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Strong minimal pair theorem for the honest polynomial degrees of Δ_2^0 low sets

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§1. Introduction.

Homer [3] has shown that assuming P=NP there is a Δ_3^0 set which is minimal with respect to the honest polynomial time Turing reducibility, $\leqq_T^h$, while it is known that the honest polynomial time Turing degrees (hp-T degrees) of recursive sets are dense. In [3], Homer raised a question whether a recursively enumerable (r.e.) set can be $\leqq_T^h$ minimal. An affirmative answer has been given by Ambos-Spies [1] (assuming P=NP). He has shown that every high r.e. Turing degree contains a $\leqq_T^h$ minimal element. Downey [2], on the other hand, has proved that no low Turing degree contains a $\leqq_T^h$ minimal set. It has also been shown there that the hp-T degrees of low r.e. sets are dense. He asks if the hp-T degrees of Δ_2^0 sets are dense. An affirmative answer evidently implies P≠NP. We notice that in contrast to the hp-T degrees, the polynomial time Turing degrees (p-T degrees) of all sets are dense, which can be proved by relativizing the proof of the density of the p-T degrees of recursive sets due to Ladner [4].

Concerning Downey's question, we shall prove the following strong minimal pair theorem which obviously implies the density of the hp-T degrees of Δ_2^0 low sets.

THEOREM. *If A and B are Δ_2^0 low sets such that $B <_T^h A$, then there are two sets C and D which satisfy the following two conditions:*

(1) $B <_T^h C <_T^h A$ *and* $B <_T^h D <_T^h A$,

(2) $\deg_T^h(B) = \deg_T^h(C) \wedge \deg_T^h(D)$.

In [5], Landweber, Lipton and Robertson have proved the strong minimal pair theorem for the p-T degrees of recursive sets. In §2, we shall give a proof of the theorem for the hp-T degrees of recursive sets. The proof is a typical example of a Ladner style "looking back" technique. In §3, we shall give a proof of the theorem for the Δ_2^0 low sets. Since our proof heavily depends on the notion of hp-T reducibility, it is not known whether the strong minimal pair theorem holds for the p-T degrees of Δ_2^0 low sets.

Given A and B with $B<_T^h A$, we may construct two sets C and D so that they look like A on some intervals of Σ^* and look like B elsewhere. The switching points $\{l_n\}_n$ are effectively computed from A and B. If A and B are recursive, the sequence $\{l_n\}_n$ turns out to be recursive. However, when A and B are non-recursive, we can not expect that the sequence is recursive any more. This causes a difficulty in proving the second condition of the theorem. If A and B are Δ_2^0 besides, then, by the limit lemma, they are approximated by recursive sets. We can use these approximations to construct recursively the switching points $\{l_n\}_n$ as in the case of recursive sets. This time, however, other difficulty occurs in proving the first condition of the theorem. Especially, it is hard to prove the inequalities $C\not\equiv_T^h B$ and $D\not\equiv_T^h B$. The lowness of A and B will resolve these difficulties.

Our notation is standard. Let $\Sigma=\{0, 1\}$. Σ^* is the set of finite strings of elements of Σ. Lower case letters $x, y, z, \cdots$ denote elements of Σ^* and capital letters $A, B, C \cdots$ denote subsets of Σ^*. $|x|$ denotes the length of x. We order Σ^* in the canonical way:

$$\lambda < 0 < 1 < 00 < 01 < 10 < 11 < 000 < 001 < 010 < \cdots .$$

We sometimes identify an integer n and the $n+1$st string z_n in this order. $A\oplus B=\{0x : x\in A\}\cup\{1x : x\in B\}$ is the effective disjoint union of A and B. For an oracle Turing machine M, $M(A, x)=1$ denotes that M with oracle A accepts x, and $M(A, x)=0$ denotes that x is refuted. Finally, let $\langle ,\rangle$ denote some polynomial time computable bijection from $\Sigma^*\times\Sigma^*$ onto Σ^* which has polynomial time computable inverses and which satisfies $x, y\leqq\langle x, y\rangle$ for all $x, y\in\Sigma^*$.

§2. The hp-T degrees of recursive sets.

An oracle Turing machine M is *polynomially honest* if there are polynomials p and q such that on input x, M halts within $p(|x|)$ steps and if M queries the oracle on a string y then $|x|\leqq q(|y|)$. We recursively enumerate all the polynomially honest oracle Turing machines and their associated polynomials, $\{(M_e, p_e, q_e) : e\in \boldsymbol{N}\}$. Let A and B be two subsets of Σ^*. A is said to be hp-T *reducible* to B, write $A\leqq_T^h B$, if there is a polynomially honest oracle Turing machine M such that $A(x)=M(B, x)$ for all $x\in\Sigma^*$. A and B have the same honest polynomial time Turing degrees (hp-T degrees), $A\equiv_T^h B$, if $A\leqq_T^h B$ and $B\leqq_T^h A$. The hp-T degree of A is denoted by $\deg_T^h(A)$. If $A\leqq_T^h B$ and $A\not\equiv_T^h B$, we write $A<_T^h B$. The greatest lower bound of $\deg_T^h(A)$ and $\deg_T^h(B)$, if exists, is written $\deg_T^h(A)\wedge\deg_T^h(B)$.

As Ambos-Spies points out in [1], most results on the structure of the polynomial time Turing degrees (p-T degrees) of the recursive sets hold for the hp-T degrees. We support his observation in proving the strong minimal pair theorem for the hp-T degrees of recursive sets. The theorem for the p-T degrees has given by Landweber, Lipton and Robertson [**5**].

THEOREM 2.1. *Given recursive sets A and B, if $B <_T^h A$, then there exist recursive sets C and D such that*

(1) $B <_T^h C <_T^h A$ *and* $B <_T^h D <_T^h A$,

(2) $\deg_T^h(B) = \deg_T^h(C) \wedge \deg_T^h(D)$.

PROOF. Since $A \oplus B \equiv_T^h A$, we may assume $B = A \cap 1\Sigma^*$. We effectively construct a strictly increasing sequence $\{l_n : n \in \boldsymbol{N}\}$ as follows.

Stage 0. $l_0 = 0$.

Stage $6e+i+1$ ($i=0, 3$). Let $n = 6e+i$. Since $B <_T^h A$, there is an x with $l_n \leqq |x|$ such that $A(x) \neq M_e(B, x)$. We take the least such x, and let $l_{n+1} = l_n + 2^{l_n} + m$, where m is the number of steps performed to find x and verify the inequality $A(x) \neq M_e(B, x)$. This means that if we perform the construction in l_{n+1} steps then we can find the least x such that $l_n \leqq |x|$ and $A(x) \neq M_e(B, x)$.

Stage $6e+i+1$ ($i=1, 2, 4, 5$). We let $l_{6e+i+1} = l_{6e+i} + 2^{l_{6e+i}}$.

We define C and D as follows:

$$C(x) = \begin{cases} A(x) & \text{if } l_{6e} \leqq |x| < l_{6e+1} \text{ for some } e, \\ B(x) & \text{otherwise.} \end{cases}$$

$$D(x) = \begin{cases} A(x) & \text{if } l_{6e+3} \leqq |x| < l_{6e+4} \text{ for some } e, \\ B(x) & \text{otherwise.} \end{cases}$$

C looks like A on the intervals $\{x : l_{6e} \leqq |x| < l_{6e+1}\}$ and looks like B on other intervals, while D looks like A on the intervals $\{x : l_{6e+3} \leqq |x| < l_{6e+4}\}$ and B elsewhere.

Since $B = C \cap 1\Sigma^*$, B is hp-T computable from C in the obvious way. By the definition of l_{6e+1}, there is an x such that $l_{6e} \leqq |x| < l_{6e+1}$ and $A(x) \neq M_e(B, x)$. A and C agree on this interval. Thus, we see that $C(x) \neq M_e(B, x)$. It follows that C is not hp-T reducible to B. Therefore, we have $B <_T^h C$.

To see that $C \leqq_T^h A$, suppose $x \in \Sigma^*$. By performing the construction of $\{l_n\}_n$ in $|x|$ steps, we can compute the n such that $l_n \leqq |x| < l_{n+1}$, and then compute $C(x)$ from A in several more steps. This "looking back" algorithm gives an hp-T reduction of C to A. In the same manner, we can show that $B <_T^h D \leqq_T^h A$.

Suppose $M_i(C) = M_j(D) = Z$ to see that C and D satisfies the condition (2). We must show that Z is hp-T reducible to B. First, take a sufficiently large n_0 so that

$$(\forall n)[n_0 \leqq n \Longrightarrow p_i(l_{n-1}),\ q_i(l_{n-1}),\ p_j(l_{n-1}),\ q_j(l_{n-1})<l_n].$$

For each x with $|x|\geqq l_{n_0}$, $Z(x)$ is computed from B as follows. Given x with $|x|\geqq l_{n_0}$, find the unique n such that $l_n\leqq|x|<l_{n+1}$ by looking back the construction of the sequence $\{l_n : n\in \boldsymbol{N}\}$ in $|x|$ steps. In the case where $n=6e$, $6e+1$ or $6e+5$ for some e, B and D agree on the interval $\{z : l_{n-1}\leqq|z|<l_{n+2}\}$ by the definition of D. If M_j queries D on a string y during the computation of $M_j(D, x)$, then y must be in the interval $\{z : l_{n-1}\leqq|z|<l_{n+2}\}$ by the choice of n_0, and therefore the query is answered by B. Thus, in this case, we may compute $M_j(B, x)$ to obtain the value of $Z(x)$. Similarly, if $n=6e+2$, $6e+3$ or $6e+4$ for some e, then we can compute $M_i(B, x)$ to obtain the value of $Z(x)$. □

§3. The hp-T degrees of Δ_2^0 low sets.

A set A is *low* if the Turing jump A' of A has the least possible Turing degree, namely that of ϕ'. Δ_2^0 sets are approximated by recursive sets (see [8, Limit lemma]): if A is Δ_2^0, then there is a recursive function $f(x, s)$ such that $f(x, s)\leqq 1$ and $A(x)=\lim_s f(x, s)$.

THEOREM 3.1. *The strong minimal pair theorem holds for the hp-T degrees of the Δ_2^0 low sets: for all Δ_2^0 low sets A and B with $B<_T^h A$, there are two sets C and D that satisfy the following conditions.*

(1) $B <_T^h C <_T^h A$ *and* $B <_T^h D <_T^h A$,

(2) $\deg_T^h(B) = \deg_T^h(C)\wedge\deg_T^h(D)$.

PROOF. Suppose A and B are given low sets with $B<_T^h A$. We may assume that $B=A\cap 1\Sigma^*$ as before. As A and B are Δ_2^0, there are recursive functions $f(x, s)$ and $g(x, s)$ with $f(x, s)$, $g(x, s)\leqq 1$ such that

$$\lim_s f(x, s) = A(x), \qquad \lim_s g(x, s) = B(x).$$

Let $A_s(x)=f(x, s)$ and $B_s(x)=g(x, s)$.

The basic idea of the proof is essentially the same as that of Theorem 2.1. We will construct a strictly increasing sequence $\{l_n : n\in\boldsymbol{N}\}$ as before but use the approximation A_s and B_s instead, and then define C, D from the sequence as in the proof of Theorem 2.1:

$$C(x) = \begin{cases} A(x) & \text{if } l_{6e}\leqq|x|<l_{6e+1} \text{ for some } e, \\ B(x) & \text{otherwise,} \end{cases}$$

$$D(x) = \begin{cases} A(x) & \text{if } l_{6e+3}\leqq|x|<l_{6e+4} \text{ for some } e, \\ B(x) & \text{otherwise.} \end{cases}$$

The sequence $\{l_n : n \in \boldsymbol{N}\}$ will be constructed so that given x, the unique n with $l_n \leqq |x| < l_{n+1}$ is calculated in $|x|$ steps by looking back the construction. It, then, follows that $B \leqq_T^h C \leqq_T^h A$ and $B \leqq_T^h D \leqq_T^h A$. The condition (2) of the theorem will be verified in the same way as in the preceding section since $\{l_n\}_n$ will be constructed to satisfy $2^{l_n} \leqq l_{n+1}$ for all n.

To ensure that $C \not\leqq_T^h B$ and $D \not\leqq_T^h B$, we require the following:

$$(R_{2e}) \qquad C \neq M_e(B),$$

$$(R_{2e+1}) \qquad D \neq M_e(B).$$

At stage $6e+1$, we will try to meet the first requirement R_{2i} $(i \leqq e)$ that is not certified at the point entering this stage by searching for some x with $|x| \geqq l_{6e}$ such that $A_s(x) \neq M_i(B_s, x)$ at some $s \geqq l_{6e}$. Such an x exists since we are assuming $B <_T^h A$. At this point we would know R_{2i} is met. However, at later stage, this disagreement might be injured, because it might happen that $A_t(x) \neq A_s(x)$ or $B_t \upharpoonright p_i(|x|) \neq B_s \upharpoonright p_i(|x|)$ at some point $t > s$. Then, we must attack R_{2i} again. We can not expect that R_{2i} is injured only finitely often. The lowness of A and B will resolve this difficulty. We use a variation of the method of Robinson [**6**] known as the "Robinson trick".

We fix a recursive enumeration $\{\sigma_k : k \in \boldsymbol{N}\}$ of the finite functions σ such that

$$\mathrm{dom}(\sigma) = \{z \in \Sigma^* : |z| \leqq l\} \text{ for some } l, \text{ and } \mathrm{rng}(\sigma) \subseteqq \{0, 1\}.$$

Let $\mathrm{lh}(\sigma)$ denote the maximum length of the strings in $\mathrm{dom}(\sigma_k)$. $B \upharpoonright l$ and $B_s \upharpoonright l$ denote the restrictions of B and B_s to $\{x : |x| \leqq l\}$ respectively. Note that for each l, we can effectively find a k such that $B_s \upharpoonright l = \sigma_k$. Define H and $\hat{H}$ by

$$H = \{e : (\exists \langle x, k \rangle \in W_e)[x \in A \ \& \ \sigma_k = B \upharpoonright \mathrm{lh}(\sigma_k)]\},$$

$$\hat{H} = \{e : (\exists \langle x, k \rangle \in W_e)[x \notin A \ \& \ \sigma_k = B \upharpoonright \mathrm{lh}(\sigma_k)]\}.$$

Since A and B are low, these sets are both Δ_2^0. Let $h(e, s)$ and $\hat{h}(e, s)$ be recursive functions with $h(x, s)$, $\hat{h}(x, s) \leqq 1$ such that

$$\lim_s h(e, s) = H(e), \qquad \lim_s \hat{h}(e, s) = \hat{H}(e).$$

We will build recursive sequences $\{V_{i,s}\}_{i,s \in \boldsymbol{N}}$ and $\{\hat{V}_{i,s}\}_{i,s \in \boldsymbol{N}}$ during the construction. Let $V_i = \bigcup_s V_{i,s}$ and $\hat{V}_i = \bigcup_s \hat{V}_{i,s}$. Then, V_i and $\hat{V}_i$ are recursively enumerable. By the Recursion Theorem we may assume that we have in advance an index $\theta(i)$ of V_i and $\hat{\theta}(i)$ of $\hat{V}_i$ with some recursive functions θ and $\hat{\theta}$.

DEFINITION 3.2. Suppose i and s are given.

(1) R_i is *h-certified* at s if $h(\theta(i), s) = 1$ and there is a $\langle x, k \rangle \in V_{i,s}$ such

that $A_s(x)=1$ and $B_s\upharpoonright \mathrm{lh}(\sigma_k)=\sigma_k$,

(2) R_i is *$\hat{h}$-certified* at s if $\hat{h}(\hat{\theta}(i), s)=1$ and there is a $\langle x, k\rangle \in \hat{V}_{i,s}$ such that $A_s(x)=0$ and $B_s\upharpoonright \mathrm{lh}(\sigma_k)=\sigma_k$,

(3) R_i is *certified* at s if R_i is either h-certified or $\hat{h}$-certified at s.

We now give the construction of $\{l_n\}_n$. We use s as a variable that counts the steps of the construction. $V_{i,s}$ represents the finite set of elements enumerated into V_i up to step s during the construction. Similar for $\hat{V}_{i,s}$.

Stage 0. Let $l_0=0$ and $V_{i,0}=\hat{V}_{i,0}=\emptyset$ for all i.

Stage $6e+i+1$ ($i=1, 2, 4, 5$). We let $n=6e+i$. Let $l_{n+1}=l_n+2^{l_n}$. No new elements are enumerated into V_i and $\hat{V}_i$ for all i at this stage: $V_{i,s+1}=V_{i,s}$; $\hat{V}_{i,s+1}=\hat{V}_{i,s}$ for all i and s with $l_n \leqq s < l_{n+1}$.

Stage $6e+1$. Take the least $i \leqq e$ such that R_{2i} is not certified at l_{6e}. We say that R_{2i} is *attacked* at this stage. Our construction in this stage consists of one main routine and 5 subroutines. No new elements are enumerated into V_j and $\hat{V}_j$ for all j with $j \neq 2i$. We enumerate some new elements into V_{2i} or into $\hat{V}_{2i}$ only when the construction enters Subroutine 1 below.

MAIN ROUTINE. We set $s:=l_{6e}$. Go to Subroutine 1.

CLAIM 1. *For every s, there exist $t>s$, $x\in\Sigma^*$ and k with $l_{6e}\leqq |x| \leqq |\langle x, k\rangle|$ $\leqq t$ such that $\sigma_k=B_t\upharpoonright p_i(|x|)$ and such that one of the following holds:*

(1.1) $A_t(x)=1$ & $M_i(B_t, x)=0$,

(1.2) $A_t(x)=0$ & $M_i(B_t, x)=1$.

PROOF OF CLAIM 1. Since $B<_T^h A$, there is an x with $l_{6e}\leqq |x|$ such that $A(x)\neq M_i(B, x)$. Take a sufficiently large $s_0>s$ such that $|x|\leqq s_0$ and

$$(\forall t)[s_0 \leqq t \Longrightarrow A_t(x)=A(x) \ \&\ B_t\upharpoonright p_i(|x|)=B\upharpoonright p_i(|x|)].$$

Let k be an integer such that $B\upharpoonright p(|x|)=\sigma_k$, and take a $t\geqq s_0$ so that $|\langle x, k\rangle|$ $\leqq t$. If $A(x)=1$ then (1.1) holds, and if $A(x)=0$ then (1.2) holds. □

SUBROUTINE 1. Suppose that the construction enters this subroutine with s. We take the least t that satisfies Claim 1. Let $\langle x, k\rangle$ be the least pair which satisfies the conditions of the claim. If (1.1) holds, then enumerate $\langle x, k\rangle$ into V_{2i}, set $s:=t$, and go to Subroutine 2. If (1.2) holds, then enumerate $\langle x, k\rangle$ into $\hat{V}_{2i}$, set $s:=t$, and go to Subroutine 3.

CLAIM 2. *Given s, suppose that $V_{2i,t}=V_{2i,s}$ for all $t\geqq s$. Then there is a $t>s$ such that one of the following holds:*

(2.1) *R_{2i} is h-certified at t,*

(2.2) $h(\theta(2i), t)=0$ & $(\forall \langle x, k\rangle \in V_{2i,t})[A_t(x)=0 \vee B_t\upharpoonright \mathrm{lh}(\sigma_k)\neq\sigma_k]$.

PROOF OF CLAIM 2. Note that $V_{2i,t}=V_{2i}$ for all $t\geqq s$, and therefore V_{2i} is finite. Take a sufficiently large $t>s$ so that $h(\theta(2i), t)=H(\theta(2i))$ and

$$(\forall\langle x, k\rangle\in V_{2i})[A_t(x)=A(x)\ \&\ B_t\restriction \mathrm{lh}(\sigma_k)=B\restriction \mathrm{lh}(\sigma_k)].$$

Suppose $H(\theta(2i))=1$. By the definition of H, we have

$$(\exists\langle x, k\rangle\in V_{2i})[A(x)=1\ \&\ B\restriction \mathrm{lh}(\sigma_k)=\sigma_k].$$

Then, we have $h(\theta(2i), t)=1$ and

$$(\exists\langle x, k\rangle\in V_{2i,t})[A_t(x)=1\ \&\ B_t\restriction \mathrm{lh}(\sigma_k)=\sigma_k].$$

Thus, R_{2i} is h-certified at t. Similarly, if $H(\theta(2i))=0$, then (2.2) holds at t. □

Similarly, we have the following.

CLAIM 3. *Suppose that $\hat{V}_{2i,t}=\hat{V}_{2i,s}$ for all t with $t\geqq s$. Then there is a $t>s$ such that one of the following holds:*

(3.1) *R_{2i} is $\hat{h}$-certified at t,*

(3.2) $\hat{h}(\hat{\theta}(2i), t)=0\ \&\ (\forall\langle x, k\rangle\in\hat{V}_{2i,t})[A_t(x)=1\vee B_t\restriction \mathrm{lh}(\sigma_k)\neq\sigma_k]$.

SUBROUTINE 2. Suppose we enter this subroutine with s. Set $t:=s$, and repeat $t:=t+1$ until either (2.1) or (2.2) of Claim 2 holds. Set $s:=t$. If (2.1) holds at t then go to Subroutine 5, and if (2.2) holds then go to Subroutine 4.

SUBROUTINE 3. Similar to Subroutine 2.

CLAIM 4. *Given s, suppose that $V_{2i,t}=V_{2i,s}$ and $\hat{V}_{2i,t}=\hat{V}_{2i,s}$ for all $t>s$. Then there is a $t\geqq s$ such that one of the following holds:*

(4.1) *R_{2i} is h-certified at t,*

(4.2) *R_{2i} is $\hat{h}$-certified at t,*

(4.3) $h(\theta(2i), t)=0\ \&\ (\forall\langle x, k\rangle\in V_{2i,t})[A_t(x)=0\vee B_t\restriction \mathrm{lh}(\sigma_k)\neq\sigma_k]$ *and*
$\hat{h}(\hat{\theta}(2i), t)=0\ \&\ (\forall\langle x, k\rangle\in\hat{V}_{2i,t})[A_t(x)=1\vee B_t\restriction \mathrm{lh}(\sigma_k)\neq\sigma_k]$.

PROOF OF CLAIM 4. By the assumption, for all sufficiently large t, $V_{2i,t}=V_{2i}$ and $\hat{V}_{2i,t}=\hat{V}_{2i}$. Take a sufficiently large t with $t>s$ which satisfies the following:

(a) $h(\theta(2i), t)=H(\theta(2i))$ and $\hat{h}(\hat{\theta}(2i), t)=\hat{H}(\hat{\theta}(2i))$,

(b) $(\forall\langle x, k\rangle\in V_{2i})[A_t(x)=A(x)\ \&\ B_t\restriction \mathrm{lh}(\sigma_k)=B\restriction \mathrm{lh}(\sigma_k)]$,

(c) $(\forall\langle x, k\rangle\in\hat{V}_{2i})[A_t(x)=A(x)\ \&\ B_t\restriction \mathrm{lh}(\sigma_k)=B\restriction \mathrm{lh}(\sigma_k)]$.

As in the proof of Claim 2, we see that if $H(\theta(2i))=1$ then R_{2i} is h-certified at t, and if $\hat{H}(\hat{\theta}(2i))=1$ then R_{2i} is $\hat{h}$-certified. Similarly, if $H(\theta(2i))=0$ and $\hat{H}(\hat{\theta}(2i))=0$, then (4.3) holds. □

SUBROUTINE 4. Similar to Subroutine 2. Suppose that the construction enters this subroutine with s. We wait for the least $t>s$ that satisfies one of the conditions of Claim 4. Set $s:=t$. If (4.1) or (4.2) holds, then go to Subroutine 5. Otherwise, go to Subroutine 1.

SUBROUTINE 5. Suppose we reach this subroutine with s. Let $l_{6e+1}=l_{6e}+2^{l_{6e}}+m$, where m is the number of steps in which the construction up to this point is performed. Set $s:=l_{6e+1}$ and exist from the main routine.

Stage $6e+4$. Similar to Stage $6e+1$. Take the least $i\leqq e$ such that R_{2i+1} is not certified at l_{6e+3}. The requirement R_{2i+1} is attacked in this stage. We leave the details to the reader.

Thus, we complete the construction of $\{l_n\}_{n\in N}$.

LEMMA 3.3. *l_{6e+1} and l_{6e+4} are defined.*

PROOF. We prove that l_{6e+1} is defined. It is sufficient to show that we reach Subroutine 5 while executing the main routine. Suppose not. Then, we always exit from Subroutine 2 with (2.2), Subroutine 3 with (3.2) and Subroutine 4 with (4.3). Since $B<_T^h A$, there is an x such that $A(x)\neq M_i(B, x)$. Take the least such x with $|x|\geqq l_{6e}$ and let k be the least integer with $B\restriction p_i(|x|)=\sigma_k$. Suppose, say, $A(x)=1$ and $M_i(B, x)=0$. Take s_0 large enough to satisfy

$$(\forall s)[s_0\leqq s \Longrightarrow A_s(x)=1 \ \&\ B_s\restriction p_i(|x|)=B\restriction p_i(|x|)].$$

We may assume that $|x|\leqq s_0$. If $\langle x, k\rangle$ is not enumerated into V_{2i} up to s_0, then $\langle x, k\rangle$ is witnessed each time Subroutine 1 is executed after s_0. By the assumption, we enter Subroutine 1 infinitely often. Thus, eventually, $\langle x, k\rangle$ must be enumerated into V_{2i}. Then, we have $H(\theta(2i))=1$ by the definition of H. Take sufficiently large $s_1>s_0$ so that $\langle x, k\rangle\in V_{2i,s_1}$ and $h(\theta(2i), s)=1$ for all $s\geqq s_1$. Then, R_{2i} is h-certified at all points after s_1. Thus, we reach Subroutine 5 whenever we exit from one of Subroutine 2–4 after s_1, which is a contradiction. □

LEMMA 3.4. *For all i, the requirement R_{2i} is attacked only finitely often.*

PROOF. We prove the lemma by induction on i. Suppose that no requirement R_{2j} with $j<i$ is attacked at any stage after n_0, which means that every requirement R_{2j} $(j<i)$ is certified at l_{6e} for all e with $n_0<6e$. Let $s_0\geqq n_0$ be large enough to satisfy

$$(\forall s)[s_0\leqq s \Longrightarrow h(\theta(2i), s)=H(\theta(2i)) \ \&\ \hat{h}(\hat{\theta}(2i), s)=\hat{H}(\hat{\theta}(2i))].$$

Suppose $H(\theta(2i))=1$. Then, by the definition of H, there is a $\langle x, k\rangle\in V_{2i}$ such that $A(x)=1$ and $B\restriction \mathrm{lh}(\sigma_k)=\sigma_k$. Take a sufficiently large $s_1\geqq s_0$ so that

$$(\forall s)[s_1 \leqq s \Longrightarrow \langle x, k \rangle \in V_{2i,s} \ \& \ A_s(x) = 1 \ \& \ B_s \upharpoonright \mathrm{lh}(\sigma_k) = B \upharpoonright \mathrm{lh}(\sigma_k)].$$

Then, R_{2i} is h-certified at every point s with $s \geqq s_1$. It follows that R_{2i} is not attacked at any stage n with $s_1 \leqq l_n$. Similarly, if $\hat{H}(\hat{\theta}(2i))=1$, then R_{2i} is not attacked infinitely often. Finally, suppose that $H(\theta(2i))=\hat{H}(\hat{\theta}(2i))=0$. We take a sufficiently large $s_2 \geqq s_0$ so that

$$(\forall s)[s_2 \leqq s \Longrightarrow h(\theta(2i), s) = \hat{h}(\hat{\theta}(2i), s) = 0].$$

It follows that R_{2i} is never certified after s_2. Thus, for every $e \geqq i$ with $n_0 < 6e$, if $s_2 \leqq l_{6e}$ and R_{2i} is attacked at stage $6e+1$, then we can not enter Subroutine 5 during stage $6e+1$, which contradicts Lemma 3.3. □

Similarly, we can prove the following.

LEMMA 3.5. *For all i, the requirement R_{2i+1} is attacked only finitely often.*

LEMMA 3.6. *For every i the requirements R_{2i} and R_{2i+1} are met.*

PROOF. We prove that the requirement R_{2i} is met. Take an n_0 so that R_{2i} is not attacked after n_0. Then, for all e with $n_0 < 6e$, R_{2i} is certified at l_{6e}. It follows that either $H(\theta(2i))=1$ or $\hat{H}(\hat{\theta}(2i))=1$. Suppose, say, $H(\theta(2i))=1$. Then, by the definition, there is a $\langle x, k \rangle \in V_{2i}$ such that $A(x)=1$ and $B \upharpoonright \mathrm{lh}(\sigma_k) = \sigma_k$. Suppose $\langle x, k \rangle$ is enumerated into V_{2i} during stage $6e+1$. Then, there is a t with $l_{6e} < t < l_{6e+1}$ such that $B_t \upharpoonright p_i(|x|) = \sigma_k$ and $M_i(B_t, x)=0$. Since $B_t \upharpoonright p_i(|x|)$ and $B \upharpoonright p_i(|x|)$ are both equal to σ_k, we see that $M_i(B, x)=0$. Thus we have the inequality $A(x) \neq M_i(B, x)$. A and C agree on the interval $\{z : l_{6e} \leqq |z| < l_{6e+1}\}$. Consequently, we obtain the desired inequality $C(x) \neq M_i(B, x)$. □

This completes the proof of Theorem 3.1. The method presented here can be applied to other problems on the theory of the hp-T degrees of Δ_2^0 low sets. For example, we can extend the result of Shore-Slaman [7] on the decidability of the Π_2 theory of the p-T degrees of recursive sets to the hp-T degrees of Δ_2^0 low sets.

References

[1] K. Ambos-Spies, Honest polynomial time reducibilities and $P=?NP$ problem, J. Comput. and System Sci., **39** (1989), 250-289.

[2] R. Downey, On computational complexity and honest polynomial degrees, Theoret. Comput. Sci., **78** (1991), 305-317.

[3] S. Homer, Minimal polynomial degrees for nonrecursive sets, Lecture Notes in Math., **1141**, Springer-Verlag, 1985, pp. 193-202.

[4] R.E. Ladner, On the structure of polynomial time reducibility, J. Assoc. Comput. Mech., **22** (1975), 155-171.

[5] L.H. Landweber, R.J. Lipton and E.L. Robertson, On the structure of sets in *NP*

and other complexity classes, Theoret. Comput. Sci., **15** (1981), 181-200.
[6] R. W. Robinson, Interpolation and embedding in the recursively enumerable degrees, Ann. of Math., **93** (1971), 285-314.
[7] R. A. Shore and T. A. Slaman, The p-T degrees of the recursive sets: lattice embeddings, extensions of embeddings and the two quantifier theory, in Proc. Structures in Complexity Theory, 4th Annual Conf., 1989, to appear.
[8] R. I. Soare, Recursively Enumerable Sets and Degrees, Springer-Verlag, New York, 1987.

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