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Some inequalities for minimal fibrations of surfaces of general type over curves

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Introduction.

Let $g: Y \rightarrow C$ be a surjective morphism from a smooth complex projective 3-fold onto a smooth curve. Assume that a generic fibre of g is an irreducible surface of general type. Then, composing divisorial contractions and flips, we can birationally modify Y into X, a normal, projective, $\boldsymbol{Q}$-factorial variety with only terminal singularities, in such a way that g induces a morphism $f: X \rightarrow C$, with K_X being f-nef [**Mo2**], [**Ka3**]. We call f a (relatively) minimal fibration of surfaces of general type over C. Since X is a $\boldsymbol{Q}$-factorial 3-fold, K_X^3 is a well-defined rational number which is independent of the choice of the relatively minimal model X. The aims of this article are (1) to estimate K_X^3 from below in terms of other geometric invariants and (2) to describe the structure of X when K_X^3 is small.

MAIN THEOREM 1. *Let $f: X \rightarrow C$ be a minimal fibration of surfaces of general type over C, a smooth projective curve of genus b. Let F be a general fibre of f.*

(1) *If $p_g(F) \geqq 3$ and $|K_F|$ is not composed of a pencil, then*

$$K_X^3 \geqq \frac{4(p_g(F)-2)}{p_g(F)}\left\{\frac{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+4\chi(\mathcal{O}_F)}{2(p_g(F)-2)}(b-1)-\chi(\mathcal{O}_X)\right\}$$

or equivalently,

$$b \leqq 1+\frac{p_g(F)\left\{K_X^3+\frac{4(p_g(F)-2)}{p_g(F)}\chi(\mathcal{O}_X)\right\}}{2\{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+4\chi(\mathcal{O}_F)\}}.$$

(2) *If $|K_F|$ is composed of a pencil and F is not a surface with $K_F^2=1$, $p_g(F)=2$, $q(F)=0$, then*

$$K_X^3 \geqq \frac{4(p_g(F)-1)}{p_g(F)}\left\{\frac{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+2\chi(\mathcal{O}_F)}{2(p_g(F)-1)}(b-1)-\chi(\mathcal{O}_X)\right\}$$

or equivalently,

$$b \leqq 1+\frac{p_g(F)\left\{K_X^3+\frac{4(p_g(F)-1}{p_g(F)}\chi(\mathcal{O}_X)\right\}}{2\{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+2\chi(\mathcal{O}_F)\}}.$$

(3) *If* $K_F^2=1$, $p_g(F)=2$ *and* $q(F)=0$, *then*

$$K_X^3 \geqq 3(b-1)-\chi(\mathcal{O}_X)$$

or equivalently,

$$b \leqq 1+\frac{K_X^3+\chi(\mathcal{O}_X)}{3}.$$

(4) *If* $p_g(F)=1$, *then*

$$K_X^3 \geqq K_F^2\{(6-\chi(\mathcal{O}_F))(b-1)-\chi(\mathcal{O}_X)\}$$

or equivalently,

$$b \leqq 1+\frac{K_X^3+K_F^2\chi(\mathcal{O}_X)}{K_F^2(6-\chi(\mathcal{O}_F))}.$$

(5) *If* $p_g(F)=0$, *then*

$$K_X^3 \geqq \begin{cases} 6K_F^2(b-1)+(2/3)\cdot l(2) & \text{when } K_F^2\geqq 2 \\ 6(b-1)+(6/13)\cdot l(2) & \text{when } K_F^2=1. \end{cases}$$

When the equality holds in one of the five cases above, f is isotrivial or, equivalently, two general fibres are isomorphic. Here, $l(2)$ denotes the correction term in the plurigenera formula of Reid-Fletcher for X; for the precise definition, see [**F1**,Definition 2.6].

MAIN THEOREM 2. *With the same notation as above, assume that*

$$K_X^3 < 2(3K_F^2-2\chi(\mathcal{O}_F))(b-1)-4\chi(\mathcal{O}_X).$$

Then a general fibre F has one of the following properties:

(1) *F carries a linear pencil of curves of genus two.*

(2) $K_F^2 \leqq 2p_g(F)-1$.

(3) $K_F^2=2p_g(F)$, $p_g(F)\geqq 3$, $q(F)\leqq 2$, *and* $|K_F|$ *is not composed of a pencil.*

(4) $|K_F|$ *is not composed of a pencil and*

(4*a*) $K_F^2=8$, $p_g(F)=3$, $q(F)\leqq 1$, *or*

(4*b*) $K_F^2=9$, $p_g(F)=4$, $q(F)\leqq 1$, *or*

(4*c*) $K_F^2=7$, $p_g(F)=3$, $q(F)\leqq 2$.

(5) $K_F^2=4$ *or* 5, $p_g(F)=2$, *and the movable part of* $|K_F|$ *is a linear pencil of curves of genus three with only one base point.*

(6) $K_F^2=2$ *or* 3 *and* $p_g(F)=1$.

(7) $p_g(F)=0$.

REMARK. More precisely, we have $p_g(F)=3$, if $q(F)=2$ in (3). Indeed, suppose $p_g(F)\geqq 4$. Since we have $K_F^2<3\chi(\mathcal{O}_F)$, F has a pencil of genus 2 or 3 over a curve of genus 2 by [**Ho2**, Theorem 3.1] and we have $K_F^2\geqq 2\chi(\mathcal{O}_F)+6$, $K_F^2\geqq(8/3)\cdot(\chi(\mathcal{O}_F)+4)$ respectively, which is absurd.

Our results are three-dimensional analogues of Xiao's result [**X**] in the geography of surfaces. Let $f: S\to C$ be a surjective morphism from a smooth projective surface onto a curve of genus b. We assume that a general fibre is a connected curve of genus $g\geqq 2$ and that S is relatively minimal (i.e., all fibres of f contain no (-1)-curves).

When general fibres of f are hyperelliptic, S is realized as a double covering of a ruled surface over C from which E. Horikawa [**Ho2**] and U. Persson [**P**] independently derived the inequality:

$$K_S^2 \geqq \frac{4(g-1)}{g}\{\chi(\mathcal{O}_S)+(g+1)(b-1)\}. \tag{1}$$

For a general minimal fibration, G. Xiao [**X**] introduced a new idea to show:

$$K_{S/C}^2 \geqq \frac{4(g-1)}{g}\deg f_*\omega_{S/C} \tag{2}$$

which reduces to (1) for a hyperelliptic fibration.

Our method in this paper essentially follows Xiao's idea: the analysis of the sheaf $f_*\omega_{X/C}$ via its Harder-Narasimhan filtration. In § 1, we generalize his technical lemmas (Lemmas 1.2 and 1.3 below). With the aid of Miyaoka's lemma [**Mi**], our proof is simpler than the original one [**X**], and the same idea yields a higher-dimensional version of a theorem of Arakelov (Theorem 1.4) as well as an inequality of the Miyaoka-Yau type (Corollary 1.7), when combined with Y. Kawamata's two Theorems: the Base Point Free Theorem [**KMM**, Theorem 3-1-1] and the semipositivity theorem [**Ka1**, Theorem 1].

In § 2, we prove three-dimensional analogues of Xiao's inequality (2) (Propositions 2.1, 2.6 and 2.7), from which we derive Main Theorem 1. Note that every minimal fibration that attains the lower bound of K_X^3 is isotrivial, while this is not the case in the surface case.

In § 3, we show another inequality (Proposition 3.1) with some exceptions that are explicitly described. Main Theorem 2 is a direct consequence of this result.

Finally, we note that our results are related to a work of B. Hunt [**Hu**].

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Notation and Convention.

In this paper, we work over the complex number field $\boldsymbol{C}$ and follow the notation and terminology of [**KMM**].

Let X be a normal variety and $f: X \to C$ a proper surjective morphism onto a smooth curve C. By $K_{X/C}$, we denote the Weil divisor $K_X - f^*K_C$. For every integer m, $\omega_{X/C}^{[m]}$ denotes the double dual of the m-th tensor product of the relative dualizing sheaf $\omega_{X/C}$. We have $\omega_{X/C}^{[m]} \cong \mathcal{O}(mK_{X/C})$, the reflexive sheaf attached to $mK_{X/C}$. Let D be a Cartier divisor on X and F a general fibre of f. We denote by D_F the restriction of D to F.

For a vector bundle $\mathcal{E}$ on C, define $\delta(\mathcal{E}) \in H^2(C, \boldsymbol{Q})$ and $\mu(\mathcal{E}) \in \boldsymbol{Q}$ as follows:

$$\delta(\mathcal{E}) := \frac{c_1(\mathcal{E})}{\operatorname{rank} \mathcal{E}}$$

$$\mu(\mathcal{E}) := \deg \delta(\mathcal{E}) .$$

There is a unique filtration

$$0 = \mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n = \mathcal{E} ,$$

such that $\mathcal{E}_i/\mathcal{E}_{i-1}$ is a semistable vector bundle and that

$$\mu(\mathcal{E}_i/\mathcal{E}_{i-1}) > \mu(\mathcal{E}_{i+1}/\mathcal{E}_i)$$

for all i [**H-N**]. We call this the Harder-Narasimhan filtration of $\mathcal{E}$. We define $\delta_-(\mathcal{E}) \in H^2(C, \boldsymbol{Q})$ and $\mu_-(\mathcal{E}) \in \boldsymbol{Q}$ as follows:

$$\delta_-(\mathcal{E}) := \delta(\mathcal{E}/\mathcal{E}_{n-1})$$

$$\mu_-(\mathcal{E}) := \deg \delta_-(\mathcal{E}) .$$

Let $p_{\mathcal{E}}: P(\mathcal{E}) \to C$ be the projective bundle associated with $\mathcal{E}$ and $L_{\mathcal{E}}$ the divisor class on $P(\mathcal{E})$ associated with the tautological line bundle $\mathcal{O}_{P(\mathcal{E})}(1)$.

When $\operatorname{rank} \mathcal{E} = 1$, we identify $P(\mathcal{E})$ with C, and $\mathcal{O}_{P(\mathcal{E})}(1)$ with $\mathcal{E}$.

The following symbols will be used in this article:

$\sim_{\text{lin}}$: linear equivalence.

$\sim_{\boldsymbol{Q}}$: $\boldsymbol{Q}$-linear equivalence.

$\sim_{\text{alg}}$: algebraic equivalence.

$\sim_{\text{num}}$: numerical equivalence.

§ 1. Preliminaries.

Let X be a normal $\boldsymbol{Q}$-factorial variety of dimension d and $f: X \to C$ a proper morphism with connected fibres onto a nonsingular complete curve.

For any Weil divisor D, $f_*\mathcal{O}(D)$ is a vector bundle, since C is a curve. Assume that $f_*\mathcal{O}(D) \neq 0$ and let $\mathcal{F}$ be any non-zero vector subbundle of $f_*\mathcal{O}(D)$. The natural homomorphism $f^*\mathcal{F} \to \mathcal{O}(D)$ yields a rational section $\tilde{\phi}: X \dashrightarrow \boldsymbol{P}(f_*\mathcal{F})$ and $\phi: X \dashrightarrow \boldsymbol{P}(\mathcal{F})$ such that $p_{\mathcal{F}} \circ \phi = f$.

The indeterminacy of ϕ is described by the following lemma, the proof of which was suggested by Y. Kawamata.

LEMMA 1.1. *In the above situation, there is a desingularization $\mu: Y \to X$ such that* (1) $\lambda := \phi \circ \mu : Y \to P(\mathcal{F})$ *is a morphism and that* (2) $\lambda^* L_{\mathcal{F}} \sim_{\boldsymbol{Q}} \mu^*(D-Z) - E$, *where Z is an effective divisor on X and E is an effective $\boldsymbol{Q}$-divisor on Y exceptional with respect to μ.*

PROOF. Take a Weil divisor $\boldsymbol{Z}$ on X such that the homomorphism $f^*\mathcal{F} \to \mathcal{O}(D-Z)$ is surjective in codimension 1, and take a positive integer m such that $m(D-Z) \in \operatorname{Div}(X)$. We note here that the induced homomorphism $S^m(f^*\mathcal{F}) \to \mathcal{O}(m(D-Z))$ is surjective in codimension 1 and corresponds to the rational map $\phi \circ i : X \dashrightarrow P(S^m(\mathcal{F}))$, where $i: P(\mathcal{F}) \to P(S^m(\mathcal{F}))$ is the relative m-uple embedding. Take any desingularization $\mu_1 : Y_1 \to X$ and take an effective divisor E_1 on Y_1 which is exceptional with respect to μ_1 such that the homomorphism $\mu_1^* S^m(f^*\mathcal{F}) \to \mu_1^* \mathcal{O}(m(D-Z)) \otimes \mathcal{O}(-E_1)$ is surjective in codimension 1. By Hironaka's Theorem ([**Hi**]), there are a projective birational morphism $\mu_2 : Y_2 \to Y_1$ from a nonsingular variety Y_2 and an effective divisor E_2 on Y_2 which is exceptional with respect to μ_2 such that the induced homomorphism

$$\mu_2^* \mu_1^* S^m(f^*\mathcal{F}) \longrightarrow \mu_2^* \mu_1^* \mathcal{O}(m(D-Z)) \otimes \mathcal{O}(-\mu_2^* E_1) \otimes \mathcal{O}(-E_2)$$

is surjective. Define E, $M \in \operatorname{Div}(Y_2) \otimes \boldsymbol{Q}$ as follows:

$$E := \frac{1}{m}(E_2 + \mu_2^* E_1), \qquad M := \mu^*(D-Z) - E,$$

where $\mu := \mu_1 \circ \mu_2$.

Corresponding to the surjection $\mu^* f^* S^m(\mathcal{F}) \to \mathcal{O}(mM)$, we have a morphism $\rho : Y_2 \to P(S^m(\mathcal{F}))$ such that $\mathcal{O}_{Y_2}(mM) = \rho^* L_{S^m(\mathcal{F})}$. If necessary, we may blow up Y_2 and assume that the induced rational map $\lambda : Y_2 \dashrightarrow P(\mathcal{F})$ is a morphism. We note here $\rho = i \circ \lambda$ and $i^* L_{S^m(\mathcal{F})} = m L_{\mathcal{F}}$. Hence

$$mM = \rho^* L_{S^m(\mathcal{F})} = \lambda^* i^* L_{S^m(\mathcal{F})} = m \lambda^* L_{\mathcal{F}},$$

so we obtain $\lambda^* L_{\mathcal{F}} \sim_{\boldsymbol{Q}} M$, which is the desired result if we take Y_2 as Y. □

DEFINITION. In the above situation, put

$$M_Y(D, \mathcal{F}) := \lambda^* L_{\mathcal{F}} \in \mathrm{Div}\,(Y)$$

$$Z_Y(D, \mathcal{F}) := \mu^* Z + E \in \mathrm{Div}\,(Y) \otimes \boldsymbol{Q}$$

$$N_Y(D, \mathcal{F}) := M_Y(D, \mathcal{F}) - g^* \delta_-(\mathcal{F}) \in \mathrm{Div}\,(Y) \otimes \boldsymbol{Q}$$

where $g := f \circ \mu$.

REMARK.

1. We note that $Z_Y(D, \mathcal{F})$ is effective and for any nonzero vector subbundle $\mathcal{F}'$ of $\mathcal{F}$, we have $Z_Y(D, \mathcal{F}') \geqq Z_Y(D, \mathcal{F})$.
2. By Lemma 1.1,

$$Z_Y(D, \mathcal{F}) \sim_{\boldsymbol{Q}} \mu^* D - \lambda^* L_{\mathcal{F}},$$

$$N_Y(D, \mathcal{F}) \sim_{\boldsymbol{Q}} \mu^* D - Z_Y(D, \mathcal{F}) - g^* \delta_-(\mathcal{F}).$$

3. Let W be the image of λ. Since we have

$$g_* \mathcal{O}(M_Y(D, \mathcal{F})) = (p_{\mathcal{F}}|_W)_* (\mathcal{O}(L_{\mathcal{F}})|_W \otimes (\lambda|_W)_* \mathcal{O}_Y),$$

there is an inclusion:

$$\mathcal{F} \cong (p_{\mathcal{F}}|_W)_* \mathcal{O}(L_{\mathcal{F}})|_W \longrightarrow g_* \mathcal{O}(M_Y(D, \mathcal{F})),$$

induced by the natural inclusion $\mathcal{O}_W \to (\lambda|_W)_* \mathcal{O}_Y$. In particular, we have

$$h^0(\mathcal{O}_{F_1}(M_Y(D, \mathcal{F})_{F_1})) \geqq \mathrm{rank}\, \mathcal{F},$$

where F_1 is a general fibre of g.

The following Lemmas 1.2, 1.3 are generalizations of [**X**, Lemma 2, Lemma 3].

LEMMA 1.2. *Let Y and F_1 be as above, and let $\tilde{D}$ be a $\boldsymbol{Q}$-divisor on Y. Let*

$$Z_1 \geqq Z_2 \geqq \cdots \geqq Z_{n+1} := 0$$

be a sequence of effective $\boldsymbol{Q}$-divisors on Y, and let

$$\mu_1 \geqq \mu_2 \geqq \cdots \geqq \mu_{n+1} := 0$$

be a sequence of rational numbers such that $N_i := \tilde{D} - Z_i - \mu_i F_1$ is nef $\boldsymbol{Q}$-divisors for every i. Then, $\tilde{D}^2 - \sum_{i=1}^n (N_i F_1 + N_{i+1} F_1)(\mu_i - \mu_{i+1})$ satisfies

$$\Big(\tilde{D}^2 - \sum_{i=1}^n (N_i F_1 + N_{i+1} F_1)(\mu_i - \mu_{i+1})\Big) A_1 A_2 \cdots A_{d-2} \geqq 0$$

for arbitrary $d-2$ nef divisors $A_1, A_2, \cdots, A_{d-2}$.

PROOF. A similar argument as in [**X**, proof of Lemma 2] applies. □

LEMMA 1.3. *$N_Y(D, \mathcal{F})$ is nef.*

PROOF. When rank $\mathcal{F} \geqq 2$, $L_{\mathcal{F}} - p_{\mathcal{F}}^* \delta_-(\mathcal{F})$ is nef by [**Mi**, Corollary 3.5]. Hence

$$N_Y(D, \mathcal{F}) = M_Y(D, \mathcal{F}) - g^* \delta_-(\mathcal{F}) = \lambda^*(L_{\mathcal{F}} - p_{\mathcal{F}}^* \delta_-(\mathcal{F}))$$

is nef. When rank $\mathcal{F}=1$, we have $M_Y(D, \mathcal{F}) = g^* c_1(\mathcal{F})$, hence

$$N_Y(D, \mathcal{F}) = g^* c_1(\mathcal{F}) - g^* \delta_-(\mathcal{F}) \sim_{\text{num}} 0$$

which is obviously nef. □

THEOREM 1.4. *Let X be a projective, normal, $\boldsymbol{Q}$-factorial variety of dimension d with only terminal singularities and $f: X \to C$ a proper surjective morphism with connected fibres onto a nonsingular complete curve C. Assume that K_X is f-nef, and general fibres of f are of general type. Then there is a positive integer m_0 such that for any positive integer $m \geqq m_0$, $mK_{X/C} - f^* \delta_-(f_* \omega_{X/C}^{[m]})$ is nef. In particular, $K_{X/C}$ is nef.*

PROOF. By the Base Point Free Theorem (cf. [**KMM**, Theorem 3-1-1]), there is a positive integer m_0 such that a natural homomorphism $f^* f_* \mathcal{O}(mK_{X/C}) \to \mathcal{O}(mK_{X/C})$ is surjective for all $m \geqq m_0$. Thus Lemma 1.3 applies with $D := mK_{X/C}$, $Y := X$, $\mathcal{F} := f_* \omega_{X/C}^{[m]}$, $Z_Y(D, \mathcal{F}) := 0$ to show that $mK_{X/C} - f^* \delta_-(f_* \omega_{X/C}^{[m]})$ is nef. As for the last statement, we only have to show $\mu_-(f_* \omega_{X/C}^{[m]}) \geqq 0$ for all positive integer m, which is the semipositivity of $f_* \omega_{X/C}^{[m]}$([**Ka1**, Theorem 1]). □

REMARK. When $d=2$, the last assertion of Theorem 1.4 is known as Arakelov's Theorem (see [**Be2**]).

COROLLARY 1.5. *Let things be as in Theorem* 1.4. *Then*

$$K_X^d \geqq 2d(b-1) K_F^{d-1},$$

where b is genus of C, and F is a general fibre of f. When equality holds, f is isotrivial.

PROOF. The inequality is a direct consequence of Theorem 1.4, if one notes that $K_X = K_{X/C} + f^* K_C$, $K_F = K_X|_F + F|_F$. The second statement follows from:

LEMMA 1.6 ([**Ko4**]). *Let $f: X \to C$ as in Theorem* 1.4, *and assume that f is non-isotrivial. Then there is a positive integer m, such that for any positive integer k, $f_* \omega_{X/C}^{[km]}$ is ample. In particular, we have $\mu_-(f_* \omega_{X/C}^{[km]}) > 0$.*

PROOF. See for example [**Ko4**], [**Mo1**]. □

Combined with [**Mi**, Theorem 1.1], Theorem 1.4 implies the following:

COROLLARY 1.7. *Assume that* $d=3$, $b\geqq 1$. *Let* $\rho: Y\to X$ *be a desingularization and* g *the induced morphism* $Y\to C$. *Put* $c_2(X)K_X:=c_2(Y)\rho^*K_X$. *Then*

$$K_X^3\leqq 3c_2(X)K_X-2(b-1)(3c_2(F)-K_F^2).$$

PROOF. Let F_1 be a general fibre of g. Since the singular locus of X is isolated and the normal bundle of F_1 is trivial, we have $c_2(Y)F_1=c_2(F_1)$, and ρ induces an isomorphism between F_1 and F. Hence

$$\begin{aligned}0 &\leqq (3c_2(Y)-c_1(Y)^2)\rho^*K_{X/C}=(3c_2(Y)-c_1(Y)^2)\rho^*K_X-(3c_2(Y)-c_1^2(Y))g^*K_C\\ &=3c_2(Y)\rho^*K_X-c_1(Y)^2\rho^*K_X-3c_2(Y)F_1(2b-2)+c_1(Y)^2F_1(2b-2)\\ &=3c_2(X)K_X-K_X^3-2(b-1)(3c_2(F)-K_F^2)\end{aligned}$$

which is the desired inequality. □

From the surface theory, we need the following lemma, which is essentially a direct consequence of a classical theorem of Clifford.

LEMMA 1.8 ([**Ho1**, Lemma 7.6], [**G**, Lemma 3.2]). *Let* S *be a smooth complete surface with* $\kappa(S)\geqq 0$, *and* M *a nef divisor so that* $|M|$ *is non-empty and not composed of a pencil, then we have:*

$$M^2\geqq 2h^0(\mathcal{O}(M))-4.$$

§2. Proof of the Main Theorem 1.

§2.1. Cases (1), (2), (3).

First we prove the following:

PROPOSITION 2.1. *Let the notation be as in the Main Theorem* 1.
(1) *If* $p_g(F)\geqq 3$ *and* $|K_F|$ *is not composed of a pencil, then*

$$K_{X/C}^3\geqq\frac{4(p_g(F)-2)}{p_g(F)}\deg f_*\omega_{X/C}.$$

(2) *If* $|K_F|$ *is composed of a pencil and* F *is not a surface with* $K_F^2=1$, $p_g(F)=2$, $q(F)=0$, *then*

$$K_{X/C}^3\geqq\frac{4(p_g(F)-1)}{p_g(F)}\deg f_*\omega_{X/C}.$$

(3) *If* $K_F^2=1$, $p_g(F)=2$ *and* $q(F)=0$, *then*

$$K_{X/C}^3\geqq\deg f_*\omega_{X/C}.$$

When equality in one of (1), (2) *and* (3) *holds,* f *is isotrivial.*

Let

$$0=:\mathcal{E}_0\subset\mathcal{E}_1\subset\cdots\subset\mathcal{E}_n:=f_*\omega_{X/C}$$

be the Harder-Narasimhan filtration of $f_*\omega_{X/C}$. For each $\mathcal{E}_i$, take a resolution of the indeterminary $\mu_i: Y_i\to X$ as in Lemma 1.1. Let Y be a nonsingular projective 3-fold which birationally dominates all the Y_i and μ the induced morphism from Y to X. Put

$$\begin{aligned} r_i &:= \operatorname{rank}\mathcal{E}_i\in \boldsymbol{N} & N_i &:= N_Y(K_{X/C},\mathcal{E}_i)\in \operatorname{Div}(Y)\otimes\boldsymbol{Q}\\ \mu_i &:= \mu_-(\mathcal{E}_i)\in\boldsymbol{Q} & Z_i &:= Z_Y(K_{X/C},\mathcal{E}_i)\in \operatorname{Div}(Y)\otimes\boldsymbol{Q}\\ & & M_i &:= M_Y(K_{X/C},\mathcal{E}_i)\in \operatorname{Div}(Y)\,. \end{aligned}$$

Then

$$\mu_1>\mu_2>\cdots>\mu_n\geqq 0\,,\qquad Z_1\geqq Z_2\geqq\cdots\geqq Z_n\geqq 0\,.$$

Applying Lemma 1.2 to the sequences

$$\{\mu_1,\mu_2,\cdots,\mu_n,\mu_{n+1}:=0\}\,,\qquad \{Z_1,Z_2,\cdots,Z_n,Z_{n+1}:=0\}\,,$$

we see that

$$(\mu^*K_{X/C})^2-\sum_{i=1}^n(\mu_i-\mu_{i+1})(N_iF_1+N_{i+1}F_1)$$

is pseudo-effective. Here F_1 is a general fibre of the induced morphism $g: Y\to C$. By Theorem 1.4, there is m_0 such that $mK_{X/C}-f^*\delta_-(f_*\omega_{X/C}^{[m]})$ is nef for any $m\geqq m_0$. Thus

$$(*)\qquad K_{X/C}^3\geqq\frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+\sum_{i=1}^n(\mu_i-\mu_{i+1})(\mu^*K_{X/C}N_iF_1+\mu^*K_{X/C}N_{i+1}F_1)$$

$$\geqq\sum_{i=1}^n(\mu_i-\mu_{i+1})(\tau^*K_FM_{iF_1}+\tau^*K_FM_{i+1F_1})\,,$$

where τ is the restriction of μ to F_1.

In the above situation, we have the following two lemmas.

LEMMA 2.2. *If $|M_{iF_1}|$ is composed of a pencil for some $i\leqq n$, then we have*

$$M_{iF_1}\tau^*K_F\geqq 2(r_i-1)$$

except for the case where F is a surface with $K_F^2=1$, $p_g(F)=2$, $q(F)=0$. When equality holds, either F has a linear pencil of genus two free from base points, or F is a surface with $K_F^2=2$, $p_g(F)=2$, $q(F)=0$.

PROOF. By the assumption, there are a smooth irreducible curve B_i on F_1, and positive integer a_i, so that $M_{iF_1}\sim_{\mathrm{alg}}a_iB_i$. We note here that $a_i\geqq r_i-1$ and if $a_i=r_i-1$, then $|B_i|$ is a linear pencil. Put $B_i':=\tau_*B_i$. We divide the proof into the following two cases: (*a*) $B_i'^2\geqq 1$, (*b*) $B_i'^2=0$.

First we consider the case (*a*). By the Hodge index theorem, we have

$(B_i\tau^*K_F)^2=(B_i'K_F)^2\geqq B_i'^2K_F^2$. If $K_F^2=1$, then we have $q(F)=0$ and $2\geqq p_g(F)\geqq h^0(M_{iF_1})\geqq 2$ ([**Bo**, Theorem 9, 11]), which was excluded. Hence $K_F^2\geqq 2$ and $B_i\tau^*K_F\geqq 2$, so that

$$M_{iF_1}\tau^*K_F = a_iB_i\tau^*K_F \geqq (r_i-1)B_i\tau^*K_F \geqq 2(r_i-1).$$

If $M_{iF_1}\tau^*K_F=2(r_i-1)$, then $B_i\tau^*K_F=B_i'K_F=2$ and $a_i=r_i-1$. In this case, we have $B_i'^2K_F^2\leqq 4$. From $K_F^2\geqq 2$, we have $B_i'^2\leqq 2$, but the case $B_i'^2=1$ is excluded, since $K_FB_i'+B_i'^2\equiv 0 \pmod 2$. Hence we obtain $K_F^2=2$, $B_i'^2=2$. Applying the Hodge index theorem again, we have $K_F\sim_{\mathrm{num}}B_i'$, but noting $K_F\geqq B_i'$, we deduce $K_F\sim_{\mathrm{lin}}B_i'$. Hence $p_g(F)=2$, and $q(F)=0$ ([**Bo**, Theorem 12]).

In the case (*b*), $|B_i'|$ is free from base points, so we may assume $\tau=id$, $F_1=F$, $B_i'=B_i$. Noting F is a surface of general type, we see that $K_FB_i=2g(B_i)-2\geqq 2$, where $g(B_i)$ is genus of B_i. Hence $M_{iF_1}\tau^*K_F\geqq 2(r_i-1)$.

When the equality holds, we have $g(B_i)=2$ and $a_i=r_i-1$. Thus we have proved Lemma 2.2. □

LEMMA 2.3. *Assume $|M_{iF_1}|$ is non-empty and not composed of a pencil for some $i\leqq n$.*

(1) *If $i=n$,*

$$M_{nF_1}\tau^*K_F \geqq 2r_n-4 .$$

When the equality holds, F is a surface with $K_F^2=2p_g(F)-4$.

(2) *If $i<n$,*

$$M_{iF_1}\tau^*K_F \geqq 2(r_i-1).$$

When the equality holds, F is a surface either with $K_F^2\leqq 2p_g(F)-1$ or with $K_F^2=8$, $p_g(F)=4$ and $r_i=3$.

PROOF. Noting $\tau^*K_F\geqq M_{iF_1}$, we can immediately get

$$M_{iF_1}\tau^*K_F \geqq M_{iF_1}^2 \geqq 2h^0(\mathcal{O}_{F_1}(M_{iF_1}))-4 \geqq 2r_i-4$$

by Lemma 1.8.

Suppose the equality holds in the case (1). Then by the Hodge index theorem, we have

$$(2r_i-4)^2 = (M_{iF_1}\tau^*K_F)^2 \geqq M_{iF_1}^2K_F^2 \geqq (2r_i-4)K_F^2 = (2p_g(F)-4)K_F^2$$

which implies $K_F^2=2p_g(F)-4$ by Noether's inequality.

Suppose that $i<n$ and that $M_{iF_1}\tau^*K_F\leqq 2r_i-3$. When $M_{iF_1}\tau^*K_F=2r_i-4$, we have

$$(2r_i-4)^2 = (M_{iF_1}\tau^*K_F)^2 \geqq M_{iF_1}^2K_F^2 \geqq (2r_i-4)K_F^2 .$$

Since $r_i\geqq 3$, we obtain $2r_i-4\geqq K_F^2$, but this contradicts Noether's inequality because $r_i<p_g(F)$ if $i<n$.

If $M_{iF_1}\tau^*K_F=2r_i-3$, put $M'_{iF}:=\tau_*M_{iF_1}$. From Lemma 1.8, we have

$$M'^2_{iF} \geqq 2h^0(\mathcal{O}_F(M'_{iF}))-4=2h^0(\mathcal{O}_{F_1}(M_{iF_1}))-4 \geqq 2r_i-4\,.$$

We claim that $M'^2_{iF}=2r_i-4$. Indeed, suppose $M'^2_{iF}\geqq 2r_i-3$. Applying the Hodge index theorem to M'_{iF} and K_F, we get

$$(2r_i-3)^2=(M_{iF_1}\tau^*K_F)^2=(M'_{iF}K_F)^2 \geqq M'^2_{iF}K_F^2 \geqq (2r_i-3)K_F^2\,.$$

Hence $2r_i-3\geqq K_F^2$, which contradicts $K_F^2\geqq 2r_i-2$.

On the other hand, we have $M'^2_{iF}+M'_{iF}K_F\equiv 0 \pmod 2$, which is incompatible with the hypothesis that $M'^2_{iF}=2r_i-4$ and $M'_{iF}K_F=2r_i-3$.

Therefore we conclude $M_{iF_1}\tau^*K_F\geqq 2(r_i-1)$ if $i<n$, which is the desired inequality.

If $M_{iF_1}\tau^*K_F=2(r_i-1)$ and $K_F^2\geqq 2p_g(F)$, then since $(M_{iF_1}\tau^*K_F)^2\geqq M^2_{iF_1}K_F^2$, we have $(2r_i-2)^2\geqq(2r_i-4)(2r_i+2)$. Hence $r_i=3$, $p_g(F)=r_i+1=4$ and $K_F^2=2p_g(F)=8$. Thus we have proved Lemma 2.3. □

PROOF OF PROPOSITION 2.1. (1) From (*) and Lemma 2.3, we get

$$\begin{aligned}
K^3_{X/C} &\geqq \sum_{i=1}^{n-2}(\mu_i-\mu_{i+1})(4r_i-2)+(\mu_{n-1}-\mu_n)(2r_{n-1}-2+2r_n-4)+(2r_n-4+K_F^2)\mu_n\\
&\geqq \sum_{i=1}^{n-2}(\mu_i-\mu_{i+1})(4r_i-2)+(\mu_{n-1}-\mu_n)(4r_{n-1}-4)+(2r_n-4+K_F^2)\mu_n\\
&=4\sum_{i=1}^{n} r_i(\mu_i-\mu_{i+1})-2\mu_1-2(\mu_{n-1}-\mu_n)-\mu_n(4r_n-2)+(2r_n-4+K_F^2)\mu_n\\
&=4\deg f_*\omega_{X/C}-2\mu_1-2\mu_{n-1}-(2r_n-K_F^2)\mu_n\\
&\geqq 4\deg f_*\omega_{X/C}-4\mu_1-(2p_g(F)-K_F^2)\mu_n\\
&=4\deg f_*\omega_{X/C}-4\Big(\mu_1+\frac{2p_g(F)-K_F^2}{4}\mu_n\Big),
\end{aligned}$$

since $r_n\geqq r_{n-1}+1$ and $\mu_1\geqq\mu_{n-1}$. Apply Lemma 1.2 to the sequence $\{\mu_1, \mu_n, 0\}$, $\{Z_1, K_n, 0\}$, to get

$$\begin{aligned}
K^3_{X/C} &\geqq \frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+(\tau^*K_FM_{1F_1}+\tau^*K_FM_{nF_1})(\mu_1-\mu_n)+(\tau^*K_FM_{nF_1}+K_F^2)\mu_n\\
&\geqq (2p_g(F)-4)(\mu_1-\mu_n)+(2p_g(F)-4+K_F^2)\mu_n\\
&=(2p_g(F)-4)\Big(\mu_1+\frac{K_F^2}{2p_g(F)-4}\mu_n\Big).
\end{aligned}$$

By Noether's inequality, we have

$$\frac{K_F^2}{2p_g(F)-4}-\frac{2p_g(F)-K_F^2}{4}\geqq 0\,,$$

so that

$$\mu_1+\frac{K_F^2}{2p_g(F)-4}\mu_n \geqq \mu_1+\frac{2p_g(F)-K_F^2}{4}\mu_n .$$

Suppose that

$$\mu_1+\frac{K_F^2}{2p_g(F)-4}\mu_n \leqq \frac{2}{p_g(F)}\deg f_*\omega_{X/C} .$$

Then

$$K_{X/C}^3 \geqq 4\deg f_*\omega_{X/C}-4\frac{2}{p_g(F)}\deg f_*\omega_{X/C}=\frac{4(p_g(F)-2)}{p_g(F)}\deg f_*\omega_{X/C} .$$

If

$$\mu_1+\frac{K_F^2}{2p_g(F)-4}\mu_n \geqq \frac{2}{p_g(F)}\deg f_*\omega_{X/C} ,$$

then we also obtain

$$K_{X/C}^3 \geqq (2p_g(F)-4)\frac{2}{p_g(F)}\deg f_*\omega_{X/C}=\frac{4(p_g(F)-2)}{p_g(F)}\deg f_*\omega_{X/C} .$$

Thus we have proved (1).

(2) From (*) and Lemma 2.2, we have

$$\begin{aligned} K_{X/C}^3 &\geqq \sum_{i=1}^{n-1}(\mu_i-\mu_{i+1})(2r_i-2+2r_{i+1}-2)+(2r_n-2+K_F^2)\mu_n \\ &\geqq \sum_{i=1}^{n-1}(\mu_i-\mu_{i+1})(4r_i-2)+(2r_n-2+K_F^2)\mu_n \\ &= 4\sum_{i=1}^{n} r_i(\mu_i-\mu_{i+1})-2\mu_1-\mu_n(4r_n-2)+(2r_n-2+K_F^2)\mu_n \\ &= 4\deg f_*\omega_{X/C}-2\mu_1-(2r_n-K_F^2)\mu_n \\ &= 4\deg f_*\omega_{X/C}-2\Big(\mu_1+\frac{2p_g(F)-K_F^2}{2}\mu_n\Big), \end{aligned}$$

since $r_{i+1}\geqq r_i+1$ for all i and $\deg f_*\omega_{X/C}=\sum_{i=1}^{n} r_i(\mu_i-\mu_{i+1})$.

On the other hand, applying Lemma 1.2 to the sequences $\{\mu_1, \mu_n, 0\}$, $\{Z_1, Z_n, 0\}$, we find that

$$\begin{aligned} &(\mu^*K_{X/C})^2-(\mu_1-\mu_n)(\mu^*K_{X/C}N_1F_1+\mu^*K_{X/C}N_nF_1) \\ &\qquad +\mu_n(\mu^*K_{X/C}N_nF_1+\mu^*K_{X/C}N_{n+1}F_1) \end{aligned}$$

is pseudo-effective. Hence for every $m\geqq m_0$, we have

$$
\begin{aligned}
K_{X/C}^3 &\geqq \frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+(M_{1F_1}\tau^*K_F+M_{nF_1}\tau^*K_F)(\mu_1-\mu_n)+(M_{nF_1}\tau^*K_F+K_F^2)\mu_n \\
&\geqq (2p_g(F)-2)(\mu_1-\mu_n)+(2p_g(F)-2+K_F^2)\mu_n \\
&= (2p_g(F)-2)\Big(\mu_1+\frac{K_F^2}{2p_g(F)-2}\mu_n\Big).
\end{aligned}
$$

Since $K_F^2\geqq 2$, we have $K_F^2\geqq 2p_g(F)-2$ (see [**Bo**, proof of Theorem 9]), so that

$$
\frac{K_F^2}{2p_g(F)-2}-\frac{2p_g(F)-K_F^2}{2}\geqq 0, \quad \mu_1+\frac{K_F^2}{2p_g(F)-2}\mu_n\geqq\mu_1+\frac{2p_g(F)-K_F^2}{2}\mu_n .
$$

If

$$
\mu_1+\frac{K_F^2}{2p_g(F)-2}\mu_n\leqq\frac{2}{p_g(F)}\deg f_*\omega_{X/C},
$$

then

$$
K_{X/C}^3\geqq 4\deg f_*\omega_{X/C}-2\frac{2}{p_g(F)}\deg f_*\omega_{X/C}=\frac{4(p_g(F)-1)}{p_g(F)}\deg f_*\omega_{X/C} .
$$

If

$$
\mu_1+\frac{K_F^2}{2p_g(F)-2}\mu_n\geqq\frac{2}{p_g(F)}\deg f_*\omega_{X/C},
$$

then

$$
K_{X/C}^3\geqq(2p_g(F)-2)\frac{2}{p_g(F)}\deg f_*\omega_{X/C}\geqq\frac{4(p_g(F)-1)}{p_g(F)}\deg f_*\omega_{X/C},
$$

which proves (2).

(3) If $f_*\omega_{X/C}$ is semistable, then $n=1$ and from (*), we have

$$
K_{X/C}^3\geqq(M_{1F_1}\tau^*K_F+K_F^2)\mu_1=2\mu_1=\deg f_*\omega_{X/C}
$$

since $M_{1F_1}\tau^*K_F=K_F^2=1$. If $f_*\omega_{X/C}$ is unstable, then

$$
\begin{aligned}
K_{X/C}^3 &\geqq (M_{1F_1}\tau^*K_F+M_{2F_1}\tau^*K_F)(\mu_1-\mu_2)+(M_{2F_1}\tau^*K_F+K_F^2)\mu_2 \\
&= (\mu_1-\mu_2)+2\mu_2=\mu_1+\mu_2=\operatorname{def} f_*\omega_{X/C} .
\end{aligned}
$$

Thus we have proved the desired inequality. The last statement easily follows from Lemma 1.6. □

To prove the case (1), (2) and (3) of the Main Theorem 1, we need the following two lemmas.

LEMMA 2.4. *Let X be a projective normal 3-fold with only canonical singularities and let $f: X\to C$ be a proper morphism with connected fibres onto a complete nonsingular curve C. Then we have*

$$\deg f_*\omega_{X/C} - \deg R^1 f_*\omega_{X/C} = \chi(\mathcal{O}_F)\chi(\mathcal{O}_C) - \chi(\mathcal{O}_X)$$

where F is a general fibre of f.

PROOF. The lemma follows from the spectral sequence:

$$E_2^{p,q} := H^p(C, R^q f_*\omega_X) \Longrightarrow E^{p+q} := H^{p+q}(X, \omega_X),$$

and Grothendieck duality (cf. [**Ha2**], [**Ko1**, Proposition 7.6]). □

The author was informed by Y. Kawamata that the following lemma follows from [**Ka1**] and [**N**], but we give here an alternate proof.

LEMMA 2.5. *Let notation be as in Lemma 2.4. Then, for all i, $R^i f_*\omega_{X/C}$ is semipositive. In particular, we have* $\deg R^i f_*\omega_{X/C} \geqq 0$ *for all i.*

PROOF. Let $\mu: Y \to X$ be a proper birational morphism from a nonsingular projective 3-fold Y, and put $g := f \circ \mu$. Consider the following spectral sequence:

$$E_2^{p,q} := R^p f_*(R^q \mu_*\omega_Y) \Longrightarrow R^{p+q} g_*\omega_Y$$

which degenerates at E_2-term. Since canonical singularities are rational ([**E**]), $R^p g_*\omega_Y = R^p f_*\omega_X$ for all p, and we may assume that X is nonsingular. Then by [**Ko2**, Corollary 2.24], for each p, there is a smooth projective variety Z, which has a proper surjective morphism $h: Z \to C$ such that $R^p f_*\omega_X$ is a direct summand of $h_*\omega_Z$. Since $h_*\omega_{Z/C}$ is semipositive, it's direct summand $R^p f_*\omega_{X/C}$ is also semipositive. □

Now we prove the cases (1), (2) and (3) of the Main Theorem 1.

In the case (1), we have

$$\begin{aligned} K_{X/C}^3 &\geqq \frac{4(p_g(F)-2)}{p_g(F)} \deg f_*\omega_{X/C} \\ &\geqq \frac{4(p_g(F)-2)}{p_g(F)} (\deg f_*\omega_{X/C} - \deg R^1 f_*\omega_{X/C}) \\ &= \frac{4(p_g(F)-2)}{p_g(F)} (\chi(\mathcal{O}_F)\chi(\mathcal{O}_C) - \chi(\mathcal{O}_X)), \end{aligned}$$

by Proposition 2.1 (1) and Lemmas 2.4, 2.5. Hence

$$\begin{aligned} K_X^3 &\geqq 6K_F^2(b-1) - \frac{4(p_g(F)-2)}{p_g(F)}\chi(\mathcal{O}_F)(b-1) - \frac{4(p_g(F)-2)}{p_g(F)}\chi(\mathcal{O}_X) \\ &= \frac{4(p_g(F)-2)}{p_g(F)} \left\{ \frac{(3K_F^2 - 2\chi(\mathcal{O}_F))p_g(F) + 4\chi(\mathcal{O}_F)}{2(p_g(F)-2)}(b-1) - \chi(\mathcal{O}_X) \right\}. \end{aligned}$$

The second inequality in (1) follows from

$$(3K_F^2 - 2\chi(\mathcal{O}_F))p_g(F) + 4\chi(\mathcal{O}_F) \geqq 2(p_g(F)-2)(2p_g(F)+q(F)-1) > 0\,.$$

In the case (2), we have

$$\begin{aligned} K_{X/C}^3 &\geqq \frac{4(p_g(F)-1)}{p_g(F)} \deg f_*\omega_{X/C} \\ &\geqq \frac{4(p_g(F)-1)}{p_g(F)}(\deg f_*\omega_{X/C}-\deg R^1f_*\omega_{X/C}) \\ &= \frac{4(p_g(F)-1)}{p_g(F)}(\chi(\mathcal{O}_F)\chi(\mathcal{O}_C)-\chi(\mathcal{O}_X)), \end{aligned}$$

by Proposition 2.1 (2) and Lemmas 2.4, 2.5. Hence we get

$$\begin{aligned} K_X^3 &\geqq 6(b-1)K_F^2-\frac{4(p_g(F)-1)}{p_g(F)}\chi(\mathcal{O}_F)(b-1)-\frac{4(p_g(F)-1)}{p_g(F)}\chi(\mathcal{O}_X) \\ &= \frac{4(p_g(F)-1)}{p_g(F)}\left\{\frac{(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+2\chi(\mathcal{O}_F)}{2(p_g(F)-1)}(b-1)-\chi(\mathcal{O}_X)\right\}. \end{aligned}$$

The second inequality in (2) follows from

$$(3K_F^2-2\chi(\mathcal{O}_F))p_g(F)+2\chi(\mathcal{O}_F) \geqq 2(p_g(F)-1)(2p_g(F)+q(F)-1) > 0 .$$

In the case (3), we have

$$K_{X/C}^3 \geqq \deg f_*\omega_{X/C} \geqq \deg f_*\omega_{X/C}-\deg R^1f_*\omega_{X/C} = \chi(\mathcal{O}_F)\chi(\mathcal{O}_C)-\chi(\mathcal{O}_X),$$

by Proposition 2.1 (3) and Lemmas 2.4, 2.5.

Noting $K_{X/C}^3=(K_X-f^*K_C)^3=K_X^3-6(b-1)K_F^2$, $K_F^2=1$ and $\chi(\mathcal{O}_F)=3$, we obtain

$$\begin{aligned} K_X^3 &\geqq 6K_F^2(b-1)-\chi(\mathcal{O}_F)(b-1)-\chi(\mathcal{O}_X) \\ &= (6K_F^2-\chi(\mathcal{O}_F))(b-1)-\chi(\mathcal{O}_X) = 3(b-1)-\chi(\mathcal{O}_X), \end{aligned}$$

$$b \leqq 1+\frac{K_X^3+\chi(\mathcal{O}_X)}{3}.$$

Thus we have proved the case (1), (2) and (3) of the Main Theorem 1.

§ 2.2. Case (4).

PROPOSITION 2.6. *If* $p_g(F)=1$, *then*

$$K_{X/C}^3 \geqq K_F^2 \deg f_*\omega_{X/C} .$$

When the equality holds, f is isotrivial.

PROOF. $f_*\omega_{X/C}$ is an invertible sheaf in this case, so that $f_*\omega_{X/C}=\mathcal{O}_C(\delta)$ for some divisor δ on C. Since the natural homomorphism $f^*\mathcal{O}_C(\delta)=f^*f_*\omega_{X/C} \to\omega_{X/C}$ is non-zero, $K_{X/C}-f^*\delta$ is effective. Therefore, by Theorem 1.4, we have

$$(K_{X/C}-f^*\delta)\left\{K_{X/C}-\frac{f^*\delta_-(f_*\omega_{X/C}^{[m]})}{m}\right\}^2 \geqq 0,$$

for sufficiently large m. Hence

$$K_{X/C}^3 \geqq \frac{2\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+K_{X/C}^2 f^*\delta \geqq K_F^2 \deg \delta = K_F^2 \deg f_*\omega_{X/C}$$

for sufficiently large m, which is the desired inequality. The last statement follows from Lemma 1.6. $\square$

Now, we prove the case (4) of the Main Theorem 1. By Proposition 2.6 and Lemma 2.4, 2.5, we have

$$\begin{aligned} K_{X/C}^3 &\geqq K_F^2 \deg f_*\omega_{X/C} \geqq K_F^2(\deg f_*\omega_{X/C}-\deg R^1 f_*\omega_{X/C}) \\ &= K_F^2(\chi(\mathcal{O}_F)\chi(\mathcal{O}_C)-\chi(\mathcal{O}_X)), \\ K_X^3 &\geqq 6K_F^2(b-1)-K_F^2\chi(\mathcal{O}_F)(b-1)-K_F^2\chi(\mathcal{O}_X) \\ &= K_F^2\{(6-\chi(\mathcal{O}_F))(b-1)-\chi(\mathcal{O}_X)\}. \end{aligned}$$

As for the second inequality in (4), we have $6-\chi(\mathcal{O}_F)=4+q(F)>0$, and hence

$$b \leqq \frac{K_X^3+K_F^2\chi(\mathcal{O}_X)}{K_F^2(6-\chi(\mathcal{O}_F))}+1.$$

Thus we have proved the case (4) of the Main Theorem 1.

§ 2.3. Case (5).

PROPOSITION 2.7.
(1) *If* $K_F^2=1$, $p_g(F)=0$, $q(F)=0$, *then*

$$K_{X/C}^3 \geqq \frac{3}{8}\deg f_*\omega_{X/C}^{[2]}.$$

(2) *If* $K_F^2 \geqq 2$, $p_g(F)=0$, $q(F)=0$, *then*

$$K_{X/C}^3 \geqq \frac{1}{2}\deg f_*\omega_{X/C}^{[2]}.$$

When the equality in each case holds, f is isotrivial.

PROOF. Let

$$0=:\mathcal{E}_0 \subset \mathcal{E}_1 \subset \cdots \subset \mathcal{E}_n := f_*\omega_{X/C}^{[2]}$$

be the Harder-Narasimhan filtration of $f_*\omega_{X/C}^{[2]} \neq 0$ and take Y, μ and τ in the same way as in the proof of Proposition 2.1. Put

$$r_i := \operatorname{rank} \mathcal{E}_i \in \boldsymbol{N} \qquad N_i := N_Y(2K_{X/C}, \mathcal{E}_i) \in \operatorname{Div}(Y)\otimes \boldsymbol{Q}$$

$$\mu_i := \mu_-(\mathcal{E}_i) \in \boldsymbol{Q} \qquad Z_i := Z_Y(2K_{X/C}, \mathcal{E}_i) \in \mathrm{Div}\,(Y) \otimes \boldsymbol{Q}$$
$$M_i := M_Y(2K_{X/C}, \mathcal{E}_i) \in \mathrm{Div}\,(Y)\,.$$

CLAIM. *$M_{iF_1}\tau^*K_F \geqq r_i - 1$ for all i.*

PROOF OF CLAIM. If $r_i=1$, we have nothing to prove. If $r_i \geqq 2$ and $|M_{iF_1}|$ is composed of a pencil, then there is a smooth irreducible curve B_i on F_1 and a positive integer a_i, such that $M_{iF_1} \sim_{\mathrm{alg}} a_i B_i$. Since $\tau^*K_F B_i > 0$ and $a_i \geqq r_i - 1$, we get the result. If $r_i \geqq 3$ and $|M_{iF_1}|$ is not composed of a pencil, then we have $2\tau^*K_F \geqq M_{iF_1}$ and

$$M_{iF_1}\tau^*K_F \geqq \frac{1}{2} M_{iF_1}^2 \geqq r_i - 2$$

by Lemma 1.8. Suppose $M_{iF_1}\tau^*K_F = r_i - 2$, then

$$(r_i-2)^2 = (M_{iF_1}\tau^*K_F)^2 \geqq M_{iF_1}^2 K_F^2 \geqq (2r_i-4)K_F^2$$

by the Hodge's index theorem. Hence we get $r_i - 2 \geqq 2K_F^2$, which contradicts $r_i \leqq P_2(F) = K_F^2 + 1$. Thus we have proved the claim. □

PROOF OF PROPOSITION 2.7 continued. Applying Lemma 1.2 to the sequences:

$$\{\mu_1, \mu_2, \cdots, \mu_n, \mu_{n+1} := 0\}, \qquad \{Z_1, Z_2, \cdots, Z_n, Z_{n+1} := 0\},$$

we see that

$$4(\mu^*K_{X/C})^2 - \sum_{i=1}^{n} (\mu_i - \mu_{i+1})(N_i F_1 + N_{i+1} F_1)$$

is pseudo-effective. Let m_0 be as in Theorem 1.4. Then for any positive integer $m \geqq m_0$,

$$\begin{aligned}
4K_{X/C}^2 &\geqq \frac{4\mu_-(f_*\omega_{X/C}^{[m]})}{m} K_F^2 + \sum_{i=1}^{n} (\mu_i - \mu_{i+1})(M_{iF_1}\tau^*K_F + M_{i+1F_1}\tau^*K_F) \\
&\geqq 2\sum_{i=1}^{n-1} r_i(\mu_i - \mu_{i+1}) + (M_{nF_1}\tau^*K_F + 2K_F^2)\mu_n \\
&= 2\sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) - 2r_n\mu_n + (M_{nF_1}\tau^*K_F + 2K_F^2)\mu_n \\
&= 2\deg f_*\omega_{X/C}^{[2]} - 2r_n\mu_n + (r_n - 1 + 2K_F^2)\mu_n \\
&= 2\deg f_*\omega_{X/C}^{[2]} - (P_2(F) + 1 - 2K_F^2)\mu_n\,.
\end{aligned}$$

In the case (5), we have $P_2(F)+1-2K_F^2 = 2-K_F^2 = 1$ and $\mu(f_*\omega_{X/C}^{[2]}) \geqq \mu_n$, so that

$$4K_{X/C}^2 \geqq 2\deg f_*\omega_{X/C}^{[2]} - \mu_n \geqq 2\deg f_*\omega_{X/C}^{[2]} - \mu(f_*\omega_{X/C}^{[2]}) = \frac{3}{2}\deg f_*\omega_{X/C}^{[2]}\,.$$

In the case (6), noting that $P_2(F)+1-2K_F^2 \leqq 0$, we have

$$4K_{X/C}^3 \geqq 2\deg f_*\omega_{X/C}^{[2]},$$

since $P_2(F)+1-2K_F^2 \leqq 0$. The last statement follows from Lemma 1.6. This completes the proof of Proposition 2.7. □

In order to prove the case (5) of the Main Theorem 1, we need the following:

LEMMA 2.8. *Let notation be as in the Main Theorem* 1. *Then we have*

$$\deg f_*\omega_{X/C}^{[2]} = \frac{1}{2}K_{X/C}^3 - 3\chi(\mathcal{O}_X) - 3(b-1)\chi(\mathcal{O}_F) + l(2).$$

PROOF. Since $R^i f_*\omega_{X/C}^{[2]}=0$ for all i (cf. [**KMM**, Theorem 1-2-5]), we have $\chi(\omega_X^{[2]})=\chi(f_*\omega_X^{[2]})$. From Reid-Fletcher's plurigenera formula (see [**F1**, Theorem 2.5]), we obtain

$$\begin{aligned}\chi(\omega_X^{[2]}) &= \frac{1}{2}K_X^3 - 3\chi(\mathcal{O}_X) + l(2)\\ &= \frac{1}{2}K_{X/C}^3 + 3K_F^2(b-1) - 3\chi(\mathcal{O}_X) + l(2).\end{aligned}$$

On the other hand, by Riemann-Roch on C, we have

$$\begin{aligned}\chi(f_*\omega_X^{[2]}) &= \deg f_*\omega_X^{[2]} - P_2(F)(b-1)\\ &= \deg f_*\omega_{X/C}^{[2]} + P_2(F)(4b-4) - P_2(F)(b-1)\\ &= \deg f_*\omega_{X/C}^{[2]} + 3P_2(F)(b-1).\end{aligned}$$

Hence

$$\begin{aligned}\deg f_*\omega_{X/C}^{[2]} &= \frac{1}{2}K_{X/C}^3 + 3K_F^2(b-1) - 3\chi(\mathcal{O}_X) - 3(K_F^2+\chi(\mathcal{O}_F))(b-1) + l(2)\\ &= \frac{1}{2}K_{X/C}^3 - 3(b-1)\chi(\mathcal{O}_F) - 3\chi(\mathcal{O}_X) + l(2),\end{aligned}$$

which is the desired inequality. □

The inequalities in the case Main Theorem 1 follow immediately from Proposition 2.7, Lemma 2.8, and thus we have completed the proof of the Main Theorem 1. □

§ 3. Proof of the Main Theorem 2.

To prove the Main Theorem 2, we only have to show the following:

PROPOSITION 3.1. *Let notation be as in the Main Theorem* 1. *Assume that*

F is not any of the surfaces (1), (2), (3), (4), (5), (6) *and* (7) *of the Main Theorem* 2. *Then*

$$K_{X/C}^2 \geqq 4 \deg f_*\omega_{X/C} .$$

When the equality holds, f is isotrivial.

PROOF. We may assume $p_g(F) \geqq 2$. We use the same notation as in the proof of the cases (1), (2) and (3) of the Main Theorem 1.

CLAIM. *For every i, the inequality*

$$(A_i) \qquad M_{iF_1}\tau^*K_F + M_{i+1F_1}\tau^*K_F \geqq 4r_i$$

holds, unless $r_1=1$ and $i=1$.

Assume the claim. Then if $r_1>1$ and m is very large, we have:

$$\begin{aligned} K_{X/C}^2 &\geqq \frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m} K_F^2 + \sum_{i=1}^{n} (M_{iF_1}\tau^*K_F + M_{i+1F_1}\tau^*K_F)(\mu_i - \mu_{i+1}) \\ &\geqq 4 \sum_{i=1}^{n} r_i(\mu_i - \mu_{i+1}) = 4 \deg f_*\omega_{X/C} , \end{aligned}$$

which is the desired equality.

PROOF OF CLAIM. We divide the proof into the following two cases:
(a) $K_F^2 \geqq 2p_g(F)+1$.
(b) $K_F^2 = 2p_g(F)$.
CASE (a). We have two subcases:
SUBCASE (a-1). $|K_F|$ is composed of a pencil.
By Lemma 2.2, if (A_i) does not hold for some i with $r_i>1$, then we are in the case (1) or (2), which was excluded.
SUBCASE (a-2). $|M_F|$ is not composed of a pencil.
In view of Lemma 2.3, we only have to prove:

$$M_{nF_1}\tau^*K_F \geqq 2r_n - 1 .$$

Note that $M_{nF_1}\tau^*K_F \geqq 2r_n-3$, by Lemma 2.3. Suppose that $M_{nF_1}\tau^*K_F = 2r_n-3$. Then by the Hodge index theorem,

$$(2r_n-3)^2 \geqq (2r_n-4)(2r_n+1) ,$$

so that $r_n \leqq 2$, which is absurd. Suppose that $M_{nF_1}\tau^*K_F = 2r_n-2$. Then

$$(2r_n-2)^2 \geqq M_{nF}'^2 K_F^2 \geqq (2r_n-4)(2r_n+1) ,$$

and hence $r_n \leqq 4$. If $r_n=4$, then we have:

$$M'_{nF}K_F = 6, \qquad M_{nF}'^2 = 4, \qquad K_F^2 = 9 .$$

Since

$$\frac{M'_{nF}K_F+M'^2_{nF}}{2} \geqq 2p_g(F)-4+q(F)$$

(see [**Bo**, proof of Theorem 9]), we deduce $q(F)\leqq 1$. Thus we are in the excluded case (4b).

If $r_n=3$, then from $M_{nF_1}\tau^*K_F=4$ and $16\geqq M'^2_{nF}K^2_F\geqq MM'_{nF}$, we derive $M'^2_{nF}\leqq 2$. On the other hand, we have $M'^2_{nF}\geqq 2r_n-4=2$, so we get $M'^2_{nF}=2$, $K^2_F=7$ or 8. Since

$$\frac{M'_{nF}K_F+M'^2_{nF}}{2} = 3 \geqq 2p_g(F)-4+q(F) = 2+q(F)\,,$$

we have $q(F)\leqq 1$. Thus we are in the excluded case (4a) or (4c). Thus we have proved the case (a).

CASE (b).

SUBCASE (b-1). $|K_F|$ is composed of a pencil.

By our assumption, there is a smooth irreducible curve B_n on F_1 and a positive integer a_n such that $M_{nF_1}\sim_{\mathrm{alg}}a_nB_n$. By Lemma 2.2, $M_{nF_1}\tau^*K_F\geqq 2r_n-1$. We claim $M_{nF_1}\tau^*K_F\geqq 2r_n$. Indeed, if $M_{nF_1}\tau^*K_F=2r_n-1$, then

$$2r_n-1 = a_nB'_nK_F \geqq (r_n-1)B'_nK_F\,,$$

and

$$B'_nK_F \leqq \frac{2r_n-1}{r_n-1} = 2+\frac{1}{r_n-1} \leqq 3\,,$$

where $B'_n:=\tau_*B_n$. If $B'_nK_F=3$, then $r_n=p_g(F)=2$, $K^2_F=4$ and $a_n=r_n-1=1$, which implies that $|B'_n|$ is a linear pencil. By the Hodge index theorem, we have $9\geqq B'^2_nK^2_F=4B'^2_n$, hence $B'^2_n=1$. So we may assume that τ is the blowing up of the unique base point of $|B'_n|$. Let E be the exceptional divisor of τ. Then

$$\tau^*B'_n = B_n+E \qquad K_{F_1} = \tau^*K_F+E\,,$$

so that

$$B_nK_{F_1} = (\tau^*B'_n-E)(\tau^*K_F+E) = K_FB'_n-E^2 = 4\,.$$

This implies that $g(B_n)=3$. Thus we are in the case (4a), which was excluded. Since $B'_nK_F+B'^2_n\equiv 0$ (mod 2), we have $B'^2_n=1$. But in this case, we have $1\geqq B'^2_nK^2_F$ and $K^2_F=1$. So we arrive at the case (2).

SUBCASE (b-2). $|K_F|$ is not composed of a pencil.

We claim $M_{nF_1}\tau^*K_F\geqq 2r_n$. Indeed, if $M_{nF_1}\tau^*K_F\leqq 2r_n-3$, then

$$(2r_n-3)^2 \geqq (2r_n-4)2r_n\,,$$

hence $r_n\leqq 2$, which is absurd. If $M_{nF_1}\tau^*K_F=2r_n-2$, then

$$(2r_n-2)^2 \geqq M'^2_{nF}K_F^2 \geqq 2r_nM'_{nF}\ ,$$

and $M'^2_{nF} \leqq 2r_n-3$. Noting that $M'_{nF}K_F+M'^2_{nF} \equiv 0 \pmod 2$, we get $M'^2_{nF}=2r_n-4$. Thus

$$\frac{M'_{nF}K_F+M'^2_{nF}}{2} = 2r_n-3 \geqq 2p_g(F)-4+q(F)\ ,$$

which implies $q(F) \leqq 1$. So we are in the excluded case (3).

If $M_{nF_1}\tau^*K_F=2r_n-1$, then

$$(2r_n-1)^2 \geqq M'^2_{nF}2r_n\ .$$

Hence $M'^2_{nF} \leqq 2r_n-2$, yielding $M'^2_{nF}=2r_n-3$. So we obtain

$$\frac{M'_{nF}K_F+M'^2_{nF}}{2} = 2r_n-2 \geqq 2p_g(F)-4+q(F)\ ,$$

which implies $q(F) \leqq 2$. So we are again in the case (3). Thus we get $M_{nF_1}\tau^*K_F \geqq 2r_n$, and from Lemma 2.3, we deduce that (A_i) holds for all i except for the case $r_1=1$ and $i=1$. Thus we have proved the claim. □

PROOF OF PROPOSITION 3.1 CONTINUED. In what follows, we may suppose $r_1=1$. And we may also assume $M_{2F_1}\tau^*K_F=3$ and $r_2=2$, since $M_{2F_1}\tau^*K_F \geqq 2r_2-1$. Let B_2 be a smooth irreducible curve on F_1 and a_2 a positive integer such that $M_{2F_1} \sim_{\mathrm{alg}} a_2B_2$, and put $B'_2 := \tau_*B_2$. Since $a_2B_2\tau^*K_F=a_2B'_2K_F=3$, we have $a_2=1$ or 3. If $a_2=3$, then $B'_2K_F=1$ and $K_F^2=1$, which is absurd. If $a_2=1$, then $|B'_2|$ is a linear pencil. Since $K_F^2 \geqq 2p_g(F)=4$ and $9 \geqq B'^2_2K_F^2 \geqq 4B'^2_2$, we have $B'^2_2=1$ or 2. But the case $B'^2_2=2$ is excluded since $B'_2K_F+B'^2_2 \equiv 0 \pmod 2$. Hence $B'^2_2=1$ and $g(B_2)=3$.

Suppose that $p_g(F)=2$. Then we have $K_F^2 \geqq 6$; otherwise we have $K_F^2=4$ or 5 and we are in the excluded case (4*a*). Let m be a sufficiently large integer. If $\mu_1-\mu_2 \leqq \mu_2$, then

$$\begin{aligned} K^2_{X/C} &\geqq \frac{\mu_-(f_*\omega^{[m]}_{X/C})}{m}K_F^2+3(\mu_1-\mu_2)+9\mu_2 \geqq 3(\mu_1-\mu_2)+9\mu_2 \\ &\geqq 4(\mu_1-\mu_2)+8\mu_2 = 4\deg f_*\omega_{X/C}\ . \end{aligned}$$

On the other hand, if $\mu_1-\mu_2 \geqq \mu_2$, then

$$\begin{aligned} K^2_{X/C} &\geqq \frac{\mu_-(f_*\omega^{[m]}_{X/C})}{m}K_F^2+K_F^2(\mu_1-0) \geqq 6\mu_1 \\ &= 6(\mu_1-\mu_2)+6\mu_2 \geqq 4(\mu_1-\mu_2)+8\mu_2 = 4\deg f_*\omega_{X/C}\ . \end{aligned}$$

Suppose that $p_g(F) \geqq 3$. In this case we have $M_{3F_1}\tau^*K_F \geqq 2r_3-1 \geqq 5$. If $M_{3F_1}\tau^*K_F \geqq 6$, then we can deduce the desired inequality in the same way as in [**X**, p. 456]. So we may assume $M_{3F_1}\tau^*K_F=5$ and $r_3=3$. When $|M_{3F_1}|$ is composed of a pencil, there are a smooth irreducible curve B_2 on F_1 and a positive integer a_3 such that $M_{3F_1}\sim_{\mathrm{alg}}a_3B_3$ and $a_3\geqq r_3-1=2$. Put $B_3':=\tau_*B_3$. Since $a_3B_3'K_F=5$, we have $a_3=5$ and $B_3'K_F=1$. Hence we get $K_F^2=1$, which is absurd. So we may assume $|M_{3F_1}|$ is not composed of a pencil. Then

$$25=M_{3F}'^2K_F^2\geqq 2r_nM_{3F}'^2\geqq 6M_{3F}'^2,$$

so that $M_{3F}'^2\leqq 4$. Noting $M_{3F}'K_F+M_{3F}'^2\equiv 0 \pmod 2$ and $M_{3F}'^2\geqq 2r_3-4=2$, we get $M_{3F}'^2=3$ and $K_F^2=6, 7, 8$. And since F has a genus 3 linear pencil, we have $q(F)\geqq 2$. Therefore, if $K_F^2=6, 7$, then we are in the case (3), (4c) respectively. If $K_F^2=8$ and $p_g(F)=4$, then we are in the case (3), which is absurd. So we may assume $K_F^2=8$ and $p_g(F)=3$. Then, if $\mu_1-\mu_2\leqq\mu_3$, we get

$$\begin{aligned}K_{X/C}^3&\geqq\frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+3(\mu_1-\mu_2)+8(\mu_2-\mu_3)+13\mu_3\\&\geqq 4(\mu_1-\mu_2)+8(\mu_2-\mu_3)+12\mu_3=4\deg f_*\omega_{X/C}.\end{aligned}$$

If $\mu_1-\mu_2\geqq\mu_3$,

$$\begin{aligned}K_{X/C}^3&\geqq\frac{\mu_-(f_*\omega_{X/C}^{[m]})}{m}K_F^2+8(\mu_1-0)=8(\mu_1-\mu_2)+8(\mu_2-\mu_3)+8\mu_3\\&\geqq 4(\mu_1-\mu_2)+8(\mu_2-\mu_3)+12\mu_3=4\deg f_*\omega_{X/C}.\end{aligned}$$

Thus we have proved the proposition and the Main Theorem 2. □

EXAMPLE. Let C be a smooth complete curve and δ a divisor on C with degree $d>0$, such that $|\delta|$ is free from base points. Put $S:=P_C(\mathcal{O}_C\oplus\mathcal{O}_C(\delta))$ and $P:=P_S(\mathcal{O}_S\oplus\mathcal{O}_S(-e))$ for some integer $e\geqq 2$. Let $\pi_1: S\to C$ and $\pi_2: P\to S$ be the natural projections. Put $p:=\pi_1\circ\pi_2$. Let $\Sigma\in|\mathcal{O}_P(1)|$ and $\Sigma_0\in|\mathcal{O}_P(1)+\pi_2^*\mathcal{O}_S(e)|$ be the sections of π_2 which correspond to the natural surjections $\mathcal{O}_S\oplus\mathcal{O}_S(-e)\to\mathcal{O}_S(-e)$ and $\mathcal{O}_S\oplus\mathcal{O}_S(-e)\to\mathcal{O}_S$ respectively. Let $L\in|\mathcal{O}_S(1)|$ and $L_0\in|\mathcal{O}_S(1)-\pi_1^*\delta|$ be the sections of π_1 which correspond to the natural surjections $\mathcal{O}_C\oplus\mathcal{O}_C(\delta)\to\mathcal{O}_C(\delta)$ and $\mathcal{O}_C\oplus\mathcal{O}_C(\delta)\to\mathcal{O}_C$ respectively. Since $L_0+\pi_1^*|\delta|\subset|L|$, $|L|$ is free from base points. And since $\Sigma+\pi_2^*|eL|\subset|\Sigma_0|$, $|\Sigma_0|$ and hence $|6\Sigma_0|$ are also free from base points. Let $R\in|6\Sigma_0|$ be a smooth general member, and put $\mathcal{L}:=\mathcal{O}_P(3\Sigma_0)$. Since $\mathcal{O}_P(R)=\mathcal{L}^{\otimes 2}$, we have a irreducible smooth double covering $\sigma: X\to P$ branched along R. Put $f:=p\circ\sigma$. Noting $\omega_{X/C}=\sigma^*(\omega_{P/C}\otimes\mathcal{L})$, we have

$$K_{X/C}=\sigma^*\{p^*\delta+(2e-2)\pi_2^*L+\Sigma\}=\sigma^*\{p^*\delta+(e-2)\pi_2^*L+\Sigma_0\}. \tag{1}$$

Since L and Σ_0 is nef, $K_{K/C}$ is also nef. From (1), we get $K_F^2=K_{X/C}|_F^2=2(3e-4)$, where F is a general fibre of f. Noting $\sigma_*\omega_{X/C}=\omega_{P/C}\oplus(\omega_{P/C}\otimes\mathcal{L})$, we have

$$f_*\omega_{X/C}=p_*(\omega_{P/C}\otimes\mathcal{L})=\bigoplus_{i=1}^{2e-1}\mathcal{O}_C(i\delta)\oplus\bigoplus_{i=1}^{e-1}\mathcal{O}_C(i\delta). \tag{2}$$

From (2), we get $p_g(F)=\text{rank}\, f_*\omega_{X/C}=3e-2$ and hence $K_F^2=2p_g(F)-4$. Thus f is a minimal fibration of surfaces of general type in the Noether line. From (1), we get $K_{X/C}^3=2(K_{P/C}+\mathcal{L})^3=2e(7e-9)d$. And from (2), we get $\deg f_*\omega_{X/C}=(1/2)\cdot e(5e-3)d$. Thus

$$K_{X/C}^3/\deg f_*\omega_{X/C}=\frac{4(7e-9)}{5e-3}<4 \Leftrightarrow e=2\,,$$

and when $e=2$, i.e., $(K_F^2, p_g(F), q(F))=(4, 4, 0)$, f gives an example which satisfies the inequality in the Main Theorem 2.

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