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Almost coinciding families and gaps in $\mathcal{P}(\omega)$

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1. Introduction.

In [7], S. Mardesic and A. Prasolov translate the calculation of the k-dimensional strong homology of $Y^{(k+1)}$ (the discrete sum of countably many copies of the $(k+1)$-dimensional Hawaiian earring) into a condition which is the existence of a certain family of functions on ω. They showed that the k-dimensional strong homology of $Y^{(k+1)}$ is nontrivial if and only if there exists such a nontrivial family. After that, Dow, Simon and Vanghan [4] named these families almost coinciding families and showed the following Proposition 1~3.

PROPOSITION 1. [4, Theorem 2.4] *If $\mathbf{d}=\omega_1$, then there exists a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.*

PROPOSITION 2. [4, Theorem 3.1] *The proper forcing axiom implies that every almost coinciding family indexed by ${}^{\omega}\omega$ is trivial.*

PROPOSITION 3. [4, Theorem 4.1, Lemma 4.2, 4.3] *If there exists a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$, then there exists an unfilled $(\mathbf{b}, \mathbf{b})$-gap in $\mathcal{P}(\omega)$. So, in Kunen's model of "*$\mathbf{ZFC}$+*Martin's Axiom* ($\mathbf{MA}$)$+2^{\omega}=\omega_2+$*there are no unfilled $(2^{\omega}, 2^{\omega})$-gaps", there does not exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.*

By Propositions 1 and 2, the existence of nontrivial almost coinciding families is independent from the negation of the Continuum Hypothesis ($\mathbf{CH}$). It is an interesting problem to consider whether certain set theoretical axioms imply the existence of nontrivial almost coinciding families. In this paper, we shall show

THEOREM 1. *Let P be the partially ordered set (poset for short) which adjoins ω_2 Cohen reals. Then, in V^P, these does not exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.*

THEOREM 2. *Let $\omega_1<\kappa=\kappa^{<\kappa}$. Then, there is a poset P with the countable chain condition such that, in V^P, $2^{\omega}=\kappa+$*$\mathbf{MA}$*+there exists an unfilled (κ, κ)-gap+*

there does not exist a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$.

Since, in Theorem 1, it holds that $\Vdash$ "$\boldsymbol{b}=\omega_1+\boldsymbol{d}\geqq\omega_2$", the assumption $\boldsymbol{d}=\omega_1$ in Proposition 1 cannot be replaced by $\boldsymbol{b}=\omega_1$. By Theorem 2, $\mathbf{MA}+\neg\mathbf{CH}+$the existence of unfilled $(2^{\omega}, 2^{\omega})$-gaps does not imply the existence of nontrivial almost coinciding families indexed by ${}^{\omega}\omega$.

Theorems 1 and 2 are results about the non-existence of the families. And Proposition 1 is the only result about the existence of the families which I know. The author failed to construct a model of **ZFC** such that $\boldsymbol{d}>\omega_1+$ *there exists a nontrivial almost coinciding family.*

Question. *Is* "$\mathbf{ZFC}+\boldsymbol{d}>\omega_1+$ *there is a nontrivial almost coinciding family indexed by* ${}^{\omega}\omega$" *consistent*?

Theorems 1 and 2 are proved in §§3 and 4, respectively.

The author thanks to the referee for improving and giving short proofs of Lemmas B.1, B.2 and C.1. (Especially, Lemma B.2 is due to the referee and much stronger than the original one.)

2. Notation and Definitions.

The notation used in this paper is a standard one. For the notation on forcing see e.g. [**6**]. Let ω be the set of natural numbers and ${}^{\omega}\omega$ the set of all functions on ω. $\forall^{\infty}x(\cdots x\cdots)$ means that $\{x:\text{not}\cdots x\cdots\}$ is finite. $\exists^{\infty}x(\cdots x\cdots)$ means that $\{x:\cdots x\cdots\}$ is infinite. Define the pseudo-ordering $\prec$ on ${}^{\omega}\omega$ by

$$f\prec g \qquad \text{iff } \forall^{\infty}n<\omega(f(n)<g(n)).$$

Let F be a subset of ${}^{\omega}\omega$. F is said to be *bounded*, if there exists a $g\in{}^{\omega}\omega$ such that $\forall f\in F(f\prec g)$. F is called a *dominating* family if, for any $g\in{}^{\omega}\omega$, there exists an $f\in F$ such that $g\prec f$. The cardinals $\boldsymbol{b}$ and $\boldsymbol{d}$ are defined by

$$\boldsymbol{b}=\min\{|F|:F\text{ is not bounded}\},$$

$$\boldsymbol{d}=\min\{|F|:F\text{ is a dominating family}\}.$$

For $f\in{}^{\omega}\omega$, L_f denotes the set $\{(n,m)\in\omega\times\omega:m\leqq f(n)\}$. Define the quasi-ordering $\subset^*$ and the equivalence relation $\sim$ by

$$X\subset^* Y \qquad \text{iff } X\setminus Y \text{ is finite,}$$

$$X\sim Y \qquad \text{iff } X\triangle Y \text{ is finite.}$$

Let $\mathcal{A}$, $\mathcal{B}$ be families of sets. $\mathcal{A}\perp\mathcal{B}$ means that $A\cap B\sim\emptyset$, for any $A\in\mathcal{A}$ and $B\in\mathcal{B}$. $\mathcal{A}\ll\mathcal{B}$ means that $A\subset^*B$, for any $A\in\mathcal{A}$ and $B\in\mathcal{B}$. $\mathcal{A}$ and $\mathcal{B}$ can be *separated*, if there is an X such that $\mathcal{A}\ll\{X\}$ and $\mathcal{B}\perp\{X\}$.

A κ-sequence $\langle X_\alpha | \alpha<\kappa \rangle$ of subsets of ω is called a *κ-tower,* if $X_\alpha \subset^* X_\beta$, for any $\alpha<\beta<\kappa$. A κ-sequence $\langle (X_\alpha, Y_\alpha) | \alpha<\kappa \rangle$ is called a *(κ, κ)-gap,* if $\langle X_\alpha | \alpha<\kappa \rangle$ and $\langle Y_\alpha | \alpha<\kappa \rangle$ are towers and $\{X_\alpha : \alpha<\kappa\} \perp \{Y_\alpha : \alpha<\kappa\}$. A (κ, κ)-gap $\langle (X_\alpha, Y_\alpha) | \alpha<\kappa \rangle$ is *unfilled,* if $\{X_\alpha : \alpha<\kappa\}$ and $\{Y_\alpha : \alpha<\kappa\}$ cannot be separated. Finally, an indexed set $\langle \psi_f | f \in F \rangle$ is called an *almost coinciding* family indexed by F, if

(i) for any $f \in F$, $\psi_f : L_f \to \omega$,

(ii) for any $f, g \in F$, $\psi_f \upharpoonright (L_f \cap L_g) \sim \psi_g \upharpoonright (L_f \cap L_g)$.

An almost coinciding family $\langle \psi_f | \psi \in F \rangle$ is *nontrivial,* if there does not exist a $\sigma : \omega \times \omega \to \omega$ such that $\{\psi_f : f \in F\} \ll \{\sigma\}$.

3. A proof of Theorem 1.

To prove Theorem 1, we need the following lemma which is a little modification of Lemma 4.3 in [4] and is easily verified by using Fact 3.2 which appears below.

LEMMA 3.1. *Let $F \subset S \subset {}^\omega\omega$. Suppose that $\langle \psi_f | f \in S \rangle$ is a nontrivial almost coinciding family indexed by S and that F is an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions. Then, $\langle \psi_f | f \in F \rangle$ is nontrivial.*

FACT 3.2 (well-known/clear). *Suppose that F is an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions. Then, it holds that, for any infinite subset A of ω,*

$$\forall f \in {}^\omega\omega \ \exists g \in F \ \exists^\infty n \in A \, (f(n) < g(n)) .$$

Let P be the poset which adjoins ω_2 Cohen reals (i.e., $P = \{p : \exists x \subset \omega_2 (|x| < \omega \ \& \ p : x \to 2)\}$) and Q the poset $\{q : \exists n < \omega \ (q : n \to \omega)\}$.

LEMMA 3.3. *Suppose that S is an unbounded subset of ${}^\omega\omega$ which consists strictly increasing functions and $\langle \psi_f | f \in S \rangle$ is a nontrivial almost coinciding family indexed by S. Let $\dot{g}$ be the Q-name of the canonical generic function in ${}^\omega\omega$. Then, in $V^{Q \times P}$, $\langle \psi_f | f \in S \rangle$ can not be extended to an almost coinciding family indexed by $S \cup \{\dot{g}\}$.*

PROOF. To get a contradiction, assume that there exist $(q, p) \in Q \times P$, $Q \times P$-name $\dot{\psi}$ such that

$$\Vdash_{Q \times P} \dot{\psi} : L_{\dot{g}} \longrightarrow \omega \,, \tag{1}$$

$$(q, p) \Vdash_{Q \times P} \forall f \in S \ \forall^\infty x \in L_f \cap L_{\dot{g}} (\dot{\psi}(x) = \psi_f(x)) \,. \tag{2}$$

Because $Q \times P$ satisfies the countable chain condition, there exists an $A \subset \omega_2$ such that

$$|A| \leqq \omega \text{ and } p \in P{\restriction}A \text{ and } \dot{\psi} \text{ is a } Q\times P{\restriction}A\text{-name.}$$

By using (2), for each $f \in S$, take an $n_f < \omega$ and $(q_f, p_f) \in Q \times P{\restriction}A$ such that

(3) $$\operatorname{dom}(q_f) \subset n_f \text{ and } (q_f, p_f) \leqq (q, p),$$

(4) $$(q_f, p_f) \Vdash_{Q\times P} \forall x \in L_f \cap L_{\dot{g}} \setminus (n_f \times \omega)(\dot{\psi}(x) = \psi_f(x)).$$

Since $|Q\times P{\restriction}A| \leqq \omega$ and S is unbounded in ${}^\omega\omega$, there exist an $n' < \omega$, $(q', p') \in Q \times P{\restriction}A$ and a subset F of S such that

(5) $$F \text{ is unbounded in } {}^\omega\omega,$$

(6) $$\forall f \in F\ (n_f = n' \text{ and } q_f = q' \text{ and } p_f = p').$$

By (5) and Lemma 3.1,

(7) $$\langle \psi_f \mid f \in F \rangle \text{ is nontrivial.}$$

CLAIM 1. *$\forall x \in L_f \cap L_h \setminus (n' \times \omega)(\psi_f(x) = \psi_h(x))$, for any $f, h \in F$.*

PROOF OF CLAIM 1. Let $f, h \in F$ and $x = (m, k) \in L_f \cap L_h$ and $n' \leqq m$. Take $q'' \in Q$ such that

$$q'' \leqq q' \text{ and } m \in \operatorname{dom}(q'') \text{ and } q''(m) > k.$$

Since $(q'', p') \Vdash$ "$x \in L_{\dot{g}} \cap L_f \setminus (n' \times \omega)$", it holds that

$$(q'', p') \Vdash \text{“}\dot{\psi}(x) = \psi_f(x)\text{”}.$$

Similary, $(q'', p') \Vdash \dot{\psi}(x) = \psi_h(x)$. Hence, $\psi_f(x) = \psi_h(x)$. QED of Claim 1

By Claim 1, it holds that $\cup\{\Phi_f{\restriction}(L_f \setminus (n' \times \omega)) : f \in F\}$ is a function. So, $\langle \psi_f \mid f \in F \rangle$ is trivial. This contradicts (7). □

PROOF OF THEOREM 1. To get a contradiction, assume that

$$\Vdash_P \text{“}\langle \psi_f \mid f \in {}^\omega\omega \rangle \text{ is a nontrivial almost coinciding family indexed by } {}^\omega\omega\text{”}.$$

Since $\Vdash_P$ "$\boldsymbol{b} = \omega_1$", we can take an $A \subset \omega_2$ and a $P{\restriction}A$-name $\dot{S}$ such that $|A| \leqq \omega_1$ and $\Vdash_P$ "$\dot{S}$ is an unbounded subset of ${}^\omega\omega$ consisting of increasing functions and $|\dot{S}| = \omega_1$". Since P satisfies the countable chain condition, there exists a $\mathcal{B} \subset \omega_2$ such that

$$A \subset B \text{ and } |B| \leqq \omega_1 \text{ and } \langle \dot{\psi}_f \mid f \in \dot{S} \rangle \text{ is a } P{\restriction}B\text{-name.}$$

Since $\Vdash_P$ "$\dot{S}$ is unbounded and consists of increasing functions", by Lemma 3.1,

$$\Vdash_P \text{“}\langle \dot{\psi}_f \mid f \in \dot{S} \rangle \text{ is nontrivial”}.$$

From this and the fact that the formula "x is nontrivial" is Π_1, it holds that

$$\Vdash_{P\restriction B} \text{“}\langle\dot{\phi}_f \mid f\in\dot{S}\rangle \text{ is nontrivial”}.$$

Since $P\restriction(\omega_2\setminus B)$ is isomorphic to P, by replacing the ground model V with $V^{P\restriction B}$, we can assume that $\dot{S}$ and $\langle\dot{\phi}_f \mid f\in\dot{S}\rangle$ are sets in V. Since $\mathrm{ro}(P)$ is isomorphic to $\mathrm{ro}(Q\times P)$, by Lemma 3.3, in V^P, $\langle\dot{\phi}_f \mid f\in\dot{S}\rangle$ cannot be extended to an almost coinciding family indexed by ${}^{\omega}\omega$. But this contradicts the fact that, in V^P, $\langle\dot{\phi}_f \mid f\in{}^{\omega}\omega\rangle$ is an almost coinciding family. □

4. A proof of Theorem 2.

LEMMA 4.1. *The following* (a), (b) *and* (b′) *are equivalent.*

(a) *There exists a nontrivial almost coinciding family indexed by* ${}^{\omega}\omega$.

(b) *There exist a dominating family* $F\subset{}^{\omega}\omega$ *and an indexed set* $\langle(A_f, B_f)\mid f\in F\rangle$ *such that*

(b.1) *for any* $f\in F$, (A_f, B_f) *is a partition of* L_f,

(b.2) $\{A_f : f\in F\}$ *and* $\{B_f : f\in F\}$ *cannot be separated,*

(b.3) *for any* $f, g\in F$, *if* $f\lessdot g$ *then* $A_f\subset^* A_g$ *and* $B_f\subset^* B_g$.

(b′) *For any dominating family* $F\subset{}^{\omega}\omega$, *there exists an indexed set* $\langle(A_f, B_f)\mid f\in F\rangle$ *which satisfies* (b.1)$\sim$(b.3).

PROOF. It is easy to see that (b) and (b′) are equivalent to the following (c) and (c′), respectively.

(c) There exists a dominating family $S\subset{}^{\omega}\omega$ and a nontrivial almost coinciding family $\langle\phi_f \mid f\in S\rangle$ such that, for every $f\in S$, $\phi_f : L_f\to 2$.

(c′) For any dominating family $S\subset{}^{\omega}\omega$, there exists a nontrivial almost coinciding family $\langle\phi_f \mid f\in S\rangle$ such that, for every $f\in S$, $\phi_f : L_f\to 2$.

Also, it is easy to see that (c) and (c′) are equivalent. So, it suffices to show that (c) and (a) are equivalent. The implication from (c) to (a) is clear. To show from (a) to (c), let $\langle\phi_f \mid f\in{}^{\omega}\omega\rangle$ be a nontrivial almost coinciding family indexed by ${}^{\omega}\omega$. For each finite sequence $s=\langle a_i \mid i<n\rangle : n\to\omega$, s^* denotes the finite sequence $\langle 0, \underbrace{1, \cdots, 1}_{a_1\text{ times}}, 0, 1, \cdots, 1, 0, \underbrace{1, \cdots, 1}_{a_{n-1}\text{ times}}, 0\rangle$. For each $g:\omega\to\omega$, $\phi: L_g\to\omega$ and $n<\omega$, let $s_{\phi,n}$ denotes $\langle\phi(n, i)\mid i<g(n)\rangle$. For each $f:\omega\to\omega$, define $\tilde{f}:\omega\to\omega$ and $\Psi_{\tilde{f}} : L_{\tilde{f}}\to 2$ by

$$\tilde{f}(n) = \text{the length of } (s_{\phi_f, n})^*,$$

$$\Psi_{\tilde{f}} = \text{the unique } \Phi : L_{\tilde{f}}\to 2 \text{ such that, for any } n<\omega,\ (s_{\phi_f, n})^* = s_{\Phi, n}.$$

Then, it is easy to see that $\{\tilde{f} : f\in{}^{\omega}\omega\}$ is a dominating subset of ${}^{\omega}\omega$ and $\langle\Psi_{\tilde{f}} \mid f\in{}^{\omega}\omega\rangle$ is a nontrivial almost coinciding family indexed by $\{\tilde{f} : f\in{}^{\omega}\omega\}$. □

The next lemma is due to Kunen (see [**1**, p. 931, Theorem 4.2]).

LEMMA 4.2. *If T is an unfilled (ω_1, ω_1)-gap, then there is a poset P with the countable chain condition such that, $|P|=\omega_1$ and, in V^P, T remains unfilled for any generic extension preserving ω_1.*

We shall show that any finite product of such posets which are described in the above lemma satisfies the countable chain condition (Theorem 4 in Appendix A). By this and Lemma 4.2, we have the following lemma (a proof is given in Appendix A).

LEMMA 4.3. *There is a poset Q such that*

(i) *Q satisfies the countable chain condition and $|Q|\leqq 2^{\omega_1}$,*

(ii) *for any unfilled (ω_1, ω_1)-gap T, $\Vdash_Q$ "T remains unfilled for any generic extension preserving ω_1".*

The next lemma follows from Lemma 4.3 and the standard forcing arguments.

LEMMA 4.4. *Let $\omega_1<\kappa=\kappa^{<\kappa}$ and $\delta<\kappa$. Suppose that $\mathcal{A}=\langle A_\alpha | \alpha<\kappa\rangle$, $\mathcal{B}=\langle B_\xi | \xi<\delta\rangle$ are towers and $\mathcal{A}\perp\mathcal{B}$. Then, there exist a poset Q and Q-names $\dot{f}$, $\dot{B}$ such that*

$$Q \text{ satisfies the countable chain condition and } |Q|=\kappa, \tag{8}$$

$$\Vdash_Q \text{"}2^\omega=\kappa \text{ and } \mathbf{MA}\text{"}, \tag{9}$$

$$\Vdash_Q \text{"}\dot{f}\in{}^\omega\omega\text{" and } \Vdash_Q \text{"}h<\dot{f}\text{", for any } h\in{}^\omega\omega, \tag{10}$$

$$\Vdash_Q \text{"}\mathcal{A}\perp\{\dot{B}\} \text{ and } \mathcal{B}\ll\{\dot{B}\}\text{"}, \tag{11}$$

$$\text{whenever } X\subset\omega \text{ and } \mathcal{A}\perp\{X\},\ \Vdash_Q \text{"}\dot{B}\not\subset^* X\text{"}, \tag{12}$$

(13) *if T is an unfilled (ω_1, ω_1)-gap, then, in V^Q, T remains unfilled for any generic extension preserving ω_1.*

(Outline of a proof) Let Q_1 be the poset as in Lemma 4.3. Since $|Q_1|\leqq\kappa$ and Q_1 satisfies the countable chain condition, it holds that $\Vdash_{Q_1}$ "$\kappa=\kappa^{<\kappa}$". So, in V^{Q_1}, take a poset $\dot{Q}_2$ such that $\dot{Q}_2$ satisfies the countable chain condition and (9)$\sim$(12) except for $\Vdash_{\dot{Q}_2}$ "**MA**". Then, in $V^{Q_1*Q_2}$, take a poset $\dot{Q}_3$ such that $\dot{Q}_3$ satisfies the countable chain condition and $\Vdash_{\dot{Q}_3}$ "$\kappa=\kappa^{<\kappa}$ and **MA**". (Such a poset exists under the assumption that $\kappa=\kappa^{<\kappa}>\omega_1$ (see e.g., [**2**, Remark after Lemma 3.5, p. 16])). Then, the poset $Q=Q_1*\dot{Q}_2*\dot{Q}_3$ is as required. □

To prove Theorem 2, assume that $\omega_1<\kappa=\kappa^{<\kappa}$. By replacing the ground model with a certain generic extension, we may assume that there exists a κ-tower $\mathcal{A}=\langle A_\alpha | \alpha<\kappa\rangle$ in $\mathcal{P}(\omega)$. By using Lemma 4.4, we can construct a κ-stage finite support iteration P_α, $\dot{Q}_\alpha$ and P_α-names $\dot{f}_\alpha$, $\dot{B}_\alpha$ (for $\alpha<\kappa$) such that, in V^{P_α}, (8′) $\dot{Q}_\alpha$ satisfies the countable chain condition and $|\dot{Q}_\alpha|=\kappa$,

(9′) $\langle \dot{B}_\xi | \xi<\alpha\rangle$ is a tower and $\mathcal{A}\perp\{\dot{B}_\xi : \xi<\alpha\}$,

(10′) for any $X\subset\omega$, if $\mathcal{A}\perp\{X\}$, then $\Vdash_{\dot{Q}_\alpha}$ "$\dot{B}_\alpha\not\subset^* X$",

(11′) $\dot{Q}_\alpha$ forces "$2^\omega=\kappa+\mathbf{MA}$",

(12′) $\dot{Q}_\alpha$ forces "$\dot{f}_\alpha\in{}^\omega\omega$ and $g\prec\dot{f}_\alpha$", for any $g\in{}^\omega\omega$,

(13′) if T is an unfilled (ω_1, ω_1)-gap (in V^{P_α}), then $\dot{Q}_\alpha$ forces that "T remains unfilled for any generic extension preserving ω_1".

Let P be the direct limit of $(P_\alpha | \alpha<\kappa)$. It is easy to see that P satisfies the requirement in Theorem 2 except for

$\Vdash_P$ "these does not exist a nontrivial almost coinciding family indexed by ${}^\omega\omega$".

To show this by a contradiction, assume that $p_0\in P$ forces the existence of a nontrivial almost coinciding family indexed by ${}^\omega\omega$. Then, by Lemma 4.1, there exist P-names $\langle(\dot{X}_\alpha, \dot{Y}_\alpha) | \alpha<\kappa\rangle$ such that

(14) $\Vdash_P$ "$(\dot{X}_\alpha, \dot{Y}_\alpha)$ is a partition of $L_{\dot{f}_\alpha}$",

(15) $\Vdash_P$ "$\dot{X}_\alpha\subset^*\dot{X}_\beta$ and $\dot{Y}_\alpha\subset^*\dot{Y}_\beta$", if $\alpha<\beta<\kappa$,

(16) $p_0\Vdash_P$ "$\{\dot{X}_\alpha : \alpha<\kappa\}$, $\{\dot{Y}_\alpha : \alpha<\kappa\}$ cannot be separated".

Set $S=\{\delta<\kappa : \delta$ is a limit ordinal and $\mathrm{cf}(\delta)=\omega_1$ and $\dot{X}_\alpha$, $\dot{Y}_\alpha$ are P_δ-names, for any $\alpha<\delta\}$. Since P satisfies the countable chain condition, S is unbounded in κ and ω_1-closed. By (13′),

$$p_0\Vdash_\delta\text{"}\langle(\dot{X}_\alpha, \dot{Y}_\alpha) | \alpha<\delta\rangle \text{ is filled", for any } \delta\in S.$$

By this and the fact that P satisfies the countable chain condition, it holds that, for any $\delta\in S$, there is a $\beta<\delta$ such that

(∗) there exists a P_β-name $\dot{C}$ such that $p_0\Vdash_\delta$"$\{\dot{X}_\alpha : \alpha<\delta\}\ll\{\dot{C}\}$ and $\{\dot{Y}_\alpha : \alpha<\delta\}\perp\{\dot{C}\}$".

So, we can define the function π from S to κ by

$$\pi(\delta) = \text{the least } \beta<\delta \text{ such that (∗) holds.}$$

For each $\delta\in S$, take a $P_{\pi(\delta)}$-name $\dot{C}_\delta$ such that

$$p_0\Vdash_\delta\text{"}\{\dot{X}_\alpha : \alpha<\delta\}\ll\{\dot{C}_\delta\} \text{ and } \{\dot{Y}_\alpha : \alpha<\delta\}\perp\{\dot{C}_\delta\}\text{".}$$

Since $\pi : S\to\kappa$ is regressive, there exist a stationary set $S'\subset S$ and $\beta<\kappa$ such that

$$p_0\in P_\beta \text{ and } \pi(\delta)=\beta, \text{ for any } \delta\in S'.$$

CLAIM 2. *Let δ, $\eta\in S'$ and $\beta<\delta<\eta$. Then, it holds that*

$$p_0 \Vdash_\beta \text{“}\dot{C}_\delta \setminus (n\times\omega) = \dot{C}_\eta \setminus (n\times\omega),\ \textit{for some}\ n<\omega\text{”}.$$

PROOF OF CLAIM 2. To get a contradiction, let $\delta, \eta \in S'$ and $p_1 \leqq p_0$ such that

$$\beta < \delta < \eta \text{ and } p_1 \Vdash_\beta \forall n < \omega(\dot{C}_\delta \setminus (n\times\omega) \neq \dot{C}_\eta \setminus (n\times\omega)).$$

Take a P_β-name $\dot{g}$ such that

$$\Vdash_\beta \text{“}\dot{g} : \omega \to \omega\text{” and } p_1 \Vdash_\beta \text{“}L_{\dot{g}} \cap (\dot{C}_\delta \triangle \dot{C}_\eta) \text{ is infinite”}.$$

Since $\Vdash_{\beta+1} \text{“}\dot{g} \prec \dot{f}_\beta\text{”}$, it holds that

$$p_1 \Vdash_{\beta+1} \text{“}L_{\dot{f}_\beta} \cap (\dot{C}_\beta \triangle \dot{C}_\eta) \text{ is infinite”}.$$

But this contradicts that $p_0 \Vdash \text{“}L_{\dot{f}_\beta} \cap \dot{C}_\delta \sim \dot{X}_\beta \sim L_{\dot{f}_\beta} \cap \dot{C}_\eta\text{”}$. QED of Claim 2

Take $\delta \in S'$ such that $\beta < \delta$. By Claim 2, since S' is cofinal in κ, it holds that

$$p_0 \Vdash \dot{C}_\delta \text{ separates } \{\dot{X}_\alpha : \alpha < \kappa\} \text{ and } \{\dot{Y}_\alpha : \alpha < \kappa\}.$$

But, this contradicts (16). □

A. The posets associated with (ω_1, ω_1)-gaps.

We state some definitions. Let $T = \langle (a_\alpha, b_\alpha) \mid \alpha < \omega_1 \rangle$ be an (ω_1, ω_1)-gap. For each $\alpha < \omega_1$, set $b'_\alpha = b_\alpha \setminus a_\alpha$. Define the poset P_T by

$$P_T = \{(s, u) : u \subset \omega_1 \ \&\ |u| < \omega \ \&\ \exists n < \omega (s : n \to 2 \ \&\ \bigcup_{\alpha\in u} a_\alpha \cap \bigcup_{\alpha \in u} b'_\alpha \subset n)\},$$

$$(s, u) \leqq (t, v) \text{ iff } t \subset s \ \&\ v \subset u \ \&\ \forall k \in \mathrm{dom}(s \setminus t)((k \in \bigcup_{\alpha \in v} a_\alpha \Rightarrow s(k) = 1)$$
$$\&\ (k \in \bigcup_{\alpha\in v} b'_\alpha \Rightarrow s(k) = 0)).$$

We note that, for any V-generic filter G on P_T, $\{k < \omega : \exists (s, u) \in G(s(k) = 1)\}$ fills the gap T.

For each $\alpha < \omega_1$, set $p_\alpha = (\phi, \{\alpha\})\ (\in P_T)$. Define the poset Q_T by

$$Q_T = \{u \subset \omega_1 : |u| < \omega \ \&\ \{p_\alpha : \alpha \in u\} \text{ is an antichain of } P_T\},$$
$$u \leqq v \text{ iff } v \subset u.$$

The following theorem is due to Kunen (see [**1**, p. 931, Theorem 4.2]).

THEOREM 3. *Let T be an (ω_1, ω_1)-gap. Set $P = P_T$ and $Q = Q_T$.*
(a) *If T is filled, then P satisfies the countable chain condition.*
(b) *If T is unfilled, then*
  (b.1) *$q \Vdash_Q$ “P has an uncountable antichain”, for some $q \in Q$,*
  (b.2) *Q satisfies the countable chain condition.*

We shall show

THEOREM 4. *Let $n<\omega$ and T_i an unfilled (ω_1, ω_1)-gap, for each $i<n$. Then, the product of $\langle Q_{T_i}|i<n\rangle$ satisfies the countable chain condition.*

REMARK. Let T be an unfilled (ω_1, ω_1)-gap. Then, under the assumption of $\mathbf{MA}+\neg\mathbf{CH}$, Theorem 4 is a trivial consequence of Theorem 3, because any poset P which satisfies the countable chain condition also has property K: Any uncountable subset of P has an uncountable subset of pairwise compatible elements. The next theorem claims that the assumption of $\mathbf{MA}+\neg\mathbf{CH}$ (or some assumption like this) is necessary to show that Q_T has property K.

THEOREM 5. *There are a poset R and an R-name $\dot{X}$ such that*
(1) *R satisfies the countable chain condition and $|R|=\omega_1$,*
(2) *$\Vdash_R$ "$\dot{X}$ is an unfilled (ω_1, ω_1)-gap and $Q_{\dot{X}}$ does not have property K."*

Theorems 4 and 5 shall be proved in Appendix B and C (respectively). The rest of this appendix is

PROOF OF LEMMA 4.3. For each unfilled (ω_1, ω_1)-gap T, by using Theorem 3 (b.1), take a $q_T\in Q_T$ such that

$$q_T\Vdash_{Q_T} \text{"}P_T \text{ has an uncountable antichain"}$$

and set $Q'_T=\{q\in Q_T : q\leqq q_T\}$. Set Q=the finite support product of $\langle Q'_T|T$ is an unfilled (ω_1, ω_1)-gap$\rangle$. Then, by Theorem 4, Q is as required. □

B. The countable chain condition.

We first show the following combinatorial lemmas which are due to the referee.

LEMMA B.1. *Let $\langle(a_\alpha, b_\alpha)|\alpha<\omega_1\rangle$ be an unfilled (ω_1, ω_1)-gap, and suppose A and B are countable subsets of ω_1. Then there are uncountable $A'\subset B$ and $B'\subset B$ such that for all $\alpha\in A'$ and $\beta\in B'$, $a_\alpha\cap b_\beta\neq\emptyset$.*

PROOF. For each $\gamma<\omega_1$, choose $n_\gamma<\omega$ such that $\{\alpha\in A : a_\gamma\setminus n_\gamma\subset a_\alpha\}$ and $\{\beta\in B : b_\gamma\setminus n_\gamma\subset b_\beta\}$ are uncountable. Let $a'_\gamma=a_\gamma\setminus n_\gamma$ and $b'_\gamma=b_\gamma\setminus n_\gamma$. Since the gap is unfilled, $\bigcup_{\gamma<\omega_1} a'_\gamma\cap\bigcup_{\gamma<\omega_1} b'_\gamma\neq\emptyset$. Take γ, $\delta<\omega_1$ and $n\in a'_\gamma\cap b'_\delta$. Let $A'=\{\alpha\in A : a'_\gamma\subset b_\alpha\}$ and $B'=\{\beta\in B : b'_\gamma\subset b_\beta\}$. By the definition of n_γ and n_δ, it holds that $n\in a_\alpha\cap b_\beta$, for all $\alpha\in A'$ and $\beta\in B'$. □

LEMMA B.2. *Let $n<\omega$ and $\langle(a^i_\alpha, b^i_\alpha)|\alpha<\omega_1\rangle$ be an unfilled (ω_1, ω_1)-gap, for each $i<n$. Let A and B are uncountable subsets of ω_1. Then there are uncountable $A'\subset B$ and $B'\subset B$ such that if $\alpha\in A'$ and $\beta\in B'$ then for all $i<n$, $a^i_\alpha\cap b^i_\beta\neq\emptyset$.*

PROOF. Use Lemma B.1 n times. $\square$

Now we are ready to prove Theorem 4. The proof is similar to the proof of Theorem 3 (b.2) (in [1, p. 932]) except we need Lemma B.2. Let $n<\omega$ and $T_i=\langle(a_\alpha^i, b_\alpha^i)|\alpha<\omega_1\rangle$ an unfilled (ω_1, ω_1)-gap, for $i<n$. Set $Q=$the product of $\langle Q_{T_i}|i<n\rangle$. To get a contradiction, suppose that $\langle w_\alpha|\alpha<\omega_1\rangle$ is an antichain of Q. For each $\alpha<\omega_1$, let $w_\alpha=(w_\alpha^0, \cdots, w_\alpha^{n-1})$. By using Δ-system argument, we may assume that there are $k_0, \cdots, k_{n-1}\in\omega\setminus\{0\}$ such that, for each $i<n$,

$$|w_\alpha^i| = k_i, \text{ for each } \alpha<\omega_1,$$

$$\text{if } \alpha<\beta, \text{ then } w_\alpha^i\cap w_\beta^i = \emptyset \text{ and } \max(w_\alpha^i) < \min(w_\beta^i).$$

For each $i<n$ and $\alpha<\omega_1$, take $m_{i,\alpha}<\omega_1$ such that

$$a_\xi^i\setminus m_{i,\alpha} \subset a_\eta^i\setminus m_{i,\alpha} \text{ and } b_\xi^i\setminus m_{i,\alpha} \subset b_\eta^i\setminus m_{i,\alpha}, \text{ if } \xi, \eta\in w_\alpha^i \text{ and } \xi<\eta.$$

Again without loss of generality, we may assume that $m_{i,\alpha}=m$, for all $i<n$ and all $\alpha<\omega_1$. For each $i<n$ and $\alpha<\omega_1$, set

$$c_\alpha^i = a_\xi^i\setminus m \text{ and } d_\alpha^i = b_\xi^i\setminus m, \text{ where } \xi = \min(w_\alpha^i).$$

Then, it holds that

$$\langle(c_\alpha^i, d_\alpha^i)|\alpha<\omega_1\rangle \text{ is an unfilled } (\omega_1, \omega_1)\text{-gap, for } i<n.$$

So, by Lemma B.2, there are $\alpha, \beta<\omega_1$ such that

$$c_\alpha^i\cap d_\beta^i \neq \emptyset, \text{ for all } i<n.$$

So, ω_α and ω_β are compatible, a contradiction. $\square$

C. Property K.

A poset P has *property* K if for any uncountable $X\subset P$ there is an uncountable $Y\subset X$ such that any two members of Y are compatible. The following facts are well-known.

(1) If P has property K, then P satisfies the countable chain condition.

(2) If P has property K and Q satisfies the countable chain condition, then $P\times Q$ satisfies the countable chain condition.

(3) The product of finitely many posets with property K also has property K.

(4) **MA**$+\neg$**CH** implies the converse of (1).

There are several examples of a poset which satisfies the countable chain condition but does not have property K, under some set theoretical assumption (see e.g., [8, section 3]). Theorem 5 gives another such example.

We turn to a proof of Theorem 5.

LEMMA C.1. *Let R be a poset and $\dot{X}$ an R-name such that*

(c.1) $V^R \models$ "$\dot{X}$ *is an unfilled* (ω_1, ω_1)-*gap*" *and* $\omega_1^{V^R} = \omega_1^V$.

Suppose that there exists an R-name $\dot{Y}$ such that, in V^R,

(c.2) $\Vdash_{\dot{Y}}$ "$\dot{X}$ *is filled*" *and* $\omega_1^{V^{R*\dot{Y}}} = \omega_1^{V^R}$.

Then, it holds that, in V^R, $Q_{\dot{X}}$ does not have property K. So, R and $\dot{X}$ satisfy (1) *and* (2) *in Theorem* 5.

PROOF. Set $W = V^R$ and $W^* = W^{\dot{Y}}$. By (c.2) and Theorem 3 (a), it holds that

$$W^* \models P_{\dot{X}} \text{ satisfies the countable chain condition.}$$

Since $\omega_1^{W^*} = \omega_1^W$, it holds that

(c.3) $W \models P_{\dot{X}}$ satisfies the countable chain condition.

Since $W \models \exists q \in Q_{\dot{X}} (q \Vdash_{Q_{\dot{X}}}$ "$P_{\dot{X}}$ has an uncountable antichain"), it holds that

(c.4) $W \models Q_{\dot{X}} \times P_{\dot{X}}$ does not satisfy the countable chain condition.

By (c.3) and (c.4), $W \models Q_{\dot{X}}$ does not have property K. □

We shall construct a poset R and R-names $\dot{X}$ and $\dot{Y}$ which satisfy (c.1) and (c.2). The method for doing this is due to Hechler [**5**] and Dordal [**3**]. Hechler used it for adjoining a tower in a generic extension and later Dordal generalized it for adjoining an arbitrary partially order type of $\mathcal{P}(\omega)$/finite in a generic extension.

DEFINITION (Hechler and Dordal). Let $A = (A, <_A)$ be a partial order type. Define the poset $P(A)$ by

$$P(A) = \{p : \exists u \subset A\, \exists n < \omega (|u| < \omega \text{ and } p : u \times n \to 2)\},$$

and for any $p, q \in P(A)$ such that $p : u \times n \to 2$ and $q : v \times m \to 2$,

$$p \leqq q \quad \text{iff} \quad q \subset p \text{ and } \forall a, b \in v\, \forall k \in [m, n)(a <_A b \Rightarrow p(a, k) \leqq p(b, k)).$$

For each $a \in A$, define $P(A)$-name $\dot{H}_a$ by

$$\Vdash \text{"}\dot{H}_a = \{n < \omega : \exists p \in \dot{G}(p(a, n) = 1)\}\text{"},$$

where $\dot{G}$ is the $P(A)$-name of the canonical generic filter.

The following lemma is due to P. Dordal ([**3**, Lemma 5.4, p. 45]).

LEMMA C.2. *Let $A = (A, <_A)$ be a linear order type and B a sub-order type of A.*

(1) *$P(A)$ satisfies the countable chain condition.*

(2) *If G is a V-generic filter on $P(A)$, then $G \cap P(B)$ is a V-generic filter on $P(B)$.*

(3) *If $\dot{x}$ is a $P(A)$-name such that $\Vdash$"$\dot{x} \subset \omega$", then there exists a countable subset C of A such that $\dot{x}$ is a $P(C)$-name.*

(4) *For any* $a, b\in A$, $a<_A b$ *if and only if* $\Vdash$"$\dot{H}_a\subset^*\dot{H}_b$".

Let $\boldsymbol{Q}$ denote the set of rationals. Set $A=\boldsymbol{Q}\times\omega_1\times 2$ and $B=A\cup\{0\}$. Define the linear ordering $<_B$ on B by

$$(q, \alpha, 0) <_B 0 <_B (q, \alpha, 1), \text{ for any } q\in\boldsymbol{Q} \text{ and any } \alpha<\omega_1,$$

$$(q, \alpha, 0) <_B (r, \beta, 0), \text{ if } \alpha<\beta \text{ or } (\alpha=\beta \text{ and } q<r),$$

$$(q, \alpha, 1) <_B (r, \beta, 1), \text{ if } \alpha>\beta \text{ or } (\alpha=\beta \text{ and } q<r).$$

We regard B as the linear order type $(B, <_B)$ and A its sub-order type. Set the poset $R=P(A)$. Define R-name $\dot{a}_\alpha$, $\dot{b}_\alpha$ (for $\alpha<\omega_1$) by

$$\dot{a}_\alpha=\dot{H}_{(0,\alpha,0)} \text{ and } \dot{b}_\alpha=\omega\setminus\dot{H}_{(0,\alpha,1)}, \text{ for each } \alpha<\omega_1.$$

Set $W=V^R$. In W, set $\dot{X}=\langle(\dot{a}_\alpha, \dot{b}_\alpha)\,|\,\alpha<\omega_1\rangle$ and take the poset $\dot{Y}$ such that $W^{\dot{Y}}=V^{P(B)}$. Then, by Lemma C.2, it holds that

$$W\models\text{“}\dot{X} \text{ is an } (\omega_1, \omega_1)\text{-gap” and } W^{\dot{Y}}\models\text{“}\dot{X} \text{ is filled”}.$$

So, the next lemma completes a proof of Theorem 5. The lemma is proved by the same way in the proof of Theorem 5.3 in [3]. So, we omit a proof.

LEMMA C.3. $W\models$"$\dot{X}$ *is unfilled*."

References

[1] James E. Baumgartner, Application of proper forcing, in Handbook of Set-Theoretic Topology, (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam, 1984, pp. 913-959.
[2] James E. Baumgartner, Iterated Forcing, Survey in Set Theory, (ed. A.R.D. Mathias), London Math. Soc. Lecture Note Ser., **87**, Cambridge University Press.
[3] P. Dordal, Towers in $[\omega]^\omega$ and ${}^\omega\omega$, preprint, 1986.
[4] Alan Dow, Petr Simon and Jerry E. Vaughan, Strong homology and the proper forcing axiom, preprint.
[5] S. Hechler, Short complete nested sequences in $\beta\boldsymbol{N}\setminus\boldsymbol{N}$ and small maximal almost-disjoint families, General Topology and Appl., **2** (1972), 139-149.
[6] K. Kunen, Set Theory, North-Holland, 1980.
[7] S. Mardesic and A.V. Prasolov, Strong homology is not additive, Trans. Amer. Math. Soc., **307** (1988), 725-744.
[8] W. Weiss, Versions of Martin's Axiom, Handbook of Set-Theoretic Topology, (eds. K. Kunen and J. Vaughan), North-Holland, Amsterdam, 1984, pp. 827-886.

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