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On the cut loci of a von Mangoldt's surface of revolution

Dedicated to Professor T. Otsuki on his 75th birthday

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It is a classical problem to investigate the behavior of a geodesic on a surface of revolution, or more generally on a surface of Liouville. A pioneering and beautiful work in this field was done by von Mangoldt in 1881; He investigated the behavior of a geodesic on a hyperboloid of two sheets (or an elliptic paraboloid), which both are surfaces of Liouville. He proved in [7] that the two umbilic points on any hyperboloid of two sheets (or any elliptic paraboloid) are poles and that for any revolutionary hyperboloid of two sheets, the set of poles on the surface is a nontrivial closed ball centered at the unique umbilic point. Note that the set of poles on any surface of revolution with vertex p is a closed ball centered at p. This fact will be proved in Lemma 1.1. Here *a surface of revolution M with vertex p* means that M is a complete Riemannian manifold homeomorphic to R^2 such that the Gaussian curvature G is constant on each metric circle $S_p(t):=\{q\in M;\ d(p, q)=t\}$, $t\in[0, \infty)$, where d denotes the Riemannian distance function on M. Furthermore, by calculating the elliptic integrals defining geodesics on a revolutionary hyperboloid of two sheets, von Mangoldt explicitly determined the radius of the ball of poles. His results were extended by Elerath almost one hundred years after. By defining the class of flattening surfaces of revolution generalizing revolutionary hyperboloids of two sheets and revolutionary paraboloids, Elerath ([3]) explicitly determined the cut locus of each point on a flattening surface of revolution. Recently another extension was made by Maeda; Let M be a non-negatively curved, complete noncompact Riemannian manifold and let D_t be the diameter of the metric sphere centered at a point p with radius t. In [6] Maeda proved that the number $\limsup_{t\to\infty} D_t^2/t(=:d_0)$ does not depend on the choice of the point p and that the diameter of the set of poles on M is bounded above by d_0.

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Note that von Mangoldt did not find any geometrical upper bound for the diameter of the set of poles on a revolutionary hyperboloid of two sheets. Very recently I knew that Maeda's result was improved by Sugahara. He proved that the diameter is bounded above by the optimal constant $d_0/8$. In this article and the forthcoming one [**9**] we study the behaviors of geodesics on a surface of revolution with vertex. The main aim of this article is to determine the cut loci for a wider class of surfaces than the one of flattening surfaces of revolution, which will be defined soon later as a von Mangoldt's surface of revolution. We need some notations in order to state our Main Theorem. *Throughout this article, (M, g) always denotes a surface of revolution with vertex p and any geodesic is assumed to be parametrized by arclength.* For each $q \in M \setminus \{p\}$, let $\mu_q : [0, \infty) \to M$ be the geodesic emanating from p with $\mu_q(d(p, q)) = q$. The μ_q is called *the meridian through q*. The meridian $\hat{\mu}_q$ opposite q is defined by

$$\hat{\mu}_q(t) = \exp_p(-t\mu_q'(0))$$

for $t \in [0, \infty)$. Here $\exp_p$ denotes the exponential map on the tangent space M_p to M at p. A surface of revolution M with vertex p will be called *a von Mangoldt's surface of revolution* if $G(x)$ is not greater than $G(y)$ for any points x, y of M with $d(p, x) \geqq d(p, y)$. For each point q on M, C_q denotes the cut locus of q and $\tau_q : [0, \infty) \to M$ denotes a geodesic emanating from q through p. If q is distinct from p, τ_q is uniquely determined, because the vertex is a pole. This fact will be proved in Lemma 1.1. In this article we shall prove

MAIN THEOREM. *If M denotes a von Mangoldt's surface of revolution with vertex p, then for any point x on M, either C_x is empty or $C_x = \hat{\mu}_x[d(p, \hat{x}), \infty)$. Here $\hat{x}$ denotes the first conjugate point of x along τ_x.*

As a corollary the cut locus of each point on a von Mangoldt's surface of revolution is connected. In [**3**] Elerath constructed a surface of revolution on which the cut locus of a point is connected, but not contained in the meridian opposite the point. Note that even in the case of a surface of revolution, the cut locus is not generally known. Since a surface of revolution has a big isometry group, one might conjecture that each cut locus on any surface of revolution with vertex would be connected. But this is not true. I shall give a surface of revolution with a disconnected cut locus in [**9**]. Furthermore in [**9**], I shall give a characterization of a surface of revolution with many poles, and shall prove that the radius of the closed ball of poles on a von Mangoldt's surface of revolution is explicitly determined by a geometrical equation depending only on the function $L(t)$, the length of $S_p(t)$.

Refer to [**1**], [**2**] for classical Riemannian geometry and basic tools in Riemannian geometry.

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1. A von Mangoldt's surface of revolution.

For each point q on M, S_qM denotes the set of unit tangent vectors at q. A point q on M is called *a pole* if the differential $d\exp_q$ of the exponential map $\exp_q$ at each tangent vector of M_q is injective or equivalently if $\exp_q$ is injective. Let $f:[0, \infty)\to[0, \infty)$ be a smooth function with $f(0)=0$ such that $f(\sqrt{x^2+y^2})$ is smooth at $(x, y)=(0, 0)\in R^2$. $M(f)$ denotes the surface of revolution defined by $z=f(\sqrt{x^2+y^2})$. Then $M(f)$ is *a flattening surface of revolution* in [3] if it is a von Mangoldt's surface of revolution. A flattening surface $M(f)$ is a surface of revolution with vertex $p=(0, 0, 0)$ such that for each positive t, the closed ball $\bar{B}(p, t)$ centered at p with radius t is totally convex. In the section 2, I shall give a von Mangoldt's surface of revolution which is not a flattening surface. Revolutionary hyperboloids of two sheets and revolutionary paraboloids are typical examples of a flattening surfaces of revolution. For each $v\in S_qM$, $\gamma_v:[0, \infty)\to M$ denotes the geodesic defined by

$$\gamma_v(t) = \exp_q(tv)\,.$$

In the following lemma it will be proved a fundamental and crucial property on the set of poles on a surface of revolution.

LEMMA 1.1. *If M is a surface of revolution with vertex p, then p is a pole. Furthermore if a point q on M is a pole, then any point x with $d(p, x)\leqq d(p, q)$ is also a pole.*

PROOF. Suppose that p is not a pole. Then we can take a cut point x of p which is conjugate to p along a minimizing geodesic ([8]). Hence p is conjugate to x along a minimizing geodesic $\tau:[0, d(p, x)]\to M$ joining p to x. Since M is noncompact and complete, there exists a ray $\gamma:[0, \infty)\to M$ emanating from p. By definition γ is *a ray* if and only if $d(\gamma(t), p)=t$ for any positive t. Hence we have

$$G(\gamma(t)) = G(\tau(t))$$

for any $t\in[0, d(p, x)]$, since $d(p, \tau(t))=d(p, \gamma(t))=t$ on $[0, d(p, x)]$. Since the Jacobi fields along $\gamma|[0, d(p, x)]$ and the ones along $\tau|[0, d(p, x)]$ satisfy the same differential equation, $\gamma(d(p, x))$ is conjugate to p along γ, which contradicts the fact that γ is a ray. Hence p is a pole, and for each positive t and for any two points x, y on $S_p(t)$, M has an isometry A with $Ax=y$. To prove the latter claim we may assume $0<d(p, x)<d(p, q)$. Let A_x be the set of all $v\in S_xM$ such that γ_v is a ray. Since A_x is closed and nonempty, it is sufficient

to show that A_x is open in S_xM. Fix any $v \in A_x$. Since γ_v intersects $S_p(t_0)$, where $t_0 = d(p, q)$, at a pole y on $S_p(t_0)$, γ_{-v} is a subray of the ray emanating from y through x. Hence there exists an open subarc I containing $-v$ of S_xM such that for any w in I, γ_w intersects $S_p(t_0)$. This implies that for any w with $-w \in I$, γ_w is a subray of a ray emanating from a point on $S_p(t_0)$. Hence A_x is open in S_xM. This means $A_x = S_xM$.

It follows from Lemma 1.1 that for each surface of revolution M with vertex p, there exists a number $r(M) \in [0, \infty]$ such that the set of poles on M is the closed ball $\bar{B}(p, r(M))$. Note that M has a unique vertex unless the Gaussian curvature of M is constant.

LEMMA 1.2. *Let M be a von Mangoldt's surface of revolution with vertex p. If a point q of M satisfies that the geodesic $\tau_q : [0, \infty) \to M$ emanating from q through p has no conjugate point of q along itself, then q is a pole.*

REMARK. The above lemma was proved by von Mangoldt [7] in the case where M is a revolutionary hyperboloid of two sheets.

PROOF. Suppose that q is not a pole. Then there exists a cut point x of q which is conjugate to q along a minimizing geodesic τ joining q to x. Since p is a pole, $d(p, \tau_q(t)) = |d(p, q) - d(q, \tau(t))|$ for any $t \in [0, d(q, x)]$. Thus it follows from the triangle inequality that

$$d(p, \tau_q(t)) \leqq d(p, \tau(t))$$

on $[0, d(q, x)]$. This inequality implies that

$$G(\tau_q(t)) \geqq G(\tau(t))$$

on $[0, d(q, x)]$, since M is a von Mangoldt's surface of revolution. By the Rauch comparison theorem, there exists a conjugate point of q along τ_q. This contradicts the assumption on τ_q.

Let (N, g) be a complete Riemannian manifold homeomorphic to R^2 and $\gamma : R \to N$ denotes a geodesic such that $\gamma(0)$ is a pole. Then N is divided into two open half planes by γ, and let N_+ be a closed half plane with boundary $\gamma(R)$ and d_+ denotes the Riemannian distance function induced from $(N_+, g|N_+)$. For each $\theta \in [0, \pi]$, let $\gamma_\theta : [0, \infty) \to N_+$ be the geodesic emanating from $\gamma(0)$ such that the angle $\sphericalangle(\gamma'(0), \gamma'_\theta(0))$ made by $\gamma'(0)$ and $\gamma'_\theta(0)$ is θ. The following lemma will play an important role to prove Theorem 1.4.

LEMMA 1.3. *Let N_+, d_+, γ_θ, γ be as above. If a, b denote arbitrarily given positive real numbers, then the function $f : [0, \pi] \to R$ defined by*

$$f(\theta) = d_+(\gamma(b), \gamma_\theta(a))$$

is strictly monotone increasing on $[0, \pi]$.

PROOF. Let θ_1, θ_2 be arbitrarily given numbers satisfying $0<\theta_1<\theta_2<\pi$. Let $c:[0, f(\theta_2)]\to N_+$ be a d_+-minimizing geodesic joining $\gamma(b)$ to $\gamma_{\theta_2}(a)$. Since c is minimizing, $c(0, f(\theta_2))$ has no selfintersection with γ nor with γ_{θ_2}. Then the geodesic c intersects the geodesic γ_{θ_1} at a unique point $x=c(t_0)=\gamma_{\theta_1}(s_0)$, where $0<t_0<f(\theta_2)$ and $s_0>0$. Since $\gamma(0)$ is a pole, $|s_0-a|=d_+(x, \gamma_{\theta_1}(a))$ is less than $d_+(x, \gamma_{\theta_2}(a))=f(\theta_2)-t_0$. Hence we have

$$f(\theta_1)=d_+(\gamma(b), \gamma_{\theta_1}(a))<t_0+|s_0-a|<f(\theta_2)$$

by the triangle inequality. This inequality implies that f is strictly monotone increasing on $[0, \pi]$.

THEOREM 1.4. *If M is a von Mangoldt's surface of revolution with vertex p, then for any $x\in M\setminus\{p\}$ the cut locus C_x of x is contained in $\hat{\mu}_x[0, \infty)$.*

PROOF. Let $\gamma: R\to M$ denote the geodesic such that $\gamma(t)=\mu_x(t)$ for nonnegative t, $\gamma(t)=\hat{\mu}_x(-t)$ for negative t. Supposing the existence of a point $q_1\in C_x$ with $q_1\notin\hat{\mu}_x[0, \infty)$, we shall get a contradiction. The existence of such a point implies that there exists a cut point $y\in M\setminus\gamma(R)$ of x which is conjugate to x along a minimizing geodesic $c:[0, d(x, y)]\to M$ joining x to y. Let M_+ denote the closed half plane with boundary $\gamma(R)$ containing y. Define the curve c_2 in M_+ by

$$c_2(\theta)=\gamma_\theta(d(p, y))$$

for $\theta\in[0, \pi]$, which parametrizes the semicircle in M_+ of radius $d(p, y)$ centered at p. Let $\theta: M_+\setminus\{p\}\to[0, \pi]$ be the continuous function defined by

$$\theta(q)=\sphericalangle(\gamma'(0), \mu_q'(0)).$$

For each $\theta\in(\theta(y), \pi)$, let $e_\theta:[0, d(y, c_2(\theta))]\to M_+$ denote a minimizing geodesic joining y to $c_2(\theta)$. Fix $\theta_1\in(\theta(y), \pi)$ in such a way that $\sphericalangle(c'(d(x, y)), c_1'(0))$ is less than $\pi/2$, where $c_1=e_{\theta_1}$. It is possible to choose such a number θ_1, since $\sphericalangle(c'(d(x, y)), c_2'(\theta(y)))$ is less than $\pi/2$. Let D denote the relatively compact domain bounded by $c, c_1, \gamma_{\theta_1}|[0, d(p, y)]$ and $\gamma|[0, d(p, x)]$. Let d_D denote the Riemannian distance function on $\bar{D}$ defined by

$$d_D(x_1, x_2)=\inf\{L(c);\ c \textit{ is a piecewise smooth curve in } \bar{D} \textit{ joining } x_1 \textit{ to } x_2\},$$

where $L(c)$ denotes the length of c. If a curve e in $\bar{D}$ realizes the d_D-distance between its endpoints, then e will be called *d_D-minimizing*. We shall prove that there exists a d_D-minimizing geodesic emanating from x which is longer than c. If the inner angle of D at y is not greater than π, then there exists a d_D-minimizing geodesic β in $\bar{D}$ joining x to $c_2(\theta_1)$ which does not intersect the boundary of D except for its endpoints. Note that β is a geodesic segment in M which is longer than c by Lemma 1.3. Suppose that the inner angle of D at y is greater than π and that $d_D(x, y)=L(c)$ is less than $d_D(x, \tilde{c}(L(c)+\varepsilon_0))$ for a sufficiently small positive ε_0. Here $\tilde{c}$ denotes the geodesic extension of c. Note that $\tilde{c}(L(c), L(c)+\varepsilon)$ lies in D for all sufficiently small positive ε, since the inner angle of D at y is greater than π. If β denotes a d_D-minimizing geodesic joining x to $\tilde{c}(L(c)+\varepsilon_0)$, then β, which is longer than c, lies in D except for $x=\beta(0)$. Suppose that the inner angle of D at y is greater than π and that $d_D(x, y)=L(c)$ is not less than $d_D(x, \tilde{c}(L(c)+\varepsilon))$ for any sufficiently small positive ε. It follows from the first variation formula that $L(c_1)=d(y, c_2(\theta_1))$ is greater than $d(c_2(\theta_1), \tilde{c}(L(c)+\varepsilon_0))$ for a sufficiently small positive ε_0. By connecting a d_D-minimizing geodesic joining x to $\tilde{c}(L(c)+\varepsilon_0)$ and the d_D-minimizing geodesic joining $\tilde{c}(L(c)+\varepsilon_0)$ to $c_2(\theta_1)$, we get a piecewise smooth curve joining x to $c_2(\theta_1)$ in $\bar{D}$ which is shorter than $c\vee c_1$, where $c\vee c_1(t)=c(t)$ for $0\leqq t\leqq L(c)$ and $c\vee c_1(t)=c_1(t-L(c))$ for $L(c)\leqq t\leqq L(c)+\varepsilon_0$. This implies that there exists a d_D-minimizing geodesic β joining x to $c_2(\theta_1)$, which is longer than c by Lemma 1.3. Therefore there exists a d_D-minimizing geodesic β emanating from x such that β is longer than c and $\theta(\beta(L(\beta)))>\theta(y)$, whether or not the inner angle of D at y is greater than π. For each $s\in(0, L(c))$, let $t(s)\in(0, L(\beta))$ be the unique parameter of β satisfying

$$\theta(\beta(t(s)))=\theta(c(s)).$$

Note that both functions $\theta\circ\beta$ and $\theta\circ c$ are monotone increasing, since p is a pole. Fix $s\in(0, L(c))$ and let (a, b) be the maximal open subinterval containing $t(s)$ of $[0, L(\beta)]$ such that

$$d(p, \beta(t))<d(p, c(s))$$

for any $t\in(a, b)$. If $a=0$ (resp. $b\geqq L(c)$), then it is trivial that $s>a$ (resp. $s<b$). Hence we may assume $a>0$ (resp. $b<L(c)$) in order to show $s>a$ (resp. $s<b$). From the maximality of (a, b) we get

$$d(p, c(s))=d(p, \beta(a))=d(p, \beta(b)).$$

Since $\theta(\beta(a))<\theta(c(s))=\theta(\beta(t(s)))<\theta(\beta(b))$, it follows from Lemma 1.3 that

$$a=d_D(x, \beta(a))=d(x, \beta(a))<s=d(x, c(s))<d(x, \beta(b))=d_D(x, \beta(b))=b.$$

Note that $\beta|[0, s_1]$, where $\theta(\beta(s_1))=\theta(y)$, is minimizing in M, since p is a pole. The above inequality implies

$$d(p, \beta(s))<d(p, c(s))$$

for any $s\in(0, L(c))$. Since M is a von Mangoldt's surface of revolution,

$$G(c(s))\leqq G(\beta(s))$$

for any $s\in[0, L(c)]$. By the Rauch comparison theorem x is conjugate to $\beta(s_1)$ along β for some $s_1\in(0, L(c)]\subset(0, L(\beta))$. Therefore there exists a shorter curve joining the endpoints of β in $\bar{D}$ than β. This contradicts the fact that β is d_D-minimizing.

REMARK. The above theorem was proved for a flattening surface of revolution by Elerath [3]. Since any geodesic balls centered at the vertex on a flattening surface of revolution is totally convex, the proof is easier.

PROOF of MAIN THEOREM. If C_x is non-empty, then τ_x is not a ray by Lemma 1.2 or Theorem 1.4. Hence there exists a cut point $\tau_x(t_0)$ $(t_0>d(p, x))$ of x along τ_x. Since $\tau_x|[0, t]$ is not minimal for any $t>t_0$, there exist at least two minimizing geodesics joining x to $\tau_x(t)$, each of which is distinct from τ_x. Note that M has the reflection fixing τ_x. Hence $\tau_x(t)$ is a cut point of x for any $t>t_0$. This implies $C_x\supset\mu_x[t_0-d(p, x), \infty)$. Therefore by Theorem 1.4 $C_x=\mu_x[t_1, \infty)$, where $t_1=t_0-d(p, x)$. If $\mu_x(t_1)$ is not conjugate to x along τ_x, then there exists a minimizing geodesic $c: [0, t_0]\to M$, which is distinct from τ_x, joining x to $\tau_x(t_1)$. By Theorem 1.4, $\mu_x(t_1)$ is conjugate to x along c. Since $d(p, c(t))\geqq d(p, \tau_x(t))$ for any $t\in[0, t_0]$ by the triangle inequality, $G(c(t))\leqq G(\tau_x(t))$ on $[0, t_0]$. By the Rauch comparison theorem $\tau_x|[0, t_0]$ has a conjugate point to x along τ_x. But this is a contradiction.

A part of the above proof leads us to the following proposition.

PROPOSITION 1.5. *Let x be a point on a von Mangoldt's surface M of revolution with vertex p. If x is not a pole and if there exists a minimizing geodesic c joining x to $\hat{x}$ which is distinct from τ_x, then the Gaussian curvature G of the surface M is constant on $\bar{B}(p, r_0)$, where $r_0=\mathrm{Max}\{d(p, c(t));\ 0\leqq t\leqq d(x, \hat{x})\}$.*

PROOF. If D denotes the domain bounded by c and τ_x, then any geodesic emanating from x in $\bar{D}$ must pass through $\hat{x}$ by Theorem 1.4. Furthermore those geodesic segments joining x to $\hat{x}$ have the same length. This implies that x is conjugate to $\hat{x}$ along any geodesic segment joining x to $\hat{x}$. Since $G(c(t))\leqq G(\tau_x(t))$ on $[0, d(x, \hat{x})]$, it follows from the Rauch comparison theorem that

$$G(c(t))=G(\tau_x(t))$$

on $[0, d(x, \hat{x})]$. In particular

$$G(c(d(p, x)))=G(\tau_x(d(p, x)))=G(p).$$

Hence $G=G(p)$ on $\bar{B}(p, d(p, c(d(p, x))))$. Suppose that r_0 is greater than r_1, where $r_1:=\mathrm{Max}\{r>0;\ G=G(p) \text{ on } \bar{B}(p, r)\}$. There exist a $t_1\in(0, d(x, \hat{x}))$ and

$\varepsilon>0$ such that $d(p, c(t_1))=r_1$, $d(p, c(t))>r_1$ on $(t_1, t_1+\varepsilon)$ or $(t_1-\varepsilon, t_1)$. Note that r_1 is not less than $d(p, c(d(p, x)))$. Since $r_1=d(p, c(t_1))$ is greater than $d(p, \tau_x(t_1))$ by the triangle inequality, there exists a positive b such that for any t with $|t-t_1|<b$,

$$d(p, \tau_x(t)) < r_1$$

and hence $G(p)=G(\tau_x(t))$. Since $G(c(t))=G(\tau_x(t))$ for any $t\in[0, d(x, \hat{x})]$, there exists a $t_0\in(t_1-b, t_1+b)$ such that $d(p, c(t_0))>r_1$ and $G(p)=G(c(t_0))$. This contradicts the definition of r_1. Hence we have $r_0\leqq r_1$ and in particular G is constant on $\bar{B}(p, r_0)$.

2. Examples.

We shall give some examples of von Mangoldt's surface of revolution. Let $f:[0, \infty)\to R$ be a smooth function such that $f(\sqrt{x^2+y^2})$ is smooth at $(x, y)=(0, 0)\in R^2$, and $f(0)=0$, $f'\geqq 0$, $f''\geqq 0$ on $[0, \infty)$. The Gaussian curvature G of the surface $M(f)$ is constant on the parallel with radius r and equals

$$f'f''/r(1+f'^2)^2 \qquad \text{if } r>0$$

or

$$f''(0)^2 \qquad \text{if } r=0\,.$$

Let (r, θ) be the canonical local coordinates for $M(f)$ satisfying

$$x=(r(x)\cos\theta(x),\quad r(x)\sin\theta(x),\quad f(r(x)))$$

for $x\in M(f)\setminus\{p\}$, where $p=(0, 0, 0)$. By the integral formula (6) in [**1**], p. 258, we can show that

$$\lim_{s\to\infty}\{\theta(s)-\theta(0)\}\leqq\pi$$

for any geodesic $(r(s), \theta(s))$ emanating from q if $\int_1^\infty x^{-2}f'dx$ is finite and if q is sufficiently close to p. In particular τ_q is a ray. By Lemma 1.2 we obtain,

LEMMA 2.1. *Let $f:[0, \infty)\to R$ be a smooth function satisfying*

1) $f(0)=0$, $f'\geqq 0$, $f''\geqq 0$ *on* $[0, \infty)$,

2) *the function* $z=f(\sqrt{x^2+y^2})$ *is smooth at* $(x, y)=(0, 0)$,

3) *the function* $f'\cdot f''\cdot x^{-1}(1+f'^2)^{-2}$ *is monotone nonincreasing on* $(0, \infty)$.

Then $M(f)$ is a positively curved von Mangoldt's surface of revolution with vertex $(0, 0, 0)$. *Furthermore if $\int_1^\infty x^{-2}f'dx$ is finite, then $r(M(f))$ is positive. Here $r(M(f))$ is the number defined above Lemma* 1.2.

EXAMPLE 1. Let $f_1:[0, \infty)\to R$ be the function defined by

$$f_1(x) = a\sqrt{x^2+b} - a\sqrt{b}$$

where a, b are positive constants. The function f_1 satisfies 1), 2), 3) in Lemma 2.1. Hence $M_1 = M(f_1)$, a revolutionary hyperboloid of two sheets, is a von Mangoldt's surface of revolution with $r(M_1) > 0$. The total curvature $C(M_1)$ of M_1, the integral of the Gaussian curvature by its area element, equals

$$C(M_1) = 2\pi - 2\pi/\sqrt{1+a^2}\,.$$

EXAMPLE 2. Let M_2 be the surface defined by $z = x^2 + y^2$. It is known that the vertex is the unique pole. Since $f_2(x) = x^2$ satisfies 1), 2), 3) in Lemma 2.1, $M_2 = M(f_2)$ is a von Mangoldt's surface of revolution with total curvature 2π.

EXAMPLE 3. Let $\phi : [0, \infty) \to [0, 1]$ be a smooth monotone nondecreasing function such that

$$\phi(0) = 0 \quad \text{on } [0, 2], \qquad \phi(x) = 1 \quad \text{on } [4, \infty)\,.$$

Then a function $f_3 : [0, \infty) \to R$ is defined by

$$f_3''(x) = 2(1-\phi(x)) + \phi(x)/x, \qquad f_3 = f_3' = 0 \quad \text{at } x = 0\,.$$

In order to check that f_3 satisfies 3) in Lemma 2.1, we introduce a function g defined by

$$g(t) = t(1+t^2)^{-2}\,.$$

Since the function g is monotone decreasing on $[1/\sqrt{3}, \infty)$. $g(f_3'(x))$ is monotone nonincreasing on $[1/2\sqrt{3}, \infty)$. Thus on this interval the function $f_3' \cdot f_3'' \cdot x^{-1}(1+f_3'^2)^{-2} = f_3'' \cdot x^{-1} g(f_3'(x))$ is also monotone non-increasing. Since $f_3(x) = x^2$ on $[0, 2]$,

$$f_3' \cdot f_3'' \cdot x^{-1}(1+f_3''^2)^{-2} = 4(1+4x^2)^{-2}$$

is monotone decreasing on $[0, 2]$. Therefore $M_3 = M(f_3)$ is a von Mangoldt's surface of revolution with $r(M_3) > 0$. The total curvature of M equals 2π.

REMARK. By Maeda's result in [**5**], if a surface of revolution M admits a point q such that A_q is of Lebesgue measure zero, then $C(M)$ equals 2π if it exists. Here A_q is the set defined in the proof of Lemma 1.1. But the converse is false by Example 3. In fact for any point q on M_3, A_q contains an open arc in $S_q M_3$ since $r(M_3)$ is positive, while $C(M_3)$ equals 2π.

All surfaces M_i, $i = 1, 2, 3$ are flattening surfaces of revolution. In Example 4, we shall give a von Mangoldt's surface of revolution, but not a flattening surface. In order to construct and abstract surface of revolution, we remark the following lemma.

LEMMA 2.2. *Let* $m:(0, \infty)\rightarrow(0, \infty)$ *be a smooth function which has an extension of a smooth odd function around* 0 *with* $m'(0)=1$. *Then the Riemannian metric* $g=dr^2+m(r)^2d\theta^2$ *on* R^2 *defines a surface of revolution with vertex the origin. Here* (r, θ) *denotes geodesic polar coordinates for the Euclidean plane* (R^2, g_0).

PROOF. Let (x, y) be the canonical coordinates for (R^2, g_0), i.e., $x=r\cos\theta$, $y=r\sin\theta$. Then it follows from a direct computation that

$$g(\partial/\partial x, \partial/\partial x) = 1+y^2(m^2-r^2)/r^4$$

$$g(\partial/\partial x, \partial/\partial y) = -xy(m^2-r^2)/r^4$$

$$g(\partial/\partial y, \partial/\partial y) = 1+x^2(m^2-r^2)/r^4, \qquad \text{where } r=\sqrt{x^2+y^2}.$$

From Proposition 2.7 in [**4**], there exists a smooth function $f(x, y)$ such that $f(r\cos\theta, r\sin\theta)=(m^2-r^2)/r^4$. Hence g is smooth at the origin. Moreover each geodesic θ=constant emanating from the origin can be defined on $0\leqq r<\infty$. By the Hopf-Rinow theorem, g defines a complete Riemannian metric. Therefore (R^2, g) is a surface of revolution with vertex the origin.

EXAMPLE 4. Let $m_0:(0, \infty)\rightarrow(0, \infty)$ be a smooth function such that $m_0(r)=\sin r$ on $(0, 3\pi/4]$. Let G be a smooth monotone non-increasing function on $[0, \infty)$ such that

$$G=1 \quad \text{on } [0, 2\pi/3], \qquad G\leqq -m_0''/m_0 \quad \text{on } (0, \infty).$$

If m denotes the solution of the differential equation

$$m''+Gm=0 \qquad \text{on } [0, \infty)$$

with the initial condition $m(0)=0$, $m'(0)=1$, then $(R^2, dr^2+m(r)^2d\theta^2)$ is a von Mangoldt's surface of revolution with vertex the origin o. Since $m(r)=\sin r$ on $[0, 2\pi/3]$, $B(o, 2\pi/3)$ is isometric to an open ball with radius $2\pi/3$ in the unit sphere. Hence $B(o, t)$ $(\pi/2\leqq t\leqq 2\pi/3)$ is not totally convex. This implies the surface is not a flattening surface as we noted in the section 1.

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