}$$

PROOF. By the derivation property of $\tilde{\mu}^c_{\langle u\rangle}$ ([**11**; Theorem 2.2]), we have for any $g\in\mathfrak{F}_b$

$$\begin{aligned}\int_{\tilde{G}}\widetilde{\Phi(g)}d\tilde{\mu}^c_{\langle\Phi(f)\rangle}&=\int_{\tilde{G}}d\tilde{\mu}^c_{\langle\Phi(g)\Phi(f),\Phi(f)\rangle}-\frac{1}{2}\int_{\tilde{G}}d\tilde{\mu}^c_{\langle\Phi(f)^2,\Phi(g)\rangle}\\&=2\tilde{\mathcal{E}}^c(\Phi(g)\Phi(f),\Phi(f))-\tilde{\mathcal{E}}^c(\Phi(f)^2,\Phi(g)).\end{aligned} \tag{1.18}$$

By (1.16) the right hand side of (1.18) is equal to

$$\begin{aligned}&2\tilde{\mathcal{E}}(\Phi(g)\Phi(f),\Phi(f))-\tilde{\mathcal{E}}(\Phi(f)^2,\Phi(g))\\&=2\int_{\tilde{G}}\widetilde{\Phi(g)}\Phi(f)(-\tilde{L}\Phi(f))d\tilde{m}-\int_{\tilde{G}}\widetilde{\Phi(g)}(-\tilde{L}\Phi(f)^2)d\tilde{m},\\&=\int_{\tilde{G}}\widetilde{\Phi(g)}(\tilde{L}\Phi(f)^2-2\Phi(f)\tilde{L}\Phi(f))d\tilde{m}\\&=\int_{\tilde{G}}\widetilde{\Phi(g)}\tilde{\Gamma}(\Phi(f))d\tilde{m},\end{aligned}$$

which implies that $d\tilde{\mu}^c_{\langle\Phi(f)\rangle}=\tilde{\Gamma}(\Phi(f))d\tilde{m}$. q. e. d.

REMARK 1. In general, a regular representation $(\tilde{\mathcal{E}},\tilde{\mathfrak{F}})$ does not always become local. In fact, consider $S=d^2/dx^2$ with $\mathcal{D}[S]=C_0^\infty((0,1))$. Then the self-adjoint operator corresponding to the Dirichlet space

$$\begin{cases}\tilde{\mathfrak{F}}=W^{1,2}((0,1))\\ \tilde{\mathcal{E}}(u,v)=\int_0^1\frac{d\tilde{u}}{dx}\frac{d\tilde{v}}{dx}dx+(\tilde{u}(1)-\tilde{u}(0))(\tilde{v}(1)-\tilde{v}(0))\end{cases}$$

is an extension of S, but it is not local. Here $\tilde{u}$ and $\tilde{v}$ mean absolutely continuous versions of u and v respectively.

2. Proof of Theorem

In this section, we shall give the proof of our theorem stated in section 0. First, we have

LEMMA 5. *$(\mathcal{E}^+, \mathcal{F}^+)$ is a closed form on $L^2(G, m)$.*

PROOF. Suppose that $\mathcal{E}_1^+(u_n-u_m, u_n-u_m)\to 0$, $m, n\to\infty$. Then, there exist $u, g_i\in L^2(G, m)$ such that $u_n\to u$ and $D_i u_n\to g_i$ in $L^2(G, m)$, and consequently

$$(u, -\nabla_i\varphi-\beta_i\varphi)_m = \lim_{n\to\infty}(u_n, -\nabla_i\varphi-\beta_i\varphi)_m = \lim_{n\to\infty}(D_i u_n, \varphi)_m = (g_i, \varphi)_m .$$

Hence, $D_i u=g_i$ and $u\in\mathcal{F}^+$ accordingly. q. e. d.

Let us denote by A^+ the self-adjoint operator associated with the closed symmetric bilinear form $(\mathcal{E}^+, \mathcal{F}^+)$. Then in order to prove our theorem we must show following statements:

$$\text{(2.1)}\qquad \begin{aligned} &\text{(I)}\quad A^+\in\mathcal{A}_{\mathcal{M}}(L_\rho) \\ &\text{(II)}\quad \text{for any } (\mathcal{E}, \mathcal{F})\in\mathcal{A}_{\mathcal{M}}^{\circ}(L_\rho) \\ &\qquad \text{a)}\ \ \mathcal{F}\subset\mathcal{F}^+ \quad \text{b)}\ \ \mathcal{E}(u, u)\geqq\mathcal{E}^+(u, u) \qquad \text{for } u\in\mathcal{F}\, . \end{aligned}$$

First we prove (I). Let $G_1=\{(x_2, \cdots, x_d)\in\boldsymbol{R}^{d-1};$ there exists x_1 such that $(x_1, x_2, \cdots, x_d)\in G\}$. Let $\tilde{\rho}^{(1)}$ be a λ^d-version of ρ which is absolutely continuous in the x_1-axis for λ^{d-1}-a. e. $(x_2, \cdots, x_d)$ in G_1. Then we introduce the linear space

$$\tilde{\mathcal{F}}_1=\left\{u\in L^2(G, m);\ \begin{array}{l}\text{there exists the function } \tilde{u}^{(1)} \text{ such that i) } \tilde{u}^{(1)}=u,\\ m\text{-a. e.}\ \ \text{ii) for } \lambda^{d-1}\text{-a. e. } (x_2, \cdots, x_d)\in G_1\ \tilde{u}^{(1)}(x_1, x_2,\\ \cdots, x_d) \text{ is absolutely continuous in } x_1 \text{ on } \{x_1\in\boldsymbol{R}^1;\\ \tilde{\rho}^{(1)}(x_1, x_2, \cdots, x_d)>0\} \text{ and } \partial\tilde{u}^{(1)}/\partial x_1\in L^2(G, m)\end{array}\right\}$$

Note that the space $\tilde{\mathcal{F}}_1$ is independent of choice of version $\tilde{\rho}^{(1)}$ and $\partial\tilde{u}^{(1)}/\partial x_1$ is defined m-almost everywhere. Let us define $\tilde{\mathcal{F}}_i$ $(i=2, \cdots, d)$ by the same manner as $\tilde{\mathcal{F}}_1$. Define $\tilde{\mathcal{F}}=\bigcap_{i=1}\tilde{\mathcal{F}}_i$. For $u\in\tilde{\mathcal{F}}$ let us denote $\partial\tilde{u}^{(i)}/\partial x_i$ by $\partial_i u$ and define a symmetric bilinear form $\tilde{\mathcal{E}}$ by

$$\tilde{\mathcal{E}}(u, v)=\sum_{i=1}^d\int_G\partial_i u\cdot\partial_i v dm\, , \qquad \text{for } u, v\in\tilde{\mathcal{F}}\, .$$

Then, we obtain

LEMMA 6. *$\tilde{\mathcal{F}}=\mathcal{F}^+$ and $\partial_i=D_i$ $(i=1, \cdots, d)$.*

PROOF. Let $\tilde{u}^{(i)}$ and $\tilde{\rho}^{(i)}$ be versions of u and ρ which appear in the definition of $\bar{\mathcal{F}}_i$. Then by the integration by part

$$\begin{aligned}(u,\ -\nabla_i\varphi-\beta_i\varphi)_m &= (\tilde{u}^{(i)}(\tilde{\rho}^{(i)})^2,\ -\nabla_i\varphi)_{\lambda^d}-2(u\rho\nabla_i\rho,\ \varphi)_{\lambda^d} \\ &= (\partial_i u\rho^2+2u\rho\nabla_i\rho,\ \varphi)_{\lambda^d}-2(u\rho\nabla_i\rho,\ \varphi)_{\lambda^d} \\ &= (\partial_i u,\ \varphi)_m, \qquad \varphi\in C_0^\infty(G),\end{aligned}$$

and which yields that $\bar{\mathcal{F}}\subset\mathcal{F}^+$ and $\partial_i u=D_i u$ for $u\in\bar{\mathcal{F}}$. On the other hand, suppose that $u\in\mathcal{F}^+$. Then it follows from the definition of $\mathcal{F}^+$ that there are some functions g_i satisfying that for any $\varphi\in C_0^\infty(G)$, $(u\rho^2,\ -\nabla_i\varphi)_{\lambda^d}=(g_i\rho^2+2\rho\nabla_i\rho u,\ \varphi)_{\lambda^d}$. Note that for any compact set $K\subset G$, $\int_K u\rho^2 d\lambda^d\leqq\left(\int_K u^2\rho^2 d\lambda^d\right)^{1/2}\cdot\left(\int_K\rho^2 d\lambda^d\right)^{1/2}<\infty$, and in the same way $\int_K g_i\rho^2 d\lambda^d<\infty$. Further, since $\int_K\rho\nabla_i\rho u d\lambda^d\leqq\left(\int_K u^2\rho^2 d\lambda^d\right)^{1/2}\cdot\left(\int_K(\nabla_i\rho)^2 d\lambda^d\right)^{1/2}<\infty$, the functions $u\rho^2$ and $g_i\rho^2+2\rho\nabla_i\rho u$ are included in $L^1_{\mathrm{loc}}(G,\ \lambda^d)$. Thus, there exist λ^d-versions $\widetilde{u\rho^2}^{(i)}$ of $u\rho^2$ which are absolutely continuous in the x_i-axis. Let $\tilde{\rho}^{(i)}$ be the functions which appear in the definition of the spaces $\bar{\mathcal{F}}_i$. Then, since for λ^{d-1}-a.e. $(x_1,\cdots,x_{i-1},x_{i+1},\cdots,x_d)$

$$\frac{\partial}{\partial x_i}\left(\frac{1}{(\tilde{\rho}^{(i)})^2}\right)=-2\frac{1}{(\tilde{\rho}^{(i)})^3}\frac{\partial\tilde{\rho}^{(i)}}{\partial x_i} \qquad \text{on } \{\tilde{\rho}^{(i)}>0\}$$

and $u=\widetilde{u\rho^2}^{(i)}\cdot 1/(\tilde{\rho}^{(i)})^2$, m-a.e., we have

$$\begin{aligned}\partial_i u &= \frac{\partial}{\partial x_i}\left(\widetilde{u\rho^2}^{(i)}\cdot\frac{1}{(\rho^{(i)})^2}\right) \\ &= \frac{\partial}{\partial x_i}\left(\widetilde{u\rho^2}^{(i)}\right)\frac{1}{(\tilde{\rho}^{(i)})^2}+\widetilde{u\rho^2}^{(i)}\frac{\partial}{\partial x_i}\left(\frac{1}{(\tilde{\rho}^{(i)})^2}\right) \\ &= (g_i\rho^2+2\rho\nabla_i\rho u)\frac{1}{\rho^2}-u\rho^2\left(2\frac{1}{\rho^3}\nabla_i\rho\right) \\ &= g_i,\ m\text{-a.e.}.\end{aligned}$$

Therefore, we can conclude that $\mathcal{F}^+\subset\bar{\mathcal{F}}$, and attain this lemma. q.e.d.

LEMMA 7. *Let ρ be a function satisfying* (1.1). *Then the self-adjoint operator A^+ corresponding to $(\mathcal{E}^+,\ \mathcal{F}^+)$ belongs to $\mathcal{A}_{\mathcal{M}}(L_\rho)$.*

PROOF. By Lemma 6 we have $(\mathcal{E}^+,\ \mathcal{F}^+)=(\bar{\mathcal{E}},\ \bar{\mathcal{F}})$. On the other hand, we can easily show that $(\bar{\mathcal{E}},\ \bar{\mathcal{F}})$ has the Markov property. (*which implies that A^+ is included in $\mathcal{A}_{\mathcal{M}}(L_\rho)$.*)

Let us denote by $\{G_\alpha^+\}$ the resolvent corresponding to $(\mathcal{E}^+,\ \mathcal{F}^+)$. Then for

$f \in L^2(G, m)$ and $\varphi \in C_0^\infty(G)$

$$(f, \varphi)_m = \mathcal{E}_\alpha^+(G_\alpha^+ f, \varphi) = \sum_{i=1}^d \int_G D_i G_\alpha^+ f \cdot D_i \varphi dm + \alpha \int_G G_\alpha^+ f \cdot \varphi dm$$

$$= \sum_{i=1}^d \int_G G_\alpha^+ f \cdot (-\nabla_i \nabla_i \varphi - \beta_i \nabla_i \varphi) dm + \alpha \int_G G_\alpha^+ f \cdot \varphi dm$$

$$= \int_G G_\alpha^+ f \cdot (-L_\rho \varphi + \alpha\varphi) dm .$$

Hence $G_\alpha^+ f \in \mathcal{D}[L_\rho^*]$, and $(-L_\rho^* + \alpha) G_\alpha^+ f = f$, where L_ρ^* is the adjoint operator of L_ρ. Therefore we can conclude that A^+ is an extension of L_ρ. q.e.d.

Secondly, we shall prove the statement (II). In accordance with notations in [5] let us define $T_i^{[u]}(g) = (u, -\nabla_i g - \beta_i g)_m$, $g \in C_0^\infty(G)$. Then we have the next key lemma.

LEMMA 8. *Let* $(\mathcal{E}, \mathcal{F}) \in \mathcal{A}_{\mathfrak{M}}^\circ(L_\rho)$. *Then it holds that for* $u \in \mathcal{F}$ *and* $g \in C_0^\infty(G)$

$$T_i^{[u]}(g) \leqq \mathcal{E}(u, u)^{1/2} \|g\|_{L^2(G, m)}, \qquad i = 1, \cdots, d . \tag{2.2}$$

PROOF. For avoiding complication we prove this lemma in the case that $d=2$ and $i=1$. Since $\mathcal{F}_b$ is dense in $\mathcal{F}$, it suffices to prove (2.2) for $u \in \mathcal{F}_b$. Set $D_0 = \{f \in C_0^\infty(G);\ f(x, y) = \varphi(x)\phi(y) \text{ and } \varphi, \phi \in C_0^\infty(\boldsymbol{R}^1)\}$. Given $\varphi(x)\phi(y) \in D_0$, we take φ', φ'' and $\phi' \in C_0^\infty(\boldsymbol{R}^1)$ such that

$$\begin{aligned} &\text{i)} \quad \varphi' = 1 \quad \text{on supp}[\varphi], \qquad \varphi'' = 1 \quad \text{on supp}[\varphi'] \\ &\text{ii)} \quad \phi' = 1 \quad \text{on supp}[\phi] \\ &\text{iii)} \quad \varphi''(x)\phi'(y) \in D_0 . \end{aligned} \tag{2.3}$$

Since

$$-\nabla_1 \nabla_1 \Big(\int_0^x \varphi(\tau) d\tau \varphi''(x) \phi(y) \Big) - \beta_1 \nabla_1 \Big(\int_0^x \varphi(\tau) d\tau \varphi''(x) \phi(y) \Big)$$

$$= -\nabla_1 \Big(\varphi(x)\phi(y) + \int_0^x \varphi(\tau) d\tau \nabla_1 \varphi''(x) \phi(y) \Big) - \beta_1 \Big(\varphi(x)\phi(y) + \int_0^x \varphi(\tau) d\tau \nabla_1 \varphi''(x) \phi(y) \Big)$$

$$= -\nabla_1 \varphi(x)\phi(y) - \int_0^x \varphi(\tau) d\tau \nabla_1 \nabla_1 \varphi''(x) \phi(y) - \beta_1 \varphi(x)\phi(y) - \beta_1 \int_0^x \varphi(\tau) d\tau \nabla_1 \varphi''(x) \phi(y),$$

we have

$$\begin{aligned} &\Big(u\varphi'(x), -\nabla_1 \nabla_1 \Big(\int_0^x \varphi(\tau) d\tau \varphi''(x) \phi(y) \Big) - \beta_1 \nabla_1 \Big(\int_0^x \varphi(\tau) d\tau \varphi''(x) \phi(y) \Big) \Big)_m \\ &\qquad = (u\varphi'(x), -\nabla_1 \varphi(x)\phi(y) - \beta_1 \varphi(x)\phi(y))_m \\ &\qquad = (u, -\nabla_1 \varphi(x)\phi(y) - \beta_1 \varphi(x)\phi(y))_m \\ &\qquad = T_1^{[u]}(\varphi(x)\phi(y)) . \end{aligned} \tag{2.4}$$

On the other hand, the left hand side of (2.4) is equal to

$$\Big(u\varphi'(x)\psi(y),\ -\nabla_1\nabla_1\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)-\beta_1\nabla_1\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)_m$$

$$(2.5)\qquad =\Big(u\varphi'(x)\psi(y),\ -L_\rho\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)_m$$

$$=\mathcal{E}\Big(u\varphi'(x)\psi(y),\ \int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big).$$

Now, let $(\tilde{\mathcal{E}}, \tilde{\mathcal{F}}, \tilde{G}, \tilde{m}, \Phi)$ be a regular representation of the Dirichlet space $(\mathcal{E}, \mathcal{F})$. Then, by (1.3) and (1.16) the right hand side of (2.5) is equal to $\tilde{\mathcal{E}}^c\Big(\Phi(u)\Phi(\varphi'(x)\psi(y)),\ \Phi\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)$. Let us use notations in the previous section. Then, by virtue of the derivation property of the measure $\tilde{\mu}^c_{\langle u,v\rangle}$ we obtain

$$\tilde{\mathcal{E}}^c\Big(\Phi(u)\Phi(\varphi'(x)\psi(y)),\ \Phi\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)$$

$$(2.6)\qquad =\frac{1}{2}\int_{\tilde{G}}\Phi(u)d\tilde{\mu}^c_{\langle\Phi(\varphi'(x)\psi(y)),\Phi(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y))\rangle}$$

$$+\frac{1}{2}\int_{\tilde{G}}\Phi(\varphi'(x)\psi(y))d\tilde{\mu}^c_{\langle\Phi(u),\Phi(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y))\rangle}.$$

Note that Lemma 1 and Lemma 4 lead us to the relation

$$d\tilde{\mu}^c_{\langle\Phi(\varphi'(x)\psi(y)),\Phi(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y))\rangle}$$

$$(2.7)\qquad =\tilde{\Gamma}\Big(\Phi(\varphi'(x)\psi(y)),\ \Phi\Big(\int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)d\tilde{m}$$

$$=\Phi\Big(\Gamma\Big(\varphi'(x)\psi(y),\ \int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)\Big)d\tilde{m}.$$

Then the first term of the right hand side of (2.6) disappear because

$$\Gamma\Big(\varphi'(x)\psi(y),\ \int_0^x\varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)$$

$$=\nabla_1\varphi'(x)\psi(y)\Big(\varphi(x)\varphi''(x)\psi'(y)+\int_0^x\varphi(\tau)d\tau\nabla_1\varphi''(x)\psi'(y)\Big)$$

$$+\varphi'(x)\nabla_2\psi(y)\int_0^x\varphi(\tau)d\tau\varphi''(x)\nabla_2\psi'(y)$$

$$=0.$$

Therefore, the following key relation is established:

$$T_1^{[u]}(\varphi(x)\psi(y)) = \frac{1}{2}\int_{\tilde{G}} \Phi(\varphi'(x)\psi(y))d\tilde{\mu}^c_{\langle \Phi(u), \Phi(\int_0^x \varphi(\tau)d\tau\varphi''(x)\psi'(y))\rangle}. \tag{2.8}$$

By Lemma 5.4.3 in [8] the left hand side of (2.8) is dominated by

$$\Big(\frac{1}{2}\int_{\tilde{G}} d\tilde{\mu}^c_{\langle \Phi(u)\rangle}\Big)^{1/2}\cdot\Big(\frac{1}{2}\int_{\tilde{G}} (\Phi(\varphi'(x)\psi(y))^2 d\tilde{\mu}^c_{\langle \Phi(\int_0^x \varphi(\tau)d\tau\varphi''(x)\psi'(y))\rangle}\Big)^{1/2}$$

$$\leqq \mathcal{E}(u, u)^{1/2}\cdot\Big(\frac{1}{2}\int_G (\varphi'(x)\psi(y))^2 \Gamma\Big(\int_0^x \varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big)dm\Big)^{1/2}.$$

Since by (1.5)

$$(\varphi'(x)\psi(y))^2\Gamma\Big(\int_0^x \varphi(\tau)d\tau\varphi''(x)\psi'(y)\Big) = 2(\varphi(x)\psi(y))^2,$$

the inequality (2.2) holds for $g\in D_0$.

Next suppose that $g=\sum_{k=1}^n \varphi_k(x)\psi_k(y)$, $\varphi_k(x)\psi_k(y)\in D_0$. We choose φ_k', φ_k'' and $\psi_k'\in C_0^\infty(\boldsymbol{R}^1)$ $(k=1, \cdots, n)$ in the same way as (2.3). Then it follows from the equality (2.8) that

$$T_1^{[u]}(g) = \frac{1}{2}\sum_{k=1}^n \int_{\tilde{G}} \Phi(\varphi_k'(x)\psi_k(y))d\tilde{\mu}^c_{\langle \Phi(u), \Phi(\int_0^x \varphi_k(\tau)d\tau\varphi_k''(x)\psi_k'(y))\rangle}$$

$$= \lim_{t\downarrow 0} \frac{1}{2t}\tilde{E}_{\tilde{m}}\Big[\Big\langle \tilde{M}^{c[\Phi(u)]}, \sum_{k=1}^n \int_0^\cdot \Phi(\varphi_k'\psi_k)(\tilde{X}_s)d\tilde{M}_s^{c[\Phi(\int_0^x \varphi_k(\tau)d\tau\varphi_k''(x)\psi_k'(y))]}\Big\rangle_t\Big]$$

$$\leqq \lim_{t\downarrow 0}\Big(\frac{1}{2t}\tilde{E}_{\tilde{m}}[\langle \tilde{M}^{c[\Phi(u)]}\rangle_t]\Big)^{1/2}\cdot\Big(\frac{1}{2t}\tilde{E}_{\tilde{m}}\Big[\sum_{k,l=1}^n \int_0^t (\Phi(\varphi_k'\psi_k)\Phi(\varphi_l'\psi_l))(\tilde{X}_s) \tag{2.9}$$

$$d\Big\langle \tilde{M}^{c[\Phi(\int_0^x \varphi_k(\tau)d\tau\varphi_k''\psi_k')]}, \tilde{M}^{c[\Phi(\int_0^x \varphi_l(\tau)d\tau\varphi_l''\psi_l')]}\Big\rangle_s\Big]\Big)^{1/2}$$

$$= \mathcal{E}(u, u)^{1/2}\cdot\Big(\sum_{k,l=1}^n \int_G \varphi_k'\psi_k\varphi_l'\psi_l\Gamma\Big(\int_0^x \varphi_l(\tau)d\tau\varphi_k''\psi_k', \int_0^x \varphi_l(\tau)d\tau\varphi_l''\psi_l'\Big)dm\Big)^{1/2}.$$

Since $\varphi_k'\psi_k\varphi_l'\psi_l\Gamma\Big(\int_0^x \varphi_k(\tau)d\tau\varphi_k''\psi_k', \int_0^x \varphi_l(\tau)d\tau\varphi_l''\psi_l'\Big)=\varphi_k\varphi_l\psi_k\psi_l$, the right hand side of (2.9) is equal to $\mathcal{E}(u, u)^{1/2}\cdot\Big(\sum_{k,l=1}^n \int_G \varphi_k\varphi_l\psi_k\psi_l dm\Big)^{1/2}=\mathcal{E}(u, u)^{1/2}\cdot\|g\|_{L^2(G,m)}$.

For a general $g\in C_0^\infty(G)$ there exists a sequence of functions $g_p=\sum_{k=1}^{n_p}\varphi_k^{(p)}(x)\psi_k^{(p)}(y)$ such that $\varphi_k^{(p)}\psi_k^{(p)}\in D_0$, $\|g-g_p\|_{L^2(G,m)}\to 0$, and $\|g-g_p\|_\infty+\sum_{i=1}^2\|\nabla_i g-\nabla_i g_p\|_\infty\to 0$. Then by the approximating argument we can conclude the inequality (2.2). q.e.d.

Now the statement (II) a) in (2.1) has been established. Let $e_i=(0, \cdots, \overset{i}{1}, \cdots, 0)\in\boldsymbol{R}^d$, $f_i\in C_0^\infty(G)$ $(i=1, \cdots, d)$ and define the operator on $L^2(G, m)\otimes\boldsymbol{R}^d$ by

$$T^{[u]}\Big(\sum_{i=1}^{d} f_i\otimes e_i\Big)=\sum_{i=1}^{d} T_i^{[u]}(f_i). \tag{2.10}$$

Then, we obtain

LEMMA 9. *For any $(\mathcal{E}, \mathcal{F})\in\mathcal{A}_{\mathcal{M}}^{\circ}(L_\rho)$ and $u\in\mathcal{F}$, the inequality*

$$T^{[u]}(F)\leqq\mathcal{E}(u, u)^{1/2}\cdot\|F\|_{L^2(G,m)\otimes\boldsymbol{R}^d}, \qquad \textit{for } F\in L^2(G, m)\otimes\boldsymbol{R}^d \tag{2.11}$$

holds.

PROOF. We prove only in the case that $d=2$. Let $\varphi_1\psi_1$, $\varphi_2\psi_2\in D_0$, where D_0 is the space defined in the proof of Lemma 8. Then let us choose φ_1', φ_1'', ψ_1', φ_2', ψ_2', $\psi_2''\in C_0^\infty(\boldsymbol{R}^1)$ such that

$$\text{i)}\ \begin{aligned} &\varphi_1'=1 \quad \text{on supp}\,[\varphi_1] \\ &\varphi_1''=1 \quad \text{on supp}\,[\varphi_1'] \\ &\psi_1'=1 \quad \text{on supp}\,[\psi_1] \\ &\varphi_1''(x)\psi_1'(y)\in D_0 \end{aligned} \qquad \text{ii)}\ \begin{aligned} &\varphi_2'=1 \quad \text{on supp}\,[\varphi_2] \\ &\psi_2'=1 \quad \text{on supp}\,[\psi_2] \\ &\psi_2''=1 \quad \text{on supp}\,[\psi_2'] \\ &\varphi_2'(x)\psi_2''(y)\in D_0\,. \end{aligned}$$

By the same calculation as in Lemma 8, we get

$$\begin{aligned} &T_1^{[u]}(\varphi_1(x)\psi_1(y))+T_2^{[u]}(\varphi_2(x)\psi_2(y)) \\ &=\lim_{t\downarrow 0}\frac{1}{2t}\tilde{E}_{\tilde{m}}\Big[\Big\langle \tilde{M}^{c[\Phi(u)]}, \int_0^{\cdot}\Phi(\varphi_1'\psi_1)(\tilde{X}_s)d\tilde{M}_s^{c[\Phi(\int_0^x\varphi_1(\tau)d\tau\varphi_1''\psi_1')]} \\ &\quad +\int_0^{\cdot}\Phi(\varphi_2\psi_2')(\tilde{X}_s)d\tilde{M}_s^{c[\Phi(\varphi_2'\int_0^y\psi_2(\tau)d\tau\psi_2'')]}\Big\rangle_t\Big] \\ &\leqq\mathcal{E}(u, u)^{1/2}\cdot\Big(\int_G(\varphi_1(x)\psi_1(y))^2dm+\int_G(\varphi_2(x)\psi_2(y))^2dm\Big)^{1/2}. \end{aligned}$$

By the approximating method as in Lemma 8, we can conclude that for $F=f_1\otimes e_1+f_2\otimes e_2\in L^2(G, m)\otimes\boldsymbol{R}^2$, $T^{[u]}(F)\leqq\mathcal{E}(u, u)^{1/2}\cdot(\|f_1\|_{L^2(G,m)}^2+\|f_2\|_{L^2(G,m)}^2)^{1/2}=\mathcal{E}(u, u)^{1/2}\|F\|_{L^2(G,m)\otimes\boldsymbol{R}^2}$. q.e.d.

Following the argument in [**5**], we obtain

LEMMA 10. *Suppose that a measurable function ρ satisfies* (1.1). *If $(\mathcal{E}, \mathcal{F})\in\mathcal{A}_{\mathcal{M}}^{\circ}(L_\rho)$, then for $u\in\mathcal{F}$*

$$\mathcal{E}^+(u, u)\leqq\mathcal{E}(u, u) \qquad \textit{for } u\in\mathcal{F}\,. \tag{2.12}$$

PROOF. Let $(\mathcal{E}, \mathcal{F})\in\mathcal{A}^{\circ}_{\mathcal{M}}(L_\rho)$ and $u\in\mathcal{F}$. Then by Lemma 9 there exists $H\in L^2(G, m)\otimes\boldsymbol{R}^d$ such that $T^{[u]}(F)=(H, F)_{L^2(G,m)\otimes\boldsymbol{R}^d}$ for any $F\in L^2(G, m)\otimes\boldsymbol{R}^d$. In particular, taking $F=\varphi\otimes e_i$, $\varphi\in C_0^\infty(G)$, we see that $u\in\mathcal{F}^+$ and $D_iu=(H, e_i)_{\boldsymbol{R}^d}$. On the other hand, the Riesz theorem tells us that the norm $\|H\|_{L^2(G,m)\otimes\boldsymbol{R}^d}$ is equal to the operator norm of $T^{[u]}$. But the inequality (2.11) implies that the operator norm of $T^{[u]}$ is not greater than $\mathcal{E}(u, u)$. Therefore, the inequality

$$\begin{aligned}\mathcal{E}^+(u, u) &= \sum_{i=1}^{d}\int_G (H, e_i)^2_{\boldsymbol{R}^d}dm\\ &= \|H\|^2_{L^2(G,m)\otimes\boldsymbol{R}^d}\\ &\leqq \mathcal{E}(u, u)\end{aligned}$$

is established. q. e. d.

By virture of Lemma 8 and Lemma 10, we have shown the maximality of the self-adjoint extension A^+, and which completes the proof of our theorem.

3. Examples

In this section, we give three examples which indicate the difference between the uniqueness of Markovian extension and the essential self-adjointness.

EXAMPLE 1. Let $G=(r_0, r_1)$ with $-\infty\leqq r_0<r_1\leqq\infty$. Let ρ be a function satisfying (1.1). Moreover let us assume that the function ρ is strictly positive everywhere. Then it is easily shown that the adjoint operator L_ρ^* is given by

$$L_\rho^* f = \frac{1}{\rho^2}\frac{d}{dx}\left(\rho^2\frac{df}{dx}\right)$$

$\mathcal{D}[L_\rho^*] = \{f\in L^2(G, \rho^2dx);\ f$ is continuously differentiable on G, df/dx is absolutely continuous, and $1/\rho^2\cdot d(\rho^2df/dx)/dx\in L^2(G, \rho^2dx)\}$.

Let us define $m(x)=\int_c^x\rho^2(t)dt$ and $s(x)=\int_c^x 1/\rho^2(t)dt$ $(r_0<c<r_1)$. Then the operator L_ρ can be represented as $L_\rho=d/dm\cdot d/ds$. In the same way as in Example 1.2.2 in [8], we can show that $\mathcal{F}^+\cap\mathrm{Ker}(\alpha-L_\rho^*)=\{0\}$ if and only if both of r_1 and r_2 are not regular boundaries. Let us denote by $\mathcal{F}^\circ$ the smallest closed extension. Then, since $\mathcal{F}^+$ can be orthogonally decomposed as $\mathcal{F}^+=\mathcal{F}^\circ\oplus(\mathcal{F}^+\cap\mathrm{Ker}(\alpha-L_\rho^*))$, we can conclude that $\mathcal{F}^+=\mathcal{F}^\circ$ and consequently L_ρ satisfies the uniqueness of Markovian extension if and only if both r_0 and r_1 are not regular. On the other hand, by virtue of the table in Wielens [**18**; pp. 111] we see that the operator L_ρ is not essentially self-adjoint if r_0 or r_1 is a weak entrance boundary, i. e. $\left|\int_c^r s(t)^2dm(t)\right|<\infty$ $(r=r_0$ or $r_1)$.

EXAMPLE 2. Let $G=\boldsymbol{R}^d\setminus\{0\}$ and $\rho(x)=|x|^\gamma$. Let us denote $L_{|x|^\gamma}$ by L_γ simply. Then the operator L_γ is written as $L_\gamma\varphi=\Delta\varphi+2\gamma\sum_{i=1}^d|x|^{-2}x_i\nabla_i\varphi$, $\varphi\in C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$. If $\gamma>-d/2$, the function $|x|^\gamma$ belongs to $L_{\mathrm{loc}}^2(\boldsymbol{R}^d, \lambda^d)$ and the symmetric form $\mathcal{E}(u, v)=\sum_{i=1}^d\int_{\boldsymbol{R}^d}\nabla_i u\cdot\nabla_i v|x|^{2\gamma}d\lambda^d$, $u, v\in C_0^\infty(\boldsymbol{R}^d)$, is closable on $L^2(\boldsymbol{R}^d, |x|^{2\gamma}\lambda^d)$ by Theorem 2.4 in [4]. Let us denote by $\mathcal{F}$ the closure of $C_0^\infty(\boldsymbol{R}^d)$ with respect to $\mathcal{E}_1=\mathcal{E}+(\ ,\ )_{|x|^{2\gamma}\lambda^d}$ and by Cap the capacity associated with the Dirichlet form $(\mathcal{E}, \mathcal{F})$. Then, if $-d/2<\gamma<1-d/2$, $\mathrm{Cap}(\{0\})>0$ ([**2**; Theorem 4.1]). Consequently, we see that $\mathcal{F}^\circ\subsetneqq\mathcal{F}$, where $\mathcal{F}^\circ$ is the closure of $C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$ with respect to $\mathcal{E}_1$. In fact, if $\mathcal{F}^\circ$ is equal to $\mathcal{F}$, then for $f\in C_0^\infty(\boldsymbol{R}^d)$ with $f=1$ on $B_r=\{|x|\leqq r\}$ there exists a sequence $\{f_n\}\subset C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$ such that $\mathcal{E}_1(f-f_n, f-f_n)\to 0$ $(n\to 0)$. But since $\mathrm{Cap}(\{0\})\leqq\mathcal{E}_1(f-f_n, f-f_n)$, we have $\mathrm{Cap}(\{0\})=0$, and which is contradictory to the fact that $\mathrm{Cap}(\{0\})>0$. Therefore, noting that $\mathcal{F}\subset\mathcal{F}^+$, we can say that non-uniqueness of Markovian extension of L_γ holds.

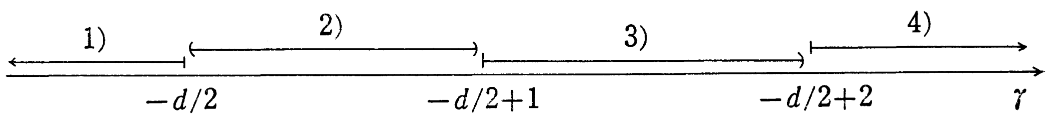
On the other hand, if $\gamma\geqq 1-d/2$, $\mathrm{Cap}(\{0\})=0$ ([**2**; Theorem 4.1]). By Problem 3.3.2 in [8], we see that $\mathrm{Cap}(\{0\})=\lim_{n\to\infty}\mathrm{Cap}(B_{1/n})=\lim_{n\to\infty}\inf_{u\in D_n}\mathcal{E}_1(u, u)$, where $D_n=\{u\in C_0^\infty;\ u\geqq 1 \text{ on } B_{1/n}\}$. Hence, there exists a sequence $\{f_n\}\subset C_0^\infty(\boldsymbol{R}^d)$ such that $f_n=1$ on $B_{1/n}$ and $\mathcal{E}_1(f_n, f_n)\to 0$ $(n\to 0)$. Moreover we assume that $f_n\to 0$, λ^d-a.e. by taking a sebsequence if necessary. Given $u\in\mathcal{F}_b^+$, define $u_n=u-uf_n\in\mathcal{F}_b^+$. Then we obtain

$$
\begin{aligned}
\mathcal{E}_1^+(u-u_n, u-u_n) &= \mathcal{E}_1^+(uf_n, uf_n)\\
&= \sum_{i=1}^d\int_G(D_i(uf_n))^2|x|^{2\gamma}d\lambda^d+\int_G u^2f_n^2|x|^{2\gamma}d\lambda^d\\
&= \sum_{i=1}^d\int_G(D_iu\cdot f_n+u\cdot\nabla_if_n)^2|x|^{2\gamma}d\lambda^d+\int_G u^2f_n^2|x|^{2\gamma}d\lambda^d\\
&\leqq 2\sum_{i=1}^d\int_G(D_iu\cdot f_n)^2|x|^{2\gamma}d\lambda^d+2\|u\|_\infty^2\mathcal{E}_1(f_n, f_n)\to 0 \quad (n\to 0).
\end{aligned}
$$

Next let us choose a sequence of $C_0^\infty(\boldsymbol{R}^d)$-functions φ_p satisfying that i) $\varphi_p(x)=\begin{cases}1 & |x|\leqq p\\ 0 & |x|\geqq p+1\end{cases}$ ii) $\|\mathrm{grad}\,\varphi_p\|\leqq 2$. Then by the same argument as above, we easily see that $\mathcal{E}_1(u_n-u_n\varphi_p, u_n-u_n\varphi_p)\to 0$ as $p\to\infty$. Moreover since $u_n\varphi_p\in W_0^{1,2}(\mathring{B}_{p+1}-B_{1/n})$, there exists a sequence $\{\psi_q\}\subset C_0^\infty(\mathring{B}_{p+1}-B_{1/n})$ such that $\|u_n\varphi_p-\psi_q\|_{1,2}\to 0$ $(q\to\infty)$ and equivalently $\mathcal{E}_1(u_n\varphi_p-\psi_q, u_n\varphi_p-\psi_q)\to 0$ $(q\to\infty)$. Therefore we can say that $\mathcal{F}_0=\mathcal{F}^+$, and which implies the uniqueness of Markovian extension of L_γ. Noting that the operator $L_\gamma\upharpoonright C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$ on $L^2(\boldsymbol{R}^d, |x|^{2\gamma}\lambda^d)$ is unitary equivalent with $\Delta-\{\gamma(\gamma+d-2)\}/|x|^2\upharpoonright C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$, it follows from Kalf-Walter-Schmincke-Simon theorem (see [**15**]) that the operator $L_\gamma\upharpoonright C_0^\infty(\boldsymbol{R}^d\setminus\{0\})$ is essentially self-adjoint if and only if $\gamma(\gamma+d+2)\geqq -d(d-4)/4$

$(\leftrightarrow \gamma \leqq -d/2,\ \gamma \geqq -d/2+2)$.

Figure 1. The operator $\Delta + 2\gamma \sum_{i=1}^{d} |x|^{-2} x_i \cdot \nabla_i \uparrow C_0^\infty(\boldsymbol{R}^d - \{0\})$



1), 4): essentially self-adjoint
2), 3): not essentially self-adjoint
2): non-unique Markovian extension
3): unique Markovian extension

REMARK 2. Let ρ be a function in $L^2_{\mathrm{loc}}(\boldsymbol{R}^d, \lambda^d)$ satisfying that there exists a closed set N of Lebesgue measure zero such that derivatives $\nabla_i \rho$ are in $L^2_{\mathrm{loc}}(\boldsymbol{R}^d \setminus N)$. Furthermore, let us suppose that $\operatorname{ess\cdot inf}_{x \in K} \rho(x) > 0$ and $\operatorname{ess\cdot sup}_{x \in K} \rho(x) < \infty$ for any compact set $K \subset \boldsymbol{R}^d \setminus N$. Then, by applying Theorem 2.4 in [**4**] we see that the symmetric form on $L^2(\boldsymbol{R}^d, \rho^2 \lambda^d)$ defined by $\mathcal{E}(u, v) = \sum_{i=1}^d \int_{\boldsymbol{R}^d} \nabla_i u \cdot \nabla_i v \rho^2 d\lambda^d$, $u, v \in C_0^\infty(\boldsymbol{R}^d)$ is closable. Let us denote by $\mathcal{F}$ the closure of $C_0^\infty(\boldsymbol{R}^d)$. Then, if the set N is not only of zero Lebesgue measure but also of zero capacity introduced by the Dirichlet form $(\mathcal{E}, \mathcal{F})$, we can prove by the same discussion as in Example 2 that the symmetric operator $L_\rho \uparrow C_0^\infty(\boldsymbol{R}^d \setminus N)$ has a unique Markovian extension.

EXAMPLE 3. Let G be a bounded Lipschitz domain and ρ be a function satisfying (1.1). Let $d(x) = \inf\{|x-y|\,;\ y \in \partial G\}$. In [**13**], Sobolev spaces with the weight function $d(x)^\mu$ (in notation $W^{k,l}(G; d, \mu)$) were investigated. In particular, it was shown that the space $W^{k,l}(G; d, \mu)$ is identified with $W_0^{k,l}(G; d, \mu)$ if $\mu \leqq -1$ or $\mu > kl-1$. Here, $W_0^{k,l}(G; d, \mu)$ is the closure of $C_0^\infty(G)$ in $W^{k,l}(G; d, \mu)$. Therefore, if the function ρ satisfies that $0 < c_1 d(x)^\mu \leqq \rho(x) \leqq c_2 d(x)^\mu$, the operator $L_\rho \uparrow C_0^\infty(G)$ has a unique Markovian extension if $\mu \leqq -1/2$ or $\mu > 1/2$. On the other hand, we see from Theorem 3.3 in [**14**] that L_ρ is not essentially self-adjoint if $-1/2 < \mu < 1/2$, and do not know when L_ρ becomes essentially self-adjoint.

REMARK 3. Even in the case treated in [**5**], [**6**], our method is efficient. But in that case, Radon measures J and k which are introduced in (1.9) and (1.10) must be replaced by cylindrical Radon measures.

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