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On the Gauss curvature of minimal surfaces

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§ 1. Introduction.

In 1952, E. Heinz showed that, for a minimal surface M in $\boldsymbol{R}^3$ which is the graph of a function $z=z(x, y)$ of class C^2 defined on a disk $\Delta_R:=\{(x, y);\ x^2+y^2<R^2\}$, there is a positive constant C not depending on each surface M such that $|K|\leqq C/R^2$ holds for the curvature K of M at the origin ([8]). This is an improvement of the classical Bernstein's theorem that a minimal surface in $\boldsymbol{R}^3$ which is the graph of a function of class C^2 defined on the total plane is necessarily a plane. Later, R. Osserman gave some generalizations of these results to surfaces which need not be of the form $z=z(x, y)$ ([**10**], [**11**]). To state one of his results, we consider a connected, oriented minimal surface M immersed in $\boldsymbol{R}^3$ and, for a point $p\in M$, we denote by $K(p)$ and $d(p)$ the Gauss curvature of M at p and the distance from p to the boundary of M respectively. He gave the following estimate of the Gauss curvature of M.

THEOREM A. *Let M be a simply-connected minimal surface immersed in $\boldsymbol{R}^3$ and assume that there is some fixed nonzero vector n_0 and a number $\theta_0>0$ such that all normals to M make angles of at least θ_0 with n_0. Then,*

$$|K(p)|^{1/2} \leqq \frac{1}{d(p)} \frac{2\cos(\theta_0/2)}{\sin^3(\theta_0/2)} \qquad (p \in M).$$

He obtained also some generalization of Theorem A to minimal surfaces immersed in $\boldsymbol{R}^m\,(m\geqq 3)$ ([**12**]).

Relating to these results, the author proved the following theorem in his paper [**4**].

THEOREM B. *Let M be a minimal surface immersed in $\boldsymbol{R}^3$ and let $G: M\to S^2$ be the Gauss map of M. If G omits mutually distinct five points $n_1, \cdots, n_5$ in S^2, then it holds that*

$$(1) \qquad |K(p)|^{1/2} \leqq \frac{C}{d(p)} \qquad (p\in M)$$

for some positive constant C depending only on n_j's.

Since $d(p)=\infty$ for any $p\in M$ in case that M is complete, Theorem B implies that the Gauss map of a complete non-flat minimal surface immersed in $\boldsymbol{R}^3$ can omit at most four points of the sphere. He obtained also a generalization of Theorem B to the case of minimal surfaces whose Gauss maps take several fixed values with high multiplicities ([**5**, Theorem II]). Recently, A. Ros added a new insight to the study of minimal surfaces satisfying the condition (1) ([**14**]).

The purpose of this paper is to give more precise estimate like the result of Theorem A for a constant C satisfying the inequality (1). We shall give an improvement of Theorem II of [**5**], which implies the following:

THEOREM C. *Let $x=(x_1, x_2, x_3): M\to \boldsymbol{R}^3$ be a minimal surface immersed in $\boldsymbol{R}^3$ and let $G: M\to S^2$ be the Gauss map of M. Assume that G omits five distinct unit vectors $n_1, \cdots, n_5\in S^2$. Let θ_{ij} be the angle between n_i and n_j and set*

$$L := \min\left\{\sin\left(\frac{\theta_{ij}}{2}\right);\ 1\leqq i<j\leqq 5\right\}.$$

Then, there exists some positive constant C not depending on each minimal surface such that

$$|K(p)|^{1/2}\leqq \frac{C}{d(p)}\frac{\log^2(1/L)}{L^3} \qquad (p\in M). \tag{2}$$

It is an interesting open problem to know whether the factor $\log^2(1/L)/L^3$ in (2) can be replaced by $1/L^3$ or not. Relating to this, in §4 we shall give an example of a family of minimal surfaces which shows that it cannot be replaced by $1/L^{3-\varepsilon}$ for any positive number ε.

§2. Sum to product estimate.

Consider the stereographic projection π of the unit sphere S^2 onto the extended complex plane $\bar{\boldsymbol{C}}:=\boldsymbol{C}\cup\{\infty\}$. For α and $\beta\in\bar{\boldsymbol{C}}$ take the unit vectors n_1 and n_2 in S^2 with $\alpha=\pi(n_1)$ and $\beta=\pi(n_2)$. Let $\theta(0\leqq\theta\leqq\pi)$ be the angle between n_1 and n_2. Define

$$|\alpha, \beta| := \sin\frac{\theta}{2}.$$

We can easily show that, if $\alpha\neq\infty$ and $\beta\neq\infty$, then

$$|\alpha, \beta| = \frac{|\alpha-\beta|}{\sqrt{1+|\alpha|^2}\sqrt{1+|\beta|^2}}$$

and, if either α or β, say β, is equal to ∞, then $|\alpha, \beta|=1/\sqrt{1+|\alpha|^2}$.

Take $q\,(\geqq 2)$ mutually distinct numbers $\alpha_1, \cdots, \alpha_q\in\bar{\boldsymbol{C}}$. Set

(3) $$L := \min_{i<j} |\alpha_i, \alpha_j|.$$

Then, we have the following:

LEMMA 1. *For all $w \in \bar{C}$ it holds that*

$$|w, \alpha_i| \geqq \frac{L}{2}$$

for all α_i except at most one.

In fact, if $|w, \alpha_i| < L/2$ for two distinct indices $i = i_1$ and $i = i_2$, then we have an absurd conclusion

$$L \leqq |\alpha_{i_1}, \alpha_{i_2}| \leqq |\alpha_{i_1}, w| + |w, \alpha_{i_2}| < L.$$

Let g be a nonconstant meromorphic function on a disc $\Delta_R := \{z; |z| < R\}$ and $\eta_1, \cdots, \eta_q$ be real numbers with $0 < \eta_j \leqq 1$. Here, we assume that

$$\gamma := \eta_1 + \cdots + \eta_q > 1.$$

The purpose of this section is to prove the following:

PROPOSITION 2. *For each ρ with $\rho > 0$ and η with $\gamma - 1 > \gamma\eta \geqq 0$, take a constant $a_0 (\geqq e^2)$ satisfying the condition*

(4) $$\frac{1}{\log^2 a_0} + \frac{1}{\log a_0} \leqq \rho'$$

for $\rho' := \rho/\gamma$. Then, it holds that

$$\Delta \log \frac{(1+|g|^2)^{\rho}}{\prod_{j=1}^{q} \log^{\eta_j}(a_0/|g, \alpha_j|^2)} \geqq C_1^2 \frac{|g'|^2}{(1+|g|^2)^2} \prod_{j=1}^{q} \left(\frac{1}{|g, \alpha_j|^2 \log^2(a_0/|g, \alpha_j|^2)} \right)^{\eta_j(1-\eta)},$$

where

(5) $$C_1 := 2\left(\frac{L}{2} \log \frac{4a_0}{L^2} \right)^{\gamma - 1 - \gamma\eta}.$$

To prove this, we need the following two lemmas.

LEMMA 3. *For an arbitrarily given $\rho' > 0$ take a number $a_0 (\geqq e)$ satisfying the condition* (4). *Then, it holds that*

$$\Delta \log \frac{1}{\log(a_0/|g, \alpha_j|^2)} \geqq \frac{4|g'|^2}{(1+|g|^2)^2} \left(\frac{1}{|g, \alpha_j|^2 \log^2(a_0/|g, \alpha_j|^2)} - \rho' \right).$$

To see this, we represent each α_j as $\alpha_j = a_{j0}/a_{j1}$ with a nonzero vector $(a_{j0}, a_{j1}) (1 \leqq j \leqq q)$ and the meromorphic function g as $g = g_0/g_1$ with holomorphic functions g_0, g_1 which have no common zero. Then, we have

$$|g, \alpha_j|^2 = \frac{|a_{j1}g_0 - a_{j0}g_1|^2}{|g_0|^2 + |g_1|^2}$$

and

$$\frac{|g'|^2}{(1+|g|^2)^2} = \frac{|g_0 g_1' - g_1 g_0'|^2}{(|g_0|^2 + |g_1|^2)^2}.$$

Therefore, Lemma 3 is a restatement of Lemma 2.2 of [**5**]. We omit the proof.

LEMMA 4. *Take nonnegative numbers $A_1, \cdots, A_q$ and a positive constant M such that $M \geqq A_j$ for all j except at most one. Then, for every η with $\gamma - 1 > \gamma\eta \geqq 0$,*

$$\eta_1 A_1 + \eta_2 A_2 + \cdots + \eta_q A_q \geqq \frac{1}{M^{\gamma-1-\gamma\eta}} (A_1^{\eta_1} A_2^{\eta_2} \cdots A_q^{\eta_q})^{1-\eta}.$$

PROOF. Without loss of generality, we may assume that

$$A_1 \geqq A_2 \geqq \cdots \geqq A_q.$$

We then have $M \geqq A_j$ for all $j = 2, 3, \cdots, q$. Set

$$\lambda_1 := \eta_1(1-\eta), \qquad \lambda_j := \frac{\eta_j}{\eta_2 + \cdots + \eta_q}(1-\lambda_1) \qquad (j=2, \cdots, q).$$

Then, we obtain the desired inequality

$$\begin{aligned} &\eta_1 A_1 + \eta_2 A_2 + \cdots + \eta_q A_q \\ &\quad \geqq \lambda_1 A_1 + \lambda_2 A_2 + \cdots + \lambda_q A_q \\ &\quad \geqq A_1^{\lambda_1} A_2^{\lambda_2} \cdots A_q^{\lambda_q} \\ &\quad = (A_1^{\eta_1} A_2^{\eta_2} \cdots A_q^{\eta_q})^{1-\eta} \frac{A_2^{\lambda_2} \cdots A_q^{\lambda_q}}{(A_2^{\eta_2} \cdots A_q^{\eta_q})^{1-\eta}} \\ &\quad \geqq (A_1^{\eta_1} A_2^{\eta_2} \cdots A_q^{\eta_q})^{1-\eta} \frac{1}{M^{\gamma-1-\gamma\eta}}. \end{aligned}$$

PROOF OF PROPOSITION 2. For brevity, we set

$$h_j := \frac{1}{|g, \alpha_j|} \qquad (1 \leqq j \leqq q).$$

Take $a_0 (\geqq e^2)$ satisfying the condition (4) for $\rho' := \rho/\gamma$. By Lemma 3 we see

$$\begin{aligned} &\Delta \log \frac{(1+|g|^2)^\rho}{\prod_{j=1}^q \log^{\eta_j}(a_0 h_j^2)} \\ &\quad \geqq \frac{4|g'|^2}{(1+|g|^2)^2} \Big(\rho + \sum_{j=1}^q \eta_j \Big(\frac{h_j^2}{\log^2(a_0 h_j^2)} - \frac{\rho}{\gamma} \Big)\Big) \qquad (6) \\ &\quad = \frac{4|g'|^2}{(1+|g|^2)^2} \sum_{j=1}^q \frac{\eta_j h_j^2}{\log^2(a_0 h_j^2)}. \end{aligned}$$

On the other hand, for each $z\in\Delta_R$ it follows from Lemma 1 that $|g(z), \alpha_j|\geqq L/2$ for all α_j except at most one. Therefore, since $x^2/\log^2(a_0x^2)$ is monotone increasing for $x\geqq 1$, we have

$$\frac{h_j^2}{\log^2(a_0h_j^2)}\leqq\frac{4}{L^2\log^2(4a_0/L^2)}$$

for such α_j's. Setting $A_j:=h_j^2/\log^2(a_0h_j^2)$ and $M:=4/(L^2\log^2(4a_0/L^2))$, we apply Lemma 4 to show that

$$\sum_{j=1}^{q}\frac{\eta_j h_j^2}{\log^2(a_0h_j^2)}\geqq\left(\frac{L}{2}\log\frac{4a_0}{L^2}\right)^{2(\gamma-1-\gamma\eta)}\prod_{j=1}^{q}\left(\frac{h_j^2}{\log^2(a_0h_j^2)}\right)^{\eta_j(1-\eta)}.$$

In view of (6) this concludes Proposition 2.

§3. An application of Ahlfors-Schwarz lemma.

We shall next prove the following:

PROPOSITION 5. *Let g be a nonconstant meromorphic function Δ_R. Assume that, for some fixed distinct points $\alpha_1, \alpha_2, \cdots, \alpha_q$ in $\bar{C}$ and integers $m_1, m_2, \cdots, m_q$ not less than two, g does not take the values α_j with multiplicities less than m_j for each j and that*

$$\gamma:=\sum_{j=1}^{q}\left(1-\frac{1}{m_j}\right)>2.$$

Then, for η_0 with $\gamma-2>\gamma\eta_0>0$ there is a constant $a_0\geqq e^2$ depending only on γ and η_0 such that, for an arbitrary positive constant $\eta\leqq\eta_0$, it holds that

$$\frac{|g'|}{1+|g|^2}\prod_{j=1}^{q}\left(\frac{1}{|g,\alpha_j|\log(4a_0/|g,\alpha_j|^2)}\right)^{(1-1/m_j)(1-\eta)}\leqq\frac{1}{C_1(1-\eta)^{1/2}}\frac{2R}{R^2-|z|^2},$$

where L and C_1 are given by (3) *and* (5) *respectively.*

This will be proved by the use of the following Ahlfors-Schwarz lemma.

LEMMA 6 (cf. [1], [2]). *If a continuous nonnegative function v on Δ_R is of class C^2 on the set $\{z\in\Delta_R;\ v(z)>0\}$ and satisfies the condition*

$$\Delta\log v\geqq v^2$$

there, then

$$v(z)\leqq\frac{2R}{R^2-|z|^2}\qquad(z\in\Delta_R).$$

PROOF OF PROPOSITION 5. For brevity, after a change of g by a suitable Möbius transformation, we assume $\alpha_q=\infty$. Set

$$\eta_j := 1 - \frac{1}{m_j}, \quad h_j := \frac{1}{|g, \alpha_j|} (1 \leqq j \leqq q), \quad \rho := \frac{\gamma - 2 - \gamma\eta}{2(1-\eta)}.$$

Consider the function

$$v := C_1(1-\eta)^{1/2} \frac{|g'|}{1+|g|^2} \prod_{j=1}^{q} \left(\frac{h_j}{\log(a_0 h_j^2)}\right)^{\eta_j(1-\eta)},$$

where C_1 and a_0 are the constants as in Proposition 2 for the above γ, $\eta(\leqq \eta_0)$ and $\rho' := (\gamma - 2 - \gamma\eta_0)/(2\gamma(1-\eta_0))(\leqq \rho/\gamma)$. Setting

$$w := \begin{cases} 0 & \text{if } g(z) = \alpha_j \text{ for some } j \\ |g'| \prod_{j=1}^{q-1} \left(\frac{(1+|\alpha_j|^2)^{1/2}}{|g - \alpha_j|}\right)^{\eta_j(1-\eta)} & \text{otherwise}, \end{cases}$$

we rewrite v as

$$v = C_1(1-\eta)^{1/2} w \left(\frac{(1+|g|^2)^\rho}{\prod_{j=1}^{q} \log^{\eta_j}(a_0 h_j^2)}\right)^{1-\eta}.$$

Then, v is continuous on Δ_R and $\log w$ is harmonic on $\{z \in \Delta_R;\ w(z) > 0\}$. In fact, for a point $z_0 \in \Delta_R$, we can write v as $v = |z - z_0|^a \tilde{v}$ with a nonnegative function $\tilde{v}$ in some neighborhood of z_0, where

$$a = m - 1 - m\left(1 - \frac{1}{m_j}\right)(1-\eta) > 0$$

when $g - \alpha_j$ has a zero of order m at z_0, and

$$\begin{aligned} a &= (\gamma - \eta_q) m (1-\eta) - m - 1 - 2m\rho(1-\eta) \\ &= m - 1 - m\eta_q(1-\eta) > 0 \end{aligned}$$

when g has a pole of order m at z_0. Therefore, the function v is continuous and, by Proposition 2, it satisfies the condition

$$\begin{aligned} \Delta \log v &= (1-\eta) \Delta \log \left(\frac{(1+|g|^2)^\rho}{\prod_{j=1}^{q} \log^{\eta_j}(a_0 h_j^2)}\right) \\ &\geqq (1-\eta) C_1^2 \frac{|g'|^2}{(1+|g|^2)^2} \prod_{j=1}^{q} \left(\frac{h_j^2}{\log^2(a_0 h_j^2)}\right)^{\eta_j(1-\eta)} = v^2. \end{aligned}$$

Proposition 5 is a consequence of Lemma 6.

COROLLARY 7. *Let g be a nonconstant meromorphic function on Δ_R satisfying the same assumption as in Proposition 5. Then, for arbitrary positive constants η and δ with $\gamma - 2 > \gamma\eta + \gamma\delta$, it holds that*

$$\frac{|g'|}{1+|g|^2} \frac{1}{\left(\prod_{j=1}^{q} |g, \alpha_j|^{1-1/m_j}\right)^{1-\eta-\delta}} \leqq C_2 \frac{2R}{R^2 - |z|^2},$$

where C_2 *is given by*

$$C_2 := \frac{a_0^{\gamma\delta/2} C_3}{\delta^{\gamma(1-\eta)}((L/2)\log(4a_0/L^2))^{\gamma-1-\gamma\eta}}$$

for some constant C_3 *depending only on* γ.

PROOF. The function

$$\varphi(x) := \frac{\log^{1-\eta}(a_0 x^2)}{x^\delta} \qquad (1 \leqq x < +\infty)$$

takes the maximum at a point $x_0 := \max((e^{2(1-\eta)/\delta}/a_0)^{1/2}, 1)$. Therefore, we have

$$\begin{aligned} &\frac{|g'|}{1+|g|^2} \frac{1}{\prod_{j=1}^q |g, \alpha_j|^{\eta_j(1-\eta-\delta)}} \\ &\quad = \frac{|g'|}{1+|g|^2} \prod_{j=1}^q \left(\frac{h_j}{\log(a_0 h_j^2)}\right)^{\eta_j(1-\eta)} \prod_{j=1}^q \left(\frac{\log^{1-\eta}(a_0 h_j^2)}{h_j^\delta}\right)^{\eta_j} \\ &\quad \leqq \frac{\varphi(x_0)^\gamma}{C_1(1-\eta)^{1/2}} \frac{2R}{R^2 - |z|^2} \end{aligned}$$

by the use of Proposition 5. Since $0 \leqq \eta < (\gamma-2)/\gamma$, we can find a positive constant C_3 depending only on γ such that

$$\frac{\varphi(x_0)^\gamma}{C_1(1-\eta)^{1/2}} \leqq \frac{2a_0^{\delta\gamma/2} C_3}{C_1 \delta^{\gamma(1-\eta)}}.$$

This concludes Corollary 7.

§ 4. Main results.

Consider a (connected, oriented) minimal surface $x := (x_1, x_2, x_3) : M \to \boldsymbol{R}^3$ immersed in $\boldsymbol{R}^3$ and the Gauss map $G : M \to S^2$ of M. By associating a holomorphic local coordinate $z = u + \sqrt{-1}v$ with each positive isothermal coordinate system (u, v), M is considered as a Riemann surface. Then, for the stereographic projection $\pi : S^2 \to \overline{\boldsymbol{C}}$, the function $g := \pi \cdot G : M \to \overline{\boldsymbol{C}}$ is meromorphic on M. We call the map g the Gauss map of M instead of G.

MAIN THEOREM. *Let* $x : M \to \boldsymbol{R}^3$ *be a minimal surface immersed in* $\boldsymbol{R}^3$. *Suppose that, for some fixed distinct values* $\alpha_1, \cdots, \alpha_q$ *and some positive integers* $m_1, \cdots, m_q$, *the Gauss map* g *of* M *does not take the value* α_j *with multiplicity less than* m_j *for each* j *and that*

$$\gamma := \sum_{j=1}^q \left(1 - \frac{1}{m_j}\right) > 4.$$

Set

$$L := \min\{|\alpha_i, \alpha_j| \,;\, 1 \leqq i < j \leqq q\}.$$

Then, there exists some positive constant C_4 not depending on any data M, α_j's, m_j's such that

$$(7) \qquad |K(p)|^{1/2} \leqq \frac{C_4}{d(p)} \frac{\log^2(1/L)}{L^3} \qquad (p \in M).$$

The proof of Main Theorem will be given in the next section. Here, we note that, for the proof of Main Theorem it suffices to show the existence of a constant satisfying (7) which may depend on the given data $m_1, \cdots, m_q$. For, if

$$(8) \qquad \sum_{j \in I}\left(1-\frac{1}{m_j}\right) > 4$$

for some proper subset I of $\{1, 2, \cdots, q\}$, then the assumption for $\{\alpha_j; 1 \leqq j \leqq q\}$ can be replaced by the assumption for $\{\alpha_j; j \in I\}$. Moreover, we may replace each m_j by m_j^* such that

$$(9) \qquad m_j^* \leqq m_j, \qquad \sum_j\left(1-\frac{1}{m_j^*}\right) > 4.$$

Therefore, after suitable changes of indices and m_j's, we may assume that (8) and (9) do not hold for any proper subset I of $\{1, 2, \cdots, q\}$ and any m_j^*'s which are different from m_j's. Then, we have $\gamma \leqq 4+1/2$. Because, otherwise, the maximum m_{j_0} of m_j's is not less than 3 and (9) holds for $m_{j_0}^* := m_{j_0}-1$ and $m_j^* := m_j (j \neq j_0)$. On the other hand, since $m_j \geqq 2$ for all j, we have $9/2 \geqq \gamma = \sum_j (1-1/m_j) \geqq q/2$ and so $q \leqq 9$. For each q with $4<q \leqq 9$ there are only finitely many possible cases of m_j's which satisfy the inequality $\gamma = q - \sum_{j=1}^q 1/m_j \leqq 9/2$ by virtue of the above assumption. As the desired constant C_4 satisfying the inequality (7), we can take the maximum of constants C_4's which are chosen for these finitely many cases of m_j's.

Theorem C stated in § 1 is an immediate consequence of Main Theorem. In fact, under the assumption of Theorem C, if we take $m_1 = \cdots = m_5 = 6$, then the Gauss map $g := \pi \cdot G$ satisfies all conditions in Main Theorem for the values $\alpha_j := \pi(n_j)$ and these m_j's.

Now, for an arbitrarily given $\varepsilon > 0$ we give an example of a family of minimal surfaces which shows that there is no positive constant C not depending on each minimal surface which satisfies the condition

$$(10) \qquad |K(p)|^{1/2} \leqq \frac{C}{d(p)} \frac{1}{L^{3-\varepsilon}}.$$

To this end, for each positive number $R (\geqq 1)$ we take five points

$$\alpha_1 := R, \quad \alpha_2 := \sqrt{-1}R, \quad \alpha_3 := -R, \quad \alpha_4 := -\sqrt{-1}R, \quad \alpha_5 := \infty$$

in $\bar{C}$. Consider Enneper surface M whose domain of definiton is restricted to the disc of radius R. Namely, for the functions $f(z)\equiv 1$ and $g(z)=z$ on the disc $\Delta_R:=\{z;\ |z|<R\}$ setting

$$x_1 := \mathrm{Re}\int_0^z f(1-g^2)dz, \quad x_2 := \mathrm{Re}\int_0^z \sqrt{-1}f(1+g^2)dz, \quad x_3 := 2\mathrm{Re}\int_0^z fg dz,$$

we define the surface $x=(x_1, x_2, x_3)$: $\Delta_R \to \boldsymbol{R}^3$ in $\boldsymbol{R}^3$. Then, this is a minimal surface immersed in $\boldsymbol{R}^3$ whose Gauss map is the function g and whose metric is given by $ds^2=(1+|z|^2)^2|dz|^2$ (cf. [**13**]). Consider the quantities $K(p)$ and $d(p)$ as in Main Theorem at the point $p=0$. We have

$$d(0) = \int_0^R (1+x^2)dx = R+\frac{1}{3}R^3$$

and

$$|K(0)|^{1/2} = \frac{2|g'(0)|}{|f(0)|(1+|g(0)|^2)^2} = 2.$$

On the other hand, the quantity L for the points in S^2 corresponding to α_j's is given by $L=1/\sqrt{1+R^2}$ and so

$$|K(0)|^{1/2}d(0)L^{3-\varepsilon} = \frac{2(R+(1/3)R^3)}{(1+R^2)^{(3-\varepsilon)/2}},$$

which converges to $+\infty$ as R tends to $+\infty$. Therefore, there is no positive constant satisfying the condition (10) which does not depend on each minimal surface.

§ 5. The proof of Main Theorem.

We consider a minimal surface $x:=(x_1, x_2, x_3)$: $M \to \boldsymbol{R}^3$ immersed in R^3 whose Gauss map $g: M \to \bar{\boldsymbol{C}}$ satisfies the assumption of Main Theorem for $\alpha_1, \cdots, \alpha_q$ and integers $m_1, \cdots, m_q$ with $m_j \geqq 2$. We may assume that M is non-flat, or g is not a constant. For, otherwise, Main Theorem is trivial. Moreover, we may assume $\alpha_q=\infty$.

Taking a holomorphic local coordinate z, we set $\phi_i:=(\partial/\partial z)x_i (i=1, 2, 3)$. Then we have $g=\phi_3/(\phi_1-\sqrt{-1}\phi_2)$ and the induced metric on M is given by $ds^2=|f_z|^2(1+|g|^2)^2|dz|^2$ for the holomorphic function $f_z:=\phi_1-\sqrt{-1}\phi_2$, where f_z has a zero of order $2m$ at each point where g has a pole of order m (cf. e.g., [**13**]).

Now, we choose some δ such that

$$\gamma-4 > 2\gamma\delta > 0 \tag{11}$$

and set

$$\eta := \frac{\gamma-4-2\gamma\delta}{\gamma}, \qquad \tau := \frac{2}{2+\gamma\delta}.$$

Then, if we choose a sufficiently small positive δ depending only on γ, for the constant $\varepsilon_0 := (\gamma-4)/2\gamma$ we have

$$(12) \qquad 0 < \tau < 1, \qquad \frac{\varepsilon_0 \tau}{1-\tau} > 1 .$$

We consider a new metric

$$(13) \qquad d\sigma^2 = |f_z|^{2/(1-\tau)} \Big(\frac{1}{|g'_z|} \prod_{j=1}^{q-1} \Big(\frac{|g-\alpha_j|}{(1+|\alpha_j|^2)^{1/2}} \Big)^{\eta_j(1-\eta-\delta)} \Big)^{2\tau/(1-\tau)} |dz|^2 ,$$

where $\eta_j := 1-1/m_j$ and g'_z denotes the derivative of g with respect to the holomorphic local coordinate z. This is a well-defined metric on the set

$$M' := \{p \in M ;\ g'_z(p) \neq 0 \ \text{ and } \ g(p) \neq \alpha_j \ \text{ for all } j\} .$$

In fact, if we choose another holomorphic local coordinate ζ, we have $f_z = f_\zeta d\zeta/dz$ and $g'_z = g'_\zeta d\zeta/dz$ and therefore $d\sigma^2$ remains unchanged.

Our purpose is to show the inequality (7) for each point $p \in M$. We may assume that $p \in M'$. Since $d\sigma^2$ is flat on M', there is a map Φ of Δ_R onto a neighborhood U of p which is an isometry with respect to the metrics $|dz|^2$ on Δ_R and $d\sigma^2$ on U. We take the largest $R (\leqq +\infty)$ such that there is a local isometry Φ of Δ_R onto an open set in M' with $\Phi(0)=p$. For brevity, we denote here the function $g \cdot \Phi$ on Δ_R by g. According to Corollary 7, we have

$$(14) \qquad R \leqq 2C_2 \frac{1+|g(0)|^2}{|g'_z(0)|} \prod_{j=1}^{q} |g(0), \alpha_j|^{\eta_j(1-\eta-\delta)} < +\infty$$

for the constant C_2 given in Corollary 7. Then, there is some point w_0 with $|w_0| = R$ such that, for the line segment

$$\Gamma :\ w = t w_0 \qquad (0 \leqq t < 1),$$

the image $\gamma := \Phi(\Gamma)$ tends to the boundary of M' as t tends to 1. In this situation, suppose that γ tends to a point p_0 where $g'(p_0)=0$ or $g(p_0)=\alpha_j$ for some j. Taking a holomorphic local coordinate ζ with $\zeta(p_0)=0$ in a neighborhood of p_0, we write the metric $d\sigma^2$ as $d\sigma^2 = |\zeta|^{2a\tau/(1-\tau)} w |d\zeta|^2$ with some positive C^∞ function w and some real number a. If $g-\alpha_j$ has a zero of order $m (\geqq m_j)$ at p_0 for some $j \leqq q-1$, then g'_z has a zero of order $m-1$ at p_0 and $f_z(p_0) \neq 0$. In this case,

$$a = m\Big(1-\frac{1}{m_j}\Big)(1-\eta-\delta) - (m-1) \leqq -(\eta+\delta) \leqq -\varepsilon_0 .$$

For the case where g has a pole of order $m (\geqq m_q)$ at p_0, g'_z has a pole of order $m+1$ and f_z has a zero of order $2m$ at p_0. Then, we have also

$$a = \frac{2m}{\tau} + m + 1 - m(\gamma-\eta_q)(1-\eta-\delta) \leqq -\varepsilon_0 .$$

Moreover, for the case where $g_z'(p_0)=0$ and $g(p_0)\neq\alpha_j$ for any j, then $a\leqq-1$. Therefore, $d\sigma\geqq C_4|\zeta|^{-\varepsilon_0\tau/(1-\tau)}|d\zeta|$ for a positive constant C_4 in a neighborhood of p_0. By (12) we have

$$R=\int_\Gamma d\sigma\geqq C_4\int_\Gamma\frac{1}{|\zeta|^{\varepsilon_0\tau/(1-\tau)}}|d\zeta|=+\infty,$$

which contradicts (14). So, γ tends to the boundary of M as t tends to 1.

To estimate the length of γ, we shall study the metric Φ^*ds^2 on Δ_R. For local considerations, the coordinate z on Δ_R may be considered as a holomorphic local coordinate on M' and so we may write $d\sigma^2=|dz|^2$. By (13) we obtain

$$1=|f_z|^{2/(1-\tau)}\Big(\frac{1}{|g_z'|}\prod_{j=1}^{q-1}\Big(\frac{|g-\alpha_j|}{(1+|\alpha_j|^2)^{1/2}}\Big)^{\eta_j(1-\eta-\delta)}\Big)^{2\tau/(1-\tau)}$$

and hence

$$|f_z|=\Big(|g_z'|\prod_{j=1}^{q-1}\Big(\frac{(1+|\alpha_j|^2)^{1/2}}{|g-\alpha_j|}\Big)^{\eta_j(1-\eta-\delta)}\Big)^{\tau}. \tag{15}$$

By the use of Corollary 7 we have

$$\begin{aligned}\Phi^*ds&=|f_z|(1+|g|^2)|dz|\\&=\Big(|g_z'|(1+|g|^2)^{1/\tau}\prod_{j=1}^{q-1}\Big(\frac{(1+|\alpha_j|^2)^{1/2}}{|g-\alpha_j|}\Big)^{\eta_j(1-\eta-\delta)}\Big)^{\tau}|dz|\\&=\Big(\frac{|g_z'|}{1+|g|^2}\frac{1}{\prod_{j=1}^{q}|g,\alpha_j|^{\eta_j(1-\eta-\delta)}}\Big)^{\tau}|dz|\\&\leqq C_2^{\tau}\Big(\frac{2R}{R^2-|z|^2}\Big)^{\tau}|dz|.\end{aligned}$$

This yields that

$$\begin{aligned}d(p)&\leqq\int_\gamma ds=\int_\Gamma\Phi^*ds\leqq C_2^{\tau}\int_\Gamma\Big(\frac{2R}{R^2-|z|^2}\Big)^{\tau}|dz|\\&=C_2^{\tau}\int_0^R\Big(\frac{2R}{R^2-x^2}\Big)^{\tau}dx\leqq\frac{(2C_2)^{\tau}R^{1-\tau}}{1-\tau}.\end{aligned}$$

By (14) we obtain

$$d(p)\leqq\frac{2C_2}{1-\tau}\Big(\frac{(1+|g(0)|^2)\prod_{j=1}^{q}|g(0),\alpha_j|^{\eta_j(1-\eta-\delta)}}{|g_z'(0)|}\Big)^{1-\tau}.$$

On the other hand, in view of (15) the curvature at p is given by

$$\begin{aligned}|K(p)|^{1/2}&=\frac{2|g_z'(0)|}{|f_z|(1+|g(0)|^2)^2}\\&=\frac{2|g_z'(0)|}{(1+|g(0)|^2)^2}\Big(\frac{(1+|g(0)|^2)^{\gamma(1-\eta-\delta)/2}\prod_{j=1}^{q}|g(0),\alpha_j|^{\eta_j(1-\eta-\delta)}}{|g_z'(0)|}\Big)^{\tau}.\end{aligned}$$

Since $|g, \alpha_j| \leqq 1$, we can easily conclude that

$$|K(p)|^{1/2} d(p) \leqq C_5 := \frac{4C_2}{1-\tau}.$$

By the definition of C_2 and τ, we see

$$C_5 = \frac{4a_0^{\gamma\delta/2} C_3(2+\gamma\delta)}{\delta^{\gamma(1-\eta)}\gamma\delta((L/2)\log(4a_0/L^2))^{\gamma-1-\gamma\eta}}.$$

Now, take a sufficiently small L_0 such that (11) and (12) hold for the constant $\delta = 1/\log(4a_0/L_0^2)$. For each positive $L(\leqq 1)$ we set $\delta := 1/\log(4a_0/L^2)$ if $L \leqq L_0$ and $\delta := \delta_0$ for some δ_0 satisfying the conditions (11) and (12) if $L_0 < L \leqq 1$. We can apply the above-mentioned arguments to these δ's. Then, we can estimate the constant C_5 as

$$C_5 \leqq 2^{\gamma-\gamma\eta} C_3 a_0^{\gamma\delta/2} \max(1, A_0) \frac{\log^2(4a_0/L^2)}{L^{\gamma-1-\gamma\eta}},$$

where

$$A_0 := \sup_{L_0 \leqq x \leqq 1} \left(\frac{1}{\delta_0 \log(4a_0/x^2)}\right)^{\gamma+1-\gamma\eta}.$$

Since a_0 can be chosen so as to be between two positive constants depending only on γ, we can conclude

$$C_5 \leqq \frac{C_6 \log^2(4a_0/L_2)}{L^{\gamma-1-\gamma\eta}} \leqq C_7 \frac{\log^2(1/L)}{L^3 L^{2\gamma\delta}}$$

for positive constants C_6 and C_7 depending only on γ_1. On the other hand, the factor $L^{2\gamma\delta}$ is bounded from below by a positive constant not depending on each L because $\log L^{2\gamma\delta} = 2\gamma \log L/\log(4a_0/L^2)$ has a limit as L tends to zero. This shows that C_7 can be replaced by a positive constant depending only on m_j's. The proof of Main Theorem is complete.

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