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A remark on the Ochanine genus

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Introduction.

In a celebrated paper [13], Witten asserted the rigidity theorem for the elliptic genus, which is an extension of the Atiyah-Hirzebruch theorem concerning the $\hat{A}$-genus. The elliptic genus is a formal power series in q whose coefficients are the indices of the Dirac operators coupled with certain vector bundles associated to the tangent bundle. After some partial results of Landweber, Stong and Ochanine, Bott and Taubes gave a mathematically rigorous proof to the rigidity theorem [3].

Recently Ochanine defined the KO-version of the elliptic genus [8]. Bendersky and Ochanine proved that the Ochanine genus vanishes for spin manifolds admitting S^1-actions of odd type [2], [9]. In a previous paper [11], we proved that the α-invariant (which is called the Atiyah invariant in [2]) vanishes for spin manifolds admitting S^1-actions of odd type. The proof in [2], [9] is based on purely topological argument and we dealt with the analytical index of the real Dirac operator in [11]. The purpose of this note is to prove the vanishing result for the Ochanine genus by the method in [11]. Our method can also be applied to spin manifolds admitting involutions of odd type.

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1. The Ochanine genus.

In this section we review the definition of the Ochanine genus [8] and the rigidity theorem of Witten [3], [13].

For a real vector bundle $E \to X$, we set

$$\Lambda_t(E) := \sum_{i \geq 0} \Lambda^i(E) t^i , \qquad S_t(E) := \sum_{i \geq 0} S^i(E) t^i$$

where $\Lambda^i(E)$ and $S^i(E)$ denote the i-th exterior product bundle and symmetric power bundle of E respectively. The Witten characteristic class is defined by

$$\Theta_q(E) := \bigotimes_{n\geq 1}(\Lambda_{-q^{2n-1}}(E)\otimes S_{q^{2n}}(E)).$$

Since $\Theta_q(E\oplus F)=\Theta_q(E)\cdot\Theta_q(F)$ holds, we get

$$\Theta_q : KO(X) \longrightarrow KO(X)[[q]].$$

From the consideration of the loop space, Witten [13] considered the Dirac operator twisted by $\Theta_q(TM)$ and asserted the following

THE RIGIDITY THEOREM OF WITTEN ([3], [13]). *Let M be a closed spin manifold admitting an S^1-action of even type. Then the S^1-equivariant index of the Dirac operator twisted by $\Theta^i(TM)$ is constant as virtual character, where $\Theta^i(TM)$ denotes the coefficient of q^i in $\Theta_q(TM)$.*

For a m-dimensional closed spin manifold M, Ochanine defined $\beta_q[M]=\Theta_q(TM-m)[M]\in KO_m(point)[[q]]$, where $[M]\in KO_m(M)$ is the fundamental class of M in KO-theory. We call $\beta_q[M]$ the Ochanine genus of M. It is essentially the elliptic genus in the case that the dimension is divisible by 4 and the α-invariant is the 0-th order coefficient of the Ochanine genus.

2. Circle actions of odd type and the Ochanine genus.

Bendersky [2] and Ochanine [9] proved the following

THEOREM (2.1). *If a closed spin manifold M admits an S^1-action of odd type, the Ochanine genus of M vanishes.*

In this section we give a different proof using the method of [11]. First of all we recall the fundamental property of the real Dirac operator (see [1], [7]).

Let $P\to M$ be the principal $Spin(m)$-bundle which defines the spin structure of M, and Cl_m the Clifford algebra corresponding to $\boldsymbol{R}^m$ with the standard inner product. Set $V:=P\times_{Spin(m)}Cl_m$, where $Spin(m)$ acts on Cl_m by the multiplication from the left. V is naturally a $\boldsymbol{Z}/2\boldsymbol{Z}$-graded vector bundle $V^+\oplus V^-$ and carries the action of Cl_m from the right. For a real vector bundle F over M, the real Dirac operator $D\otimes F$ twisted by F is defined by $\sum e_i\cdot\nabla_{e_i}$, where $\{e_i\}$ is a local orthonormal frame field and ∇ denotes the tensor product connection on $V\otimes F$, and it commutes with the action of Cl_m from the right. In particular, $\operatorname{Ker} D\otimes F$ is a $\boldsymbol{Z}/2\boldsymbol{Z}$-graded Cl_m-module.

From now on, we assume that $m=8k+1$ or $8k+2$. The value of $F\in KO(M)$ evaluated on the fundamental class $[M]\in KO_m(M)$ is given by

$$\langle F, [M]\rangle \equiv 0 \pmod 2 \quad \text{if } \operatorname{Ker} D\otimes F \text{ can be a } Cl_{m+1}\text{-module.}$$

$$\langle F, [M]\rangle \equiv 1 \pmod 2 \quad \text{otherwise.}$$

By the periodicity of Clifford algebras, $Cl_{8k+l}\cong Cl_{8k}\otimes Cl_l$ and Cl_{8k} is a simple graded algebra. Let $W=W^+\oplus W^-$ be the irreducible $\boldsymbol{Z}/2\boldsymbol{Z}$-graded left Cl_{8k}-module. Since $Cl_{8k}=W\otimes W^*$ as a left and right Cl_{8k}-module, we get $V^\pm=E^\pm\otimes W^*$, where $E^\pm$ denotes the real spinor bundle $P\times_{Spin(m)}(W^\pm\otimes Cl_l)$. The twisted real Dirac operator $D\otimes F$ is Cl_{8k+l}-linear and can be restricted to the operator acting on sections of $E^\pm$. We also call the restricted operator as the twisted real Dirac operator. Recalling that $(Cl_1)^{\mathrm{ev}}=\boldsymbol{R}$ and $(Cl_2)^{\mathrm{ev}}=\boldsymbol{C}$, we obtain the following fact.

FACT (2.2). Let M be a closed m-dimensional spin manifold and F a real vector bundle over M. The evaluation of F with the KO-theoretic fundamental class $[M]\in KO_m(M)$ is given by

$$\dim_{\boldsymbol{R}} \operatorname{Ker}(D\otimes F\colon \Gamma(E^+\otimes F)\longrightarrow \Gamma(E^-\otimes F)) \text{ in case that } m=8k+1$$

$$\dim_{\boldsymbol{C}} \operatorname{Ker}(D\otimes F\colon \Gamma(E^+\otimes F)\longrightarrow \Gamma(E^-\otimes F)) \text{ in case that } m=8k+2$$

modulo 2, where D denotes the real Dirac operator on M.

We show the following

LEMMA (2.3). *Let M be a closed m-dimensional spin manifold admitting an S^1-action of odd type.*

(1) *In case that $m=8k+1$, we have*

$$\dim_{\boldsymbol{R}} \operatorname{Ker} D\otimes\Theta^i(TM) \equiv 0 \pmod 2.$$

(2) *In case that $m=8k+2$, we have*

$$\dim_{\boldsymbol{C}} \operatorname{Ker} D\otimes\Theta^i(TM) \equiv 0 \pmod 2.$$

PROOF. By taking the double covering action, the operator $D\otimes\Theta^i(TM)$ can be considered as an S^1-invariant differential operator.

(1) Decompose the vector space $\operatorname{Ker} D\otimes\Theta^i(TM)$ into the direct sum of S^1-modules $\oplus_l W_l$, where l is the weight of W_l as a real S^1-module. W_l is even dimensional for $l\neq 0$, therefore it is sufficient to show that $W_0=0$. Since the original S^1-action is of odd type, the action of $-1\in S^1$ on spinor bundles is the multiplication by -1 and the action of -1 on $\Theta^i(TM)$ is the identity. Hence there are no sections of $E^0\otimes\Theta^i(TM)$ which are invariant by the action of -1 and we get $W_0=0$.

(2) Decompose the vector space $\operatorname{Ker} D\otimes\Theta^i(TM)$ into the direct sum of complex S^1-modules $\oplus_k N_k$, where k is the weight of N_k as a complex

S^1-module. By Lemma (2.4) below, $\dim_C N_k = \dim_C N_{-k}$ for $k \neq 0$. Hence $\dim_C \operatorname{Ker} D \otimes \Theta^i(TM) \equiv \dim_C N_0$. The rest of the proof goes as in (1).

LEMMA (2.4). $\dim_C N_k = \dim_C N_{-k}$ for $k \neq 0$.

PROOF. In case that $m = 8k+2$, the real spinor bundles E^+ and E^- are nothing but the complex spinor bundles S^+ and S^-, which are complex conjugate each other, and the real Dirac operator D is the usual Dirac operator D^+ acting on sections of complex spinor bundles. The adjoint of $D^+ \otimes \Theta^i(TM)$ is $D^- \otimes \Theta^i(TM)$. Recall that $\operatorname{Ker} D^+ \otimes \Theta^i(TM)$ is decomposed into $\bigoplus_k N_k$. Decompose the vector space $\operatorname{Ker} D^- \otimes \Theta^i(TM)$ into the direct sum of complex S^1-modules $\bigoplus_k M_k$, where k is the weight of M_k as a complex S^1-module. The rigidity theorem of Witten implies that N_k and M_k are equi-dimensional for $k \neq 0$. Since S^+ and S^- are complex conjugate, N_k and M_{-k} are equi-dimensional. Hence we get the conclusion.

PROOF OF THEOREM (2.1). Note that the coefficient of q^i in $\Theta_q(TM-m)$ is a linear combination of $\Theta^j(TM)$ for $0 \leqq j \leqq i$. The assertion follows from Fact (2.2) and Lemma (2.3).

3. Involutions of odd type and the Ochanine genus.

$8k+1$-dimensional case: Theorem (2.1) is strengthened as follows.

PROPOSITION (3.1). *Let M be an $8k+1$-dimensional closed spin manifold admitting an involution of odd type. For a $\boldsymbol{Z}/2\boldsymbol{Z}$-equivariant real vector bundle F over M, $\langle F, [M] \rangle$ vanishes. In particular, the Ochanine genus of M vanishes.*

PROOF. The involution τ on M is lifted to an automorphism $\tilde{\tau}$ of the principal spin bundle, which defines the spin structure. Since τ is of odd type, $\tilde{\tau}$ is a periodic mapping of order 4. Therefore the vector space $\operatorname{Ker} D \otimes F$ is a $\boldsymbol{Z}/4\boldsymbol{Z}$-module and the element of order 2 acts as multiplication by -1. The rest of the proof goes in a similar way as in Theorem (2.1).

$8k+2$-dimensional case: For a torus $T^2 = S^1 \times S^1$ and the product spin structure, $\tau \times \tau$ is an involution of odd type, where τ is a reflection of S^1, i.e. $(x, y) \mapsto (-x, y)$ on the unit circle S^1 in $\boldsymbol{R}^2$, and the α-invariant does not vanish. Hence Proposition (3.1) is not true in this dimension. Although we can show the following

PROPOSITION (3.2). *Let M be an $8k+2$-dimensional closed spin manifold admitting a $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$ action. If the number of involutions of odd type is* 1 *or* 3, *$\langle F, [M] \rangle$ vanishes for a $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$-equivariant real vector bundle F over M. In particular, the Ochanine genus of M vanishes.*

PROOF. For a group Γ acting on a spin manifold M, there is a central extension $\tilde{\Gamma}$ of Γ by $\boldsymbol{Z}/2\boldsymbol{Z}$ such that the action of $\tilde{\Gamma}$ on M is lifted to the principal spin bundle $P \to M$. Central extensions of $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$ are the following four groups.

1) $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/2\boldsymbol{Z}$

2) $\boldsymbol{Z}/2\boldsymbol{Z} \times \boldsymbol{Z}/4\boldsymbol{Z}$

3) $\boldsymbol{D}_4$, the dihedral group of order 8

4) $\{\pm 1, \pm i, \pm j, \pm k\}$ where i, j, k are the standard generators of the quaternion algebra.

If the number of involutions of odd type is 1, the extension is the group in the case 3). The unique involution τ of odd type is lifted to an automorphism $\tilde{\tau}$ of P of period 4. The group $\tilde{\Gamma}$ acts on the vector space $\mathrm{Ker}\, D \otimes F$. Restricting to the subgroup generated by $\tilde{\tau}$, the character is invariant under the inner automorphism of $\tilde{\Gamma}$. We can show that $\tilde{\tau}^2$ acts on sections of spinor bundles by multiplication by -1, hence the value of the character at $\tilde{\tau}$ is 0. Thus $\dim_{\boldsymbol{C}} \mathrm{Ker}\, D \otimes F$ must be even and we get the result.

If the number of involutions of odd type is 3, the extension is the group in the case 4). The rest of the proof goes in a similar way as above.

Appendix. Involutions of odd type on $8k$-dimensional spin manifolds.

For an $8k+4$-dimensional closed spin manifold M, the Rochlin-Ochanine theorem [**10**] states that the signature of M is divisible by 16. This is not true in dimension of $8k$. For example, the signature of the quaternionic projective plane $\boldsymbol{H}P^2$ is 1. We can show the following

PROPOSITION. *Let M be an $8k$-dimensional spin maifold admitting an involution τ of odd type. Then the equivariant signature* $\mathrm{Sign}(\tau, M)$ *is divisible by 16. In particular, the signature of M is even. Moreover any other coefficient of the q-expansion of the equivariant elliptic genus $\Phi_q(\tau, X)$ at the signature cusp is divisible by* 2^9.

PROOF. Let $F = \cup F_i$ be the fixed point set of the involution τ and $\nu_i \to F_i$ the normal bundle of F_i. The equivariant signature $\mathrm{Sign}(\tau, M)$ equals the sum of the signatures of the self-intersection S_i of F_i. The normal bundle of S_i is $\nu_i \oplus \nu_i|_{S_i}$. Since ν_i is orientable [**4**], [**12**], the second Stiefel-Whitney class of $\nu_i \oplus \nu_i$ vanishes. Thus S_i is spin. Since τ is of odd type, codim F_i is congruent to 2 modulo 4 and codim S_i is congruent to 4 modulo 8. By the Rochlin-Ochanine theorem, $\mathrm{Sign}(S_i)$ is divisible by 16. Hence $\mathrm{Sign}(\tau, M)$ is divisible by 16. The rest of the statement follows from the following results.

THEOREM [**5**]. *Let X be an $8k+4$-dimensional closed spin manifold. Then any coefficient of the q-expansion of the elliptic genus $\Phi_q(X)$ at the signature cusp* *is divisible* 2^9 *except the constant term, i.e.* Sign(X).

THEOREM [6]. *Let* X *be a closed oriented manifold with an orientation preserving involution* τ. *Then we have*

$$\Phi_q(\tau, X) = \Phi_q(X^\tau \circ X^\tau),$$

where $\Phi_q(\tau, X)$ *is the q-expansion of the equivariant elliptic genus at the signature cusp and* $\Phi_q(X^\tau \circ X^\tau)$ *is the q-expansion of the elliptic genus of the self-intersection submanifold* $X^\tau \circ X^\tau$ *at the signature cusp.*

REMARK 1. Under the same assumption in the Proposition above, we can show that the $\hat{A}$-genus of M is even.

REMARK 2. Hirzebruch [5] also proved that any coefficient of the q-expansion of $\Phi_q(X)$ is divisible by 2^{12} except the constant term.

Addendum (added on February 4, 1992).
K. Liu obtained the same result in $8k+1$-dimensional case.
K. Liu, On some vanishing theorems for elliptic genera, Harvard Univ., December 1991, preprint

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