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On the integrated density of states for the Schrödinger operators with certain random electromagnetic potentials

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1. Introduction.

Let $\{V(x, \omega);\ x \in \boldsymbol{R}^d\}$ and $\{A(x, \omega);\ x \in \boldsymbol{R}^d\}$ be mutually independent, stationary random fields with values in $\boldsymbol{R}$ and in $\boldsymbol{R}^d$, respectively, defined on a probability space $(\Omega, \mathcal{F}, P)$. In this article we are interested in a family $\{H(\omega)\}_{\omega \in \Omega}$ of the Schrödinger operators with magnetic fields which are defined by

$$H(\omega) = \frac{1}{2} \sum_{j=1}^{d} \left(\sqrt{-1} \frac{\partial}{\partial x^j} - A_j^{\omega} \right)^2 + V^{\omega}, \tag{1.1}$$

where A_j^{ω} and V^{ω} are the multiplication operators given by $A_j^{\omega}(x) = A_j(x, \omega)$ and $V^{\omega}(x) = V(x, \omega)$, respectively. $A^{\omega} = (A_1^{\omega}, A_2^{\omega}, \cdots, A_d^{\omega})$ is called the (magnetic) vector potential and is often identified with the differential 1-form $\theta^{\omega} = \sum_{j=1}^{d} A_j^{\omega}(x) dx^j$. Then the corresponding magnetic field is given by the differential 2-form $d\theta^{\omega}$ and, if $d=1$, $H(\omega)$ is unitary equivalent to the operator $-(1/2)d^2/dx^2 + V^{\omega}$, for which fairly satisfactory theory has been developed. Therefore we throughout assume that $d \geqq 2$. As usual we call V^{ω} the (electric) scalar potential.

Such Schrödinger operators arise in quantum models of disordered systems (see, e.g., [5], [9], [11], [13], [17] and references therein) and the spectral properties are one of the main targets of the study. Among them we are interested in the integrated density of states (IDS for short).

The IDS is defined as follows. Let Λ_r, $r=0, 1, 2, \cdots$ be a sequence of hypercubes increasing to $\boldsymbol{R}^d$ and $H_{\Lambda_r}^D(\omega)$ be the restriction of $H(\omega)$ given by (1.1) to $L^2(\Lambda_r)$ with the Dirichlet boundary condition. We denote by $\rho_{\Lambda_r}^D(\lambda, \omega)$ the number of the eigenvalues less than λ of $H_{\Lambda_r}^D(\omega)$. Then the IDS ρ^D is defined by

$$\rho^D(\lambda) = \lim_{r \to \infty} \frac{1}{|\Lambda_r|} \rho_{\Lambda_r}^D(\lambda, \omega), \tag{1.2}$$

where $|A|$ is the volume of a subset A in $\boldsymbol{R}^d$. Under some natural assumptions (see Section 2), it can be proved that the limit in the right hand side of (1.2) exists with probability 1 and is independent of ω.

The main purpose of this article is to study the asymptotic behavior of the IDS $\rho^D(\lambda)$ as $\lambda \to -\infty$ (Lifshitz-type asymptotics) for two types of the random Schrödinger operators, while the asymptotic behavior as $\lambda \to \infty$ will be also discussed shortly.

Firstly we will assume that the scalar potential $\{V(x, \omega)\}$ is a Gaussian random field. But, as will be seen in Section 3, the assumption that the tail probability like $P(V(0, \omega) < \lambda)$ decays exponentially as $\lambda \to -\infty$ is essential. Secondly we will consider the case that $\{V(x, \omega)\}$ is given by

$$V(x, \omega) = q_i(\omega)\chi(x-i) \quad \text{for} \quad x \in \Delta_0 + i, \quad i \in \boldsymbol{Z}^d, \tag{1.3}$$

where χ is a positive function on the unit cube $\Lambda_0 = [-1/2, 1/2)^d$ and $\{q(i); i \in \boldsymbol{Z}^d\}$ is an independent identically distributed sequence of real valued random variables. In the second problem we will also assume that the magnetic field is uniform and is non-random.

We will see that in both problems the effect of the scalar potentials are strong for the asymptotic behavior of the IDS $\rho^D(\lambda)$ as $\lambda \to -\infty$ and that the magnetic fields do not affect the leading asymptotics. This is one of the main results of this article.

As in the case without magnetic fields, there are two methods for the study of the asymptotic behavior of the IDS's: the path integral approach and the functional analytic approach if we follow the terminology of [3]. For the studies on the case that $A=0$, see Nakao [**12**] and Pastur [**13**] for the first approach and Kirsch-Martinelli [**7**] and the references therein for the second one.

We will use the functional analytic methods. As will be seen in Sections 3-5, the Cwikel-Lieb-Rosenbljum bound (if $d \geqq 3$) and the Lieb-Thirring bound (if $d=2$) for the number of the negative eigenvalues of Schrödinger operators will play important roles in order to prove the upper estimates for the IDS $\rho^D(\lambda)$. Moreover, when the magnetic field is uniform, the lemma due to Colin de Verdière will be useful.

Recently Ueki [**16**], following the path integral approach, has studied a similar problem. In particular, he has proved the same result as that for our first problem (Theorem 3.1) using the explicit representation of the Laplace transform of the IDS.

This article is organized as follows. In the next section we will show the existence of the IDS and discuss its independence of the boundary conditions following [**7**] and using the ergodic theorem due to Akcoglu-Krengel [**1**]. In Section 3, we will study the asymptotic behavior of the IDS $\rho^D(\lambda)$ as $\lambda \to -\infty$ under the assumption that V^ω is a Gaussian random field. In Section 4, we will consider the asymptotics of $\rho^D(\lambda)$ as $\lambda\to-\infty$ when V^ω is given by (1.2) and the magnetic field is uniform. Also in Section 5, we will consider a uniform magnetic field. We will study there the asymptotic behavior of $\rho^D(\lambda)$ as $\lambda\to\infty$.

2. The integrated density of states.

In this section, we show the existence of the integrated density of states. For this we follow Kirsch-Martinelli [7] using the ergodic theorem for superadditive processes due to Akcoglu and Krengel [1].

Let $d\geqq 2$ be a fixed integer. Let $\{V(x, \omega);\ x\in\boldsymbol{R}^d\}$ be a jointly measurable real-valued random field on $\boldsymbol{R}^d$, which is defined on a complete probability space $(\Omega, \mathcal{F}, P)$.

We throughout assume the following.

(Ass. 1) (i) There exists a group of measure preserving transformation $T=\{T_i\}_{i\in I}$, $I=\boldsymbol{R}^d$ or $\boldsymbol{Z}^d$, on Ω such that

$$V(x, T_i\omega)=V(x-i, \omega) \tag{2.1}$$

holds for every $x\in\boldsymbol{R}^d$, $\omega\in\Omega$ and $i\in I$;

(ii) T is metrically transitive in the sense that, if $T_iU=U$ for all $i\in I$, $U\in F$, then U is a trivial set, i.e., $P(U)=0$ or 1;

(iii) If $d\geqq 3$, there exists $p\geqq d/2$ such that

$$E[|V(x, \omega)|^p]<\infty \tag{2.2}$$

if $I=\boldsymbol{R}^d$ and

$$E\left[\int_{\Lambda_0}|V(x, \omega)|^p dx\right]<\infty \tag{2.3}$$

if $I=\boldsymbol{Z}^d$, where E denotes the expectation with respect to P and $\Lambda_0=[-1/2, 1/2)^d$. If $d=2$, there exists $p>1$ such that (2.2) or (2.3) holds.

When we assume that the magnetic field is also a random field, we assume the following.

(Ass. 2) (i) $\{A(x, \omega);\ x\in\boldsymbol{R}^d\}$ is a random field with values in $\boldsymbol{R}^d$ which satisfies

$$A(x, T_i\omega)=A(x-i, \omega)$$

for every $x\in\boldsymbol{R}^d$, $\omega\in\Omega$ and $i\in I$, where the group $T=\{T_i\}_{i\in I}$ is the same as in (Ass. 1);

(ii) $\{A(x, \omega)\}$ is C^2 in x almost surely;

(iii) A and V are statistically independent.

When we do not consider a random magnetic field, we assume that the magnetic field is uniform and that the vector potential A is given by $A=Bx/2$ for some real skew-symmetric $d\times d$ matrix B.

Now we proceed to the introduction of the integrated density of states. Let us consider the operator ∇_A defined by

$$\nabla_A f=\nabla f+\sqrt{-1}Af\,.$$

Letting Λ be a hypercube in $\boldsymbol{R}^d$, we consider the quadratic form $q_{\omega,0}^{\Lambda;D}$ defined by

$$q_{\omega,0}^{\Lambda;D}(f,\,g)=\int_\Lambda(\nabla_A f,\,\nabla_A g)(x)dx\,,\qquad f,\,g\in C_0^\infty(\Lambda)\,,$$

where dx is the d-dimensional Lebesgue measure and for $\xi,\,\eta\in\boldsymbol{C}^d$, $(\xi,\,\eta)=\sum_{i=1}^d\bar{\xi}^i\eta^i$. It can be seen by the assumptions that $q_{\omega,0}^{\Lambda;D}$ is closable with probability 1 and we denote by $H_{\Lambda_0}^D(\omega)$ the self-adjoint operator on $L^2(\Lambda)$ corresponding to the closure.

Moreover we let

$$H_\Lambda^D(\omega)=H_{\Lambda_0}^D(\omega)+V^\omega$$

be defined as the self-adjoint operator arising from the sums of forms. It is also easy by the assumptions that $H_\Lambda^D(\omega)$ is well defined with probability one (cf. [**14**]).

Since $V^\omega\in L^p(\Lambda)$ almost surely, $H_\Lambda^D(\omega)$ has compact resolvents (cf. [**15**]). Therefore the spectrum of $H_\Lambda^D(\omega)$ consists of a discrete set of eigenvalues, which we denote by $\lambda_1(H_\Lambda^D(\omega))\leqq\lambda_2(H_\Lambda^D(\omega))\leqq\cdots$ counting the multiplicity. Let $\rho_\Lambda^D(\lambda,\,\omega)$ be the number of the eigenvalues less than λ of $H_\Lambda^D(\omega)$. Then, by the same way as in [**7**], we see that $\rho_\Lambda^D(\lambda,\,\omega)$ defines a superadditive process in the sense of Akcoglu-Krengel [**1**], which we now mention.

Let $\mathcal{J}$ be the class of subsets $[a,\,b)$ of $\boldsymbol{R}^d$ of the form

$$[a,\,b)=\{x\in\boldsymbol{R}^d\,;\ a^i\leqq x^i<b^i,\ i=1,\,2,\,\cdots,\,d\ \text{and}\ a,\,b\in\boldsymbol{R}^d\}\,.$$

A family of sets $\{\Lambda_r\}_{r=0}^\infty$ in $\mathcal{J}$ is called regular if there exists another family $\{\Lambda'_r\}_{r=0}^\infty$ in $\mathcal{J}$ such that

(i) $\Lambda_r\subset\Lambda'_r$ for every r;

(ii) $\Lambda'_r\subset\Lambda'_s$ if $r<s$;

(iii) $0<|\Lambda'_r|\leqq C|\Lambda_r|$ for any r and some constant $C>0$.

Furthermore, if $\cup_r\Lambda'_r=\boldsymbol{R}^d$, then we write $\lim_{r\to\infty}\Lambda_r=\boldsymbol{R}^d$.

Now let $T=\{T_i\}_{i\in I}$ be the group of measure preserving transformations on Ω mentioned in (Ass. 1). A set function $F\colon\mathcal{J}\to L^1(\Omega,\,P)$ is called a superadditive process with respect to T if

(i) $F_\Lambda(T_i\omega)=F_{\Lambda+i}(\omega)$ for every $\Lambda\in\mathcal{J}$ and $i\in I$;

(ii) if $\Lambda_1,\,\Lambda_2,\,\cdots,\,\Lambda_n$ are disjoint sets in $\mathcal{J}$ and if $\Lambda=\cup_{i=1}^n\Lambda_i$ is also in

$\mathcal{J}$, then

$$F_\Lambda \geqq \sum_{i=1}^{n} F_{\Lambda_i};$$

(iii) $\sup\left(\frac{1}{|\Lambda|}\int F_\Lambda(\omega)dP(\omega);\ I\in\mathcal{J},\ |\Lambda|>0\right)\equiv\gamma(F)<\infty.$

Letting $\mathcal{J}_1$ be the class of subsets of the form

$$[a, b) = \{x \in \boldsymbol{R}^d;\ a^i \leqq x^i < b^i,\ a^i,\ b^i \in \boldsymbol{Z}+1/2,\ i=1, 2, \cdots, d\},$$

regular families in $\mathcal{J}_1$ and superadditive processes on $\mathcal{J}_1$ are defined in the same way if we replace $\mathcal{J}$ with $\mathcal{J}_1$.

It is a matter of course that the family $\{[-r-1/2, r+1/2)^d\}_{r=0}^{\infty}$ is a regular family increasing to $\boldsymbol{R}^d$. In the sequel it makes the arguments clear to consider only this regular family.

For a superadditive process, Akcoglu and Krengel [**1**] has shown the following ergodic theorem.

THEOREM 2.1. (1) *Let $I=\boldsymbol{R}^d$ and let F be a superadditive process with respect to the group T. Assume that there exists an $\tilde{F}\in L^1(\Omega, P)$ such that $F_{[a,b)} \leqq \tilde{F}$ for every a, b with $|a^i|, |b^i|\leqq 1$ and $a^i, b^i\in\boldsymbol{Q}$, $i=1, 2, \cdots, d$. Then, for every regular family $\{\Lambda_r\}_{r=0}^{\infty}$ in $\mathcal{J}$ with $\lim_{r\to\infty}\Lambda_r=\boldsymbol{R}^d$,*

$$\lim_{r\to\infty}\frac{1}{|\Lambda_r|}F_{\Lambda_r}(\omega)=\gamma(F) \tag{2.4}$$

holds for almost all $\omega\in\Omega$.

(2) *Let $I=\boldsymbol{Z}^d$ and let F be a superadditive process with respect to T. Then, for every regular family $\{\Lambda_r\}_{r=0}^{\infty}$ in $\mathcal{J}_1$ with $\lim_{r\to\infty}\Lambda_r=\boldsymbol{R}^d$, (2.4) holds for almost all $\omega\in\Omega$.*

By using this ergodic theorem, the following fundamental theorem, which assures the existence of the IDS, can be proved by the same way as in [7]. We show only the statement without the proof.

We set

$$\rho^D(\lambda)=\sup\left\{\frac{1}{|\Lambda|}E[\rho_\Lambda^D(\lambda, \omega)]\right\},$$

where the supremum is taken over all $\Lambda\in\mathcal{J}$ or $\mathcal{J}_1$, according to $I=\boldsymbol{R}^d$ or $\boldsymbol{Z}^d$, with $|\Lambda|>0$. The finiteness of the right hand side is seen from the Cwikel-Lieb-Rosenbljum bound and the Lieb-Thirring bound for the number of the negative eigenvalues of Schrödinger operators. For these facts, see the next section.

Then we have the following:

THEOREM 2.2. *Let $\{V(x, \omega)\}$ be a random field satisfying* (Ass. 1). *Let $\{A(x, \omega)\}$ be a random field satisfying* (Ass. 2) *or let $A(x)$ be given by $A(x)=Bx/2$ for some $d\times d$ real skew-symmetric matrix B. Then it holds that*

$$\lim_{r\to\infty}\frac{1}{|\Lambda_r|}\rho_{\Lambda_r}^D(\lambda, \omega)=\rho^D(\lambda)$$

almost surely for every $\lambda\in\boldsymbol{Q}$ and for every regular family $\{\Lambda_r\}_{r=0}^\infty$ in $\mathcal{J}$ if $I=\boldsymbol{R}^d$ and in $\mathcal{J}_1$ if $I=\boldsymbol{Z}^d$ increasing to $\boldsymbol{R}^d$.

The non-decreasing function ρ^D obtained above and the non-negative measure $d\rho^D$ on $\boldsymbol{R}$ is called the integrated density of states and the density of states, respectively, for the family $\{H(\omega)\}$ of the Schrödinger operators with random potential.

So far we have imposed the Dirichlet boundary condition for the Schrödinger operators on hypercubes. It is an important and interesting problem to consider whether or not the function ρ^D depends on the boundary condition.

To mention a result in this direction, let $H_{\Lambda_0}^N(\omega)$ be the self-adjoint operator on $L^2(\Lambda)$ corresponding to the closable form $q_{\omega,0}^N$ defined by

$$q_{\omega,0}^N(f, g)=\int_\Lambda(\nabla_A f, \nabla_A g)(x)dx, \qquad f, g\in\boldsymbol{H}^1(\Lambda),$$

where $\boldsymbol{H}^1(\Lambda)$ is the set of functions in $L^2(\Lambda)$ whose derivatives of first order in the sense of distribution are also in $L^2(\Lambda)$. We also define $H_\Lambda^N(\omega)$, $\lambda_n(H_\Lambda^N(\omega))$, $\rho_\Lambda^N(\lambda, \omega)$ as before and set

$$\rho^N(\lambda)=\inf_\Lambda\left\{\frac{1}{|\Lambda|}E[\rho_\Lambda^N(\lambda, \omega)]\right\}.$$

Then, using the Feynman-Kac-Ito's formula and tracing the argument in [7], we can prove the following.

THEOREM 2.3. *Let $\{V(x, \omega)\}$ and $\{A(x, \omega)\}$ be random fields as in Theorem 2.2. Assume that $E[\mathrm{Tr}\exp(-t(-\Delta_{\Lambda_0}^N+qV))]$ is finite for some $t>0$ and some $q>1$. Moreover, if $I=\boldsymbol{R}^d$, assume that $\sup\rho_{[a,b)}^N(\lambda, \omega)$ belongs to $L^1(\Omega, P)$ for each $\lambda\in\boldsymbol{R}$, where the supremum is taken over all $a, b\in\boldsymbol{R}^d$ with $|a^i|\leqq 1$, $|b^i|\leqq 1$, $a^i, b^i\in\boldsymbol{Q}$, $i=1, 2, \cdots, d$. Then*

$$\lim_{r\to\infty}\frac{1}{|\Lambda_r|}\rho_{\Lambda_r}^N(\lambda, \omega)=\rho^N(\lambda)$$

holds almost surely for every $\lambda\in\boldsymbol{Q}$ and for every regular family $\{\Lambda_r\}_{r=0}^\infty$ in $\mathcal{J}$ if $I=\boldsymbol{R}^d$ and in $\mathcal{J}_1$ if $I=\boldsymbol{Z}^d$ increasing to $\boldsymbol{R}^d$. Furthermore $\rho^N(\lambda)=\rho^D(\lambda)$ holds for almost every $\lambda\in\boldsymbol{R}$.

For the operators we will consider in the sequel, the assumptions of

Theorem 2.3 are not necessarily satisfied. Therefore we always denote the IDS by $\rho^D(\lambda)$, which is obtained from the Schrödinger operators on hypercubes with the Dirichlet boundary conditions.

We have not mentioned the essential self-adjointness of $H(\omega)$ itself and the spectral property of its self-adjoint extension because they will not be necessary in the following sections. But we should mention them here. Under more restrictive assumptions than those above, in particular, under the assumption on the existence of higher order moments of $|V(0, \omega)|$, it can be proved that $H(\omega)$ is essentially self-adjoint on $C_0^\infty(\boldsymbol{R}^d)$ with probability 1 (see [3], [8] and references therein for the case of $A=0$). Moreover it can also be proved that the spectrum of $H(\omega)$ does not depend on ω with probability 1 and that it coincides with the support of the density of states $d\rho^D(\lambda)$.

3. Asymptotic behavior of the IDS at $-\infty$, I.

In this section, we study the asymptotic behavior of $\rho^D(\lambda)$ as $\lambda\to-\infty$ when $\{V(x, \omega);\ x\in\boldsymbol{R}^d\}$ is a Gaussian random field. We can prove a similar assertion (see Theorem 3.2 below) if the tail probability $P(V(0, \omega)<\lambda)$ decays exponentially as $\lambda\to-\infty$. But we will mainly consider the problem under the assumption (Ass. 3) below since it is typical.

(Ass. 3) $\{V(x, \omega);\ x\in\boldsymbol{R}^d\}$ is a stationary Gaussian random field with mean m and covariance $h(x-y)=E[(V(x, \omega)-m)(V(y, \omega)-m)]$, $x, y\in\boldsymbol{R}^d$. Moreover h is a continuous function.

REMARK. Under (Ass. 3), the assumptions of Theorem 2.3 are satisfied. Therefore $\rho^D(\lambda)=\rho^N(\lambda)$ holds for almost every $\lambda\in\boldsymbol{R}$ in this case.

The main result in this section is the following.

THEOREM 3.1. *Let $\{V(x, \omega)\}$ and $\{A(x, \omega)\}$ be random fields satisfying* (Ass. 1), (Ass. 2) *and* (Ass. 3). *Let $\rho^D(\lambda)$ be the corresponding IDS. Then it holds that*

$$\lim_{\lambda\to-\infty} -\frac{1}{\lambda^2}\log\rho^D(\lambda)=\frac{1}{2h(0)}.$$

In order to show the upper estimate for $\rho^D(\lambda)$, the following bounds for the number of the negative eigenvalues of the Schrödinger operators, known as the Cwikel-Lieb-Rosenbljum bound and as the Lieb-Thirring bound, are useful. Since they will be used also in the next section, we give the statements here.

Let V be a multiplication operator which is $-\Delta$-form bounded. We assume that $V_-=\min(0, V)$ belongs to $L^{d/2}(\boldsymbol{R}^d)$ if $d\geqq 3$ and to $L^p(\boldsymbol{R}^2)$ for some $p>1$

if $d=2$. For $A\in C^2(\boldsymbol{R}^d\to\boldsymbol{R}^d)$, we denote by $H_{(A,V)}$ the Schrödinger operator defined by

$$\frac{1}{2}\sum_{j=1}^{d}\left(\sqrt{-1}\frac{\partial}{\partial x^j}-A_j\right)^2+V\,.$$

Then Avron-Herbst-Simon ([2], Theorem 2.15) has proved the following:

LEMMA 3.2. ([2]) *Let $d\geqq 3$ and denote by $N_0(A, V)$ the number of the negative eigenvalues of $H_{(A,V)}$. Then there exists a constant C_d, depending only on d, such that*

$$N_0(A, V)\leqq C_d\int_{\boldsymbol{R}^d}|V_-(x)|^{d/2}dx\,.$$

If $d=2$, we have the following.

LEMMA 3.3. *Let $d=2$, $p>1$ and denote by $e_1, e_2, \cdots$, the negative eigenvalues of $H_{(A,V)}$. Then there exists a constant C such that*

$$\sum_j|e_j|^{p-1}\leqq C\int_{\boldsymbol{R}^2}|V_-(x)|^p dx\,.$$

PROOF. If we note that

$$((\sqrt{-1}\nabla-A)^2+\alpha)^{-1}\leqq(-\Delta+\alpha)^{-1}$$

holds for any $\alpha\geqq 0$, the proof of Theorem 1 in Lieb-Thirring [10] goes through without any change.

From Lemma 3.3, we obtain the following corollary which, in fact, will be used in the proof of Theorem 3.1 and also in the next section.

COROLLARY 3.4. *Let $p>1$, Λ be a hypercube in $\boldsymbol{R}^2$ and I_Λ be its indicator function. For $\eta>0$, denote by $N_{-\eta}(A, V)$ the number of the eigenvalues less than $-\eta$ for the Schrödinger operator*

$$\frac{1}{2}\sum_{j=1}^{2}\left(\sqrt{-1}\frac{\partial}{\partial x^j}-A_j\right)^2+(V-\eta)_-I_\Lambda\,.$$

Then, it holds that

$$N_{-\eta}(A, V)\leqq C\eta^{1-p}\int_\Lambda|(V-\eta)_-(x)|^p dx\,.$$

PROOF. We have only to note

$$N_{-\eta}(A, V)\leqq\eta^{1-p}\sum_j|e_j|^{p-1}$$

and to use Lemma 3.3.

We are now in the position to give the proof of Theorem 3.1.

PROOF OF THEOREM 3.1. (the upper estimate) In the sequel, we denote by $\lambda_k(H)$, $k=1, 2, \cdots$, the eigenvalues of a self-adjoint operator H.

At first, we fix a hypercube Λ in $\boldsymbol{R}^d$. Then, by using the min-max principle repeatedly, we get

$$\begin{aligned}\rho_\Lambda^D(\lambda, \omega) &\equiv {}^\#\{k \in \boldsymbol{N};\ \lambda_k(H_\Lambda^D(\omega)-\lambda) < 0\} \\ &\leqq {}^\#\{k \in \boldsymbol{N};\ \lambda_k(H_{\Lambda_0}^D(\omega)+(V^\omega-\lambda)_-) < 0\} \\ &\leqq {}^\#\{k \in \boldsymbol{N};\ \lambda_k(H_0(\omega)+I_\Lambda(V^\omega-\lambda)_-) < 0\},\end{aligned}$$

where the Schrödinger operators $H_\Lambda^D(\omega)$ and $H_{\Lambda_0}^D(\omega)$ on $L^2(\Lambda)$ were defined in Section 2 and $H_0(\omega)$, which is an operator on $L^2(\boldsymbol{R}^d)$, is defined by

$$H_0(\omega) = \frac{1}{2}\sum_{j=1}^d \left(\sqrt{-1}\frac{\partial}{\partial x^j} - A_j^\omega\right)^2.$$

At the last line we have used the fact that the essential spectrum of $H_0(\omega)+I_\Lambda(V^\omega-\lambda)_-$ is contained in $[0, \infty)$. This can be seen by noting that $I_\Lambda(V^\omega-\lambda)_- \in L^p(\boldsymbol{R}^d)$ almost surely for every $p\geqq 1$ and it is a relatively compact perturbation of $H_0(\omega)$.

Now we use Lemma 3.2 and Corollary 3.4. Then we get

$$\rho_\Lambda^D(\lambda, \omega) \leqq C_d\int_\Lambda |(V^\omega-\lambda)_-(x)|^{d/2}dx$$

if $d\geqq 3$ and

$$\rho_\Lambda^D(\lambda, \omega) \leqq C\eta^{1-p}\int_\Lambda |(V^\omega-\lambda-\eta)_-(x)|^p dx$$

if $d=2$ for any $\eta>0$ and for some constants C_d and C.

Moreover, using the stationarity of V, we get

$$\frac{1}{|\Lambda|}E[\rho_\Lambda^D(\lambda, \omega)] \leqq C_d E[|(V(0, \omega)-\lambda)_-|^{d/2}]$$

if $d\geqq 3$ and

$$\frac{1}{|\Lambda|}E[\rho_\Lambda^D(\lambda, \omega)] \leqq C\eta^{1-p}E[|(V(0, \omega)-\lambda-\eta)_-|^p] \tag{3.1}$$

if $d=2$. Therefore, by Theorem 2.2, we get

$$\rho^D(\lambda) \leqq C_d E[|(V(0, \omega)-\lambda)_-|^{d/2}]$$

and the assumption that $V(0, \omega)$ is a Gaussian random variable implies

$$\liminf_{\lambda\to-\infty} -\frac{1}{|\lambda|^2}\log \rho^D(\lambda) \geqq \frac{1}{2h(0)} \tag{3.2}$$

when $d\geqq 3$. The proof of (3.2) in the case that $d=2$ is similar if we use (3.1).

PROOF OF THEOREM 3.1. (the lower estimate) Fix a hypercube Λ containing 0 and set

$$\lambda_1(H_\Lambda^D(\omega)) = \inf \operatorname{spec}(H_\Lambda^D(\omega)).$$

By the min-max principle,

$$\lambda_1(H_\Lambda^D(\omega)) = \inf \int_\Lambda \{|\nabla\varphi + \sqrt{-1}\varphi A^\omega|^2 + V^\omega|\varphi|^2\}\,dx,$$

where the infimum is taken over all $\varphi \in C_0^\infty(\Lambda)$ with $\|\varphi\|_2 = 1$. Therefore, for any such φ, it holds that

$$E[\rho_\Lambda^D(\lambda, \omega)] \geqq P(\lambda_1(H_\Lambda^D(\omega)) < \lambda) \geqq P(\xi_\varphi < \lambda - \eta_\varphi),$$

where the random variables ξ_φ and η_φ are given by

$$\xi_\varphi = \int_\Lambda V^\omega(x)|\varphi(x)|^2 dx \quad \text{and} \quad \eta_\varphi = \int_\Lambda |\nabla\varphi + \sqrt{-1}\varphi A^\omega|^2 dx,$$

respectively. By (Ass. 3), ξ_φ obeys the Gaussian distribution with mean m and variance σ_φ:

$$\sigma_\varphi = \int_\Lambda \int_\Lambda |\varphi(x)|^2 |\varphi(y)|^2 h(x-y)\,dx\,dy.$$

Now, denoting by F_φ the distribution function of η_φ, we get

$$\rho^D(\lambda) \geqq \frac{1}{|\Lambda|} E[\rho_\Lambda^D(\lambda, \omega)] \geqq \frac{1}{|\Lambda|} \int_0^\infty dF_\varphi(u) \int_{-\infty}^{\lambda-u} \frac{1}{\sqrt{2\pi\sigma_\varphi}} e^{-(v-m)^2/2\sigma_\varphi} dv$$

by the independence of A and V. Moreover the Jensen's inequality implies

$$-\frac{1}{|\lambda|^2} \log \rho^D(\lambda) \leqq \frac{1}{|\lambda|^2} \log|\Lambda| + J_\varphi(\lambda),$$

where

$$J_\varphi(\lambda) = \int_0^\infty dF_\varphi(u) \left\{ -\frac{1}{|\lambda|^2} \log \int_{-\infty}^{\lambda-u} \frac{1}{\sqrt{2\pi\sigma_\varphi}} e^{-(v-m)^2/2\sigma_\varphi} dv \right\}.$$

For $J_\varphi(\lambda)$, the Fatou's lemma implies

$$\limsup_{\lambda\to-\infty} J_\varphi(\lambda) \leqq \frac{1}{2\sigma_\varphi}.$$

Therefore we have proved

$$\limsup_{\lambda\to-\infty} -\frac{1}{|\lambda|^2} \log \rho^D(\lambda) \leqq \frac{1}{2\sigma_\varphi}.$$

Finally, by letting φ approach the δ-function, we get

$$\limsup_{\lambda\to-\infty} -\frac{1}{|\lambda|^2} \log \rho^D(\lambda) \leqq \frac{1}{2h(0)}.$$

The proof of Theorem 3.1 is now completed.

Tracing the arguments of the proof of Theorem 3.1 above, we can prove the following.

THEOREM 3.2. *Under the assumptions* (Ass. 1) *and* (Ass. 2),

(1) *if there exist* α, $\beta>0$ *such that*

$$\limsup_{\lambda\to-\infty}\frac{1}{\lambda^{\alpha}}\log P(V(0,\omega)<\lambda)\leqq-\beta,$$

then it holds that

$$\limsup_{\lambda\to-\infty}\frac{1}{\lambda^{\alpha}}\log\rho^{D}(\lambda)\leqq-\beta;$$

(2) *if there exist* γ, $\delta>0$ *such that*

$$\liminf_{\lambda\to\infty}\frac{1}{\lambda^{\gamma}}\log P(\sup_{x\in\Lambda_0}V(0,\lambda)<\lambda)\geqq-\delta,$$

where Λ_0 *is the unit cube* $(-1/2, 1/2]^d$, *then it holds that*

$$\liminf_{\lambda\to\infty}\frac{1}{\lambda^{\gamma}}\log\rho^{D}(\lambda)\geqq-\delta.$$

4. Asymptotic behavior of the IDS at $-\infty$, II.

In this section, we study the asymptotic behavior of the IDS $\rho^D(\lambda)$ as $\lambda\to-\infty$ when the scalar potential is given in the form (1.3).

We assume that the magnetic field is uniform and deterministic. Moreover for the scalar potential, we assume the following.

(Ass. 4) Let $\{q_i(\omega)\}_{i\in Z^d}$ be a sequence of real-valued independent identically distributed random variables such that $E[|q_0(\omega)|^p]<\infty$ for some p as in (Ass. 1). We assume that

$$\lim_{\lambda\to-\infty}|\lambda|^{\alpha}P(q_0(\omega)<\lambda)=\beta$$

holds for some $\alpha>d/2$ and $\beta>0$ and that V is written in the following form;

$$V^{\omega}(x)=V(x,\omega)=\sum_{i\in Z^d}q_i(\omega)\chi(x-i)$$

for some measurable function χ on $\boldsymbol{R}^d$ which vanishes outside the unit cube $\Lambda_0=[-1/2, 1/2)^d$. Moreover we assume that χ is bounded and uniformly positive in Λ_0, that is,

$$0<\underline{\chi}\equiv\inf_{x\in\Lambda_0}\chi(x)\leqq\sup_{x\in\Lambda_0}\chi(x)\equiv\bar{\chi}<\infty.$$

We consider a family $\{H(\omega)\}$ of the Schrödinger operators defined by

$$
\begin{aligned}
&H(\omega) = H_{0A}+V^{\omega}\,, \\
(4.1)\qquad & H_{0A} = \frac{1}{2}\sum_{j=1}^{d}\Big(\sqrt{-1}\frac{\partial}{\partial x^j}-A_j\Big)^2,
\end{aligned}
$$

where $A=(A_1, A_2, \cdots, A_d)$ is given by

$$
A(x) = \frac{1}{2}Bx \tag{4.2}
$$

for some real skew-symmetric $d\times d$ matrix B which is not equivalent to the zero matrix. Let ρ^D be the IDS for $\{H(\omega)\}$.

The main result of this section is the following.

THEOREM 4.1. *Let $\{V(x, \omega)\}$ be a random field satisfying* (Ass. 1) *and* (Ass. 4) *and let the vector potential A be defined by* (4.2). *Then there exist positive constants K_1 and K_2, depending on $\underline{\gamma}$, $\bar{\gamma}$, α, d and independent of A, such that*

$$
K_1\beta \leqq \liminf_{\lambda\to-\infty}|\lambda|^{\alpha-d/2}\rho^D(\lambda) \leqq \limsup_{\lambda\to-\infty}|\lambda|^{\alpha-d/2}\rho^D(\lambda) \leqq K_2\beta\,. \tag{4.3}
$$

Since K_1 and K_2 can be chosen to be independent of A, we may say that the magnetic field does not contribute to the leading asymptotics of the IDS $\rho^D(\lambda)$ at $-\infty$.

For the proof of the upper estimate, we will use the Cwikel-Lieb-Rosenbljum bound (Lemma 4.2) and the Lieb-Thirring bound (Corollary 4.4) as in the proof of Theorem 3.1. For the lower estimate, the following lemma due to Colin de Verdière [4] is useful.

Let Λ_R be the hypercube $[-R, R)^d$ in $\boldsymbol{R}^d$ and denote by $H_{0A}^D(\Lambda_R)$ the restriction of H_{0A} to $L^2(\Lambda_R)$ with the Dirichlet boundary condition. We denote by r half of the rank of the skew-symmetric matrix B and by $\pm\sqrt{-1}b_1, \cdots,$ $\pm\sqrt{-1}b_r$ the non-zero eigenvalues of B. Without loss of generality, we assume that $b_1, \cdots, b_r$ are positive.

As in [4], we set

$$
\nu_B(\lambda) = \frac{\gamma_k}{(2\pi)^{k+r}}b_1\cdots b_r\sum_{n_i\geqq 0}\Big(2\lambda-\sum_{i=1}^{r}(2n_i+1)b_i\Big)_+^{k/2},
$$

where $f_+=\max(f, 0)$, $k=d-2r$ and γ_k is the volume of the unit ball in $\boldsymbol{R}^k$. If $k=0$, f_+^0 is by definition the Heaviside function:

$$
f_+^0 = \begin{cases} 1 & \text{if } f\geqq 0 \\ 0 & \text{if } f<0. \end{cases}
$$

Then the following is known.

LEMMA 4.2. ([4]) *Let* $N_{0A}^{D}(\mu;\ \Lambda_R)$ *be the number of the eigenvalues of* $H_{0A}^{D}(\Lambda_R)$ *less than* $\mu>0$. *Then*

$$N_{0A}^{D}(\mu;\ \Lambda_R) \leqq (2R)^d \nu_B(\mu) \tag{4.4}$$

holds and there exists a constant C_d, *depending only on* d, *such that*

$$N_{0A}^{D}(\mu;\ \Lambda_R) \geqq (2R-k)^d \nu_B(\mu - C_d k^{-2}) \tag{4.5}$$

holds for all k *with* $0<k<R$.

We will use the lower estimate (4.5) in this section. (4.4) will be useful in the next section.

Now we give the proof of Theorem 4.1.

PROOF OF THEOREM 4.1. (the upper estimate) We prove only the case that $d\geqq 3$, because the proof for $d=2$ is similar if we replace the Cwikel-Lieb-Rosenbljum bound with the Lieb-Thirring bound in the following arguments. Moreover we suppose that the regular family $\{\Lambda_r\}_{r=0}^{\infty}$ is chosen to be $\Lambda_r=[-r-1/2,\ r+1/2)^d$.

At first fix r. Then, as in the proof of Theorem 3.1, the Cwikel-Lieb-Rosenbljum bound (Lemma 3.2) implies

$$\begin{aligned}\frac{1}{|\Lambda_r|}E[\rho_{\Lambda_r}^{D}(\lambda,\ \omega)] &\leqq \frac{C_1}{|\Lambda_r|}E\left[\int_{\Lambda_r}|(V(x,\ \omega)-\lambda)_-|^{d/2}dx\right]\\ &= C_1 E\left[\int_{\Lambda_0}|(q_0(\omega)\chi(x)-\lambda)_-|^{d/2}dx\right],\end{aligned}$$

where C_1 is a constant depending only on d. Therefore we get

$$\begin{aligned}\rho^D(\lambda) &\leqq C_1\int_{\Lambda_0}E[|q_0(\omega)\chi(x)-\lambda|^{d/2}I_{(q_0(\omega)\chi(x)<\lambda)}]dx\\ &\leqq C_1 E[(\lambda-\bar{\chi}q_0(\omega))_+^{d/2}]\end{aligned}$$

for $\lambda<0$, where I_A is the indicator function of $A\in\mathcal{F}$.

Now fix any $\varepsilon>0$ and choose $\lambda_0=\lambda_0(\varepsilon)<0$ such that

$$P(q_0(\omega)<\lambda) \leqq (\beta+\varepsilon)|\lambda|^{-\alpha}$$

for any $\lambda<\lambda_0\bar{\chi}$ by using (Ass. 4). Then we get

$$\rho^D(\lambda) \leqq C_1(\beta+\varepsilon)\int_{-\infty}^{\lambda/\bar{\chi}}(\lambda-t\bar{\chi})^{d/2-1}t^{-\alpha}dt,$$

which proves the upper part of (4.3).

PROOF OF THEOREM 4.1. (the lower estimate) At first we note that the min-max principle implies

$$\begin{aligned}\rho^{D}(\lambda) &\geqq E[\rho^{D}_{\Lambda_0}(\lambda, \omega)]\\ &= E[{}^{\#}\{n\in \boldsymbol{N};\ \lambda_n(H^{D}_{0A}(\Lambda_0)+q_0(\omega)\chi(x)) < \lambda\}]\\ &\geqq E[{}^{\#}\{n\in \boldsymbol{N};\ \lambda_n(H^{D}_{0A}(\Lambda_0)) < \lambda-\chi q_0(\omega)\}]\end{aligned}$$

for $\lambda<0$, where $\lambda_n(H)$'s denote the eigenvalues of a self-adjoint operator H. Then, by Lemma 4.2, we get

$$\rho^{D}(\lambda) \geqq \left(\frac{1}{2}-k\right)^{d} E[\nu_B(\lambda-\chi q_0(\omega)-C_2k^{-2})]$$

for any k with $0<k<1/2$ and some constant C_2. Therefore

$$\lambda^{-d/2}\rho^{D}(\lambda) \geqq C_3E\left[\frac{1}{|\lambda|^{r}}\sum_{n_i\geq 0} f(n_1, \cdots, n_r;\ |\lambda|)\right],$$

where

$$C_3=\left(\frac{1}{2}-k\right)^{d}\frac{\gamma_k b_1\cdots b_r 2^{k/2}}{(2\pi)^{k+r}}$$

and

$$f(n_1, \cdots, n_r;\ |\lambda|)=\left(-1-\frac{C_2k^{-2}+\chi q_0(\omega)}{|\lambda|}-\sum_{i=1}^{r}\left(\frac{n_i}{|\lambda|}+\frac{1}{2|\lambda|}\right)b_i\right)_{+}^{k/2}.$$

It is easy to see that there exists a constant C_4 such that

$$\left|\frac{1}{|\lambda|^{r}}\sum_{n_i\geq 0}\left(a-\sum_{i=1}^{r}\frac{n_i}{|\lambda|}b_i\right)_{+}^{k/2}-\frac{C}{b_1\cdots b_r}a^{r+k/2}\right| \leqq C_4\frac{1}{|\lambda|}\frac{a^{-1+r+k/2}}{b_1\cdots b_r}$$

for any $a>0$ and sufficiently large $|\lambda|$, where the constant $C=C(k, r)$ is given by

$$C=\prod_{j=1}^{r}\left(\frac{r}{2}+j\right)^{-1}=\int_{\{0\leq\sum_{i=1}^{r}t_i\leq 1\}}\left(1-\sum_{i=1}^{r}t_i\right)^{k/2}dt_1\cdots dt_r.$$

Therefore, since $2r+k=d$, we get

$$\liminf_{\lambda\to-\infty}|\lambda|^{\alpha-d/2}\rho^{D}(\lambda) \geqq C_5\liminf_{\lambda\to-\infty}|\lambda|^{\alpha-d/2}E[(\lambda-\chi q_0(\omega))_{+}^{d/2}].$$

for some $C_5>0$. The rest is similar to the last part of the proof for the upper part of (4.3). We have now completed the proof of Theorem 4.1.

REMARK. When the tail probability $P(q_0(\omega)<\lambda)$ decays exponentially as $\lambda\to-\infty$, a similar estimate to (4.3) holds if we replace $\rho^{D}(\lambda)$ with $\log\rho^{D}(\lambda)$.

5. Asymptotic behavior of the IDS at ∞.

In this last section we consider the asymptotic behavior of the IDS $\rho^{D}(\lambda)$ as $\lambda\to\infty$. We assume that (Ass. 1) holds for the scalar potential and that the magnetic field is uniform. We will show the result by using Lemma 4.2 (due to Colin de Verdière).

The asymptotic behavior of $\rho^D(\lambda)$ as $\lambda\to\infty$ is connected with the asymptotics of its Laplace transform $L(\rho^D, t)$ as $t\downarrow 0$ by virtue of the Tauberian theorem. This reminds us the studies on the asymptotic distributions of the eigenvalues for the Laplacian on bounded domains (the Weyl's problem) and for the Schrödinger operators with compact resolvents. For these operators, the so-called M. Kac's strategy is important. In fact, Ueki [**16**] has shown a similar result to ours in this section under more restrictive assumptions on V which is necessary to use the Feynman-Kac's formula.

By using the functional analytic methods, we will show the following in this section:

THEOREM 5.1. *Let $\{V(x, \omega);\ x\in\boldsymbol{R}^d\}$ be a random field satisfying* (Ass. 1) *and A be defined by $A(x)=Bx/2$ for some real skew-symmetric $d\times d$ matrix B. Denote by ρ^D the IDS for the family $\{H(\omega)\}$ of the Schrödinger operators defined by* (1.1). *Then it holds that*

$$\lim_{\lambda\to\infty}\lambda^{-d/2}\rho^D(\lambda)=(2\pi)^{-d/2}\Gamma\left(\frac{d}{2}+1\right)^{-1}.$$

PROOF. We can prove

$$\liminf_{\lambda\to\infty}\lambda^{-d/2}\rho^D(\lambda)\geqq(2\pi)^{-d/2}\Gamma\left(\frac{d}{2}+1\right)^{-1}$$

by the same way as in [7].

At first fix a hypercube Λ in $\boldsymbol{R}^d$. Theorem 2.2 implies

$$\rho^D(\lambda)\geqq\frac{1}{|\Lambda|}E[\rho^D_\Lambda(\lambda, \omega)].$$

Then, by using the Fatou's lemma and the Weyl's formula for the asymptotic distribution of the eigenvalues, we get

$$\begin{aligned}\liminf_{\lambda\to\infty}\lambda^{-d/2}\rho^D(\lambda)&\geqq\frac{1}{|\Lambda|}E[\liminf_{\lambda\to\infty}\lambda^{-d/2}\rho^D_\Lambda(\lambda, \omega)]\\&=(2\pi)^{-d/2}\Gamma\left(\frac{d}{2}+1\right)^{-1}.\end{aligned}$$

For the proof of the opposite inequality:

$$\limsup_{\lambda\to\infty}\lambda^{-d/2}\rho^D(\lambda)\leqq(2\pi)^{-d/2}\Gamma\left(\frac{d}{2}+1\right)^{-1}, \tag{5.1}$$

we modify the arguments in [7] and use Lemma 4.2. We prove the case that $d\geqq3$ as in the proof of Theorem 3.1 by the same reason. Moreover we assume that the regular family in Theorem 2.2 is chosen to be $\{[-r-1/2, r+1/2)^d\}_{r=0}^{\infty}$.

We fix a hypercube $\Lambda=\Lambda_r$ and choose any $\alpha>1$ and any ε with $0<\varepsilon<1$. Setting

$$W_1^\omega(x, \lambda) = (V(x, \omega)-\lambda)I_{(V(x,\omega)\leqq(1-\alpha)\lambda)}$$

and

$$W_2^\omega(x, \lambda) = (V(x, \omega)-\lambda)I_{(V(x,\omega)>(1-\alpha)\lambda)},$$

we define the Schrödinger operators $H_{\Lambda,1}^{D,\varepsilon}(\omega)$ and $H_{\Lambda,2}^{D,\varepsilon}(\omega)$ on $L^2(\Lambda)$ by

$$H_{\Lambda,1}^{D,\varepsilon}(\omega)=\varepsilon H_{0A}^D(\Lambda)+W_1^\omega \quad\text{and}\quad H_{\Lambda,2}^{D,\varepsilon}(\omega)=(1-\varepsilon)H_{0A}^D(\Lambda)+W_2^\omega,$$

respectively, with the Dirichlet boundary condition, where $H_{0A}^D(\Lambda)$ is the same as in Section 4.

Then, since $H_\Lambda^D(\omega)=H_{\Lambda,1}^{D,\varepsilon}(\omega)+H_{\Lambda,2}^{D,\varepsilon}(\omega)$, the min-max principle implies

$$\begin{aligned}\frac{1}{|\Lambda|}E[\rho_\Lambda^D(\lambda, \omega)] &= \frac{1}{|\Lambda|}E[N_0(H_\Lambda^D(\omega)-\lambda)] \\ &\leqq \frac{1}{|\Lambda|}E[N_0(H_{\Lambda,1}^{D,\varepsilon}(\omega))]+\frac{1}{|\Lambda|}E[N_0(H_{\Lambda,2}^{D,\varepsilon}(\omega))],\end{aligned}\tag{5.2}$$

where $N_0(H)$ denotes the number of the negative eigenvalues of a selfadjoint operator H as before.

For the first term of the right hand side of (5.2), we apply the Cwikel-Lieb-Rosenbljum bound. Then we get

$$\begin{aligned}\frac{1}{|\Lambda|}E[N_0(H_{\Lambda,1}^{D,\varepsilon}(\omega))] &\leqq \frac{C_1}{|\Lambda|}E\left[\int_\Lambda |\varepsilon^{-1}W_1(x, \lambda, \omega)|^{d/2}dx\right] \\ &\leqq C_1\varepsilon^{-d/2}E[|V(0, \omega)-\lambda|^{d/2}I_{(V(0,\omega)<(1-\alpha)\lambda)}]\end{aligned}\tag{5.3}$$

by the definition of W_1 for some constant C_1. We have used the stationarity of V at the last line.

To estimate the second term of the right hand side of (5.2), we first note the following trivial inequality as quadratic forms:

$$(1-\varepsilon)^{-1}H_{\Lambda,2}^{D,\varepsilon}(\omega) \geqq H_{0A}^D(\Lambda)-(1-\varepsilon)^{-1}\alpha\lambda.$$

Then the min-max principle and Lemma 4.2 imply

$$\begin{aligned}\frac{1}{|\Lambda|}E[N_0(H_{\Lambda,2}^{D,\varepsilon}(\omega))] &\leqq \frac{1}{|\Lambda|}\#\{n\in\boldsymbol{N};\ \lambda_n(H_{0A})<(1-\varepsilon)\alpha\lambda\} \\ &\leqq \nu_B((1-\varepsilon)\alpha\lambda),\end{aligned}\tag{5.4}$$

where $\lambda_n(H_{0A})$'s denote the eigenvalues of H_{0A} and the function ν_B was defined in Section 4.

Therefore we have proved that, writing Λ_r for Λ,

$$\frac{1}{|\Lambda_r|}E[\rho^D_{\Lambda_r}(\lambda,\omega)]\leqq C_1\varepsilon^{-d/2}E[|V(0,\omega)-\lambda|^{d/2}I_{(V(0,\omega)<(1-\alpha)\lambda)}]$$
$$+\nu_B((1-\varepsilon)^{-1}\alpha\lambda)$$

holds for any r and, therefore, that

$$\rho^D(\lambda)\leqq C_1\varepsilon^{-d/2}E[|V(0,\omega)-\lambda|^{d/2}I_{(V(0,\omega)<(1-\alpha)\lambda)}]+\nu_B((1-\varepsilon)\alpha\lambda). \tag{5.5}$$

By (Ass. 1), it is easy to see that

$$\lim_{\lambda\to\infty}\lambda^{-d/2}E[|V(0,\omega)-\lambda|^{d/2}I_{(V(0,\omega)<(1-\alpha)\lambda)}]=0. \tag{5.6}$$

Moreover, since $d=2r+k$,

$$\lim_{\lambda\to\infty}\lambda^{-d/2}\nu_B((1-\varepsilon)^{-1}\alpha\lambda)=\lim_{\lambda\to\infty}\frac{2^{k/2}\gamma_k b_1\cdots b_r}{(2\pi)^{r+k}}\frac{1}{\lambda^r}\sum_{n_i\geq 0}\Big(\frac{\alpha}{1-\varepsilon}-\sum_{i=1}^r\Big(\frac{n_i}{\lambda}+\frac{1}{2\lambda}\Big)b_i\Big)_+^{k/2} \tag{5.7}$$
$$=\frac{b_1\cdots b_r}{(2\pi)^{r+k/2}\Gamma(k/2+1)}\int_{R^r}\Big(\frac{\alpha}{1-\varepsilon}-\sum_{i=1}^r t_ib_i\Big)_+^{k/2}dt_1\cdots dt_r$$
$$=(2\pi)^{-d/2}\Gamma\Big(\frac{d}{2}+1\Big)^{-1}\Big(\frac{\alpha}{1-\varepsilon}\Big)^{d/2}.$$

Now, combining (5.5) with (5.6) and (5.7), we get

$$\limsup_{\lambda\to\infty}\lambda^{-d/2}\rho^D(\lambda)\leqq(2\pi)^{-d/2}\Gamma\Big(\frac{d}{2}+1\Big)^{-1}\Big(\frac{\alpha}{1-\varepsilon}\Big)^{d/2}.$$

Letting $\alpha\downarrow 1$ and $\varepsilon\downarrow 0$, we have proved (5.1) and have completed the proof of Theorem 5.1.

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