J. Math. Soc. Japan
Vol. 45, No. 2, 1993

Perturbation formula of eigenvalues in a singularly perturbed domain

By Shuichi JIMBO

(Received May 6, 1992)

§1. Introduction.

In this paper, we will deal with the eigenvalues of Laplacian (Neumann B. C.) in a singularly perturbed domain and consider their elaborate characterization. The eigenvalue problem of Laplacian has been an important subject since many years ago and has been studied from the view point of physics, geometry, PDE and other fields of mathematics. In particular, the eigenvalue problem on singularly perturbed domains arise in several real phenomena of physcical situations. We study the Dumbbell shaped domain (cf. Fig. 1) which is related with sound phenomena of wind instruments and is also a simplest case of partial degeneration of domain. Beale [4] has first studied a spectral property of such domain. Actually he characterized the set of the eigenfrequencies and the scattering frequencies. Several related results on the eigenvalue problem have been obtained afterwards (see Hale and Vegas [12], Anné [1], Jimbo [16], Fang [9], Jimbo and Morita [18] and other papers in the references). With the aid of the results and methods in [4], we can easily see that the set of the eigenvalues (Neumann B. C.) of $\Omega(\zeta)$ (see Fig. 1) is divided into two parts (in some sense). One is associated with the fixed region $D=D_1\cup D_2$ and the other is with degenerating region $Q(\zeta)$. That is, the set of eigenvalue $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ can be expressed as follows,

$$\{\mu_k(\zeta)\}_{k=1}^{\infty}=\{\omega_k(\zeta)\}_{k=1}^{\infty}\cup\{\lambda_k(\zeta)\}_{k=1}^{\infty} \tag{1.1}$$

where $\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k$ and $\lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k$. Here $\{\omega_k\}_{k=1}^{\infty}$ is the set of eigenvalues of $-\Delta$ on D (Neumann B. C.) and $\{\lambda_k\}_{k=1}^{\infty}$ is that of $-d^2/dz^2$ on $L=\bigcap_{\zeta>0}Q(\zeta)$ with Dirichlet condition (see also [1], [16]). From $\omega_1=\omega_2=0$, $\omega_3>0$, $\lambda_1>0$ and (1.1), one can easily see that $\mu_2(\zeta)$ goes to 0 while $\mu_3(\zeta)$ is bounded away from 0 when $\zeta\to 0$. In [9], Fang obtained an elaborate convergence rate of $\mu_2(\zeta)$. In [18], we extended this result to more general cases and moreover we obtained some useful properties of the corresponding eigenfunctions which will be effectively used in this paper. In both papers [9] and [18], they dealt with only eigenvalues tending to 0, whose corresponding eigenfunctions are "almost constant" in each D_i. In this paper, we are concerned with the extension of the above results, that is, we deal with cases for higher eigenvalues. Precisely speaking, we consider elaborate behaviors of $\omega_k(\zeta)$, for general $k\geqq 1$, that is, we will obtain the limit,

$$\lim_{\zeta\to 0}\frac{\omega_k(\zeta)-\omega_k}{\tau_{n-1}\zeta^{n-1}}$$

for all $k\geqq 1$, where τ_{n-1} is the volume of the unit ball in $\boldsymbol{R}^{n-1}$. In the proof, we rely on some earlier results ([**15**], [**18**]) which are concerned with characterization of the eigenfunctions.

There have been several studies on other topics on such domain as in this paper. Some of them are concerning solutions and their structure of elliptic equations and reaction diffusion equations. See the papers in the references. There have been also studies on eigenvalues of domains of other type of singular perturbation. Among them, Ozawa [**24**], [**25**], [**26**] dealt with "domain with small holes" and deduced elaborate perturbation formula of eigenvalues. The result in this paper is partially motivated from his nice work.

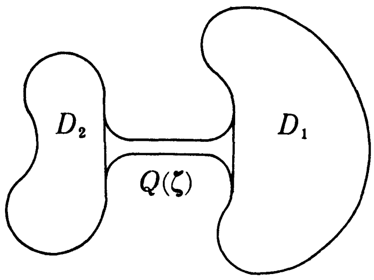

Fig. 1: $\Omega(\zeta)=D_2\cup Q(\zeta)\cup D_1$.

§2. Main result.

We first specify the domain $\Omega(\zeta)$ as in Fig. 1 which is the same domain as that in our previous work [**15**], [**16**]. We put,

$$\Omega(\zeta)=D_1\cup D_2\cup Q(\zeta),$$

where D_1, D_2 and $Q(\zeta)$ satisfy the following conditions where $x'=(x_2, x_3, \cdots, x_n)\in\boldsymbol{R}^{n-1}$.

(I) D_1 and D_2 are bounded domains in $\boldsymbol{R}^n$ where $\bar{D}_1\cap\bar{D}_2=\emptyset$ and each D_i has a smooth boundary and the following conditions hold for some positive constant $\zeta_*>0$.

$$\bar{D}_1\cap\{x=(x_1, x')\in\boldsymbol{R}^n\,|\,x_1\leqq 1,\ |x'|<3\zeta_*\}=\{(1, x')\in\boldsymbol{R}^n\,|\,|x'|<3\zeta_*\},$$

$$\bar{D}_2\cap\{x=(x_1, x')\in\boldsymbol{R}^n\,|\,x_1\geqq -1,\ |x'|<3\zeta_*\}=\{(-1, x')\in\boldsymbol{R}^n\,|\,|x'|<3\zeta_*\}.$$

$$\text{(II)} \qquad Q(\zeta) = R_1(\zeta)\cup R_2(\zeta)\cup\Gamma(\zeta)$$

$$R_1(\zeta) = \{(x_1,\ x')\in\boldsymbol{R}^n \mid 1-2\zeta<x_1\leqq 1,\ |x'|<\zeta\rho((x_1-1)/\zeta)\},$$

$$R_2(\zeta) = \{(x_1,\ x')\in\boldsymbol{R}^n \mid -1\leqq x_1<1-2\zeta,\ |x'|<\zeta\rho((-1-x_1)/\zeta)\},$$

$$\Gamma(\zeta) = \{(x_1,\ x')\in\boldsymbol{R}^n \mid -1+2\zeta\leqq x_1\leqq 1-2\zeta,\ |x'|<\zeta\},$$

where $\rho\in C^0((-2,\ 0])\cap C^\infty((-2,\ 0))$ is a positive function such that $\rho(0)=2$, $\rho(\xi)=1$ for $\xi\in(-2,\ -1)$, $d\rho/d\xi>0$ for $\xi\in(-1,\ 0)$, and the inverse function ρ^{-1}: $(1,\ 2)\to(-1,\ 0)$ satisfies $\lim_{\xi\to 2-0}d^k\rho^{-1}/d\xi^k=0$ holds for any positive integer $k\geqq 1$. We denote $L=\bigcap_{\zeta>0}Q(\zeta)$, $p_1=(1,\ 0,\ \cdots,\ 0)$, $p_2=(-1,\ 0,\ \cdots,\ 0)$ and $D=D_1\cup D_2$. We can identify as $L=(-1,\ 1)$ and $\partial L=\{p_1,\ p_2\}$. By this definition, $\Omega(\zeta)$ is a bounded domain with a smooth boundary for any $\zeta\in(0,\ \zeta_*)$.

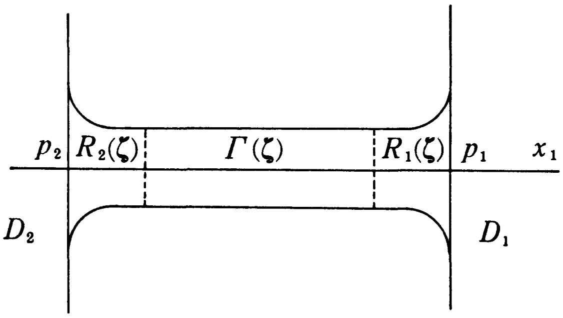

Fig. 2: $Q(\zeta)=R_2(\zeta)\cup\Gamma(\zeta)\cup R_1(\zeta)$.

To state the main results, we need to define several notations.

DEFINITION. Let $\{\mu_k(\zeta)\}_{k=1}^\infty$ and $\{\Phi_{k,\zeta}\}_{k=1}^\infty$ be, respectively, the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of orthonormalized eigenfunctions of the following eigenvalue problem,

$$\text{(2.1)} \qquad \begin{cases} \Delta\Phi+\mu\Phi=0\,, & \text{in } \Omega(\zeta)\,, \\ \dfrac{\partial\Phi}{\partial\nu}=0\,, & \text{on } \partial\Omega(\zeta)\,. \end{cases}$$

DEFINITION. Let $\{\omega_k\}_{k=1}^\infty$ and $\{\phi_k\}_{k=1}^\infty$ be, respectively, the eigenvalues arranged in increasing order (counting multiplicity) and the complete system of orthonormalized eigenfunctions of the eigenvalue problem,

$$\text{(2.2)} \qquad \begin{cases} \Delta\phi+\omega\phi=0\,, & \text{in } D\,, \\ \dfrac{\partial\phi}{\partial\nu}=0\,, & \text{on } \partial D\,, \end{cases}$$

such that

$$(2.3)\qquad (\phi_i\cdot\phi_j)_{L^2(D)}=\begin{cases}1 & \text{for } i=j\,,\\ 0 & \text{for } i\neq j\,.\end{cases}$$

We put $\{\lambda_k\}_{k=1}^{\infty}$ by $\lambda_k=(\pi k/2)^2$ which are eigenvalues of $-d^2/dz^2$ in L with Dirichlet boundary condition.

We recall some characterization of $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ (cf. [1], [4], [16]).

PROPOSITION 2.1. *The set of eigenvalues* $\{\mu_k(\zeta)\}_{k=1}^{\infty}$ *is divided as follows,*

$$(2.4)\qquad \{\mu_k(\zeta)\}_{k=1}^{\infty}=\{\omega_k(\zeta)\}_{k=1}^{\infty}\cup\{\lambda_k(\zeta)\}_{k=1}^{\infty}\,,$$

where $\lim_{\zeta\to 0}\omega_k(\zeta)=\omega_k$, $\lim_{\zeta\to 0}\lambda_k(\zeta)=\lambda_k$, for $k\geqq 1$.

The convergence of $\omega_k(\zeta)$ and $\lambda_k(\zeta)$ are not uniform in k and so the way of decomposition is not unique for each fixed $\zeta>0$.

In this paper, we will work under the following assumption.

$$(\mathrm{A})\qquad \{\omega_k\}_{k=1}^{\infty}\cap\{\lambda_k\}_{k=1}^{\infty}=\emptyset,\quad\text{and}\quad n\geqq 3\,.$$

For the statement of the main results, we prepare a sequence of functions $\{V_k\}_{k=1}^{\infty}\subset C^{\infty}([-1,1])$, which are determined uniquely (from the condition (A)) by the following system of ordinary differential equations,

$$(2.5)\qquad \begin{cases}\dfrac{d^2V_k}{dz^2}+\omega_k V_k=0\,, & \text{in } L=(-1,1)\,,\\ V_k(1)=\phi_k(p_1)\,, & V_k(-1)=\phi_k(p_2)\,.\end{cases}$$

Now we present the main results of this paper.

THEOREM 2.2. *Assume* (A) *and the condition* $(*)$ ω_k *is simple for* $k\geqq 3$, *then we have*

$$(2.6)\qquad \lim_{\zeta\to 0}\frac{\omega_k(\zeta)-\omega_k}{\tau_{n-1}\zeta^{n-1}}=\int_{-1}^{1}\left(\left(\frac{dV_k}{dz}\right)^2-\omega_k V_k^2\right)dz\qquad (k\geqq 1)\,,$$

where $\tau_{n-1}=\dfrac{\pi^{(n-1)/2}}{\Gamma((n+1)/2)}$ (*the volume of unit ball in* $\boldsymbol{R}^{n-1}$).

The next theorem is a generalization of Theorem 2.2. To deal with a case that ω_k $(k\geqq 3)$, are not necessary simple, we define a sequence of natural integers $\{k_j\}_{j=1}^{\infty}$ inductively. Let $k_1=1$. After defining $k_1, k_2, \cdots, k_j$, let k_{j+1} be the smallest integer k satisfying $\omega_k>\omega_{k_j}$.

THEOREM 2.3. *Assume the condition* (A). *Then we have,*

$$(2.7)\qquad \lim_{\zeta\to 0}\frac{\omega_k(\zeta)-\omega_k}{\tau_{n-1}\zeta^{n-1}}=\alpha_j(k-k_j+1)\qquad (k_j\leqq k<k_{j+1},\ j\geqq 1)\,,$$

where $\alpha_j(1), \alpha_j(2), \cdots, \alpha_j(k_{j+1}-k_j)$, *are the eigenvalues of the matrix* $A_j=[a_j(r,q)]_{rq}$ *whose entries are defined by*

$$a_j(r,q)=\int_{-1}^{1}\Big(\frac{dV_{r+k_j-1}}{dz}\frac{dV_{q+k_j-1}}{dz}-\omega_{k_j}V_{r+k_j-1}V_{q+k_j-1}\Big)dz\,, \tag{2.8}$$

$$(1\leqq r,\ q\leqq k_{j+1}-k_j,\ j\geqq 1)\,.$$

REMARK 2.4. $\{V_k\}_{k=1}^{\infty}$ and the matrix A_j, $j\geqq 1$ in the theorems depend on the choice of the orthonormalized eigenfunctions $\{\phi_k\}_{k=1}^{\infty}$. However, all the values in the right hand side of the expressions (2.6) and (2.7) are well-defined. Because if we took another system of orthonormalized eigenfunctions, A_j would be another matrix which is unitary equivalent to the original one.

We carry out the proofs using only real valued functions and hence all the functions and function spaces in the following sections are real valued.

§ 3. Proof of Theorem 2.2.

Before the proof of the theorems, we recall some results (in the following propositions) which were obtained in the earlier work (Fang [**10**], Jimbo [**16**], Jimbo and Morita [**18**]). Let $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ be any complete system of eigenfunctions for (2.4) orthonormalized in $L^2(\Omega(\zeta))$.

$$(\Phi_{k,\zeta}\cdot\Phi_{j,\zeta})_{L^2(\Omega(\zeta))}=\begin{cases}1 & \text{for } k=j\\ 0 & \text{for } k\neq j\end{cases}\qquad (k,\ j\geqq 1)\,. \tag{3.1}$$

According to the decomposition of the set of eigenvalues (2.4), we can express $\{\Phi_{k,\zeta}\}_{k=1}^{\infty}$ as follows,

$$\{\Phi_{k,\zeta}\}_{k=1}^{\infty}=\{\phi_{k,\zeta}\}_{k=1}^{\infty}\cup\{\psi_{k,\zeta}\}_{k=1}^{\infty}\,, \tag{3.2}$$

where $\phi_{k,\zeta}$ and $\psi_{k,\zeta}$ correspond to $\omega_k(\zeta)$ and $\lambda_k(\zeta)$, respectively. We have the following characterization for these eigenfunctions.

PROPOSITION 3.1 ([**16**]). *Assume* (A). *Then we have,*

$$\begin{cases}\limsup_{\zeta\to 0}\|\phi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}<\infty\,,\\ \lim_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}=\infty\,,\\ \limsup_{\zeta\to 0}\|\psi_{k,\zeta}\|_{L^1(\Omega(\zeta))}\zeta^{-(n-1)/2}<\infty\,.\end{cases} \tag{3.3}$$

PROPOSITION 3.2 ([**16**]). *Assume* (A). *Then, for any sequence of positive values* $\{\zeta_m\}_{m=1}^{\infty}$ *with* $\lim_{m\to\infty}\zeta_m=0$, *there exist a subsequence* $\{\sigma_m\}_{m=1}^{\infty}\subset\{\zeta_m\}_{m=1}^{\infty}$ *and complete orthonormal system of eigenfunctions* $\{\phi_k\}_{k=1}^{\infty}$ *in* (2.2) *such that*

(3.4) $$\limsup_{m\to\infty}\sup_{x\in D}|\phi_{k,\sigma_m}(x)-\phi_k(x)|=0\,,\qquad (k\geqq 1),$$

(3.5) $$\limsup_{m\to\infty}\sup_{(x_1,x')\in Q(\sigma_m)}|\phi_{k,\sigma_m}(x_1,x')-V_k(x_1)|=0\,,\qquad (k\geqq 1),$$

where V_k *is defined through* (2.5) *by the above* ϕ_k.

PROPOSITION 3.3 ([**10**]).

(3.6) $$\omega_1(\zeta)=0\,,\qquad \lim_{\zeta\to 0}\frac{\omega_2(\zeta)}{\tau_{n-1}\zeta^{n-1}}=\frac{1}{2}\Big(\frac{1}{|D_1|}+\frac{1}{|D_2|}\Big).$$

REMARK 3.4. We remark that Theorem 2.3 generalizes Proposition 3.3. By taking ϕ_1, ϕ_2, V_1, V_2 as follows,

$$\phi_1(x)=(|D_1|+|D_2|)^{-1/2}\qquad \text{in } D_1\cup D_2\,,$$

$$\phi_2(x)=\begin{cases}|D_2|^{1/2}/|D_1|^{1/2}|D|^{1/2} & \text{in } D_1\,,\\ -|D_1|^{1/2}/|D_2|^{1/2}|D|^{1/2} & \text{in } D_2\,,\end{cases}$$

$$V_1(z)=(|D_1|+|D_2|)^{-1/2}\,,$$

$$V_2(z)=\frac{|D_2|^{1/2}}{2|D_1|^{1/2}|D|^{1/2}}(z+1)-\frac{|D_1|^{1/2}}{2|D_2|^{1/2}|D|^{1/2}}(1-z)\,,$$

we see that (2.7), (2.8) of Theorem 2.3 for $j=1$ coincide with (3.6). We denoted the volume of the regions D_1, D_2, D by $|D_1|$, $|D_2|$, $|D|$. From $D_1\cap D_2=\emptyset$, we have $|D|=|D_1|+|D_2|$.

PROPOSITION 3.5 ([**18**]). *Let* ϕ_1, ϕ_2 *be functions defined in the above Remark,*

(3.7) $$\lim_{\zeta\to 0}\int_D|\nabla\phi_{k,\zeta}|^2dx/\zeta^{n-1}=0\,,\qquad (k=1,\,2)\,.$$

(3.8) $$\begin{cases}\lim_{\zeta\to 0}\|\phi_{i,\zeta}-\sum_{k=1}^{2}(\phi_{i,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|^2_{H^1(D)}/\zeta^{n-1}=0\,, & (i=1,\,2),\\ \lim_{\zeta\to 0}\|\phi_i-\sum_{k=1}^{2}(\phi_i\cdot\phi'_{k,\zeta})_{L^2(D)}\phi'_{k,\zeta}\|^2_{H^1(D)}/\zeta^{n-1}=0\,, & (i=1,\,2),\end{cases}$$

where $\phi'_{k,\zeta}$ *is defined as follows,*

$$\phi'_{k,\zeta}(x)=\phi_{k,\zeta}(x)/\|\phi_{k,\zeta}\|_{L^2(D)}\,,\qquad x\in\Omega(\zeta)\quad (k\geqq 1)\,.$$

We define some notation which we need in the proof.

$$\mathcal{D}_\zeta(\varphi)=\int_{\Omega(\zeta)}|\nabla_x\varphi|^2dx,\qquad \mathcal{H}_\zeta(\varphi)=\int_{\Omega(\zeta)}|\varphi|^2dx\,.$$

Let r_j be the largest natural number r satisfying $\lambda_r<\omega_{k_j}$. If there is no such number, we put $r_j=0$ for convenience. The key in the proof of the main results is to make full use of the following approximate eigenfunctions,

$$\varphi_{k,\zeta}(x)=\begin{cases}\phi_k(x) & \text{in } D,\\ V_k(x_1) & \text{in } \Gamma(\zeta),\\ (x_1-(1-2\zeta))/2\zeta\,\phi_k(1,x')+(1-x_1)/2\zeta\,V_k(x_1), & \text{in } R_1(\zeta),\\ (x_1+1)/2\zeta\,V_k(x_1)+(-1+2\zeta-x_1)/2\zeta\,\phi_k(-1,x'), & \text{in } R_2(\zeta),\end{cases}$$

$(k\geqq 1)$.

It is easy to see that

$$\varphi_{k,\zeta}\in C^0(\overline{\Omega(\zeta)})\cap H^1(\Omega(\zeta))\cap W^{1,\infty}(\Omega(\zeta))$$

and there exists a constant $c_1(k)>0$ for each k such that

$$|\varphi_{k,\zeta}(x)|+|\nabla_x\varphi_{k,\zeta}(x)|\leqq c_1(k)\qquad \text{a.e. } x\in\Omega(\zeta),\ (0<\zeta<\zeta_*).$$

In the proof, we will prove that $\varphi_{k,\zeta}$ is a nice approximation of the true eigenfunction $\phi_{k,\zeta}$. Theorem 2.2 is only a special case of Theorem 2.3. But we will first prove Theorem 2.2 so that we see the important point of the proof avoiding the complicated notation. In this section, we assume that (A) and ω_k is simple for $k\geqq 3$ and hence it means that $k_1=1$ and $k_j=j+1$, $(j\geqq 2)$. Let $\{\phi_k\}_{k=1}^{\infty}\subset L^2(D)$ be any complete orthonormal system of eigenfunctions of (2.2). From the above assumption, by replacing $\phi_{k,\zeta}$ by $-\phi_{k,\zeta}$ if necessary we can assume

$$\lim_{\zeta\to 0}\|\phi_{k,\zeta}-\varphi_{k,\zeta}\|_{L^\infty(\Omega(\zeta))}=0,\tag{3.9}$$

for $k\geqq 1$ (see Proposition 3.2).

We will prove the following, inductively, in $j\geqq 2$,

$$\lim_{\zeta\to 0}\frac{\omega_j(\zeta)-\omega_j}{\tau_{n-1}\zeta^{n-1}}=\int_{-1}^{1}\Big(\Big(\frac{dV_j}{dz}\Big)^2-\omega_j V_j^2\Big)dz,\tag*{(3.10)$_j$}$$

$$\lim_{\zeta\to 0}\|\phi_{j,\zeta}-\sum_{k=1}^{j}(\phi_{j,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|^2_{L^2(D)}/\zeta^{n-1}=0,\tag*{(3.11)$_j$}$$

$$\lim_{\zeta\to 0}\|\phi_j-\sum_{k=1}^{j}(\phi'_{k,\zeta}\cdot\phi_j)_{L^2(D)}\phi'_{k,\zeta}\|^2_{L^2(D)}/\zeta^{n-1}=0.\tag*{(3.12)$_j$}$$

As we mentioned just before in the Remark, the case $j=2$ is true from Proposition 3.5. Thus we assume $(3.10)_j$, $(3.11)_j$ and $(3.12)_j$ for $j=2,\cdots,l-1$. To prove $(3.10)_l$, we use the variational characterization of the eigenvalue,

$$\omega_l(\zeta)\leqq\frac{\mathcal{D}_\zeta(\varphi)}{\mathcal{H}_\zeta(\varphi)}\qquad\text{for any } \varphi\in H^1(\Omega(\zeta)) \text{ with}$$

$$(\varphi\cdot\phi_{j,\zeta})_{L^2(\Omega(\zeta))}=(\varphi\cdot\psi_{h,\zeta})_{L^2(\Omega(\zeta))}=0,\qquad(1\leqq j\leqq l-1,\ 1\leqq h\leqq r_l).$$

Put

$$\varphi=\varphi_{l,\zeta}-\sum_{j=1}^{l-1}(\varphi_{l,\zeta}\cdot\phi_{j,\zeta})_{L^2(\Omega(\zeta))}\phi_{j,\zeta}-\sum_{h=1}^{r_l}(\varphi_{l,\zeta}\cdot\psi_{h,\zeta})_{L^2(\Omega(\zeta))}\psi_{h,\zeta}$$

and then we have,

$$\mathcal{D}_\zeta(\varphi)=\int_{\Omega(\zeta)}|\nabla\varphi_{l,\zeta}|^2dx-\sum_{j=1}^{l-1}\omega_j(\zeta)(\varphi_{l,\zeta}\cdot\phi_{j,\zeta})^2_{L^2(\Omega(\zeta))}-\sum_{h=1}^{r_l}\lambda_h(\zeta)(\varphi_{l,\zeta}\cdot\psi_{h,\zeta})^2_{L^2(\Omega(\zeta))}\tag{3.13}$$

$$\mathcal{H}_\zeta(\varphi)=\int_{\Omega(\zeta)}|\varphi_{l,\zeta}|^2dx-\sum_{j=1}^{l-1}(\varphi_{l,\zeta}\cdot\phi_{j,\zeta})^2_{L^2(\Omega(\zeta))}-\sum_{h=1}^{r_l}(\varphi_{l,\zeta}\cdot\psi_{h,\zeta})^2_{L^2(\Omega(\zeta))}\,.\tag{3.14}$$

We can evaluate the first (main) terms in $\mathcal{D}_\zeta(\varphi)$, $\mathcal{H}_\zeta(\varphi)$ exactly modulo $O(\zeta^n)$. From the definition of $\varphi_{l,\zeta}$, we have,

$$\begin{aligned}\int_{\Omega(\zeta)}|\nabla\varphi_{l,\zeta}|^2dx&=\int_D|\nabla\phi_l|^2dx+\int_{Q(\zeta)}|\nabla\varphi_{l,\zeta}|^2dx\\&=\omega_l+\tau_{n-1}\zeta^{n-1}\int_{-1}^1\left(\frac{dV_l}{dz}\right)^2dz+O(\zeta^n)\end{aligned}\tag{3.15}$$

$$\begin{aligned}\int_{\Omega(\zeta)}|\varphi_{l,\zeta}|^2dx&=\int_D|\phi_l|^2dx+\int_{Q(\zeta)}|\varphi_{l,\zeta}|^2dx\\&=1+\tau_{n-1}\zeta^{n-1}\int_{-1}^1V_l^2dz+O(\zeta^n).\end{aligned}\tag{3.16}$$

Applying (3.3), (3.8) and $(3.11)_j$, $(2\leqq j\leqq l-1)$ and the orthonormality of $\{\phi_k\}\subset L^2(D)$, we estimate the remainder terms as follows,

$$(\varphi_{l,\zeta}\cdot\phi_{j,\zeta})_{L^2(D)}=(\phi_l\cdot\phi_{j,\zeta})_{L^2(D)}=o(\zeta^{(n-1)/2}),\qquad(1\leqq j\leqq l-1),\tag{3.17_1}$$

$$(\varphi_{l,\zeta}\cdot\phi_{j,\zeta})_{L^2(Q(\zeta))}=O(\zeta^{(n-1)}),\qquad(1\leqq j\leqq l-1).\tag{3.17_2}$$

To estimate the latter half of the remainder terms, we do as follows. So we obtain, from Proposition 3.1,

$$\begin{aligned}(\varphi_{l,\zeta}\cdot\psi_{h,\zeta})_{L^2(\Omega(\zeta))}&=(\varphi_{l,\zeta}-\phi_{l,\zeta}\cdot\psi_{h,\zeta})_{L^2(\Omega(\zeta))}\\&\leqq\|\varphi_{l,\zeta}-\phi_{l,\zeta}\|_{L^\infty(\Omega(\zeta))}\|\psi_{h,\zeta}\|_{L^1(\Omega(\zeta))}=o(1)O(\zeta^{(n-1)/2})=o(\zeta^{(n-1)/2}).\end{aligned}\tag{3.18}$$

From all the calculations, we have,

$$\omega_l(\zeta) \leqq \frac{\omega_l+\tau_{n-1}\zeta^{n-1}\int_{-1}^{1}(dV_l/dz)^2 dz+o(\zeta^{n-1})}{1+\tau_{n-1}\zeta^{n-1}\int_{-1}^{1}V_l^2 dz+o(\zeta^{n-1})} \tag{3.19}$$

$$=\omega_l+\tau_{n-1}\zeta^{n-1}\int_{-1}^{1}\left(\left(\frac{dV_l}{dz}\right)^2-\omega_l V_l^2\right)dz+o(\zeta^{n-1}).$$

On the other hand, we estimate $\omega_l(\zeta)$ from below. Using above estimate, we have,

$$\begin{aligned}\omega_l(\zeta) &= \int_{\Omega(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx = \int_D|\nabla\phi_{l,\zeta}|^2 dx+\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx \\ &= \int_D\left|\nabla\Big(\phi_{l,\zeta}-\sum_{j=1}^{l-1}(\phi_{l,\zeta}\cdot\phi_j)_{L^2(D)}\phi_j\Big)\right|^2 dx \\ &\quad+\sum_{j=1}^{l-1}\omega_j(\phi_{l,\zeta}\cdot\phi_j)^2_{L^2(D)}+\int_{Q(\zeta)}|\nabla\varphi_{l,\zeta}|^2 dx \\ &\geqq \omega_l\int_D\left|\phi_{l,\zeta}-\sum_{j=1}^{l-1}(\phi_{l,\zeta}\cdot\phi_j)_{L^2(D)}\phi_j\right|^2 dx \\ &\quad+\sum_{j=1}^{l-1}\omega_j(\phi_{l,\zeta}\cdot\phi_j)^2_{L^2(D)}+\int_{Q(\zeta)}|\nabla\varphi_{l,\zeta}|^2 dx \\ &= \omega_l\int_D|\phi_{l,\zeta}|^2 dx+\sum_{j=1}^{l-1}(\omega_j-\omega_l)(\phi_{l,\zeta}\cdot\phi_j)^2_{L^2(D)}+\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx \\ &= \omega_l\Big(1-\int_{Q(\zeta)}|\phi_{l,\zeta}|^2 dx\Big)+o(\zeta^{n-1})+\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx \\ &= \omega_l\Big(1-\tau_{n-1}\zeta^{n-1}\int_{-1}^{1}V_l^2 dz\Big)+o(\zeta^{n-1})+\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx.\end{aligned} \tag{3.20}$$

In the above process, we used (3.8), $(3.11)_j$, $(2\leqq j\leqq l-1)$. From (3.19), we get

$$\limsup_{\zeta\to 0}\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx/\tau_{n-1}\zeta^{n-1} \leqq \int_{-1}^{1}\left(\frac{dV_l}{dz}\right)^2 dz. \tag{3.21}$$

To estimate the left side of the above expression, we define

$$\bar\phi_{l,\zeta}(y_1, y') = \phi_{l,\zeta}(y_1, \zeta y'), \qquad \text{in } I=\{(y_1, y')\in \boldsymbol{R}^n \mid |y_1|\leqq 1,\ |y'|\leqq 1\},$$

and we have,

$$\int_I\left(\left(\frac{\partial\bar\phi_{l,\zeta}}{\partial y_1}\right)^2+|\nabla_{y'}\bar\phi_{l,\zeta}|^2/\zeta^2\right)dy \leqq \int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx/\zeta^{n-1} \tag{3.22}$$

and hence it yields $\{\bar\phi_{l,\zeta}\}_\zeta$ is bounded in $H^1(I)$ from (3.22) and it is also weakly relatively compact. From (3.9), we see that $\phi_{l,\zeta}$ weakly converges to V_l in H^1. Hence by the lower semicontinuity of the weak convergence of $\bar\phi_{l,\zeta}$ in $H^1(I)$, we have,

$$\liminf_{\zeta\to 0}\int_I\Big(\frac{\partial\bar{\phi}_{l,\zeta}}{\partial y_1}\Big)^2 dy \geqq \int_I\Big(\frac{\partial V_l(y_1)}{\partial y_1}\Big)^2 dy\,. \tag{3.23}$$

From (3.21), (3.22) and (3.23), we get,

$$\lim_{\zeta\to 0}\int_{Q(\zeta)}|\nabla\phi_{l,\zeta}|^2 dx/\tau_{n-1}\zeta^{n-1} = \int_{-1}^{1}\Big(\frac{dV_l}{dz}\Big)^2 dz\,. \tag{3.24}$$

With (3.19), (3.20), we get,

$$\lim_{\zeta\to 0}\frac{\omega_l(\zeta)-\omega_l}{\tau_{n-1}\zeta^{n-1}} = \int_{-1}^{1}\Big(\Big(\frac{dV_l}{dz}\Big)^2-\omega_l V_l^2\Big)dz\,. \tag{3.25}$$

Applying this result to (3.20), we have,

$$\frac{\int_D|\nabla\phi_{l,\zeta}|^2 dx}{\int_D|\phi_{l,\zeta}|^2 dx}-\omega_l = o(\zeta^{n-1})\,. \tag{3.26}$$

To prove $(3.11)_l$, we calculate as follows. From

$$\omega_{l+1}\int_D|\phi_{l,\zeta}-\sum_{k=1}^{l}(\phi_{l,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k|^2 dx$$
$$\leqq \int_D|\nabla(\phi_{l,\zeta}-\sum_{k=1}^{l}(\phi_{l,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k)|^2 dx\,,$$

we have,

$$(\omega_{l+1}-\omega_l)\int_D|\phi_{l,\zeta}-\sum_{k=1}^{l}(\phi_{l,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k|^2 dx$$
$$\leqq \int_D|\nabla\phi_{l,\zeta}|^2 dx-\omega_l\int_D|\nabla\phi_{l,\zeta}|^2 dx+\sum_{k=1}^{l-1}(\omega_l-\omega_k)(\phi_{l,\zeta}\cdot\phi_k)^2_{L^2(D)}\,.$$

From (3.17), (3.26) and $\omega_{l+1}>\omega_l$, the value of the right hand side is $o(\zeta^{n-1})$ and $(3.11)_l$ is proved. $(3.12)_l$ follows from

$$\|\phi_l-(\phi_l\cdot\phi'_{l,\zeta})_{L^2(D)}\phi'_{l,\zeta}\|^2_{L^2(D)} = \|\phi'_{l,\zeta}-(\phi'_{l,\zeta}\cdot\phi_l)_{L^2(D)}\phi_l\|^2_{L^2(D)}$$
$$= \|\phi_{l,\zeta}-(\phi_{l,\zeta}\cdot\phi_l)_{L^2(D)}\phi_l\|^2_{L^2(D)}/\|\phi_{l,\zeta}\|^2_{L^2(D)} = o(\zeta^{n-1})\,.$$

Thus $(3.12)_l$ is true and this completes the induction and the proof of Theorem 2.2.

§ 4. Proof of Theorem 2.3.

In this section, we will prove Theorem 2.3. The proof is a more complicated because of the multiplicity of the eigenvalue, but the idea is essentially same as that of Theorem 2.2. We will proceed under the same notation as in

§ 3. We will show $(2.7)_j$ and the following $(4.1)_j$, $(4.2)_j$ by induction in j. As is mentioned in § 3, the first step $j=1$ of the induction corresponds to Proposition 3.3, Proposition 3.5 and hence we assume that $(2.7)_j$, $(4.1)_j$ and $(4.2)_j$ are true for j such that $1 \leqq j \leqq p-1$ and prove $(2.7)_p$, $(4.1)_p$, and $(4.2)_p$.

$$(4.1)_j \qquad \|\phi_{s,\zeta}-\sum_{k=1}^{k_{j+1}-1}(\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|_{L^2(D)}^2=o(\zeta^{n-1}), \qquad (1\leqq s\leqq k_{j+1}).$$

$$(4.2)_j \qquad \|\phi_s-\sum_{k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}\phi'_{k,\zeta}\|_{L^2(D)}^2=o(\zeta^{n-1}), \qquad (1\leqq s\leqq k_{j+1}).$$

To prove $(2.7)_p$, $(4.1)_p$ and $(4.2)_p$, we take arbitrary sequence of positive values $\{\zeta_m\}_{m=1}^\infty$ with $\lim_{m\to\infty}\zeta_m=0$. From Proposition 3.2, we can choose some subsequence $\{\sigma_m\}_{m=1}^\infty\subset\{\zeta_m\}_{m=1}^\infty$ and complete orthonormal system of eigenfunctions $\{\phi_k\}_{k=1}^\infty$ of (2.2) and solutions $\{V_k\}_{k=1}^\infty$ of (2.5) so that all statements in Proposition 3.2 hold and we have only to prove $(2.7)_p$, $(4.1)_p$ and $(4.2)_p$ with these functions, because the value of the right hand side of $(2.7)_k$ and the meanings of $(4.1)_j$ and $(4.2)_j$ do not depend on choices of such $\{\phi_k\}_{k=1}^\infty$. To avoid very complicated notation, we put, $N=k_{p+1}-k_p$ and a quadratic form,

$$(4.3) \qquad G(\xi)=(A_p\xi\cdot\xi)_{\boldsymbol{R}^N}, \qquad \xi={}^t(\xi_1,\cdots,\xi_N)\in\boldsymbol{R}^N.$$

G is also expressed as

$$G(\xi)=G_1(\xi)-\omega_{k_p}G_2(\xi), \qquad \text{when } G_i(\xi)=\sum_{1\leq r,q\leq N}a^i(r,q)\xi_r\xi_q,$$

for $i=1, 2$ and

$$a_p^1(r,q)=\int_L\frac{dV_{k_p+r-1}}{dz}\frac{dV_{k_p+q-1}}{dz}dz, \qquad a_p^2(r,q)=\int_L V_{k_p+r-1}V_{k_p+q-1}dz.$$

$$a_p(r,q)=a_p^1(r,q)-\omega_{k_p}a_p^2(r,q).$$

To prove $(2.7)_p$, $(4.1)_p$, $(4.2)_p$, we will prove the following $(4.4)_s$, $(4.5)_s$, $(4.6)_s$, $(1\leqq s\leqq N)$ by induction in s.

$$(4.4)_s \qquad \lim_{m\to\infty}\frac{\omega_{k_p+r-1}(\sigma_m)-\omega_{k_p+r-1}}{\tau_{n-1}\sigma_m^{n-1}}=\alpha_p(r), \qquad (1\leqq r\leqq s),$$

$$(4.5)_s \qquad \|\phi_{k_p+r-1,\sigma_m}-\sum_{k=1}^{k_{p+1}-1}(\phi_{k_p+r-1,\sigma_m}\cdot\phi_k)_{L^2(D)}\phi_k\|_{L^2(D)}^2=o(\sigma_m^{n-1}),$$

$$\text{as } m\to\infty, \qquad (1\leqq r\leqq s),$$

$$(4.6)_s \qquad G(e_r)=\alpha_p(r), \qquad (1\leqq r\leqq s),$$

where e_q is the vector in $\boldsymbol{R}^N$ whose q-th element is 1 and others are 0. For later use, we take an orthonormal system of eigenvectors $b_1, b_2, \cdots, b_N\in\boldsymbol{R}^N$ of the symmetric matrix $A_p=(a_p(r,q))_{1\leq r,q\leq N}$ corresponding to the eigenvalues

$\alpha_p(1) \leqq \alpha_p(2) \leqq \cdots \leqq \alpha_p(N)$. We define,

$$\tilde{\varphi}_{r,\zeta}(x) = \sum_{q=1}^{N} b_{qr}\varphi_{k_p+q-1,\zeta}(x), \qquad x \in \Omega(\zeta), \quad (1 \leqq r \leqq N),$$

where we denote the q-th element of the vector $b_r \in \boldsymbol{R}^N$ by b_{qr}.

However the first step $(4.4)_1$, $(4.5)_1$, $(4.6)_1$ can be proved by a similar argument as in the proof of Theorem 2.2 with the aid of $\tilde{\varphi}_{1,\zeta}$ and the spectral gap between ω_{k_p-1} and ω_{k_p}. We assume $(4.4)_s$ and prove the next step. To prove $(4.4)_{s+1}$, we choose a non-zero element

$$\phi_\zeta \in \mathrm{L.\,h.}\,[\tilde{\varphi}_{1,\zeta}, \tilde{\varphi}_{2,\zeta}, \cdots, \tilde{\varphi}_{s+1,\zeta}] \subset L^2(\Omega(\zeta))$$

such that

$$(4.7) \qquad (\phi_\zeta \cdot \phi_{k_p+r-1,\zeta})_{L^2(\Omega(\zeta))} = 0, \qquad (1 \leqq r \leqq s),$$

where L. h. $[W]$ is the subspace spanned by the set W. By multiplying by an appropriate constant, we can express ϕ_ζ as follows,

$$\phi_\zeta = \sum_{q=1}^{s+1} c_q(\zeta)\tilde{\varphi}_{q,\zeta} \qquad \sum_{q=1}^{s+1} c_q(\zeta)^2 = 1.$$

Putting

$$\phi_\zeta^* = \phi_\zeta - \sum_{k=1}^{k_p-1} (\phi_\zeta \cdot \phi_{k,\zeta})_{L^2(\Omega(\zeta))}\phi_{k,\zeta} - \sum_{k=1}^{r_p} (\phi_\zeta \cdot \psi_{k,\zeta})_{L^2(\Omega(\zeta))}\psi_{k,\zeta}$$

we have,

$$\mathcal{D}_\zeta(\phi_\zeta^*) = \int_{\Omega(\zeta)} |\nabla\phi_\zeta|^2 dx - \sum_{k=1}^{k_p-1} \omega_k(\zeta)(\phi_\zeta \cdot \phi_{k,\zeta})_{L^2(\Omega(\zeta))}^2$$

$$- \sum_{k=1}^{r_p} \lambda_k(\zeta)(\phi_\zeta \cdot \psi_{k,\zeta})_{L^2(\Omega(\zeta))}^2$$

$$\mathcal{H}_\zeta(\phi_\zeta^*) = \int_{\Omega(\zeta)} \phi_\zeta^2 dx - \sum_{k=1}^{k_p-1} (\phi_\zeta \cdot \phi_{k,\zeta})_{L^2(\Omega(\zeta))}^2 - \sum_{k=1}^{r_p} (\phi_\zeta \cdot \psi_{k,\zeta})_{L^2(\Omega(\zeta))}^2.$$

Applying a similar argument as that to prove $(3.17)_1$, $(3.17)_2$ and (3.18), we see that the remainder terms can be estimated from $(4.1)_{p-1}$ as follows,

$$(\phi_{\sigma_m} \cdot \phi_{k,\sigma_m})_{L^2(\Omega(\sigma_m))} = o(\sigma_m^{(n-1)/2}), \qquad (1 \leqq k < k_p),$$

$$(\phi_{\sigma_m} \cdot \psi_{k,\sigma_m})_{L^2(\Omega(\sigma_m))} = o(\sigma_m^{(n-1)/2}), \qquad (1 \leqq k \leqq r_p).$$

Using these estimates, we have,

$$(4.8) \qquad \begin{cases} \mathcal{H}_{\sigma_m}(\phi_{\sigma_m}^*) = 1 + \tau_{n-1}\sigma_m^{n-1}G_2(\xi(\sigma_m)) + o(\sigma_m^{n-1}), \\ \mathcal{D}_{\sigma_m}(\phi_{\sigma_m}^*) = \omega_{k_p} + \tau_{n-1}\sigma_m^{n-1}G_1(\xi(\sigma_m)) + o(\sigma_m^{n-1}), \end{cases}$$

$$\text{where } \xi(\zeta) = \sum_{q=1}^{s+1} c_q(\zeta)b_q \in \boldsymbol{R}^N.$$

From the Taylor expansion theorem, we get

$$\frac{\mathcal{D}_{\sigma_m}(\phi^*_{\sigma_m})}{\mathcal{H}_{\sigma_m}(\phi^*_{\sigma_m})}=\omega_{k_p}+\tau_{n-1}\sigma_m^{n-1}(G_1(\xi(\sigma_m))-\omega_{k_p}G_2(\xi(\sigma_m)))+o(\sigma_m^{n-1}).$$

From the fact that $b_1, \cdots, b_s$ and b_{s+1} correspond to the eigenvalues $\alpha_p(1), \cdots, \alpha_p(s)$ and $\alpha_p(s+1)$ respectively and the minimax characterization of the eigenvalues of a real symmetric matrix, we have,

$$\alpha_p(s+1)=G(b_{s+1})\geqq G(\xi(\zeta))=G_1(\xi(\zeta))-\omega_{k_p}G_2(\xi(\zeta)). \tag{4.9}$$

Consequently we have,

$$\omega_{k_p+s}(\sigma_m)\leqq\omega_{k_p}+\tau_{n-1}\sigma_m^{n-1}\alpha_p(s+1)+o(\sigma_m^{n-1}). \tag{4.10}$$

Thus we conclude the estimate from the above.

On the other hand, to obtain an estimate from below, we use,

$$\omega_{k_p+s}(\zeta)=\int_D|\nabla\phi_{k_p+s,\zeta}|^2dx+\int_{\Omega(\zeta)}|\nabla\phi_{k_p+s,\zeta}|^2dx. \tag{$*$}$$

To estimate the right hand side from the below, we see,

$$\|\nabla(\phi_{k_p+s,\zeta}-\sum_{k=1}^{k_p-1}(\phi_{k_p+s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k)\|^2_{L^2(D)}$$

$$\geqq\omega_{k_p}\|\phi_{k_p+s,\zeta}-\sum_{k=1}^{k_p-1}(\phi_{k_p+s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|^2_{L^2(D)}$$

or equivalently,

$$\|\nabla\phi_{k_p+s,\zeta}\|^2_{L^2(D)}\geqq\omega_{k_p}\|\phi_{k_p+s,\zeta}\|^2_{L^2(D)}+\sum_{k=1}^{k_p-1}(\omega_k-\omega_{k_p})(\phi_{k_p+s,\zeta}\cdot\phi_k)^2_{L^2(D)}.$$

The second term is $o(\sigma_m^{n-1})$ from Proposition 3.1 and $(4.2)_{p-1}$. To see this, we use (3.2) and $(4.2)_{p-1}$ as follows,

$$(\phi_{k_p+s,\zeta}\cdot\phi_k)_{L^2(D)}=(\phi_{k_p+s,\zeta}\cdot\sum_{r=1}^{k_p-1}(\phi_k\cdot\phi'_{r,\zeta})_{L^2(D)}\phi'_{r,\zeta})_{L^2(D)}+o(\zeta^{(n-1)/2}) \tag{4.11}$$

$$=(\phi_{k_p+s,\zeta}\cdot\sum_{r=1}^{k_p-1}(\phi_k\cdot\phi'_{r,\zeta})_{L^2(D)}\phi'_{r,\zeta})_{L^2(\Omega(\zeta))}+o(\zeta^{(n-1)/2})=o(\zeta^{(n-1)/2}).$$

Thus using the above argument in $(*)$, we get,

$$\omega_{k_p+s}(\sigma_m)=\int_D|\nabla\phi_{k_p+s,\sigma_m}|^2dx+\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx \tag{4.12}$$

$$\geqq\omega_{k_p}\|\phi_{k_p+s,\sigma_m}\|^2_{L^2(D)}+o(\sigma_m^{n-1})+\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx$$

$$=\omega_{k_p}(\|\phi_{k_p+s,\sigma_m}\|^2_{L^2(\Omega(\sigma_m))}-\|\phi_{k_p+s,\sigma_m}\|^2_{L^2(Q(\sigma_m))})+o(\sigma_m^{n-1})$$
$$+\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx$$
$$=\omega_{k_p}\Big(1-\tau_{n-1}\sigma_m^{n-1}\int_L V_{k_p+s}^2dz\Big)+o(\sigma_m^{n-1})+\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx\,.$$

Consequently, we get, with (4.10),

$$\limsup_{m\to\infty}\frac{1}{\tau_{n-1}\sigma_m^{n-1}}\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx\leqq\alpha_p(s+1)+\omega_{k_p}\int_L V_{k_p+s}^2dz. \tag{4.13}$$

We can apply similar arguments as in the §3 to the stretched functions

$$\{\bar{\phi}_{k_p+s,\sigma_m}\}_{m\geqq 1}\qquad(\text{cf. }\bar{\phi}_{k,\zeta}(y_1,\,y')=\phi_{k,\zeta}(y_1,\,\zeta y'),\ k\geqq 1),$$

defined in the region $I=\{(y_1,\,y')\in\boldsymbol{R}^n\,|\,|y_1|\leqq 1,\ |y'|\leqq 1\}$ (see (3.21)∼(3.24)) and from (4.13), we deduce $H^1(I)$ a-priori estimate to this family $\{\bar{\phi}_{k_p+s,\sigma_m}\}_m$ and with the aid of Proposition 3.2, we have that $\bar{\phi}_{k_p+s,\sigma_m}(y)$ converges to $V_{k_p+s}(y_1)$ uniformly in I and weakly in $H^1(I)$ when $m\to\infty$ and we obtain,

$$\liminf_{m\to\infty}\frac{1}{\tau_{n-1}\sigma_m^{n-1}}\int_{Q(\sigma_m)}|\nabla\phi_{k_p+s,\sigma_m}|^2dx\geqq\int_L\Big(\frac{dV_{k_p+m}}{dz}\Big)^2dz\,. \tag{4.14}$$

From (4.12), we have,

$$\omega_{k_p+s}(\sigma_m)\geqq\omega_{k_p}+\tau_{n-1}\sigma_m^{n-1}\int_L\Big(\Big(\frac{dV_{k_p+s}}{dz}\Big)^2-\omega_{k_p}V_{k_p+s}^2\Big)dz+o(\sigma_m^{n-1}) \tag{4.15}$$
$$=\omega_{k_p}+\tau_{n-1}\sigma_m^{n-1}G(e_{s+1})+o(\sigma_m^{n-1})\,.$$

From (4.10) and (4.15), we have,

$$\tau_{n-1}\sigma_m^{n-1}\alpha_p(s+1)\geqq\tau_{n-1}\sigma_m^{n-1}G(e_{s+1})+o(\sigma_m^{n-1})\,,$$

and so by taking $m\to\infty$, we have $G(e_{s+1})\leqq\alpha_p(s+1)$.
By applying the min-max principle using the orthogonality of $e_1,\,e_2,\,\cdots,\,e_{s+1}$ and $(4.6)_s$ to the quadratic form G we conclude $G(e_{s+1})=\alpha_p(s+1)$ and at the same time, we see, from (4.10), (4.12), (4.15),

$$\omega_{k_p+s}(\sigma_m)=\omega_{k_p}+\tau_{n-1}\sigma_m^{n-1}\alpha_p(s+1)+o(\sigma_m^{n-1})\,.$$

Thus we have proved that $(4.4)_{s+1}$ is true. By using (4.12)-(4.15), we see,

$$\frac{\int_D|\nabla\phi_{k_p+s,\sigma_m}|^2dx}{\int_D\phi_{k_p+s,\sigma_m}^2dx}-\omega_{k_p}=o(\sigma_m^{n-1})\,. \tag{4.16}$$

On the other hand, we have,

$$\|\nabla(\phi_{k_{p}+s,\sigma_m}-\sum_{k=1}^{k_{p+1}-1}(\phi_{k_{p}+s,\sigma_m}\cdot\phi_k)_{L^2(D)}\phi_k)\|^2_{L^2(D)}$$

$$\geqq \omega_{k_{p+1}}\|\phi_{k_{p}+s,\sigma_m}-\sum_{k=1}^{k_{p+1}-1}(\phi_{k_{p}+s,\sigma_m}\cdot\phi_k)_{L^2(D)}\phi_k\|^2_{L^2(D)}$$

or equivalently,

$$(\omega_{k_{p+1}}-\omega_{k_p})\|\phi_{k_{p}+s,\sigma_m}-\sum_{k=1}^{k_{p+1}-1}(\phi_{k_{p}+s,\sigma_m}\cdot\phi_k)_{L^2(D)}\phi_k\|^2_{L^2(D)}$$

$$\leqq \int_D|\nabla\phi_{k_{p}+s,\sigma_m}|^2dx-\omega_{k_p}\int_D\phi^2_{k_{p}+s,\sigma_m}dx+\sum_{k=1}^{k_p-1}(\omega_{k_p}-\omega_k)(\phi_{k_{p}+s,\sigma_m}\cdot\phi_k)^2_{L^2(D)}.$$

In the above, we used $\omega_{k_p}=\omega_{k_{p}+1}=\cdots=\omega_{k_{p+1}-1}$. The last two terms of the right hand side are $o(\sigma_m^{n-1})$ from (4.11) and (4.16). From this, we complete the proof of $(4.4)_{s+1}$, $(4.5)_{s+1}$, $(4.6)_{s+1}$ and at the same time $(4.4)_s$, $(4.5)_s$, $(4.6)_s$ are true for $s=1, 2, \cdots, N$. The whole sequence $\{\zeta_m\}_{m=1}^{\infty}$ is arbitrary and so these mean that $(2.7)_p$ and $(4.1)_p$ are true. We will prove $(4.2)_p$. In the process of the above arguments, we also have,

$$\sum_{s=1}^{k_{p+1}-1}\|\phi_s-\sum_{k=1}^{k_{p+1}-1}(\phi_s\cdot\phi'_{k,\sigma_m})_{L^2(D)}\phi'_{k,\sigma_m}\|^2_{L^2(D)}$$

$$=\sum_{s=1}^{k_{p+1}-1}\{\|\phi_s\|^2_{L^2(D)}-2\sum_{k=1}^{k_{p+1}-1}(\phi_s\cdot\phi'_{k,\sigma_m})^2_{L^2(D)}+\sum_{k=1}^{k_{p+1}-1}(\phi_s\cdot\phi'_{k,\sigma_m})^2_{L^2(D)}\}$$

$$+2\sum_{s=1}^{k_{p+1}-1}\sum_{1\leq q<r<k_{p+1}}(\phi_s\phi'_{q,\sigma_m})_{L^2(D)}(\phi_s\phi'_{r,\sigma_m})_{L^2(D)}(\phi'_{q,\sigma_m}\phi'_{r,\sigma_m})_{L^2(D)}$$

$$=\sum_{k=1}^{k_{p+1}-1}\{\|\phi'_{k,\sigma_m}\|^2_{L^2(D)}-\sum_{s=1}^{k_{p+1}-1}(\phi_s\cdot\phi'_{k,\sigma_m})^2_{L^2(D)}\}+o(\sigma_m^{n-1})$$

$$=\sum_{k=1}^{k_{p+1}-1}\|\phi'_{k,\sigma_m}-\sum_{s=1}^{k_{p+1}-1}(\phi_s\cdot\phi'_{k,\sigma_m})_{L^2(D)}\phi_s\|^2_{L^2(D)}+o(\sigma_m^{n-1})$$

$$=o(\sigma_m^{n-1}),\qquad \text{as } m\to\infty.$$

In this calculation, we used

$$(**)\qquad\left\{\begin{array}{l}(\phi'_{q,\zeta}\cdot\phi'_{r,\zeta})_{L^2(D)}=-(\phi'_{q,\zeta}\cdot\phi'_{r,\zeta})_{L^2(Q(\zeta))}=O(\zeta^{n-1}),\\ \displaystyle\sum_{s=1}^{k_{p+1}-1}(\phi_s\phi'_{q,\zeta})_{L^2(D)}(\phi_s\phi'_{r,\zeta})_{L^2(D)}=o(1),\qquad 1\leqq q\leqq r<k_{p+1},\end{array}\right.$$

which follow from Proposition 3.2. This calculation concludes $(4.2)_p$ and the induction is completed and the proof of Theorem 2.3 has been completed.

REMARK 4.1. We can prove estimates which are stronger than $(4.1)_j$ and $(4.2)_j$. Actually the following convergences are true.

$$\frac{\int_D |\nabla\phi_{s,\zeta}|^2 dx}{\int_D \phi_{s,\zeta}^2 dx} - \omega_s = o(\zeta^{n-1}) \qquad (s \geqq 1), \tag{4.17}$$

$$\|\phi_{s,\zeta} - \sum_{k=1}^{k_{j+1}-1} (\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|_{H^1(D)}^2 = o(\zeta^{n-1}), \qquad (1 \leqq s \leqq k_{j+1}). \tag{4.18$_j$}$$

$$\|\phi_s - \sum_{k=1}^{k_{j+1}-1} (\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}\phi'_{k,\zeta}\|_{H^1(D)}^2 = o(\zeta^{n-1}), \qquad (1 \leqq s < k_{j+1}), \tag{4.19$_j$}$$

for any $j \geqq 1$.

(4.17) follows from (4.16). We give proofs of $(4.18)_j$ and $(4.19)_j$.

$$\begin{aligned}
&\sum_{s=k_j}^{k_{j+1}-1} \|\nabla(\phi_{s,\zeta} - \sum_{k=1}^{k_{j+1}-1}(\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k)\|_{L^2(D)}^2 \\
&= \sum_{s=k_j}^{k_{j+1}-1} \Big(\|\nabla\phi_{s,\zeta}\|_{L^2(D)}^2 - \sum_{k=1}^{k_{j+1}-1} \omega_k(\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}^2\Big) \\
&= \sum_{s=k_j}^{k_{j+1}-1} (\|\nabla\phi_{s,\zeta}\|_{L^2(D)}^2 - \omega_s\|\phi_{s,\zeta}\|_{L^2(D)}^2) \\
&\quad + \sum_{s=k_j}^{k_{j+1}-1} \omega_s\|\phi_{s,\zeta} - \sum_{k=1}^{k_{j+1}-1}(\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k\|_{L^2(D)}^2 \\
&\quad + \sum_{s=k_j}^{k_{j+1}-1}\sum_{k=1}^{k_j-1} (\omega_{k_j} - \omega_k)(\phi_{s,\zeta}\cdot\phi_k)_{L^2(D)}^2
\end{aligned}$$

From (4.17) and $(4.1)_j$, each term in the right hand side is $o(\zeta^{n-1})$. $(4.18)_j$ is proved. To prove $(4.19)_j$, we remark

$$\int_D |\nabla\phi'_{s,\zeta}|^2 dx - \omega_s = o(\zeta^{n-1}) \qquad (s \geqq 1),$$

$$\begin{aligned}
\int_{Q(\zeta)} |\nabla\phi'_{s,\zeta}|^2 dx &= \int_{\Omega(\zeta)} |\nabla\phi'_{s,\zeta}|^2 dx - \int_D |\nabla\phi'_{s,\zeta}|^2 dx \\
&= \omega_s(\zeta)\int_{\Omega(\zeta)} |\phi'_{s,\zeta}|^2 dx - (\omega_s + o(\zeta^{n-1})) \\
&= (\omega_s + O(\zeta^{n-1}))(1 + O(\zeta^{n-1})) - (\omega_s + o(\zeta^{n-1})) = O(\zeta^{n-1}).
\end{aligned}$$

Using this estimates, we calculate as follows,

$$\begin{aligned}
&\|\nabla(\phi_s - \sum_{k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}\phi'_{k,\zeta})\|_{L^2(D)}^2 \\
&= \sum_{s=k_j}^{k_{j+1}-1}\Big(\omega_s - 2\sum_{k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}(\nabla\phi_s\cdot\nabla\phi'_{k,\zeta})_{K^2(D)} \\
&\quad + \sum_{r,k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{r,\zeta})_{L^2(D)}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}(\nabla\phi'_{r,\zeta}\cdot\nabla\phi'_{k,\zeta})_{L^2(D)}\Big)
\end{aligned}$$

$$= \sum_{s=k_j}^{k_{j+1}-1}\Big(\int_D |\nabla\phi'_{s,\zeta}|^2 dx - 2\sum_{k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}(\nabla\phi_s\cdot\nabla\phi'_{k,\zeta})_{L^2(D)}$$

$$+\sum_{k=1}^{k_{j+1}-1}(\phi_s\cdot\phi'_{k,\zeta})^2_{L^2(D)}\|\nabla\phi'_{k,\zeta}\|^2_{L^2(D)}$$

$$-2\sum_{1\le r<k<k_{j+1}}(\phi_s\cdot\phi'_{r,\zeta})_{L^2(D)}(\phi_s\cdot\phi'_{k,\zeta})_{L^2(D)}(\nabla\phi'_{r,\zeta}\cdot\nabla\phi'_{k,\zeta})_{L^2(Q(\zeta))}\Big)$$

$$= \sum_{s=k_j}^{k_{j+1}-1}\Big(\|\nabla(\phi'_{s,\zeta}-\sum_{k=1}^{k_{j+1}-1}(\phi'_{s,\zeta}\cdot\phi_k)_{L^2(D)}\phi_k)\|^2_{L^2(D)}$$

$$+\sum_{k=1}^{k_{j+1}-1}(\phi'_{s,\zeta}\cdot\phi_k)^2_{L^2(D)}(\|\nabla\phi'_{k,\zeta}\|^2_{L^2(D)}-\omega_k)\Big)+o(\zeta^{n-1}).$$

From $(4.17)_j$ and $(4.18)_j$, this term is $o(\zeta^{n-1})$. In the above calculation, we used (**).

Acknowledgement. The author is very grateful to Prof. Y. Morita (Ryukoku Univ.) for fruitful discussions.

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Shuichi JIMBO
Department of Mathematics
College of Liberal Arts and Sciences
Okayama University
Tsushima, Okayama 700
Japan