J. Math. Soc. Japan
Vol. 45, No. 1, 1993

Manifold of primitive idempotents in a Jordan-Hilbert algebra

By Takaaki NOMURA

(Received Dec. 11, 1991)

Introduction.

In this paper, we study an infinite dimensional analogue of compact Riemannian symmetric spaces of rank one by using a Jordan-algebraic method. This work is motivated by the paper [6] of U. Hirzebruch in which it is shown that the set of primitive idempotents in a finite dimensional simple formally real Jordan algebra is a compact Riemannian symmetric space of rank one and that any such space arises in this way. So, as a natural step of generalization, we consider an infinite dimensional analogue of formally real Jordan algebras and treat the set of primitive idempotents. Since it is the associative inner product, not the algebraic formally real property, that plays a significant role in [6], we shall base our study on Jordan-Hilbert algebras (for definition, see the beginning of § 1) and deal with the set $\mathfrak{I}_1$ of primitive idempotents as a Riemannian Hilbert manifold. We prove that $\mathfrak{I}_1$ is two-point homogeneous (Theorem 5.3) and derive a unified formula for the sectional curvature of $\mathfrak{I}_1$ (Theorem 6.4).

Let us explain the matters which do not occur in the finite dimensional cases. First, Jordan-Hilbert algebras do not necessarily have a unit element. Note that the adjunction of unit element does not in general agree with the Hilbert space structure. However, this lack of unit element is compensated to some extent by (a version of) McCrimmon's theorem [11]. Our version of his theorem states that in a topologically simple (non-trivial) Jordan-Hilbert algebra, the Peirce 1-space is also topologically simple (see Proposition 1.6). This enables us to carry out computations concerning idempotents as in the finite dimensional cases.

Next, infinite dimensional connected complete (in the sense of Riemannian distance) Riemannian manifolds may carry two points which cannot be joined by a minimal geodesic [4], [9, p. 127]. Because of this possibility of missing minimal geodesic, we compute the Riemannian distance on $\mathfrak{I}_1$ along the standard line of textbooks on Riemannian geometry: the inclusion map of $\mathfrak{I}_1$ into the ambient Hilbert space being an embedding, we define the canonical Levi-Civita connection, exhibit a geodesic, examine the diffeomorphism domain of the exponential mapping and make use of Gauß lemma to derive the minimality of geodesics.

We organize this paper as follows. In § 1, we give the definition of a Jordan-Hilbert algebra V and state some basic facts. In § 2, propositions involving idempotents are presented. Although these are known in the finite dimensional cases, some require a modified proof (*e.g.*, Proposition 2.2), whereas others do not (*e.g.*, Proposition 2.9). In any case, we supply each of them with a proof in order to make this paper readable.

In § 3, we introduce a Riemannian Hilbert manifold structure on the set $\mathfrak{J}_1$ of primitive idempotents in V in an explicit way. Since the automorphism group $\mathrm{Aut}(V)$ of the Jordan-Hilbert algebra V is merely a Banach-Lie group, it is not so evident that one can introduce a Hilbert manifold structure on $\mathfrak{J}_1$ through the aid of $\mathrm{Aut}(V)$. Such being the case, we shall give a concrete atlas using the maps ξ_a defined by (3.3). We also show that the inclusion $\mathfrak{J}_1 \hookrightarrow V$ is an embedding, so that $\mathfrak{J}_1$ has a natural Riemannian manifold structure.

In § 4, we compute the Riemannian distance by introducing the canonical Levi-Civita connection through the embedding $\mathfrak{J}_1 \hookrightarrow V$ as already mentioned above. The resulting formula for Riemannian distance between two points $a, b \in \mathfrak{J}_1$ is given in Theorem 4.5, showing that it is equal to the length of a geodesic segment connecting a and b.

In § 5, two-point homogeneity of $\mathfrak{J}_1$ is established. This is done by first proving the isotropy property (the stabilizer at $a \in \mathfrak{J}_1$ acts transitively on the unit sphere in the tangent space at a). Since we have a minimizing geodesic at hand, two-point homogeneity follows from this without difficulty.

In § 6, we give a formula to the sectional curvature $k_a(x, y)$ of $\mathfrak{J}_1$ (Theorem 6.4). This formula is visible enough to derive the estimate $1/2 \leqq k_a(x, y) \leqq 2$ immediately (Corollary 6.5). Of course, the formula accounts for the constancy of $k_a(x, y)$ for real projective spaces and for spheres in Hilbert spaces (see the end of § 6).

The major part of progress in this work was made while I was staying at Nancy I university in France during the second semester of 1990/91. I am grateful to all of the staff at Nancy I university for a friendly atmosphere. In particular, I express my gratitude to Professor Pierre Eymard who made my stay fruitful.

§ 1. Jordan-Hilbert algebra.

Let V be a *Jordan-Hilbert algebra* (*JH*-algebra for short). By this we mean

(1) V is a real Hilbert space with inner product $\langle \cdot | \cdot \rangle$,

(2) V is a real Jordan algebra, so that a bilinear product $x, y \mapsto xy$ is defined and one has for all $x, y \in V$,

$$\text{(i)}\quad xy = yx\,, \qquad \text{(ii)}\quad x^2(xy) = x(x^2y)\,,$$

(3) $\langle xy | z \rangle = \langle y | xz \rangle$ for all $x, y, z \in V$.

In what follows, we denote by $L(x)$ the multiplication operator by $x : L(x)y = xy$. We also use the operators

$$P(x) := 2L(x)^2 - L(x^2)\,, \qquad x \Box y := L(xy) + [L(x), L(y)]\,,$$

where $[A, B] := AB - BA$ for two operators A, B. As is well-known, we have $P(x^n) = P(x)^n$ $(n = 1, 2, \cdots)$.

LEMMA 1.1. *The Jordan product* $V \times V \ni (x, y) \mapsto xy \in V$ *is continuous.*

PROOF. Since $L(x)$ is a symmetric operator defined everywhere on V, the closed graph theorem implies $L(x) \in \boldsymbol{B}(V)$, where $\boldsymbol{B}(V)$ denotes the Banach algebra of all bounded linear operators defined everywhere on V. Consider the family $\mathcal{L} := \{L(x);\ \|x\| \leqq 1\}$ of bounded linear operators. Since

$$\|L(x)y\| = \|xy\| = \|L(y)x\| \leqq \|L(y)\| \qquad \text{for } \|x\| \leqq 1\,,$$

the family $\mathcal{L}$ is pointwise bounded. Hence Banach-Steinhaus theorem says that $\mathcal{L}$ is uniformly bounded. This implies that the linear operator $L : V \ni x \mapsto L(x) \in \boldsymbol{B}(V)$ is bounded, so that we get

$$\|xy\| = \|L(x)y\| \leqq \|L(x)\| \|y\| \leqq \|L\| \|x\| \|y\|\,. \qquad \blacksquare$$

Later we will normalize the inner product of V in a convenient way to our subject matter (see Proposition 2.2). Throughout this paper, the following basic assumptions are in force:

$$\text{(i)}\quad \dim V > 0, \qquad \text{(ii)}\quad L(x) = 0 \quad \text{implies } x = 0.$$

The assumption (ii) is necessary in order to exclude the algebra in which $xy = 0$ for all x, y. We also remark that V does not necessarily have a unit element (consider for instance the JH-algebra of symmetric Hilbert-Schmidt operators on an infinite dimensional real Hilbert space). We need the following lemma to ensure the existence of non-zero idempotents.

LEMMA 1.2. (1) $x^2 = 0$ *implies* $x = 0$.
(2) *If* $x \neq 0$, *then* $x^n \neq 0$ *for any* $n = 1, 2, \cdots$.

PROOF. (1) Since $P(x)$ is a selfadjoint operator, we have for any $y \in V$

$$\|P(x)y\|^2 = \langle P(x)y | P(x)y \rangle = \langle P(x^2)y | y \rangle = 0.$$

Thus, $2L(x)^2=P(x)+L(x^2)=0$, so that $\|L(x)y\|^2=\langle L(x)^2y\,|\,y\rangle=0$. By the basic assumption (ii), we get $x=0$.

(2) A simple inductive argument. ■

In particular, there is $x\in V$ such that $P(x)x=x^3\neq 0$. This means that, the Hilbert Jordan triple system associated to V defined by introducing the triple product $\{x, y, z\} := (x\square y)z$ satisfies the condition of [**12**, Theorem IV.2.4] (see also [**8-II**, Lemma 3.3]). Consequently, there are non-zero $c\in V$ and $\varepsilon=\pm 1$ such that $c^3=\varepsilon c$. But

$$0 < \langle c\,|\,c\rangle = \varepsilon\langle c^3\,|\,c\rangle = \varepsilon\|c^2\|^2$$

forces $\varepsilon=1$, and $(c^2)^2=c^4=c^2$ implies that c^2 is a non-zero idempotent. Thus we have shown

PROPOSITION 1.3. *V contains a non-zero idempotent.*

Now, for every idempotent $a\in V$, we have the *Peirce decomposition* of V relative to a (see [**15**, Theorem 21.2]): for $k=0, 1/2, 1$, let us put $V_k(a) := \{x\in V\,;\, ax=kx\}$, then

$$V = V_0(a)\oplus V_{1/2}(a)\oplus V_1(a) \qquad \text{(orthogonal direct sum)} \tag{1.1}$$

with the multiplication rule: (here we write V_k instead of $V_k(a)$ for simplicity)

$$\begin{cases} V_0V_0 \subset V_0, & V_0V_1 = \{0\}, \qquad V_1V_1 \subset V_1, \\ V_kV_{1/2} \subset V_{1/2} \quad (k=0, 1), & V_{1/2}V_{1/2} \subset V_0+V_1. \end{cases} \tag{1.2}$$

Moreover,

$$[L(u), L(v)] = 0 \qquad \text{for } u\in V_0(a),\ v\in V_1(a). \tag{1.3}$$

The orthogonal projections $E_k(a)$ of V onto $V_k(a)$ are expressed respectively as

$$\begin{aligned} E_1(a) &= P(a), \\ E_{1/2}(a) &= 4L(a)-4L(a)^2 = 2L(a)-2P(a), \\ E_0(a) &= I-2L(a)+P(a). \end{aligned} \tag{1.4}$$

Two idempotents $a, b\in V$ are said to be *orthogonal* if $ab=0$. This is equivalent to the orthogonality with respect to the inner product:

LEMMA 1.4. *Idempotents a, b are orthogonal if and only if $\langle a\,|\,b\rangle=0$.*

PROOF. Let $b=b_0+b_{1/2}+b_1$ be the Peirce decomposition of b relative to a, so that $b_k\in V_k(a)$. Then, the lemma is clear from $\langle a\,|\,b\rangle=\langle ab\,|\,b\rangle=\|b_{1/2}\|^2/2+\|b_1\|^2$. ■

A non-zero idempotent a is said to be *primitive* if a cannot be expressed as a sum of non-zero orthogonal idempotents. Arguing just as in the proof of [**12**, Corollary IV.2.7], one sees easily that every non-zero idempotent in V can be written as a *finite* sum of orthogonal primitive idempotents. Furthermore, every element $x\in V$ has the following *spectral decomposition*:

$$x=\sum_{k=1}^{\infty}\lambda_k a_k \qquad \text{(convergent in norm)}, \tag{1.5}$$

where the a_k are mutually orthogonal primitive idempotents and the λ_k are real numbers. This is obtained by taking a maximal orthogonal family of primitive idempotents in the *associative* closed algebra $\overline{\boldsymbol{R}[x]}$, the closure of the algebra generated by $x, x^2, \cdots$ (cf. [**12**, Corollary IV.2.9]).

LEMMA 1.5. *A non-zero idempotent a is primitive if and only if $V_1(a)=\boldsymbol{R}a$.*

PROOF. If a is not primitive, then $a=a_1+a_2$ with non-zero orthogonal idempotents a_1, a_2. Since $aa_j=a_j$ $(j=1, 2)$, one has $\dim V_1(a)\geqq 2$. Conversely, suppose a is primitive. Let $b\in V_1(a)$ be a non-zero idempotent. Then, $a-b$ is an idempotent orthogonal to b as is easily seen. Since $a=(a-b)+b$, the primitivity of a then implies $b=a$. This means that $V_1(a)$ does not contain non-zero idempotents other than a. Thus, only the idempotent a appears in the spectral decomposition (1.5) for every $x\in V_1(a)$. Hence $\dim V_1(a)=1$. ∎

We say that V is *topologically simple* if V has no non-trivial closed ideal. The following proposition compensates the lack of unit element to some extent.

PROPOSITION 1.6 [**11**]. *If V is topologically simple, then for any non-zero idempotent a, the subalgebra $V_1(a)$ is also topologically simple.*

REMARK. McCrimmon [**11**] works in the framework of quadratic Jordan algebras. An almost word for word translation of the proof of his Theorems 1.11 and 2.11 shows that if I_1 is a closed ideal $I_1\lhd V_1(a)$, then

$$I:=\overline{P(V_{1/2}(a))I_1}\oplus\overline{L(V_{1/2}(a))I_1}\oplus I_1 \quad \text{(direct sum in accordance with (1.1))}$$

is a closed ideal of V, where the bars stand for the closure and $P(V_{1/2}(a))I_1$, for example, is the subspace spanned by $P(x)u$ with $x\in V_{1/2}(a)$ and $u\in I_1$. The only thing to be noted here is the following "strong semiprimeness" (in the sense of [**11**]) of a closed ideal of $V_1(a)$.

LEMMA 1.7. *Suppose I_1 is a closed ideal $I_1\lhd V_1(a)$. Then, for $u\in V_1(a)$, one has $P(u)V_1(a)\subset I_1$ if and only if $u\in I_1$.*

PROOF. The if part being evident, we suppose $P(u)V_1(a)\subset I_1$. Then, since a is the unit element of $V_1(a)$, we have $u^2=P(u)a\in I_1$. Denote by $I_1^{\perp}$ the orthogonal complement of I_1 in $V_1(a)$. Then, we have an orthogonal direct sum of closed ideals: $V_1(a)=I_1\oplus I_1^\perp$. Let $u=u_1+u_1'$ $(u_1\in I_1,\ u_1'\in I_1^\perp)$. Then, $u^2\in I_1$ implies $u_1'^2=0$, so that we get $u_1'=0$ by Lemma 1.2. Hence $u=u_1\in I_1$. ∎

The rest of the proof of Proposition 1.6 is omitted.

§2. Pairs of primitive idempotents.

From now on, the JH-algebra V is supposed to be topologically simple. In this section, we prepare some propositions about idempotents. First of all, we note that $V_{1/2}(a)\neq\{0\}$ for every non-zero idempotent a. This is an immediate consequence of our simplicity assumption together with the multiplication rules (1.2). In the same way, we have the following

LEMMA 2.1. *Let a, b be non-zero orthogonal idempotents. Then one has*

$$V_{1/2}(a)\cap V_{1/2}(b)\neq\{0\}.$$

PROOF. Since $ab=0$, $c:=a+b$ is an idempotent. Thus (1.3) implies

$$V_1(c)=V_1(a)\oplus V_1(b)\oplus(V_{1/2}(a)\cap V_{1/2}(b)).$$

If $V_{1/2}(a)\cap V_{1/2}(b)=\{0\}$, then, $V_1(c)=V_1(a)\oplus V_1(b)$, a direct sum of non-zero closed ideals. This contradicts Proposition 1.6. ∎

Let $\mathfrak{J}_1=\mathfrak{J}_1(V)$ be the set of all primitive idempotents in V.

PROPOSITION 2.2. *All the elements in $\mathfrak{J}_1$ have the same norm.*

PROOF. Let a, $b\in\mathfrak{J}_1$ be distinct. We separate the cases $ab=0$ and $ab\neq0$.

(1) *Case* $ab=0$: By Lemma 2.1, there is non-zero $x\in V_{1/2}(a)\cap V_{1/2}(b)\subset V_1(a+b)$. Then, (1.2) shows $x^2\in V_1(a)+V_1(b)$, so that there are λ, $\mu\in\boldsymbol{R}$ with which $x^2=\lambda a+\mu b$ by virtue of Lemma 1.5. Thus,

$$(\lambda+\mu)x=2(\lambda a+\mu b)x=4x^2(xa)=4x(x^2a)=4x(\lambda a)=2\lambda x.$$

Since $x\neq0$, we get $\mu=\lambda$, so that $x^2=\lambda(a+b)$. Then, we have

$$\lambda\|a\|^2=\langle x^2|a\rangle=\langle x|ax\rangle=\|x\|^2/2=\langle x|bx\rangle=\langle x^2|b\rangle=\lambda\|b\|^2. \tag{2.1}$$

Hence $\lambda\neq0$ and $\|a\|=\|b\|$.

(2) *Case* $ab\neq0$: Since $P(a)$ is the orthogonal projection onto $V_1(a)=\boldsymbol{R}a$, Lemma 1.5 says that there are α, $\beta\in\boldsymbol{R}$ such that $P(a)b=\alpha a$ and $P(b)a=\beta b$. Then,

$$\alpha\langle a|b\rangle=\langle P(a)b|b\rangle=\langle(a\square b)a|b\rangle=\langle a|(b\square a)b\rangle=\langle a|P(b)a\rangle=\beta\langle a|b\rangle.$$

By Lemma 1.4, this implies $\alpha=\beta$ and

$$\alpha\|a\|^2 = \langle P(a)b\,|\,a\rangle = \langle b\,|\,a\rangle = \langle P(b)b\,|\,a\rangle = \langle b\,|\,P(b)a\rangle = \alpha\|b\|^2. \tag{2.2}$$

Thus $\alpha \neq 0$ and $\|a\| = \|b\|$. ■

In view of Proposition 2.2, we normalize the inner product of V so that $\|a\|=1$ for any $a \in \mathfrak{J}_1$. Since every idempotent is expressed as a finite sum of orthogonal primitive idempotents, we see that an idempotent a is primitive if and only if $\|a\|=1$. We record here some formulas used frequently in this paper.

LEMMA 2.3. *Let $a \in \mathfrak{J}_1$. Then, for $x, y \in V_{1/2}(a)$, one has*

(i) $a(xy) = \langle x\,|\,y\rangle a/2$, (ii) $x^3 = \|x\|^2 x/2$, (iii) $\|x^2\|^2 = \|x\|^4/2$.

PROOF. (i) Since $x^2 \in V_0(a)+V_1(a)$, there is $\alpha \in \boldsymbol{R}$ such that $ax^2 = \alpha a$. Then $\alpha = \langle ax^2\,|\,a\rangle = \langle x\,|\,ax\rangle = \|x\|^2/2$. Thus, $ax^2 = \|x\|^2 a/2$. Then we get (i) by polarization.

(ii) $x^3 = 2(ax)x^2 = 2(ax^2)x = \|x\|^2 ax = \|x\|^2 x/2$.

(iii) $\langle x^2\,|\,x^2\rangle = \langle x^3\,|\,x\rangle = \|x\|^2\langle x\,|\,x\rangle/2$. ■

LEMMA 2.4. *Let $a, b \in \mathfrak{J}_1$ be orthogonal. Then, one has*

$$xy = \langle x\,|\,y\rangle(a+b)/2 \qquad \textit{for } x, y \in V_{1/2}(a) \cap V_{1/2}(b).$$

PROOF. The proof (1) of Proposition 2.2 ((2.1) in particular) shows that $x^2 = \|x\|^2(a+b)/2$. Polarization of this gives the lemma. ■

LEMMA 2.5. *Let $a, b \in \mathfrak{J}_1$. Then, $\|ab\|^2 = \lambda(\lambda+1)/2$, where $\lambda := \langle a\,|\,b\rangle$.*

PROOF. The cases $a=b$ or $ab=0$ being obvious, we assume that $a \neq b$ and $ab \neq 0$. Then, the proof (2) of Proposition 2.2 ((2.2) in particular) shows $P(a)b = \lambda a$ with $\lambda := \langle a\,|\,b\rangle$. Then, we obtain

$$2\|ab\|^2 = 2\langle L(a)^2 b\,|\,b\rangle = \langle (P(a)+L(a))b\,|\,b\rangle = \lambda\langle a\,|\,b\rangle + \langle a\,|\,b\rangle. \qquad ■$$

We denote by $\mathrm{Sym}(2, \boldsymbol{R})$ the three dimensional JH-algebra of all 2×2 real symmetric matrices with inner product $\langle A\,|\,B\rangle := \mathrm{trace}(AB)$. If $a, c \in \mathfrak{J}_1$ are orthogonal, then there is $y \in V_{1/2}(a) \cap V_{1/2}(c)$ with $\|y\|^2 = 2$ such that $y^2 = a+c$ by virtue of Lemmas 2.1 and 2.4. It is evident that $\boldsymbol{R}a \oplus \boldsymbol{R}c \oplus \boldsymbol{R}y$ is a subalgebra isometrically isomorphic to $\mathrm{Sym}(2, \boldsymbol{R})$ via

$$\nu^{y}_{a,c}(\alpha a+\beta c+\xi y) := \begin{pmatrix} \alpha & \xi \\ \xi & \beta \end{pmatrix} \qquad (\alpha, \beta, \xi \in \boldsymbol{R}). \tag{2.3}$$

For $u, v \in V$, we denote by $V[u, v]$ the subalgebra generated by u, v.

PROPOSITION 2.6. *Let $a, b \in \mathfrak{J}_1$ with $a \neq b$ and $ab \neq 0$. Then,* $\dim V[a, b] = 3$

and there is an isometric Jordan algebra isomorphism $\rho_{a,b}$ *of* $V[a, b]$ *onto* $\mathrm{Sym}(2, \boldsymbol{R})$ *such that*

$$\rho_{a,b}(a) = A := \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \qquad \rho_{a,b}(b) = B := \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta \\ \cos\theta\sin\theta & \sin^2\theta \end{pmatrix}, \tag{2.4}$$

where $\cos^2\theta = \langle a \mid b\rangle$, *so that* $0 < \theta < \pi/2$.

PROOF. Put $u := ab$ and $\lambda := \langle a \mid b\rangle$. Then, the proof (2) of Proposition 2.2 shows $P(a)b = \lambda a$ and $P(b)a = \lambda b$. Since $P(a)b = 2au - u$ and $P(b)a = 2bu - u$, we obtain

$$au = (\lambda a + u)/2, \quad bu = (\lambda b + u)/2. \tag{2.5}$$

Just in the same way as [6, (1.6)], we also have

$$u^2 = \lambda(a + b + 2u)/4. \tag{2.6}$$

Now by Lemma 2.5 and Schwarz inequality, we get $0 < \lambda < 1$. Thus, there is θ $(0 < \theta < \pi/2)$ such that $\lambda = \cos^2\theta$. Let A, B be the matrices in (2.4) and consider the basis A, B, U of $\mathrm{Sym}(2, \boldsymbol{R})$, where

$$U := \frac{1}{2}(AB + BA) = \begin{pmatrix} \cos^2\theta & \cos\theta\sin\theta/2 \\ \cos\theta\sin\theta/2 & 0 \end{pmatrix}.$$

Then we see, through a straightforward computation with (2.5) and (2.6), that $\rho_{a,b}(\alpha a + \beta b + \xi u) := \alpha A + \beta B + \xi U$ $(\alpha, \beta, \xi \in \boldsymbol{R})$ defines a Jordan algebra isomorphism of $V[a, b]$ onto $\mathrm{Sym}(2, \boldsymbol{R})$. Moreover, ρ is an isometry: the equality $\|U\|^2 = \cos^4\theta + \cos^2\theta\sin^2\theta/2 = \|u\|^2$ is seen from Lemma 2.5 and the others such as $\langle a \mid u\rangle = \mathrm{trace}(AU)$ are immediate. ■

PROPOSITION 2.7. *Let* $a \in \mathfrak{J}_1$ *and* $x \in V_{1/2}(a)$. *Suppose* $x \neq 0$.

(1) $c := (2/\|x\|^2)x^2 - a$ *is a primitive idempotent orthogonal to* a.

(2) *Put* $y := \sqrt{2}\,x/\|x\|$, *so that* $y^2 = a + c$. *Then the map* $\nu^y_{a,c}$ *defined by* (2.3) *gives an isometric isomorphism of* $V[a, x]$ *onto* $\mathrm{Sym}(2, \boldsymbol{R})$ *such that*

$$\nu^y_{a,c}(x) = \frac{\|x\|}{\sqrt{2}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}. \tag{2.7}$$

PROOF. (1) The formulas in Lemma 2.3 together with $2x^4 = \|x\|^2 x^2$ yield $c^2 = c$ and $\|c\| = 1$, so that $c \in \mathfrak{J}_1$. Moreover, $ca = 0$ as is readily seen.

(2) Evident. ■

Let $GL(V)$ be the multiplicative group of invertible elements $\boldsymbol{B}(V)^\times$ of the Banach algebra $\boldsymbol{B}(V)$ of all bounded linear operators defined everywhere on V. Let $T \in \boldsymbol{B}(V)$. Then, $T \in GL(V)$ if and only if T is bijective by the open mapping theorem. The group $GL(V)$ has a natural Banach-Lie group structure in the norm topology. Let $\mathrm{Aut}(V)$ denote the automorphism group of the JH-algebra V:

$$\mathrm{Aut}(V) := \{T \in GL(V)\,;\, T(xy) = (Tx)(Ty) \quad \text{for all } x,\, y \in V\}.$$

Thus, as an algebraic subgroup of degree $\leqq 2$ of $GL(V)$, $\mathrm{Aut}(V)$ inherits a Banach-Lie group structure from $GL(V)$ ([**5**], [**15**, Theorem 7.14]). Hence $\mathrm{Aut}(V)$ is a Banach-Lie group in the norm topology. Let $O(V)$ be the orthogonal group of V:

$$O(V) := \{T \in GL(V)\,;\, {}^tTT = I\},$$

where ${}^tT$ denotes the adjoint operator of T. $O(V)$ is also a Banach-Lie group in the norm topology for the same reason.

PROPOSITION 2.8. $\mathrm{Aut}(V) \subset O(V)$.

PROOF. Suppose $T \in \mathrm{Aut}(V)$. Let $x \in V$ and $x = \sum \lambda_k a_k$ be the spectral decomposition (1.5) of x. Then, $Tx = \sum \lambda_k T a_k$. Since the a_k are mutually orthogonal primitive idempotents, so are the Ta_k. Therefore, $\|Tx\|^2 = \sum \lambda_k^2 = \|x\|^2$. Thus, $T \in GL(V)$ is an isometry, so that $T \in O(V)$. ■

We now recall the orthogonal projection $E_{1/2}(a)$ onto $V_{1/2}(a)$ for an idempotent a (see (1.4)). Then, the operator

$$T(a) := I - 2E_{1/2}(a) \tag{2.8}$$

is a *symmetry* (*i.e.*, selfadjoint involution): we have $T(a)x = x$ for $x \in V_0(a) + V_1(a)$ and $T(a)x = -x$ for $x \in V_{1/2}(a)$. It is easy to see that $T(a) \in \mathrm{Aut}(V)$ for any idempotent a.

PROPOSITION 2.9. *If $a, b \in \mathfrak{J}_1$, then there is $c \in \mathfrak{J}_1$ such that $T(c)a = b$.*

PROOF. We first assume $a \neq b$ and $ab \neq 0$. Then, Proposition 2.6 says that $V[a, b]$ is isometrically isomorphic to $\mathrm{Sym}(2, \boldsymbol{R})$ via $\rho_{a,b}$. With θ as in Proposition 2.6, let $W := \begin{pmatrix} \cos\theta & \sin\theta \\ \sin\theta & -\cos\theta \end{pmatrix}$ and $C := (I_2 - W)/2$, where I_2 is the 2×2 identity matrix. Clearly $W, C \in \mathrm{Sym}(2, \boldsymbol{R})$. We have $W^2 = I_2$ and

$$B = WAW = (I_2 - 2C)A(I_2 - 2C) = A - 2(CA + AC) + 4CAC\,. \tag{2.9}$$

Put $c := \rho_{a,b}^{-1}(C) \in V[a, b]$. Then, c is an idempotent with $\|c\| = \|C\| = 1$, so that $c \in \mathfrak{J}_1$. Applying $\rho_{a,b}^{-1}$ to (2.9), we get $b = a - 4L(c)a + 4P(c)a = T(c)a$.

For the case $ab = 0$, it suffices to argue as above with $\theta = \pi/2$ and the isometric isomorphism $\nu_{a,b}^y$ in (2.3). ■

§ 3. Primitive idempotents as a Riemannian Hilbert manifold.

In this section, we introduce a Riemannian Hilbert manifold structure on $\mathfrak{J}_1$ in such a way that the inclusion $\mathfrak{J}_1 \hookrightarrow V$ is an embedding. For finite dimensional JH-algebras treated by U. Hirzebruch [6], the manifold structure is defined as a homogeneous space of the finite dimensional compact Lie group $\mathrm{Aut}(V)$. However, since $\mathrm{Aut}(V)$ is merely a Banach-Lie group in the infinite dimensional case, we prefer to introduce a Riemannian manifold structure modelled on a Hilbert space in a more explicit manner. We will say a Riemannian manifold instead of Riemannian Hilbert manifold for simplicity.

First, it is clear that $\mathfrak{J}_1$ is a closed subset of V. So, we equip $\mathfrak{J}_1$ with the norm topology of V. Next, we note that $\mathfrak{J}_1$ is arcwise connected. This can be seen as follows. Let $a, b \in \mathfrak{J}_1$ be distinct and suppose first $ab \neq 0$. Then, Proposition 2.6 says that there is an isometric Jordan algebra isomorphism $\rho_{a,b}$ of $V[a, b]$ onto $\mathrm{Sym}(2, \boldsymbol{R})$ such that (2.4) holds. With θ as in (2.4), we let

$$\Gamma(t) := \begin{pmatrix} \cos^2 t & \cos t \sin t \\ \cos t \sin t & \sin^2 t \end{pmatrix} \quad \text{for } 0 \leqq t \leqq \theta \,, \tag{3.1}$$

and put $\gamma(t) := \rho_{a,b}^{-1}(\Gamma(t))$. Then, it is easy to see that γ is a curve in $\mathfrak{J}_1$ joining a and b. For the case $ab=0$, modify the argument as in the last part of the proof of Proposition 2.9. Later, this γ will be shown to be a geodesic in $\mathfrak{J}_1$ (see the discussion after Proposition 4.2).

For every $a \in \mathfrak{J}_1$, put

$$N_a := \{b \in \mathfrak{J}_1 \,;\, \langle b \,|\, a \rangle \neq 0\} \,. \tag{3.2}$$

It is clear that N_a is an open subset of $\mathfrak{J}_1$ containing a, so the family $\{N_a \,; a \in \mathfrak{J}_1\}$ is an open covering of $\mathfrak{J}_1$. Let us define two maps ξ_a, η_a as follows:

$$\begin{aligned} &\xi_a(b) := -2a + 2ab/\langle a \,|\, b \rangle \qquad (b \in N_a) \,, \\ &\eta_a(x) := (2 - \|x\|^2)a/(2 + \|x\|^2) + 2x/(2 + \|x\|^2) + 2x^2/(2 + \|x\|^2) \quad (x \in V_{1/2}(a)). \end{aligned} \tag{3.3}$$

PROPOSITION 3.1. *One has*

$$\mathrm{Image}(\xi_a) = V_{1/2}(a) \,, \qquad \mathrm{Image}(\eta_a) = N_a$$

and the two maps are inverses to each other:

$$\xi_a \circ \eta_a(x) = x \quad \textit{for all } x \in V_{1/2}(a) \,, \qquad \eta_a \circ \xi_a(b) = b \quad \textit{for all } b \in N_a \,.$$

PROOF. *Step* 1. Let $b \in N_a$. Since $\xi_a(a) = 0$, we assume $b \neq a$. Then, Proposition 2.6 says that there is an isometric Jordan algebra isomorphism $\rho_{a,b}$ of $V[a, b]$ onto $\mathrm{Sym}(2, \boldsymbol{R})$ such that (2.4) holds. Then,

$$\rho_{a,b}(\xi_a(b)) = \begin{pmatrix} 0 & \tan\theta \\ \tan\theta & 0 \end{pmatrix}, \tag{3.4}$$

so that $\xi_a(b) \in V_{1/2}(a)$.

Step 2. Let $x \in V_{1/2}(c)$. Since $\eta_a(0)=a$, we assume $x \neq 0$. Define $c \in \mathfrak{J}_1 \cap V_0(a)$ and $y \in V_{1/2}(a)$ as in Proposition 2.7. Then, (2.7) holds. Putting $\|x\| = \sqrt{2}\tan\theta$ $(0<\theta<\pi/2)$, we find through a simple computation that $\nu_{a,c}^y(\eta_a(x))$ equals the matrix B in (2.4). Thus, $\eta_a(x)$ belongs to $\mathfrak{J}_1$ and indeed to N_a.

Step 3. Let $b \in N_a$. We may assume $b \neq a$. Then, $x := \xi_a(b) \neq 0$. In Step 1, we have shown that $x \in V_{1/2}(a)$. By definition, x belongs to $V[a, b]$, so that $V[a, x] \subset V[a, b]$. Both subalgebras being of three dimension (Propositions 2.6 and 2.7), we obtain $V[a, x] = V[a, b]$. On the other hand, $\rho_{a,b}$ is an isometry, so we have $\|x\|^2 = 2\tan^2\theta$ by (3.4). Setting $c := y^2 - a$ with $y := \sqrt{2}x/\|x\|$ and comparing (2.7) with (3.4), we get $\rho_{a,b} = \nu_{a,c}^y$. Therefore, $\eta_a \circ \xi_a(b) = b$ by Proposition 2.6 and Step 2. In a similar way, we obtain $\xi_a \circ \eta_a(z) = z$ for all $z \in V_{1/2}(a)$. ■

We shall make use of the maps ξ_a and η_a to introduce a Hilbert manifold structure on $\mathfrak{J}_1$. Before proceeding, we note that the Peirce (1/2)-spaces $V_{1/2}(a)$ $(a \in \mathfrak{J}_1)$ are mutually unitarily isomorphic, because we have a continuous family of orthogonal projections $\mathfrak{J}_1 \ni a \mapsto E_{1/2}(a)$ and $\mathfrak{J}_1$ is arcwise connected. This isomorphism follows from the well-known fact that if P, Q are orthogonal projections on a Hilbert space such that $\|P-Q\|<1$, then the operator

$$U := [I-(P-Q)^2]^{-1/2}[QP+(I-Q)(I-P)]$$

gives a unitary equivalence $Q = UPU^{-1}$ [7, Theorem I.6.32]. Thus, take $a_0 \in \mathfrak{J}_1$ and put $\mathfrak{H} = V_{1/2}(a_0)$. Let $\Upsilon_a : V_{1/2}(a) \to \mathfrak{H}$ be the unitary isomorphism. Then, thanks to Proposition 3.1, it is now easy to see that the collection

$$\{(N_a, \Upsilon_a \circ \xi_a, \mathfrak{H})\,;\, a \in \mathfrak{J}_1\}$$

defines an atlas of $\mathfrak{J}_1$.

We shall show that the inclusion $\mathfrak{J}_1 \hookrightarrow V$ is an embedding. Let $a \in \mathfrak{J}_1$ and consider the Peirce decomposition (1.1). Let us define a map Φ_a as follows:

$$\Phi_a(x, y) := \eta_a(x) + y \qquad \text{for } x \in V_{1/2}(a) \text{ and } y \in V_0(a) \oplus V_1(a).$$

Then, $\Phi_a(0, 0) = \eta_a(0) = a$. Moreover, we have $(d/dt)\eta_a(tx)|_{t=0} = x$. Hence the Fréchet derivative $d_{(0,0)}\Phi_a$ of Φ_a at $(x, y) = (0, 0)$ is the "identity" map $(x, y) \mapsto x+y$. The inverse mapping theorem then guarantees the existence of an open neighborhood W_a of $(0, 0)$ such that $\Phi_a : W_a \to \Phi_a(W_a)$ is a diffeomorphism. The equality

$$N_a \cap \Phi_a(W_a) = \mathfrak{J}_1 \cap \Phi_a(W_a) = \Phi_a((V_{1/2}(a) \oplus \{0\}) \cap W_a)$$

shows that $\mathfrak{J}_1$ is a submanifold of the Hilbert space V. Furthermore the inclusion $\iota: \mathfrak{J}_1 \hookrightarrow V$ has the following local expression:

$$\begin{tikzcd} \mathfrak{J}_1 \cap \Phi_a(W_a) \arrow[r, "\iota"] \arrow[d, "\Phi_a^{-1}"'] & \Phi_a(W_a) \arrow[d, "\Phi_a^{-1}"] \\ V_{1/2}(a) \arrow[r] & V \end{tikzcd}$$

The topology of $\mathfrak{J}_1$ being the induced one from V, we thus see the inclusion ι is an embedding, so that the tangent space $T_a(\mathfrak{J}_1)$ at a is identified with $V_{1/2}(a)$. In this way, $\mathfrak{J}_1$ now inherits a Riemannian manifold structure as a closed submanifold of the Hilbert space V.

Finally, we mention that $\mathfrak{J}_1$ is a symmetric space. In fact, every $a \in \mathfrak{J}_1$ is an isolated fixed point of the symmetry $T(a) \in \mathrm{Aut}(V)$ introduced in (2.8).

§ 4. Riemannian distance.

In this section, we compute the Riemannian distance on the Riemannian manifold $\mathfrak{J}_1$. Our reference concerning Riemannian manifolds is the book [**9**].

Let M be an arcwise connected Riemannian manifold. By definition, the *Riemannian distance* $\mathrm{dist}: M \times M \to \boldsymbol{R}$ between two points $p, q \in M$ is

$$\mathrm{dist}(p, q) := \inf\{\mathrm{length}(\gamma)\,;\, \gamma \text{ a curve from } p \text{ to } q\}.$$

We know that the function 'dist' determines a metric on M compatible with the given topology on M. Unlike finite dimensional cases, there may exist two points on M which miss a minimizing geodesic even if M is complete with respect to the Riemannian distance 'dist'. Such an example can be found in [**4**] (see also [**9**, p. 127]).

Now, let $\mathfrak{J}_1$ be the Riemannian manifold of primitive idempotents in the JH-algebra V. We denote by $\mathfrak{X}(\mathfrak{J}_1)$ the set of vector fields on $\mathfrak{J}_1$. Since the tangent space $T_a(\mathfrak{J}_1)$ at $a \in \mathfrak{J}_1$ is identified with $V_{1/2}(a)$, we shall consider every vector field on $\mathfrak{J}_1$ as a V-valued function such that the value at $a \in \mathfrak{J}_1$ is contained in $V_{1/2}(a)$. Thus, for $Y \in \mathfrak{X}(\mathfrak{J}_1)$, d_aY $(a \in \mathfrak{J}_1)$ stands for the tangent map at a of the V-valued function Y on $\mathfrak{J}_1$, so that d_aY is a bounded linear operator $V_{1/2}(a) \to V$. Recalling the orthogonal projection $E_{1/2}(a)$ onto $V_{1/2}(a)$ introduced in (1.4), we define the *canonical connection* ∇ of $\mathfrak{J}_1$:

$$\nabla_X Y(a) := E_{1/2}(a)(d_aY(X(a))) \qquad (a \in \mathfrak{J}_1 \text{ and } X, Y \in \mathfrak{X}(\mathfrak{J}_1)). \tag{4.1}$$

Then, one can show by a standard calculation that the connection ∇ is *Levi-Civita*: the parallel transport associated to ∇ is an isometry and the torsion tensor vanishes. Furthermore, it is $\mathrm{Aut}(V)$-invariant:

$$g \cdot (\nabla_Y Z) = \nabla_{g \cdot Y}(g \cdot Z) \qquad \text{for all } g \in \mathrm{Aut}(V),$$

where $(g\cdot Y)(a):=g(Y(g^{-1}a))$ for $Y\in\mathfrak{X}(\mathfrak{J}_1)$. This can be easily proved by noting the relation $E_{1/2}(ga)=gE_{1/2}(a)g^{-1}$ for $g\in\mathrm{Aut}(V)$.

Let $\mathrm{Der}(V)$ denote the set of all continuous derivations of the JH-algebra V:

$$\mathrm{Der}(V):=\{T\in\boldsymbol{B}(V);\,T(xy)=(Tx)y+x(Ty)\quad\text{for all } x,\,y\in V\}.$$

Then, $\mathrm{Der}(V)$ is a Banach-Lie algebra as a closed Lie subalgebra of $\mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ is $\boldsymbol{B}(V)$ regarded as a Banach-Lie algebra by the usual bracket of operators $[A, B]=AB-BA$. Clearly, $\mathrm{Der}(V)$ is the Lie algebra of the Banach-Lie group $\mathrm{Aut}(V)$. For $x, y\in V$, it is known that the operator $[L(x), L(y)]$ is a derivation, called an inner derivation of V (cf. [**15**, 19.6]). For simplicity, we shall put

$$D(x,\,y):=4[L(x),\,L(y)]\in\mathrm{Der}(V)\,. \tag{4.2}$$

Then, $\exp tD(x, y)\in\mathrm{Aut}(V)$ for every $t\in\boldsymbol{R}$. Moreover, it is easy to show that

$$TD(x,\,y)T^{-1}=D(Tx,\,Ty)\quad\text{for any } T\in\mathrm{Aut}(V)\,. \tag{4.3}$$

PROPOSITION 4.1. *Let $a\in\mathfrak{J}_1$ and $x\in V_{1/2}(a)$. Then the curve $\boldsymbol{R}\ni t\mapsto\gamma_{a,x}(t)$ $=(\exp tD(x, a))a$ is a geodesic with $\gamma_{a,x}(0)=a$ and $\dot{\gamma}_{a,x}(0)=x$.*

PROOF. Put $D=D(x, a)$ and $\gamma=\gamma_{a,x}$ for brevity. Our task is to show that $E_{1/2}(\gamma(t))(\ddot{\gamma}(t))=0$ for all $t\in\boldsymbol{R}$. Since $Da=4(xa^2-a(xa))=x$, we have

$$\dot{\gamma}(t)=(\exp tD)Da=(\exp tD)x\,.$$

Therefore, $\ddot{\gamma}(t)=(\exp tD)Dx$. Here $Dx=2(x^2-\|x\|^2a)$ by (i) of Lemma 2.3, so that

$$\ddot{\gamma}(t)=2(\exp tD)x^2-2\|x\|^2\gamma(t)\,.$$

By using the explicit expression (1.4) for $E_{1/2}(\gamma(t))$, we have

$$E_{1/2}(\gamma(t))(\ddot{\gamma}(t))=8(L(\gamma(t))-L(\gamma(t))^2)(\exp tD)x^2\,,$$

because $\gamma(t)\in V_1(\gamma(t))$. Since $\exp tD\in\mathrm{Aut}(V)$, we obtain

$$E_{1/2}(\gamma(t))(\ddot{\gamma}(t))=8(\exp tD)(ax^2-a(ax^2))\,.$$

By (i) of Lemma 2.3, we have $ax^2=\|x\|^2a/2=a(ax^2)$, so that we arrive at $E_{1/2}(\gamma(t))(\ddot{\gamma}(t))=0$. ■

PROPOSITION 4.2. *Let $a\in\mathfrak{J}_1$ and $x\in V_{1/2}(a)$. If $|t|\cdot\|x\|<\pi/\sqrt{2}$, then*

$$\gamma_{a,x}(t)=\eta_a\Big(\frac{\tan(\|x\|t/\sqrt{2})}{(\|x\|/\sqrt{2})}x\Big), \tag{4.4}$$

where η_a is the map introduced in (3.3).

PROOF. Let the notation be as in the proof of Proposition 4.1. An easy induction together with Lemma 2.3 shows that for $n=1, 2, \cdots$

$$D^{2n-1}a=(-2\|x\|^2)^{n-1}x\,, \qquad D^{2n}a=2(-2\|x\|^2)^{n-1}(x^2-\|x\|^2a)\,.$$

From this, we get easily

$$\gamma_{a,x}(t)=(\cos\sqrt{2}\,\|x\|t)a+\frac{\sin(\sqrt{2}\,\|x\|t)}{\sqrt{2}\,\|x\|}x+\frac{1-\cos(\sqrt{2}\,\|x\|t)}{\|x\|^2}x^2 \tag{4.5}$$

for all $t\in\boldsymbol{R}$. Comparing (4.5) with (3.3), we find (4.4). ■

Now, given two distinct points $a, b\in\mathfrak{J}_1$ such that $ab\neq 0$, we take θ $(0<\theta<\pi/2)$ as in Proposition 2.6. Then, we have (2.4) with $\rho_{a,b}$ the isometric Jordan algebra isomorphism of $V[a, b]$ onto $\mathrm{Sym}(2, \boldsymbol{R})$. Let

$$X:=\begin{pmatrix}0 & 1\\ 1 & 0\end{pmatrix}\in\mathrm{Sym}(2, \boldsymbol{R})\,, \qquad x:=\rho_{a,b}^{-1}(X)\in V_{1/2}(a)\cap V[a, b]. \tag{4.6}$$

Then, $\|x\|=\sqrt{2}$ and $V[a, x]=V[a, b]$. Moreover, with the same notation as in Proposition 2.7, we have $\nu_{a,c}^{x}=\rho_{a,b}=:\rho$ (cf. Step 3 of the proof of Proposition 3.1). Let $\Gamma(t)$ be as in (3.1). Then, (2.7) and (4.5) show that $\rho(\gamma_{a,x}(t))=\Gamma(t)$. Hence, a and b are connected by the geodesic $\gamma_{a,x}$ with x defined by (4.6). Modification of argument for $ab=0$ is obvious.

Our next objective is to show that the geodesic segment $\gamma_{a,x}|_{[0,\theta]}$ in the above paragraph is minimal. In other words, we shall show that it is a curve attaining the infimum in the definition of the Riemannian distance. To do so, let us define the *Exponential mapping* Exp_a at $a\in\mathfrak{J}_1$ by

$$\mathrm{Exp}_a(x):=\gamma_{a,x}(1)=(\exp D(x, a))a \qquad \text{for } x\in V_{1/2}(a)\,.$$

Let B_a be the open ball in $V_{1/2}(a)$ with radius $\pi/\sqrt{2}$:

$$B_a:=\{x\in V_{1/2}(a)\,;\ \|x\|<\pi/\sqrt{2}\}\,.$$

Proposition 4.2 says that

$$\mathrm{Exp}_a(x)=\eta_a([\tan(\|x\|/\sqrt{2})/(\|x\|/\sqrt{2})]x) \qquad \text{for } x\in B_a\,. \tag{4.7}$$

Recalling the set N_a defined by (3.2), we have

PROPOSITION 4.3. *The map* Exp_a *is a diffeomorphism of* B_a *onto* N_a. *Denote by* Log_a *the inverse map of* Exp_a, *then*

$$\mathrm{Log}_a(b)=\frac{\arctan\phi(\langle a\,|\,b\rangle)}{\phi(\langle a\,|\,b\rangle)}\cdot\xi_a(b) \qquad (b\in N_a)\,, \tag{4.8}$$

where $\phi(t):=\sqrt{1-t}/\sqrt{t}$ *and* ξ_a *is the map introduced in* (3.3).

PROOF. Let us define the map Log_a by (4.8). It is clear that both Exp_a and Log_a are real analytic. We shall show that they are inverses to each other.

Step 1. Let $x \in B_a$. We set $\varphi := \|x\|/\sqrt{2}$ and $y := (\tan\varphi/\varphi)x$ for simplicity. By Proposition 3.1 and (4.7), we have $\mathrm{Exp}_a(x) \in N_a$ and

$$\xi_a(\mathrm{Exp}_a(x)) = \xi_a(\eta_a(y)) = y = (\tan\varphi/\varphi)x .$$

Thus, to show that $\mathrm{Log}_a(\mathrm{Exp}_a(x)) = x$, it suffices to prove $\psi(\langle a | \eta_a(y)\rangle) = \tan\varphi$. Now, by (3.3), we have

$$\langle a | \eta_a(y)\rangle = (2 - \|y\|^2)/(2 + \|y\|^2) + 2\langle a | y^2\rangle/(2 + \|y\|^2) .$$

Since $\|y\| = \sqrt{2}\tan\varphi$ and since $2\langle a | y^2\rangle = \|y\|^2$, we get $\langle a | \eta_a(y)\rangle = \cos^2\varphi$, so that $\psi(\langle a | \eta_a(y)\rangle) = \tan\varphi$.

Step 2. Let $b \in N_a$ and put $\lambda := \langle a | b\rangle = \cos^2\theta$ with $0 \leqq \theta < \pi/2$. By (3.4), we have $\|\xi_a(b)\|^2 = 2\tan^2\theta = 2(1-\lambda)/\lambda$. Hence, we get

$$\|\mathrm{Log}_a(b)\|^2 = 2\arctan^2\psi(\langle a | b\rangle) < \pi^2/2 .$$

This implies that if we put $x := \mathrm{Log}_a(b)$ and $\varphi := \|x\|/\sqrt{2}$, then $x \in B_a$ and $\tan\varphi = \psi(\langle a | b\rangle)$. Consequently, $\mathrm{Exp}_a(\mathrm{Log}_a(b)) = \mathrm{Exp}_a(x) = \eta_a(\xi_a(b)) = b$. ■

THEOREM 4.4. *Let $a, b \in \mathfrak{J}_1$ be distinct and $ab \neq 0$. Define $x \in V_{1/2}(a)$ by (4.6). Then, the geodesic $\gamma_{a,x}$ passing a, b is a minimizing one.*

PROOF. We have $\|x\| = \sqrt{2}$ by definition, so that Proposition 4.2 gives $\gamma_{a,x}(t) = \eta_a((\tan t)x)$ for $|t| < \pi/2$. We note $b = \gamma_{a,x}(\theta)$ with θ defined by $\langle a | b\rangle = \cos^2\theta$ $(0 < \theta < \pi/2)$. Therefore,

$$\gamma_{a,x}(t) \in N_a = \mathrm{Exp}_a(B_a) \qquad \text{for } 0 \leqq t \leqq \theta \tag{4.9}$$

by Propositions 3.1 and 4.3. Since our connection ∇ defined by (4.1) is Levi-Civita, a standard argument using Gauß' lemma ([**9**, 1.9]) together with (4.9) leads us to the inequality $\mathrm{length}(\gamma_{a,x}) \leqq \mathrm{length}(\gamma)$ for any curve γ joining a and b. ■

THEOREM 4.5. *For $a, b \in \mathfrak{J}_1$, one has*

$$\mathrm{dist}(a, b) = \sqrt{2}\arcsin(\|a-b\|/\sqrt{2}).$$

PROOF. For $a = b$, the formula is trivial. If $a \neq b$ and $ab \neq 0$, then we get by Theorem 4.4 (with the notation therein)

$$\mathrm{dist}(a, b) = \mathrm{length}(\gamma_{a,x}) = \int_0^\theta \|\dot\gamma_{a,x}(t)\| dt = \|\dot\gamma_{a,x}(0)\|\theta = \|x\|\theta = \sqrt{2}\theta.$$

Since, $\cos^2\theta = \langle a | b\rangle = 1 - \|a-b\|^2/2$, we have $\sin\theta = \|a-b\|/\sqrt{2}$.

If $ab=0$, then we let $x:=(\nu_{a,b}^{y})^{-1}(X)$ with the isometric isomorphism $\nu_{a,b}^{y}$ defined by (2.3) and the X in (4.6). We have $b=\gamma_{a,x}(\pi/2)$. Since $\langle\gamma_{a,x}(t)|a\rangle \neq 0$ for $t\in[0, \pi/2)$, the preceding argument gives

$$\mathrm{dist}(a, \gamma_{a,x}(t)) = \sqrt{2}\arcsin(\|a-\gamma_{a,x}(t)\|/\sqrt{2}) \qquad \text{for } 0\leqq t<\pi/2.$$

Letting $t\uparrow\pi/2$, we get the formula by continuity. ∎

§5. Two-point homogeneity.

First of all, we recall here the JH-algebras called spin factors. Let $\mathfrak{H}$ be a real Hilbert space with inner product $\langle\cdot|\cdot\rangle_{\mathfrak{H}}$. Let e be a symbol and we denote by $\boldsymbol{R}e$ the one-dimensional vector space spanned by e. Then, the vector space $\boldsymbol{R}e\oplus\mathfrak{H}$ becomes a JH-algebra $\mathcal{S}(e, \mathfrak{H})$, the *spin factor* associated to $\mathfrak{H}$, if we define a Jordan product and an inner product respectively by

$$(5.1)\qquad (\alpha e+u)(\beta e+v):=[\alpha\beta+\langle u|v\rangle_{\mathfrak{H}}]e+\alpha v+\beta u\,,$$

$$(5.2)\qquad \langle\alpha e+u|\beta e+v\rangle:=2(\alpha\beta+\langle u|v\rangle_{\mathfrak{H}})\,,$$

where $\alpha, \beta\in\boldsymbol{R}$ and $u, v\in\mathfrak{H}$. We note that the coefficient 2 appears on the right hand side of (5.2) in order to make the norm of the primitive idempotents equal to 1. We have

$$(5.3)\qquad \mathfrak{J}_1(\mathcal{S}(e, \mathfrak{H})) = \{e/2+u\,;\ \|u\|_{\mathfrak{H}}=1/2\}.$$

For the symmetry $T(a)$ defined by (2.8) with $a=e/2+u\in\mathfrak{J}_1(\mathcal{S}(e, \mathfrak{H}))$, the restriction $T(a)|_{\mathfrak{H}}$ is the orthogonal reflection with respect to the hyperplane $(\boldsymbol{R}u)^{\perp}$.

Now we return to our JH-algebra V.

LEMMA 5.1. *Let $a\in\mathfrak{J}_1$ and suppose non-zero $x, y\in V_{1/2}(a)$ satisfy $x^2=y^2$.*

(1) *Put $e:=2x^2/\|x\|^2$. Then, $e=2y^2/\|y\|^2$ and e is a non-zero idempotent.*

(2) *Put $c:=(2x^2/\|x\|^2)-a\in\mathfrak{J}_1$, so that $e=a+c$. Let $\langle a, x, y\rangle$ denote the subalgebra generated by a, x, y. Then, there is an isometric Jordan algebra isomorphism σ of $\langle a, x, y\rangle$ onto the spin factor $\mathcal{S}(e, \mathfrak{H})$, where $\mathfrak{H}$ is the vector space spanned by $x, y, a-c$ with inner product $\langle u|v\rangle_{\mathfrak{H}}=\langle u|v\rangle/2$.*

PROOF. (1) By (iii) of Lemma 2.3, $x^2=y^2$ implies $\|x\|=\|y\|$. The fact that e is an idempotent follows from (1) of Proposition 2.7.

(2) Note $x, y\in V_{1/2}(a)\cap V_{1/2}(c)$. Then, we have $xy=\langle x|y\rangle e/2$ by Lemma 2.4. This together with $x^2=y^2=\|x\|^2e/2$ gives the isometric isomorphism σ immediately in view of (5.1) and (5.2). ∎

For every $a\in\mathfrak{J}_1$, let S_a denote the sphere in $V_{1/2}(a)$ with radius $\sqrt{2}$:

$$S_a := \{x \in V_{1/2}(a)\,;\ \|x\| = \sqrt{2}\}.$$

It is clear that if $T \in \mathrm{Aut}(V)$ fixes $a \in \mathfrak{J}_1$, then T leaves S_a stable by Proposition 2.8. Since $V_{1/2}(a)$ is identified with the tangent space $T_a(\mathfrak{J}_1)$, the following proposition due to U. Hirzebruch [**6**] for finite dimensional V implies that the Riemannian manifold $\mathfrak{J}_1$ is *isotropic*, that is, the stabilizer of $\mathrm{Aut}(V)$ at $a \in \mathfrak{J}_1$ acts transitively on the unit sphere in the tangent space at a.

PROPOSITION 5.2. *Let $a \in \mathfrak{J}_1$. For $x, y \in S_a$, there is $T \in \mathrm{Aut}(V)$ such that $Ta=a$ and $Tx=y$. This T can be taken as a finite product of the operators $T(d)$ with $d \in \mathfrak{J}_1$.*

PROOF. Suppose first $x^2 \neq y^2$. Then, $b := x^2 - a$ and $c := y^2 - a$ are distinct primitive idempotents in $V_0(a)$ by virtue of Proposition 2.7. The subalgebra $V[b, c]$ has the unit element e: if $bc=0$ then $e=b+c$, and if $bc \neq 0$ then note $V[b, c] \cong \mathrm{Sym}(2, \boldsymbol{R})$. Since e itself is a non-zero idempotent, Proposition 1.6 says that the JH-algebra $V_1(e)$ is topologically simple. An application of Proposition 2.9 to the pair b, c within $V_1(e)$ gives $d \in \mathfrak{J}_1 \cap V_1(e) \subset V_0(a)$ such that $T_1(d)b=c$, where $T_1(d)$ is the symmetry defined by (2.8) acting on $V_1(e)$. Then, it is immediate to show that $T(d)a=a$, $T(d)b=T_1(d)b=c$. This implies $(T(d)x)^2 = T(d)x^2 = y^2$, so that we are led to the case $x^2=y^2$.

We thus assume that $x^2=y^2$. Then, (2) of Lemma 5.1 says that the subalgebra $\langle a, x, y\rangle$ generated by a, x, y is isometrically isomorphic to the spin factor $\mathcal{S}(e, \mathfrak{H})$, where $e := x^2 = y^2$ and $\mathfrak{H}$ is the vector space spanned by x, y, $a-c$ ($c := x^2 - a \in \mathfrak{J}_1$) with inner product $\langle u|v\rangle_{\mathfrak{H}} = \langle u|v\rangle/2$. Let $u := \delta(x+y) \in \langle a, x, y\rangle$, where $\delta \in \boldsymbol{R}$ is chosen so that $2\|u\|_{\mathfrak{H}}=1$. Put $d_1 := e/2 + u \in \mathfrak{J}_1$ (see (5.3)). Then, translating everything into $\mathcal{S}(e, \mathfrak{H})$, we see easily that

$$T(d_1)e = e, \qquad T(d_1)(a-c) = c-a, \qquad T(d_1)y = x\,, \tag{5.4}$$

because $\langle a-c|u\rangle=0$. The first two equalities in (5.4) yield $T(d_1)a=c$. Next, since $\|x\|_{\mathfrak{H}}=1$, we have $d_2 := (e+x)/2 \in \mathfrak{J}_1$ and

$$T(d_2)x = x, \qquad T(d_2)e = e, \qquad T(d_2)(a-c) = c-a\,. \tag{5.5}$$

The last two equalities in (5.5) imply $T(d_2)c=a$. Hence we get

$$T(d_2)T(d_1)y = x, \qquad T(d_2)T(d_1)a = a\,. \qquad \blacksquare$$

Once we have the isotropy property, two-point homogeneity follows through a somewhat general argument.

THEOREM 5.3. *The Riemannian manifold $\mathfrak{J}_1$ is two-point homogeneous: suppose* $\mathrm{dist}(a_1, b_1) = \mathrm{dist}(a_2, b_2)$ *for $a_1, a_2, b_1, b_2 \in \mathfrak{J}_1$, then there is $T \in \mathrm{Aut}(V)$ such that $Ta_1=a_2$ and $Tb_1=b_2$. This T can be chosen as a finite product of the operators $T(d)$ with $d \in \mathfrak{J}_1$.*

PROOF. Clearly we may assume $a_1=a_2=:a$ by transitivity (Proposition 2.9). Take geodesic segments γ_1, γ_2 such that γ_j realizes $\operatorname{dist}(a, b_j)$ for $j=1, 2$. This is possible by Theorem 4.4 in case $ab_j\neq 0$, and if $ab_j=0$, then one argues as in the second half of the proof of Theorem 4.5. Each of the γ_j has the form $\gamma_j(t)=(\exp tD(x_j, a))a$ with $x_j\in S_a$. Since $\operatorname{dist}(a, b_1)=\operatorname{dist}(a, b_2)$, we have $\operatorname{length}(\gamma_1)=\operatorname{length}(\gamma_2)$. Hence, $b_j=\gamma_j(\theta)$ with $\theta\in[0, \pi/2]$ independent of j. By Proposition 5.2, there is $T\in\operatorname{Aut}(V)$, which is a finite product of the operators $T(d)$ with $d\in\mathfrak{J}_1$, such that $Ta=a$ and $Tx_1=x_2$. Then, using (4.3), we get

$$Tb_1 = T\gamma_1(\theta) = T(\exp\theta D(x_1, a))a = (\exp\theta D(x_2, a))a = \gamma_2(\theta) = b_2. \quad \blacksquare$$

§6. Sectional curvature.

In this section, we compute the sectional curvature of the Riemannian manifold $\mathfrak{J}_1$. First, we recall the orthogonal projection $E_{1/2}(a)$ onto $V_{1/2}(a)$ for every idempotent a (see (1.4)). For simplicity, we write $E(a)$ instead of $E_{1/2}(a)$ till Theorem 6.4. Thus, $E:\mathfrak{J}_1\ni a\mapsto E(a)=4L(a)-4L(a)^2\in\boldsymbol{B}(V)$ is a real analytic $\boldsymbol{B}(V)$-valued function. The derivative of E at $a\in\mathfrak{J}_1$ will be denoted by $E'(a)$ in place of d_aE. Being a bounded linear operator $V_{1/2}(a)\to\boldsymbol{B}(V)$, the map $E'(a)$ will be considered as a bounded bilinear operator $V_{1/2}(a)\times V\to V$. An explicit formula for $E'(a)$ is given in the next lemma.

LEMMA 6.1. *Let $a\in\mathfrak{J}_1$ and $x\in V_{1/2}(a)$. Then,*

$$E'(a)(x, u) = \begin{cases} 2(xu)_0-\langle x\,|\,u\rangle a \in V_0(a)+V_1(a) & \text{if } u\in V_{1/2}(a), \\ 2(1-2k)xu \in V_{1/2}(a) & \text{if } u\in V_k(a)\ (k=0,1), \end{cases}$$

where $v_0:=E_0(a)v$ for any $v\in V$.

PROOF. Let c be a curve in $\mathfrak{J}_1$ such that $c(0)=a$ and $\dot{c}(0)=x$. Then, for any $u\in V$, we have

$$E'(a)(x, u) = (d/dt)E(c(t))u\,|_{t=0} = 4xu-4x(au)-4a(xu)\,.$$

Thus, if $u\in V_k(a)$, then $E'(a)(x, u)=4(1-k)xu-4a(xu)$.

Case 1. Suppose $u\in V_{1/2}(a)$. Then, $xu\in V_0(a)+V_1(a)$. Hence, by (i) of Lemma 2.3 we have

$$E_1(a)(xu) = a(xu) = \langle x\,|\,u\rangle a/2\,, \tag{6.1}$$

so that $E'(a)(x, u)=2xu-2\langle x\,|\,u\rangle a=2(xu)_0-\langle x\,|\,u\rangle a$.

Case 2. Suppose $u\in V_k(a)$ $(k=0, 1)$. Since $xu\in V_{1/2}(a)$, we have $2a(xu)=xu$, so that $E'(a)(x, u)=2(1-2k)xu$. $\blacksquare$

Recalling the connection ∇ defined by (4.1), we consider the *curvature tensor*:

$$R(X, Y) := \nabla_X \nabla_Y - \nabla_Y \nabla_X - \nabla_{[X,Y]} \qquad (X, Y \in \mathfrak{X}(\mathfrak{J}_1)).$$

In what follows, we write $Z'(a)$ for the derivative $d_a Z$ at a of vector fields Z on $\mathfrak{J}_1$ which are considered as V-valued functions such that $Z(a) \in V_{1/2}(a)$. Thus, $Z'(a)$ is a bounded linear operator $V_{1/2}(a) \to V$. We also introduce the notation

$$\bigwedge_{X,Y} f(X, Y) := f(X, Y) - f(Y, X).$$

LEMMA 6.2. *For $X, Y, Z \in \mathfrak{X}(\mathfrak{J}_1)$, one has*

$$R(X, Y)Z(a) = \bigwedge_{X,Y} E'(a)(X(a), E'(a)(Y(a), Z(a))).$$

PROOF. By a simple calculation, we have

$$R(X, Y)Z(a) = \bigwedge_{X,Y} E(a)E'(a)(X(a), Z'(a)(Y(a))). \tag{6.2}$$

Now, differentiation of the identity $E(a)^2 = E(a)$ in the direction $x \in V_{1/2}(a)$ gives

$$E(a)E'(a)(x, u) = E'(a)(x, u) - E'(a)(x, E(a)u) \qquad \text{for all } u \in V.$$

Substituting this into (6.2), we get

$$R(X, Y)Z(a) = \bigwedge_{X,Y} E'(a)(X(a), (I - E(a))Z'(a)(Y(a))).$$

Since $Z(a) \in V_{1/2}(a)$, we have $E(a)Z(a) = Z(a)$. Differentiating this in the direction $y \in V_{1/2}(a)$, we obtain

$$(I - E(a))Z'(a)(y) = E'(a)(y, Z(a)) \qquad \text{for all } y \in V_{1/2}(a),$$

whence the lemma. ∎

The previous lemma says that $R(X, Y)Z(a)$ depends only on the values $X(a)$, $Y(a)$, $Z(a)$ but not on the vector fields X, Y, Z. Thus, for $x, y \in V_{1/2}(a)$, we consider the bounded linear operator $R_a(x, y)$ on $V_{1/2}(a)$ defined by

$$R_a(x, y)z := E'(a)(x, E'(a)(y, z)) - E'(a)(y, E'(a)(x, z)) \qquad (z \in V_{1/2}(a)).$$

Let $D(x, y)$ be the operator introduced in (4.2). We note that the multiplication rules (1.2) imply that $D(x, y)$ leaves $V_{1/2}(a)$ stable if x, y are in $V_{1/2}(a)$.

PROPOSITION 6.3. *One has $R_a(x, y) = D(x, y)|_{V_{1/2}(a)}$ for all $x, y \in V_{1/2}(a)$.*

PROOF. Let $x, y, z \in V_{1/2}(a)$. Then, by Lemma 6.1, we have

$$E'(a)(y, z) = 2(yz)_0 - \langle y | z \rangle a \in V_0(a) + V_1(a),$$

so that by Lemma 6.1 again

$$E'(a)(x,\ E'(a)(y,\ z)) = 4x(yz)_0 + \langle y \,|\, z\rangle x\,.$$

Since $yz \in V_0(a)+V_1(a)$, we get by (6.1)

$$(yz)_0 = yz - E_1(a)(yz) = yz - \langle y \,|\, z\rangle a/2\,. \tag{6.3}$$

Hence we arrive at $E'(a)(x,\ E'(a)(y,\ z)) = 4x(yz)$, which gives the conclusion $R_a(x,\ y)z = 4x(yz) - 4y(xz)$. ■

Now, for $x,\ y \in V_{1/2}(a)$ with $\|x\| = \|y\| = 1$ and $\langle x \,|\, y\rangle = 0$, the *sectional curvature* $k_a(x,\ y)$ at $a \in \mathfrak{J}_1$ is defined as

$$k_a(x,\ y) := \langle R_a(x,\ y)y \,|\, x\rangle\,.$$

Using (4.3) and Proposition 6.3, we see that if $T \in \mathrm{Aut}(V)$, then on $V_{1/2}(a)$ one has

$$R_{Ta}(Tx,\ Ty)T = TR_a(x,\ y) \qquad \text{for all } x,\ y \in V_{1/2}(a)\,.$$

Hence $k_{Ta}(Tx,\ Ty) = k_a(x,\ y)$. Before giving a formula for $k_a(x,\ y)$, we note here that if $x \in V_{1/2}(a)$ with $\|x\| = 1$, then $2x^2 - a$ is a primitive idempotent orthogonal to a by virtue of Proposition 2.7.

THEOREM 6.4. *The sectional curvature of $\mathfrak{J}_1$ is given by*

$$k_a(x,\ y) = \frac{1}{2} + \frac{3}{2}\|E_{1/2}(2x^2 - a)y\|^2.$$

COROLLARY 6.5. *The sectional curvature of $\mathfrak{J}_1$ satisfies $1/2 \leqq k_a(x,\ y) \leqq 2$.*

Theorem 6.4 is a consequence of Lemmas 6.7 and 6.8 below. First of all, we need a technical lemma. Let $x,\ y \in V_{1/2}(a)$ with $\|x\| = \|y\| = 1$ and $\langle x \,|\, y\rangle = 0$. This condition for $x,\ y$ will be kept in all of the following.

LEMMA 6.6. *One has $2\|xy\|^2 = 1/4 - \langle (x^2)_0 \,|\, (y^2)_0\rangle$.*

PROOF. We have

$$\begin{aligned} 2\|xy\|^2 &= 2\langle xy \,|\, xy\rangle = 2\langle L(x)^2 y \,|\, y\rangle = \langle (P(x) + L(x^2))y \,|\, y\rangle \\ &= \langle P(x)y \,|\, y\rangle + \langle x^2 \,|\, y^2\rangle = \langle P(x)y \,|\, y\rangle + \langle (x^2)_0 \,|\, (y^2)_0\rangle + 1/4\,, \end{aligned}$$

where the last equality follows from (6.3): one has $x^2 = (x^2)_0 + a/2$ in the present situation. On the other hand, using the triple product $\{u,\ v,\ w\} := (u \square v)w$, we obtain

$$P(x)y = \{x, y, x\} = 2\{x, \{y, a, a\}, x\} = 4\{\{x, y, a\}, a, x\} - 2\{a, y, \{x, a, x\}\},$$

where we have used the Jordan triple identity [**15**, 18.2.3] to derive the third equality. Now, $2\{x, y, a\}=2(xy)a=\langle x|y\rangle a=0$, because $x, y\in V_{1/2}(a)$ are orthogonal. Thus

$$\langle P(x)y|y\rangle=-2\langle(a\square y)\{x, a, x\}|y\rangle=-2\langle\{x, a, x\}|\{y, a, y\}\rangle.$$

Since $\{x, a, x\}=x^2-ax^2=x^2-a/2=(x^2)_0$ as is seen from (i) of Lemma 2.3, we get

$$\langle P(x)y|y\rangle=-2\langle(x^2)_0|(y^2)_0\rangle,$$

whence the lemma. ■

LEMMA 6.7. *One has* $k_0(x, y)=1/2+6\langle(x^2)_0|(y^2)_0\rangle$.

PROOF. By definition and Proposition 6.3, we have

$$\begin{aligned} k_a(x, y) &= 4\langle xy^2|x\rangle-4\langle y(xy)|x\rangle=4\langle x^2|y^2\rangle-4\|xy\|^2 \\ &= 4\langle(x^2)_0|(y^2)_0\rangle+1-4\|xy\|^2 && \text{(by (6.3))} \\ &= 1/2+6\langle(x^2)_0|(y^2)_0\rangle && \text{(by Lemma 6.6).} \ ■ \end{aligned}$$

LEMMA 6.8. *One has* $4\langle(x^2)_0|(y^2)_0\rangle=\|E_{1/2}(2x^2-a)y\|^2$.

PROOF. Put $c:=2x^2-a\in\mathfrak{J}_1\cap V_0(a)$. Then, since $x^2=(a+c)/2$, we have $(x^2)_0=c/2$. Taking (1.2) and (1.3) into account, we let $y=y'_{1/2}+y'_0$ with $y'_j\in V_j(c)\cap V_{1/2}(a)$ $(j=0, 1/2)$. Then,

$$2\langle(x^2)_0|(y^2)_0\rangle=\langle c|y^2\rangle=\langle c|(y'_{1/2})^2\rangle,$$

because $y'_{1/2}y'_0\in V_{1/2}(c)$ and $(y'_0)^2\in V_0(c)$ owing to (1.2). Since $2(y'_{1/2})^2=\|y'_{1/2}\|^2(a+c)$ by Lemma 2.4, we get

$$4\langle(x^2)_0|(y^2)_0\rangle=\|y'_{1/2}\|^2=\|E_{1/2}(c)y\|^2. \quad ■$$

We end this paper by mentioning some usefulness of the unified formula for $k_a(x, y)$ given in Theorem 6.4. Let x, y be as in Theorem 6.4 and put $c:=2x^2-a\in\mathfrak{J}_1\cap V_0(a)$. Since $x\in V_{1/2}(a)\cap V_{1/2}(c)$, we have $\langle x|E_{1/2}(c)y\rangle=\langle x|y\rangle=0$. Hence,

$$E_{1/2}(c)y\in V_{1/2}(a)\cap V_{1/2}(c)\cap(\boldsymbol{R}x)^{\perp}. \tag{6.4}$$

If $\dim V_{1/2}(a)\cap V_{1/2}(c)=1$, which is the case when $V=\mathrm{Sym}_2(\mathfrak{H})$, the *JH*-algebra of symmetric Hilbert-Schmidt operators on a *real* Hilbert space $\mathfrak{H}$, then $E_{1/2}(c)y=0$ in view of (6.4). In this case, Theorem 6.4 says that $\mathfrak{J}_1$, which is in fact a (real) projective space of $\mathfrak{H}$, is of constant curvature 1/2.

On the other hand, if V is the spin factor $\mathcal{S}(e, \mathfrak{H})$, then $2x^2=e$ for all $x\in V_{1/2}(a)$ with $\|x\|=1$. Thus, $E_{1/2}(c)=E_{1/2}(e-a)=E_{1/2}(a)$. Hence, $k_a(x, y)=2$ in the spin factor case. In this case, $\mathfrak{J}_1$ is identified with the sphere in $\mathfrak{H}$ of radius 1/2 owing to (5.3).

References

[1] W. Ambrose, Structure theorem for a special class of Banach algebras, Trans. Amer. Math. Soc., **57** (1945), 364-386.

[2] P. de la Harpe, Classical Banach-Lie algebras and Banach-Lie groups of operators in Hilbert space, Lecture Notes in Math., **285**, Springer, Berlin, 1972.

[3] J. Faraut, Algèbres de Jordan et cônes symétriques, École d'été CIMPA, 1988.

[4] N. Grossman, Hilbert manifolds without epiconjugate points, Proc. Amer. Math. Soc., **16** (1965), 1365-1371.

[5] L. A. Harris and W. Kaup, Linear algebraic groups in infinite dimension, Illinois J. Math., **21** (1977), 666-674.

[6] U. Hirzebruch, Über Jordan-Algebren und kompakte Riemannsche symmetrische Räume vom Rang 1, Math. Z., **90** (1965), 339-354.

[7] T. Kato, Perturbation theory for linear operators, Springer, Berlin, 1966.

[8] W. Kaup, Über die Klassifikation der symmetrischen hermiteschen Mannigfaltigkeiten unendlicher Dimension I, Math. Ann., **257** (1981), 463-486; II, ibid., **262** (1983), 57-75.

[9] W. Klingenberg, Riemannian geometry, Walter de Gruyter, Berlin, 1982.

[10] E. Määttä, On the Frenet theory of plane fields in Hilbert space, Ann. Acad. Sci. Fenn., Dissert., **47** (1983).

[11] K. McCrimmon, Peirce ideals in Jordan algebras, Pacific J. Math., **78** (1978), 397-414.

[12] E. Neher, Jordan triple systems by the grid approach, Lecture Notes in Math., **1280**, Springer, Berlin, 1987.

[13] H. Porta and L. Recht, Minimality of geodesics in Grassmann manifolds, Proc. Amer. Math. Soc., **100** (1987), 464-466.

[14] H. Upmeier, Automorphism groups of Jordan C^*-algebras, Math. Z., **176** (1981), 21-34.

[15] H. Upmeier, Symmetric Banach manifolds and Jordan C^*-algebras, North-Holland, Amsterdam, 1985.

[16] C. Viola Devapakkiam, Hilbert space methods in the theory of Jordan algebras I, Math. Proc. Cambridge Philos. Soc., **78** (1974), 293-300.

Takaaki NOMURA
Department of Mathematics
Faculty of Science
Kyoto University
Sakyo, Kyoto 606-01
Japan