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Noncompact Liouville surfaces

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1. Introduction.

A (local) Liouville surface is by definition a surface with a Riemannian metric of the following form:

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

where $x=(x_1, x_2)$ is a local coordinate system, and f_i is a function of the single variable $x_i (i=1, 2)$. This type of metric has a special property. Define

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) are the canonical coordinates on the cotangent bundle. Then F is a first integral of the geodesic flow on the bundle, i.e., the Poisson bracket $\{F, E\}$ of F and the energy function

$$E=\frac{1}{2}\frac{1}{f_1(x_1)+f_2(x_2)}(\xi_1^2+\xi_2^2)$$

vanishes. As a matter of fact, the Liouville surfaces are characterized in terms of a first integral.

Let g be a Riemannian metric on a neighborhood U of a point $p\in \boldsymbol{R}^2$, and $E\in C^\infty(T^*U)$ the corresponding energy function. For a function $H\in C^\infty(T^*U)$ on the cotangent bundle, we denote by H_p the restriction of H to the cotangent space T_p^*U at p. The following proposition is classical.

PROPOSITION 1.1 ([**2**, Proposition 1.1]). *Assume that $F\in C^\infty(T^*U)$ satisfies the following conditions:*

(1) $\{F, E\}=0$,

(2) *F_q is a homogeneous polynomial of degree 2 for every $q\in U$,*

(3) $F_p\notin \boldsymbol{R}E_p$.

Then there is a coordinate system (x_1, x_2) on a (possibly smaller) neighborhood of p, and there are functions $f_i(x_i) (i=1, 2)$ such that

$$g=(f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2)$$

and

$$F=\frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) are the associated canonical coordinates on the cotangent bundle. Furthermore, such coordinate system (x_1, x_2) and functions f_1, f_2 are essentially unique.

Based on this fact, the second author defined a global Liouville surface in terms of a first integral of the geodesic flow and classified the equivalence classes of compact Liouville surfaces in [**2**].

DEFINITION 1.2. A triple (S, g, F) is called a Liouville surface if (S, g) is a 2-dimensional Riemannian manifold and F a C^∞ function on the cotangent bundle T^*S satisfying the following conditions:

(L.1) $\{E, F\}$ vanishes,

(L.2) F_p is a homogeneous polynomial of degree 2 for any $p\in S$,

(L.3) F is not of the form $rH+sE$, where $H\in C^\infty(T^*S)$ is fibrewise the square of a linear form, and $s, r\in \boldsymbol{R}$.

DEFINITION 1.3. Two Liouville surfaces (S, g, F) and (S', g', F') are said to be equivalent if there is an isometry $\varphi: (S, g)\to(S', g')$ and $r, s\in\boldsymbol{R}$ such that $F'\circ(\varphi^*)^{-1}=rF+sE$. In case $r=1$ and $s=0$, the two Liouville surfaces are said to be isomorphic.

The main purpose of this paper is to classify the equivalence classes of noncompact complete Liouville surfaces.

It is known that quadratic surfaces in the Euclidean 3-space $\boldsymbol{E}^3$ are Liouville surfaces. Their first integrals F can be explicitly expressed in terms of the elliptic coordinates and the behavior of geodesics on them are fully investigated (see Darboux [**1**], Klingenberg [**3**], v. Mangoldt [**5**], and Sugahara [**6**]). As v. Mangoldt studied the distribution of poles in [**5**], noncompact quadratic surfaces are good examples of Riemannian manifolds with poles. For Riemannian manifolds of nonnegative sectional curvature, M. Maeda [**4**] gave an inequality that the diameter of the set of poles does not exceed a constant given by the expanding order at infinity. The third author gave a best possible constant for the inequality in [**6**], by making use of a family of elliptic paraboloids.

Let (S, g, F) be a complete Liouville surface. As in [**2**], the set

$$\mathcal{N}=\{p\in S\,;\ F_p=rE_p \text{ for some } r\in\boldsymbol{R}\}$$

plays an important role also in this paper.

Since the first integrals F of quadratic surfaces are proportional pointwise to the second fundamental forms modulo $\boldsymbol{R}E$, their singularity sets $\mathcal{N}$ are equal to the set of umbilic points. And for noncompact quadratic surfaces, umbilic points are poles ([**5**], and [**6**]). This fact will be generalized in Theorem 2.1 in section 2.

We note that the constant r does not depend on the point of $\mathcal{N}$.

LEMMA 1.4 ([**2**, Lemma 1.2]). *The ratio* $r=F_p/E_p$ *does not depend on the point* $p\in\mathcal{N}$.

For any $r\in\boldsymbol{R}$, a triple $(S, g, F-rE)$ is also a Liouville surface which is equivalent to (S, g, F). Hence, in view of Lemma 1.4, we can assume the following condition without loss of generality as in [**2**]:

(L.4) $F_p=0$ if $p\in\mathcal{N}$.

In this paper Liouville surfaces are assumed to satisfy (L.4) unless otherwise stated.

In §2, we study the geometric properties of points of $\mathcal{N}$ and show that S is diffeomorphic to $\boldsymbol{R}^2$, a cylinder, or a Möbius band.

In §3, we give a classification of noncompact complete Liouville surfaces, using the natural coordinate system given in Proposition 1.1.

In the final section, §4, we show that quadratic surfaces in the hyperbolic 3-space $\boldsymbol{H}^3(-1)$ of constant sectional curvature -1 are also Liouville surfaces.

We would like to thank our referee for kind advice to complete our classification.

2. The set $\mathcal{N}$ and the geodesics with $F=0$.

Let (S, g, F) be a noncompact complete Liouville surface and $\mathcal{N}$ the subset of S defined in the previous section. In this section, we shall prove the following theorem which is a noncompact version of [**2**, Theorem 2.1].

THEOREM 2.1. *$\#\mathcal{N}$, the number of the points in $\mathcal{N}$, must be* 0 *or* 1 *or* 2.

If $\#\mathcal{N}>0$, then every point of $\mathcal{N}$ is a pole of S, i.e., the exponential map $\exp_p: T_pS\to S$ is a diffeomorphism. Consequently, S is diffeomorphic to $\boldsymbol{R}^2$. Moreover, there is a geodesic L which satisfies the following conditions:

(1) *L passes all points of $\mathcal{N}$,*

(2) *F_q is indefinite if $q\notin L$,*

(3) *F_q is degenerate and semidefinite if $q\in L$.*

If $\#\mathcal{N}=0$, then S is diffeomorphic to $\boldsymbol{R}^2$, a cylinder, or a Möbius band.

PROOF. First we shall consider the case where $\#\mathcal{N}=0$. Let $\pi: \tilde{S}\to S$ be the universal covering of S. The uniqueness of the coordinates (x_1, x_2) in Proposition 1.1 implies that the tangent bundle $T\tilde{S}$ of $\tilde{S}$ splits into the sum of two trivial line bundles $E_i=\boldsymbol{R}\partial/\partial x_i$ $(i=1, 2)$ and that vector fields $X_i=\partial/\partial x_i$ are globally well-defined. Let φ_t (resp. ψ_s) denote the local one-parameter subgroup of transformations generated by X_1 (resp. X_2). Let p be a point of $\tilde{S}$. Suppose that $\varphi_t(p)(a<t<b)$ are well-defined. For each $t\in(a, b)$, define

$$a_t=\inf\{s_0\,;\ \psi_s(\varphi_t(p)) \text{ is well-defined for any } s\in(s_0, 0]\}$$
$$b_t=\sup\{s_0\,;\ \psi_s(\varphi_t(p)) \text{ is well-defined for any } s\in[0, s_0)\}\,.$$

For $t\in(a, b)$ and $s\in(a_t, b_t)$, define $\Phi(t, s)=\psi_s(\varphi_t(p))$. Then

$$\Phi^*(\pi^*g)=(f_1(t)+f_2(s))(dt^2+ds^2)\,.$$

Since $\tilde{S}$ is complete, the length of the curve $s\mapsto\psi_s(\varphi_t(p))$ is infinite. More precisely, we have

$$\int_0^{b_t}\sqrt{f_1(t)+f_2(s)}\,ds=\infty\,,$$

$$\int_{a_t}^0\sqrt{f_1(t)+f_2(s)}\,ds=\infty\,,$$

for any $t\in(a, b)$. Hence a_t and b_t must not depend on t. The same argument can be applied to the integral curves of X_2. Therefore there are four real numbers a_1, a_2, b_1, and b_2 so that the map Φ: $(a_1, b_1)\times(a_2, b_2)\to\tilde{S}$ defined by $\Phi(t, s)=\psi_s(\varphi_t(p))$ is a Riemannian covering with respect to $\Phi^*(\pi^*g)$ and π^*g, and $\Phi^*(\pi^*g)$ is complete. Since $\tilde{S}$ is simply connected, Φ is an isometry. Suppose S is not simply connected. We identify $\tilde{S}$ with the rectangle $(a_1, b_1)\times(a_2, b_2)$ through the isometry Φ. Since the action of the fundamental group $\pi_1(S)$ on $\tilde{S}$ maps vector fields X_i to $\pm X_i$ $(i=1, 2)$, we may assume that $\pi_1(S)$ consists of the following maps

$$(t, s)\longrightarrow(\pm t+\alpha,\ \pm s+\beta)\,,$$

where α and β are real numbers. Since S is noncompact, it is easily seen that $\pi_1(S)$ is generated by one of the following maps

$$(t, s)\longrightarrow(t+\alpha,\ s+\beta)\,,$$
$$(t, s)\longrightarrow(t+\alpha,\ -s+\beta)\,,$$
$$(t, s)\longrightarrow(-t+\alpha,\ s+\beta)\,.$$

Hence S is diffeomorphic to a cylinder or a Möbius band.

The case where $\#\mathfrak{N}>0$ shall be treated as in [2]. We need some lemmas and a proposition in it.

For a tangent vector v to S, let $\gamma_v(t)$ denote the geodesic of (S, g) such that the initial vector $\dot{\gamma}_v(0)$ is v.

LEMMA 2.2 (cf. [**2**, Lemma 2.3]). *Suppose that* $\#\mathfrak{N}\geqq 3$, *and let* $p_i\,(i=1, 2, 3)$ *be three points in* $\mathfrak{N}$. *Assume that* p_2 *and* p_3 *are not in the cut locus* $Cut(p_1)$ *of* p_1. *Let* $\gamma_i(t)=\gamma_{v_i}(t)$ $(0\leqq t\leqq t_i,\ v_i\in S_{p_1}S)$ *be the minimizing geodesic from* p_1 *to* p_i $(i=2, 3)$ *parametrized by arc length. Then we have* $v_2+v_3=0$. *Furthermore, if we put* $w=-\dot{\gamma}_2(t_2)\in S_{p_2}S$, *then* p_3 *is the first conjugate point of* p_2 *along the geodesic* $\gamma_w(t)$ $(0\leqq t\leqq t_2+t_3)$. *In particular* $\mathfrak{N}$ *is discrete.*

PROPOSITION 2.3 (cf. [**2**, Proposition 2.6]). *If the tangential first conjugate locus* $\mathcal{T}Conj(p)$ *of* $p\in\mathfrak{N}$ *is not empty, then it is a circle of constant radius.*

In [**2**], Liouville surfaces are assumed to be compact, and the compactness derives that $\mathcal{T}Conj(p)$ is not empty for any $p\in\mathfrak{N}$. In our case Liouville surfaces are noncompact. Hence we get

COROLLARY 2.4. (1) $\mathcal{T}Conj(p)=\emptyset$ *for any* $p\in\mathfrak{N}$.

(2) $\#\mathfrak{N}\leqq 2$.

(3) S *is diffeomorphic to* $\boldsymbol{R}^2$ *if* $\mathfrak{N}\neq\emptyset$.

PROOF. (1) If $\mathcal{T}Conj(p)$ is not empty, then Proposition 2.3 implies that S is compact, which contradicts our assumption.

(2) Suppose $\#\mathfrak{N}>2$. Here we may assume that S is simply connected. If not, we may consider the universal covering space of S in place of S. Let p_i $(i=1, 2, 3)$ be three distinct points of $\mathfrak{N}$. Since $\mathcal{T}Conj(p_i)=\emptyset$, it follows that each p_i is a pole of S. Then Lemma 2.2 leads us to a contradiction.

(3) Since the exponential mapping $\exp_p: T_pS\to S$ is of maximal rank for any $p\in\mathfrak{N}$, it suffices to show that S is simply connected. Suppose S is not simply connected and let $\pi:\tilde{S}\to S$ be the universal covering of S. From (2) we get $\#\mathfrak{N}(\tilde{S})=\#\pi^{-1}(\mathfrak{N}(S))\leqq 2$. Hence the fundamental group $\pi_1(S)$ is isomorphic to $\boldsymbol{Z}_2$ and $\#\mathfrak{N}(\tilde{S})=2$. Let $\mathfrak{N}(\tilde{S})=\{\tilde{p}_1, \tilde{p}_2\}$. Let σ be the generator of $\pi_1(S)$. Then σ acts on $\tilde{S}$ as an isometry and $\mathfrak{N}(\tilde{S})$ is invariant by σ. Since $\tilde{p}_i\,(i=1, 2)$ are poles, the geodesic which connects $\tilde{p}_1$ and $\tilde{p}_2$ is unique. Hence the middle point of the geodesic is fixed by σ. Since the fundamental group acts freely, it is a contradiction. ■

In the rest of this paper we shall often regard the first integral F as a function on the tangent bundle by using the natural identification of TS with T^*S.

LEMMA 2.5. *If* $\#\mathfrak{N}>0$, *then there is a geodesic* L *which satisfies*

(1) L *passes all points of* $\mathfrak{N}$,

(2) F_q *is indefinite for* $q\notin L$,

(3) F_q *is degenerate and semidefinite for* $q\in L$.

PROOF. *Case* 1: $\mathfrak{N}=\{p_1\}$. Since the point p_1 is a pole, we consider the normal polar coordinates (r, θ) centered at p_1. Let

$$\mathcal{I} = \{q \in S\,;\ F_q \text{ is indefinite}\},$$
$$\mathcal{S} = \{q \in S\,;\ F_q \text{ is semidefinite}\}.$$

Then $\mathcal{I}$ is an open set.

Suppose $\mathcal{I} \neq \emptyset$. Since $F(\partial/\partial r)=0$, there is a unit vector $v_q \in S_q S$ for each $q \in \mathcal{I}$ with $F(v_q)=0$ which is linearly independent of $(\partial/\partial r)_q$. Then $F(\dot{\gamma}_{v_q}(t))=0$ along each geodesic $\gamma_{v_q}(t)$. Since p_1 is a pole, γ_{v_q} does not pass p_1. We note that the two geodesics γ_{v_x} and γ_{v_y} $(x, y \in \mathcal{I})$ do not intersect transversally. (If they intersect at a point q transversally, F_q has value 0 with respect to three directions $(\partial/\partial r)_q$, $\dot{\gamma}_{v_x}$ and $\dot{\gamma}_{v_y}$ which are in general direction. Hence $F_q=0$, i.e., $q=p_1$.) Therefore $\mathcal{I}$ consists of 'parallel' geodesics and its boundary is also a disjoint union of 'parallel' geodesics. Let γ be one of the boundary geodesics. Then we have

$$\lim_{q \to \gamma(t)} v_q = \pm\dot{\gamma}(t),$$
$$0 = \lim_{q \to \gamma(t)} F_q(v_q) = F_{\gamma(t)}(\pm\dot{\gamma}(t)),$$

for any $t \in \boldsymbol{R}$. Since $\mathcal{I}$ is open, $\gamma \cap \mathcal{I} = \emptyset$. Hence $\dot{\gamma}(t)$ must be linearly dependent on $(\partial/\partial r)_{\gamma(t)}$. Consequently γ passes p_1. Therefore the boundary of $\mathcal{I}$ consists of a single geodesic, which we denote by $L=\{\gamma(t)\,;\ t \in \boldsymbol{R}\}$ with $\gamma(0)=p_1$. Let q be a point of $\mathcal{I}$. Then the tangent space $T_q S$ is divided into four parts by two lines $\boldsymbol{R}v_q$ and $\boldsymbol{R}(\partial/\partial r)_q$. For a small positive number ε, let γ_+ and γ_- denote the geodesics from q to $\gamma(\varepsilon)$ and $\gamma(-\varepsilon)$ respectively. Then their initial tangent vectors $\dot{\gamma}_+(0)$ and $\dot{\gamma}_-(0) \in T_q S$ are separated by $\boldsymbol{R}(\partial/\partial r)_q$. Since $\dot{\gamma}_\pm(0)$ goes towards $(\partial/\partial r)_q$ as $\varepsilon \to 0$ and since F is indefinite at q with $F_q((\partial/\partial r)_q)=0$, $F_q(\dot{\gamma}_+(0))$ and $F_q(\dot{\gamma}_-(0))$ have different sign. Hence we may assume that $F_{\gamma(t)}$ is positive semidefinite for $t>0$ and negative semidefinite for $t<0$. Let $\tilde{q}$ be a point of $S \setminus L$. Let $\tilde{\gamma}_+$ and $\tilde{\gamma}_-$ denote geodesics from $\tilde{q}$ to $\gamma(\varepsilon)$ and $\gamma(-\varepsilon)$ with $\tilde{\gamma}_+(t_+)=\gamma(\varepsilon)$ and $\tilde{\gamma}_-(t_-)=\gamma(-\varepsilon)$ respectively. Since F is invariant by the geodesic flow, we have

$$F_{\tilde{q}}(\dot{\tilde{\gamma}}_+(0)) = F_{\gamma(\varepsilon)}(\dot{\tilde{\gamma}}_+(t_+)) > 0,$$
$$F_{\tilde{q}}(\dot{\tilde{\gamma}}_-(0)) = F_{\gamma(-\varepsilon)}(\dot{\tilde{\gamma}}_-(t_-)) < 0.$$

Hence $\tilde{q} \in \mathcal{I}$ and S is a disjoint union of $\mathcal{I}$ and L.

If $\mathcal{I}=\emptyset$, then we may assume F is positive semidefinite. Since $\dim S=2$, F is the square of some linear form, which is excluded by our condition (L.3).

Case 2: $\mathfrak{N}=\{p_1, p_2\}$. Let L denote the unique geodesic which passes p_1 and p_2. Suppose F is indefinite at some point $p \in L$. Let $v_p \in S_p S$ denote a unit tangent vector which is transversal to L with $F_p(v_p)=0$. Let q be a point on a geodesic γ_{v_p}. Then F_q has value zero for tangent vectors to geodesics from q to p_1, p_2, and p at q. Hence $F_q=0$, i.e., $\gamma_{v_p}\subset\mathfrak{N}$, which contradicts our assumption $\#\mathfrak{N}=2$. Next suppose that F is semidefinite at some point $q\notin L$. Since F_q has value zero for tangent vectors to geodesics from q to p_1 and p_2 at q which are linearly independent. Hence $F_q=0$, which also contradicts our assumption $\#\mathfrak{N}=2$. ∎

It is clear that Theorem 2.1 follows from Corollary 2.4 and Lemma 2.5. ∎

3. The natural coordinate system and the classification.

First we shall classify Liouville surfaces with $\#\mathfrak{N}>0$. We need some modification of the quadruples $(\alpha_1, \alpha_2, f_1, f_2)$ introduced in [2].

Case 1: $\#\mathfrak{N}=1$. Let us introduce sextuples $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)$ such that $\alpha_1=\alpha_2=\infty$ and $0<r_1, r_2\leqq\infty$, and that f_i is a C^∞ function on an interval $(-r_i, r_i)$ $(i=1, 2)$ satisfying the following conditions:

$$\begin{aligned} &f_i(-t)=f_i(t), \quad f_i''(0)>0, \quad f_i(0)=0,\\ &f_i(t)>0 \quad \text{if } t\neq 0 \ (i=1, 2);\\ &\text{if } f_1(t)\sim\sum_{k\geqq 1}a_k t^{2k} \ (t\to 0), \text{ then } f_2(t)\sim\sum_{k\geqq 1}(-1)^{k-1}a_k t^{2k} \ (t\to 0);\\ &\int_0^{r_1}\sqrt{f_1(t_1)+f_2(t_2)}\,dt_1=\infty \quad \text{for any } -r_2<t_2<r_2,\\ &\int_0^{r_2}\sqrt{f_1(t_1)+f_2(t_2)}\,dt_2=\infty \quad \text{for any } -r_1<t_1<r_1. \end{aligned} \tag{3.1}$$

Here the symbol $\sim$ stands for the formal Taylor expansion. Let Q_1 be the set of all such sextuples. We say that two sextuples $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)$ and $(\beta_1, \beta_2, h_1, h_2, s_1, s_2)$ in Q_1 are equivalent if there is a constant $c>0$ such that one of the following conditions is satisfied:

(1) $s_1=cr_1$, $s_2=cr_2$, and $c^2h_1(ct)=f_1(t)$, $c^2h_2(ct)=f_2(t)$;

(2) $s_1=cr_2$, $s_2=cr_1$, and $c^2h_1(ct)=f_2(t)$, $c^2h_2(ct)=f_1(t)$.

If $c=1$, then these two sextuples are said to be isomorphic.

We can construct a Liouville surface whose underlying manifold is diffeomorphic to $\boldsymbol{R}^2$ from each sextuple $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)\in Q_1$ in the following way. Let T denote an open rectangle $(-r_1, r_1)\times(-r_2, r_2)\subset\boldsymbol{R}^2$. We shall identify $\boldsymbol{R}^2$ with $\boldsymbol{C}$ by taking the complex coordinate $z=x_1+\sqrt{-1}x_2$. Let τ be an involution on T defined by $z\to -z$. We now consider the quotient space $R=T/\{\mathrm{id}, \tau\}$. Note that τ has a unique fixed point $z=0$. By taking z^2 as a coordinate, the quotient space R can be regarded as a 1-dimensional complex manifold. Clearly the quotient mapping $\Phi: T\to R$ is holomorphic. If $r_1=r_2=\infty$, then R is isomorphic to $\boldsymbol{C}$ as a complex manifold. If one of r_i's is finite, then R is isomorphic to the unit disk. By the condition (3.1) we have a unique Riemannian metric g on R such that

$$\Phi^* g = (f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

and we also have a unique C^∞ function F on T^*R such that

$$\tilde{F}\circ\Phi = F, \quad \tilde{F} = \frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2 - f_1(x_1)\xi_2^2),$$

where (x, ξ) is the canonical coordinate system of T^*T. Then (R, g, F) is a Liouville surface which satisfies the condition (L.4).

Case 2: $\#\mathfrak{N}=2$. Let us introduce sextuples $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)$ such that $0<\alpha_1<\infty$, $\alpha_2=\infty$, $r_1=\infty$, and $0<r_2\leqq\infty$ and that f_1 and f_2 are C^∞ functions defined on $\boldsymbol{R}/\alpha_1\boldsymbol{Z}$ and $(-r_2, r_2)$ respectively which satisfy the following conditions:

$$\begin{aligned}
&f_i(-t) = f_i(t), \quad f_i''(0) > 0, \quad (i = 1, 2);\\
&f_1(0) = f_1(\alpha_1/2) = f_2(0) = 0;\\
&f_1(t) > 0 \quad \text{if } t \not\equiv 0,\ \alpha_1/2 \mod \alpha_1\boldsymbol{Z};\\
&f_2(t) > 0 \quad \text{if } t \neq 0;\\
&\text{if } f_1(t) \sim \sum_{k\geqq 1} a_k t^{2k} \quad (t\to 0), \text{ then}\\
&f_2(t) \sim \sum_{k\geqq 1} (-1)^{k-1} a_k t^{2k} \quad (t\to 0),\\
&f_1(t) \sim \sum_{k\geqq 1} a_k (t-\alpha_1/2)^{2k} \quad (t\to \alpha_1/2);\\
&\int_0^{r_2} \sqrt{f_1(t_1)+f_2(t_2)}\,dt_2 = \infty \quad \text{for any } -r_1 < t_1 < r_1.
\end{aligned} \tag{3.2}$$

Let Q_2 be the set of all such sextuples. We say that two sextuples $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)$ and $(\beta_1, \beta_2, h_1, h_2, s_1, s_2)$ in Q_2 are equivalent if there is a constant $c>0$ and

$$\nu \in \{0, -\alpha_1/2\}$$

such that

$$\beta_1 = c\alpha_1,\ s_2 = cr_2, \quad \text{and} \quad c^2 h_1(ct) = f_1(t+\nu),\ c^2 h_2(ct) = f_2(t).$$

If $c=1$, these two sextuples are said to be isomorphic.

We can also construct a Liouville surface whose underlying manifold is diffeomorphic to $\boldsymbol{R}^2$ from each sextuple $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)\in Q_2$ in the following way. Let T be a cylinder $\boldsymbol{R}/\alpha_1\boldsymbol{Z}\times(-r_2, r_2)$. We shall regard T as a Riemann surface by taking the complex coordinate $z=x_1+\sqrt{-1}x_2$ as above. Let τ be an involution on T defined by $z \to -z$. We now consider the quotient space $R=T/\{\mathrm{id}, \tau\}$. Note that τ has two fixed points $z=0$, $\alpha_1/2$. Let z_0 be one of these points. By taking $(z-z_0)^2$ as a local coordinate around the point $z=z_0$, the quotient space R can be regarded as a 1-dimensional complex manifold. Clearly the quotient mapping $\Phi: T \to R$ is holomorphic. If $r_2=\infty$, then R is isomorphic to $\boldsymbol{C}$ as a complex manifold. If r_2 is finite, then R is isomorphic to the unit disk. By the condition (3.2) we have a unique Riemannian metric g on R such that

$$\Phi^* g = (f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2),$$

and we also have a unique C^∞ function F on T^*R such that

$$\tilde{F} \circ \Phi = F, \quad \tilde{F} = \frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2 - f_1(x_1)\xi_2^2).$$

Then (R, g, F) is also a Liouville surface which satisfies the condition (L.4).

THEOREM 3.1. *The constructions above give the one-to-one correspondence between the equivalence classes (resp. isomorphism classes) of sextuples in Q_i and the equivalence classes (resp. isomorphism classes) of Liouville surface with $\#\mathcal{N}=i\,(i=1, 2)$.*

PROOF. It is clear that equivalent (resp. isomorphic) sextuples yield equivalent (resp. isomorphic) Liouville surfaces. Therefore we shall give the inversecorrespondence. Let (S, g, F) be a Liouville surface which satisfies the condition (L.4). As in the proof of [**2**, Proposition 1.1], two functions $E, F \in C^\infty(T^*S)$ can be expressed in $S \setminus \mathcal{N}$ as follows:

$$E = \frac{1}{2}(V_1^2+V_2^2), \quad F = f_2 V_1^2 - f_1 V_2^2, \quad f_1, f_2 \in C^\infty(S \setminus \mathcal{N}),$$

where $V_i\,(i=1, 2)$ are fibrewise linear functions defined locally only in a simply connected domain. But their squares are globally well-defined, and $V_1^2, V_2^2 \in C^\infty(T^*(S \setminus \mathcal{N}))$. Furthermore, if we consider $V_i\,(i=1, 2)$ as vector fields, then $V_1 f_2 = V_2 f_1 = 0$. Let L denote the geodesic given in Theorem 2.1. L divides S into two domains whose closure we denote by D_1 and D_2. Then D_1 and D_2 are isomorphic to a half plane. Since F is indefinite in $S \setminus L$ and since f_1 and f_2 have no common zeros except $\mathcal{N}$, we may assume that both f_1 and f_2 are positive on $S \setminus L$.

Case 1: $\mathcal{N}=\{p_1\}$. The point p_1 divides L into two parts, say L_1 and L_2, where F is semidefinite. We note that F has different sign on L_1 and L_2. (If not, take two points $q_1 \in L_1$ and $q_2 \in L_2$ near p_1. Let v_1, v_2 and v be the tangent vectors at $q \in D_1 \setminus L$ to geodesics from q to q_1, q_2 and p_1 respectively.

Then v_1 and v_2 are near v, and v is between them. If F has the same sign in L_1 and L_2, then so do $F(v_1)$ and $F(v_2)$. Since $F(v)=0$, it follows that F is semidefinite at q.) We suppose that $F \geqq 0$ on L_1, and $F \leqq 0$ on L_2, i.e., $f_1=0$, $f_2>0$ on L_1, and $f_1>0$, $f_2=0$ on L_2. Put $f_1(p_1)=f_2(p_1)=0$. Then f_1 and f_2 become continuous in S. V_i are C^∞ in $D_1 \setminus \mathfrak{N}(i=1, 2)$, and vector fields $X_i=\sqrt{f_1+f_2}V_i$ are continuous in D_1 if we put $X_i(p_1)=0\,(i=1, 2)$. We choose V_i so that X_i point inward of D_1 on $L_i\,(i=1, 2)$.

LEMMA 3.2. *Consider the ordinary differential equation*

$$\frac{dc_i(t)}{dt} = X_i(c_i(t))$$

with the initial condition $c_i(0)=q \in D_1 \setminus L\,(i=1, 2)$. *Then there are constants* $0< a_i<\infty$ *and* $0<b_i \leqq \infty\,(i=1, 2)$ *such that*

(i) *the solution* $c_i(t)$ *exists on* $[-a_i, b_i)$,

(ii) $c_i((-a_i, b_i)) \subset D_1 \setminus L$, $c_i(-a_i) \in L_i$,

(iii) $\lim_{t \to b_i} d(c_i(t), p_1) = \infty$,

(iv) $a_i+b_i\,(\leqq \infty)$ *does not depend on the initial point* q,

where $d(\ ,\ )$ *denotes the distance function defined by the Riemannian metric.*

PROOF. Let (r, θ) be the normal polar coordinate system centered at p_1 such that $L_1=\{\theta=\pi,\ 0<r\}$, $L_2=\{\theta=0,\ 0<r\}$, and $D_1 \setminus L=\{0<\theta<\pi,\ 0<r\}$. Since the vectors X_1, X_2 are linearly independent of, and not orthogonal to the vectors defined by $F=0$ on $D_1 \setminus L$, it is easily follows that $\theta(c_1(t))$ (resp. $\theta(c_2(t))$) is a decreasing function (resp. an increasing function), and $r(c_i(t))\,(i=1, 2)$ are increasing functions. Therefore $c_i(t)\,(i=1, 2)$ can be extended until they reach L when t decreases. Since $X_1 f_2=0$ (resp. $X_2 f_1=0$), and since $f_2=0$ on L_2 (resp. $f_1=0$ on L_1), it follows that $c_1(t)$ (resp. $c_2(t)$) does not pass a neighborhood of the closure of L_2 (resp. L_1). Hence there are constants $a_i\,(i=1, 2)$ with the conditions (i) and (ii) above. We shall prove (iii) and (iv) only for $i=1$. Let $\gamma_2(t)\,(-a_2<t<b_2')$ be the solution of the equation $dc_2/dt=X_2$ with $\gamma_2(0)=q_0 \in D_1 \setminus L$. We suppose that γ_2 comes to L_2 at $t=-a_2$. Let $c_1^\tau(t)$ be the solution of $dc_1/dt=X_1$ with $c_1^\tau(0)=\gamma_2(\tau)\,(-a_2<\tau \leqq b_2')$. We denote by $-a_1^\tau$ the time when c_1^τ arrives at L_1. Suppose $[-a_1^\tau, b_1^\tau)$ be the maximum interval such that c_1^τ is defined. Let ω_i be a closed 1-form on $D_1 \setminus \mathfrak{N}$ defined by $\omega_i(X_j)=\delta_{ij}$. Let $\Gamma(t)\,(a \leqq t \leqq b)$ be a curve in $D_1 \setminus \mathfrak{N}$ with $\Gamma(a) \in L_1$, $\Gamma(b) \in \gamma_2|_{[-a_2, b_2']}$. Then the integral

$$\int_\Gamma \omega_1$$

does not depend on the choice of Γ. Hence

$$\int_{c_1^\tau|[-a_1^\tau,0]}\omega_1 = 0-(-a_1^\tau)$$

does not depend on τ. Since $X_1f_2=X_2f_1=0$ and $[X_1, X_2]=0$, $f_1(c_1^\tau(t))$ does not depend on τ, and $f_2(c_1^\tau(t))$ does not depend on t. Since the Riemannian metric g is complete, we have

$$\begin{aligned}\infty &= \lim_{t\to b_1^\tau} d(p_1, c_1^\tau(t))\\ &\leqq d(p_1, c_1^\tau(-a_1^\tau))+\mathrm{length}(c_1^\tau|_{[-a_1^\tau, b_1^\tau)})\\ &= d(p_1, c_1^\tau(-a_1^\tau))+\int_{-a_1^\tau}^{b_1^\tau}\sqrt{f_1(c_1^\tau(t))+f_2(c_1^\tau(t))}\,dt.\end{aligned}$$

Therefore b_1^τ is independent of τ. Hence so does $a_1^\tau+b_1^\tau$. ∎

Let $r_i=a_i+b_i$ and $x_i(q)=\int_{\Gamma_i}\omega_i\,(i=1, 2)$, where Γ_i is a curve from L_i to q in $D_1\setminus\mathcal{N}$ and ω_i is a 1-form defined in the proof of Lemma 3.2. We put $x_i(p_1)=0\,(i=1, 2)$. Then the functions (x_1, x_2) can be regarded as a local coordinate system with $X_i=\partial/\partial x_i\,(i=1, 2)$. Moreover, we can prove the next lemma in the same way as [**2**, Lemma 3.3].

LEMMA 3.3. *Functions (x_1, x_2) give a homeomorphism from D_1 to $[0, r_1)\times[0, r_2)$ which is a diffeomorphism from $D_1\setminus\mathcal{N}$ to $[0, r_1)\times[0, r_2)\setminus\{(0, 0)\}$.*

We now continue the proof of Theorem 3.1. In the similar way to the case of D_1, we can define vector fields X_i and functions $x_i\,(i=1, 2)$ on D_2 such that they coincide with those on D_1 along the common boundary L. Then $x_i=\mathrm{const}$ defines a simple curve and the coordinate functions (x_1, x_2) on D_1 and D_2 define a map Φ: $(-r_1, r_1)\times(-r_2, r_2)\to S$ such that

(1) Φ maps $[0, r_1)\times[0, r_2)$ homeomorphically onto D_1, and

$$x_i\circ\Phi(y_1, y_2) = y_i\ (i=1, 2)\qquad \text{for } (y_1, y_2)\in[0, r_1)\times[0, r_2),$$

(2) Φ maps $[0, r_1)\times(-r_2, 0]$ homeomorphically onto D_2, and

$$\begin{aligned}&x_1\circ\Phi(y_1, y_2) = y_1,\\ &x_2\circ\Phi(y_1, y_2) = -y_2\qquad \text{for } (y_1, y_2)\in[0, r_1)\times(-r_2, 0],\end{aligned}$$

(3) $\Phi(-y_1, -y_2)=\Phi(y_1, y_2)$.

From Lemma 3.3, we see that the map Φ is C^∞ except (0, 0) and is a two-fold covering of $S\setminus\mathcal{N}$. It follows from the definition of the coordinates (x_1, x_2) that

$$\Phi^*g = (\Phi^*f_1+\Phi^*f_2)(dy_1^2+dy_2^2)$$

holds on $(-r_1, r_1)\times(-r_2, r_2)\setminus\Phi^{-1}(\mathcal{N})$. Since the map Φ is continuous on $(-r_1, r_1)\times(-r_2, r_2)$ and conformal except at $\Phi^{-1}(\mathcal{N})$ with respect to the conformal structure induced from the inclusion $(-r_1, r_1)\times(-r_2, r_2)\subset \boldsymbol{C}$, Φ is of class C^∞ on $(-r_1, r_1)\times(-r_2, r_2)$. We can see as in the proof of [**2**, Theorem 3.1] that the functions Φ^*f_1 and Φ^*f_2 satisfy the condition (3.1). This completes the proof of Case 1.

Case 2: $\mathfrak{N}=\{p_1, p_2\}$. In this case the geodesic is divided by p_1 and p_2 into three parts L_1, L_2 and L_3, where p_1 separates L_1 and L_2, and p_2 separates L_2 and L_3. Then F is semidefinite on L and has different sign on L_i and $L_{i+1}(i=1, 2)$. Suppose $F\geqq 0$ on L_1 and L_3, and $F\leqq 0$ on L_2. Then

$$f_1=0,\quad f_2>0 \text{ on } L_1 \text{ and } L_3,$$

$$f_1>0,\quad f_2=0 \text{ on } L_2.$$

Since $F=0$ at $\mathfrak{N}$, f_1 and f_2 become continuous on S if we put $f_1|\mathfrak{N}=f_2|\mathfrak{N}=0$.

As in Case 1, we can define vector fields $X_i=\sqrt{f_1+f_2}V_i$ on D_1. We choose the direction of X_1(resp. X_2) so that V_1(resp. V_2) points inward of D_1 on L_1 (resp. L_2).

LEMMA 3.5. *Consider the ordinary differential equation*

$$\frac{dc_i(t)}{dt}=X_i(c_i(t))$$

with the initial condition $c_i(0)=q\in D_1\setminus L\,(i=1, 2)$. *Then there are constants* $0<a_1$, a_2, $b_1<\infty$ *and* $0<b_2\leqq\infty$ *such that*

(i) *the solution* $c_1(t)$ *exists on* $[-a_1, b_1]$, *and* $c_1((-a_1, b_1))\subset D_1\setminus L$, $c_1(-a_1)\in L_1$, $c_1(b_1)\in L_3$,

(ii) *the solution* $c_2(t)$ *exists on* $[-a_2, b_2)$, *and* $c_2((-a_2, b_2))\subset D_1\setminus L$, $c_2(-a_2)\in L_2$, $\lim_{t\to b_2} d(c_2(t), p_1)=\infty$,

(iii) *the value* $a_i+b_i(\leqq\infty)(i=1, 2)$ *does not depend on the initial point* q.

PROOF. Let (r_i, θ_i) denote the polar coordinates centered at $p_i(i=1, 2)$. In these coordinates $L_1=\{\theta_1=\pi,\ 0<r_1\}$, $L_2=\{\theta_1=0,\ 0<r_1<d(p_1, p_2)\}$, $L_3=\{\theta_1=0,\ d(p_1, p_2)<r_1\}$, $D_1\setminus L=\{0<\theta_1<\pi,\ 0<r_1\}$. $\theta_1(c_1(t))$ is a decreasing function and $\theta_1(c_2(t))$ is an increasing function since both X_1 and X_2 are independent of the vectors v with $F(v)=0$ and not orthogonal to them except on L. $r_1(c_i(t))(i=1, 2)$ are increasing functions on D_1 and $r_2(c_1(t))$ is a decreasing function. Hence it is clear that a_1, a_2 and b_1 exist and satisfy the conditions required above. Let $[-a_2, b_2)$ be the maximal interval where the solution of the equation $dc_2/dt=X_2$ with $c_2(0)=q$ exists. Then, as in the proof of Lemma 3.2, we can prove that a_2+b_2 does not depend on the initial condition $c_2(0)=q$, and that $\lim_{t\to b_2} d(c_2(t), p_1)=\infty$. ■

Let $r_1=\infty$, $r_2=a_2+b_2$, $\alpha_1=2(a_1+b_1)$ and $x_i(q)=\int_{\Gamma_i}\omega_i\,(i=1, 2)$, where Γ_i is a curve from L_i to q in $D_1\setminus\mathcal{N}$ and ω_i is a 1-form defined in the proof of Lemma 3.2. And we put $x_1(p_1)=x_2(p_1)=x_2(p_2)=0$, $x_1(p_2)=\alpha_1/2$. Then the functions (x_1, x_2) can be regarded as a local coordinate system with $X_i=\partial/\partial x_i\,(i=1, 2)$. Moreover, we can prove the next lemma in the same way as [**2**, Lemma 3.3].

LEMMA 3.6. *Functions* (x_1, x_2) *give a homeomorphism from* D_1 *to* $[0, \alpha_1/2]\times[0, r_2)$ *which is a diffeomorphism from* $D_1\setminus\mathcal{N}$ *to* $[0, \alpha_1/2]\times[0, r_2)\setminus\{(0, 0), (\alpha_1/2, 0)\}$.

We now continue the proof of Theorem 3.1. In the similar way to the case of D_1, we can define vector fields X_i and functions $x_i\,(i=1, 2)$ on D_2 such that they coincide with those on D_1 along the common boundary L. Then $x_i=\mathrm{const}$ defines a simple curve. Let $\Gamma=\Gamma(\alpha_1)$ denote the group of parallel translations of $\boldsymbol{R}^2$ generated by $(\alpha_1, 0)$. Then the coordinate functions (x_1, x_2) on D_1 and D_2 define a map $\Phi:\boldsymbol{R}\times(-r_2, r_2)/\Gamma\to S$ such that

(1) Φ maps $[0, \alpha_1/2]\times[0, r_2)$ homeomorphically onto D_1, and

$$x_1\circ\Phi(y_1, y_2)\equiv y_1 \mod \alpha_1\boldsymbol{Z},$$
$$x_2\circ\Phi(y_1, y_2)=y_2$$

for $(y_1, y_2)\in[0, \alpha_1/2]\times[0, r_2)$,

(2) Φ maps $[0, \alpha_1/2]\times(-r_2, 0]$ homeomorphically onto D_2, and

$$x_1\circ\Phi(y_1, y_2)\equiv y_1 \mod \alpha_1\boldsymbol{Z},$$
$$x_2\circ\Phi(y_1, y_2)=-y_2$$

for $(y_1, y_2)\in[0, \alpha_1/2]\times(-r_2, 0]$,

(3) $\Phi([-y_1, -y_2])=\Phi([y_1, y_2])$, where $[y_1, y_2]$ denotes the equivalence class which contains (y_1, y_2).

Lemma 3.6 yields that the map Φ is C^∞ except $[0, 0]$ and $[\alpha_1/2, 0]$, and is a two-fold covering of $S\setminus\mathcal{N}$. It follows from the definition of the coordinates (x_1, x_2) that

$$\Phi^*g=(\Phi^*f_1+\Phi^*f_2)(dy_1^2+dy_2^2)$$

holds on $\boldsymbol{R}\times(-r_2, r_2)/\Gamma\setminus\Phi^{-1}(\mathcal{N})$. Since the map Φ is continuous on $\boldsymbol{R}\times(-r_2, r_2)/\Gamma$, and conformal except at $\Phi^{-1}(\mathcal{N})$ with respect to the conformal structure induced from the inclusion $\boldsymbol{R}\times(-r_2, r_2)\subset\boldsymbol{C}$, Φ is of class C^∞ on $\boldsymbol{R}\times(-r_2, r_2)/\Gamma$.

We can see as in the proof of [**2**, Theorem 3.1] that the functions Φ^*f_1 and Φ^*f_2 satisfy the condition (3.2). This completes the proof of Case 2. ∎

As in [**2**], Theorem 2.1 implies the following corollaries.

COROLLARY 3.7 ([2, Corollary 3.4]). *Let (S, g, F) be a noncompact complete Liouville surface and L the geodesic given in Theorem* 2.1. *Then the reflection with respect to L is an isometry.*

COROLLARY 3.8 ([2, Corollary 3.6]). *Let (S, g, F) be a noncompact complete Liouville surface which corresponds to a sextuple $(\alpha_1, \alpha_2, f_1, f_2, r_1, r_2)$. If (S, g) is analytic, then f_1, f_2 and F are also analytic. If $\alpha_1<\infty$, then*

$$f_1(t+\alpha_1/2) = f_1(t)$$

for any $t\in\boldsymbol{R}$. If we extend f_1 as a holomorphic function about the real axis, then

$$f_2(t) = -f_1(\sqrt{-1}\,t)$$

for $t\in\boldsymbol{R}$ near 0.

We are now in position to consider the case $\#\mathfrak{N}=0$. In the first part of the proof of Theorem 2.1, we have essentially proved

PROPOSITION 3.9 ([2, Proposition 3.8]). *Let (S, g, F) be a noncompact complete Liouville surface with $\mathfrak{N}=\emptyset$. Then there is a global coordinate system (x_1, x_2) on the universal covering space $\tilde{S}$ of S with*

$$-\infty \leqq -a_1 < x_1 < b_1 \leqq \infty, \quad -\infty \leqq -a_2 < x_2 < b_2 \leqq \infty,$$

i.e., $\tilde{S}$ is diffeomorphic to $(-a_1, b_1)\times(-a_2, b_2)$. And if S is not simply connected, then its fundamental group is a cyclic group generated by one of the following transformations:

$$(x_1, x_2) \longrightarrow (x_1, x_2)+(c_1, c_2)$$
$$(x_1, x_2) \longrightarrow (x_1, -x_2)+(c_1, c_2)$$
$$(x_1, x_2) \longrightarrow (-x_1, x_2)+(c_1, c_2).$$

In this coordinate system, g and F are expressed as

$$g = (f_1(x_1)+f_2(x_2))(dx_1^2+dx_2^2)$$
$$F = \frac{1}{f_1(x_1)+f_2(x_2)}(f_2(x_2)\xi_1^2-f_1(x_1)\xi_2^2),$$

where (x, ξ) is the canonical coordinate system on T^*S.

Next Lemma is also valid in our case.

LEMMA 3.10 ([2, Lemma 3.9]). *f_1 and f_2 are not constant.*

We shall now consider complete Liouville surfaces whose underlying manifolds are diffeomorphic to a cylinder with the help of conformal structure.

Let $z=x_1+\sqrt{-1}x_2$ be the natural coordinate of $\boldsymbol{C}=\boldsymbol{R}^2$. Let Γ denote the group of parallel translations of $\boldsymbol{R}^2$ generated by (1, 0). Set

$$\Omega_r = \{z \in \boldsymbol{C}\,;\, 0 < x_2 < r\},\quad 0 < r \leqq \infty,$$

and set $S_r=\Omega_r/\Gamma$, $S_0=\boldsymbol{C}/\Gamma$. Then cylinders S_r and S_0 have natural conformal structures induced from the Riemannian metric $dx_1^2+dx_2^2$. As is easily seen, $S_r\,(0<r\leqq\infty)$ and S_0 are not mutually conformally isomorphic, and every cylinder with a conformal structure is isomorphic to one of them.

Let $\mathcal{A}_0$ be the set of pairs of lines in $\boldsymbol{R}^2$ which pass the origin and are mutually orthogonal. For $l=(l_1, l_2)\in\mathcal{A}_0$, let $\pi_i: \boldsymbol{C}\to l_i$ denote the orthogonal projection, and put $\Gamma_i=\pi_i(\Gamma)\,(i=1, 2)$. For each $l=(l_1, l_2)$, we denote by $\mathcal{F}(l)$ the set of pairs (f_1, f_2) of functions with the following properties: f_i is a C^∞ function on l_i which is invariant under the action of $\Gamma_i\,(i=1, 2)$; $f_1+f_2>0$; If $\Gamma_i=\{0\}$, i.e., $\partial/\partial x_2$ is tangent to l_i, then

$$\int_0^\infty \sqrt{f_1+f_2}\,dx_2 = \int_{-\infty}^0 \sqrt{f_1+f_2}\,dx_2 = \infty\,.$$

Here the functions f_i are identified with $\pi_i^* f_i$. We put

$$\tilde{\mathcal{B}}_0 = \{(l, f)\,;\, l \in \mathcal{A}_0,\ f \in \mathcal{F}(l)\}\,.$$

For $r\in(0, \infty]$ we denote by $\tilde{\mathcal{B}}_r$ the set of pairs $f=(f_1, f_2)$ of C^∞ functions with the following properties: f_1 is a non-constant function on $\boldsymbol{R}=\{(x_1)\}$ with period 1; f_2 is a non-constant function on the interval $\{x_2\,;\,0<x_2<r\}$; $f_1+f_2>0$;

$$\int_a^b \sqrt{f_1(x_1)+f_2(x_2)}\,dx_2 \longrightarrow \infty \quad \text{as} \quad a \longrightarrow 0 \quad \text{or} \quad b \longrightarrow r\,.$$

For each $(l, f)\in\tilde{\mathcal{B}}_0$ we assign a Liouville surface whose underlying Riemannian manifold is conformally isomorphic to S_0 as follows. Let (y_1, y_2) be an orthonormal coordinate system on $\boldsymbol{C}=\boldsymbol{R}^2$ so that $\partial/\partial y_i$ is tangent to $l_i\,(i=1, 2)$. Put

$$g = (f_1(y_1)+f_2(y_2))(dy_1^2+dy_2^2)\,,$$

$$F = \frac{1}{f_1(y_1)+f_2(y_2)}(f_2(y_2)\eta_1^2-f_1(y_1)\eta_2^2)\,,$$

where (y, η) is the associated canonical coordinate system on $T^*\boldsymbol{C}$. Clearly g and F are invariant under the action of Γ. By regarding them as a Riemannian metric on $\boldsymbol{C}/\Gamma=S_0$ and a function on T^*S_0 respectively, we obtain a Liouville surface (S_0, g, F).

In the same way, for each $f=(f_1, f_2)\in\tilde{\mathcal{B}}_r$ we can assign a Liouville surface whose underlying Riemannian manifold is conformally isomorphic to S_r.

We say that two elements (l, f) and (m, h) of $\tilde{\mathcal{B}}_0$ are equivalent if there are a conformal transformation φ of $\boldsymbol{C}$ with $\varphi_o(\Gamma)=\Gamma$ and a constant $a\in\boldsymbol{R}$

such that one of the following conditions is satisfied:

(1) $\varphi_o(l_i)=m_i,\ \varphi^*h_i=f_i+(-1)^i a,\ (i=1, 2)$,

(2) $\varphi_o(l_1)=m_2,\ \varphi_o(l_2)=m_1,\ \varphi^*h_1=f_2+a,\ \varphi^*h_2=f_1-a$.

Here φ_o stands for the linear part of φ. Similarly, two elements f and h of $\tilde{\mathscr{B}}_r$ are said to be equivalent if there are a conformal transformation φ of $\boldsymbol{C}$ with $\varphi_o(\Gamma)=\Gamma$ and $\varphi(\Omega_r)=\Omega_r$ and a constant $a\in\boldsymbol{R}$ such that

$$\varphi^*h_i=f_i+(-1)^i a \quad (i=1, 2).$$

It is easy to see that mutually equivalent elements give mutually equivalent Liouville surfaces. Let $\mathscr{B}_0$ (resp. $\mathscr{B}_r$) denote the set of all equivalence classes in $\tilde{\mathscr{B}}_0$ (resp. $\tilde{\mathscr{B}}_r$).

THEOREM 3.11. *The assignment above gives the one-to-one correspondence between the set $\mathscr{B}_0$ (resp. $\mathscr{B}_r$) and the set of equivalence classes of complete Liouville surfaces whose underlying Riemannian manifolds are conformally isomorphic to the cylinder S_0 (resp. S_r).*

PROOF. In view of Proposition 3.9, it is clear that the assignment described above is surjective. And the theorem can be proved by the same argument with that of [**2**, Theorem 3.11]. ■

Next we shall consider the case where the underlying manifold is diffeomorphic to the Möbius band. For $r\in[0, \infty)$ we put

$$\tilde{c}_r(z)=\bar{z}+\sqrt{-1}\,r+\frac{1}{2}, \quad z\in\boldsymbol{C}.$$

Clearly $\tilde{c}_r$ induces a conformal transformation c_r of S_r satisfying $c_r^2=\mathrm{id}$, $(0\leqq r<\infty)$. Put

$$M_r=S_r/\{\mathrm{id}, c_r\}, \quad (0\leqq r<\infty).$$

Then M_r is diffeomorphic to the Möbius band which possesses the conformal structure induced from that on S_r.

For $r\in[0, \infty)$ we define a subset $\mathcal{C}_r$ of $\mathscr{B}_r$ as follows. $\mathcal{C}_0$ is the set of all elements of $\mathscr{B}_0$ whose representatives (l, f) satisfy $l_1=\{x_2=0\}$, $l_2=\{x_1=0\}$, $f_1(x_1+1/2)=f_1(x_1)$ and $f_2(-x_2)=f_2(x_2)$. For $r>0$, $\mathcal{C}_r$ is the set of all elements of $\mathscr{B}_r$ whose representatives f satisfy $f_1(x_1+1/2)=f_1(x_1)$ and $f_2(r-x_2)=f_2(x_2)$.

As we have already seen, an element of $\mathcal{C}_r$ induces a Liouville surface whose underlying manifold is diffeomorphic to a cylinder S_r, and the mapping c_r acts on it as an automorphism of a Liouville surface. Hence, dividing it by the action of $\{\mathrm{id}, c_r\}$, we get a complete Liouville surface whose underlying Riemannian manifold is conformally isomorphic to M_r, and we have

PROPOSITION 3.12. *The assignment above gives a one-to-one correspondence between the set C_r and the equivalence classes of complete Liouville surfaces whose underlying Riemannian manifolds are conformally isomorphic to M_r.*

Finally we shall consider simply connected Liouville surfaces with $\#\mathfrak{N}=0$. For $0 \leqq r_1 \leqq r_2 \leqq \infty$, let

$$\mathcal{H}_{r_1, r_2} = \begin{cases} \{(x_1, x_2)\,;\ -\infty < x_1 < +\infty,\ -\infty < x_2 < +\infty\} & \text{if } r_1 = r_2 = 0, \\ \{(x_1, x_2)\,;\ -\infty < x_1 < +\infty,\ 0 < x_2 < r_2\} & \text{if } r_1 = 0 < r_2, \\ \{(x_1, x_2)\,;\ 0 < x_1 < r_1,\ 0 < x_2 < r_2\} & \text{if } 0 < r_1 \leqq r_2. \end{cases}$$

For $0 \leqq r_1 \leqq r_2 \leqq \infty$, we denote by $\tilde{\mathcal{D}}_{r_1, r_2}$ the set of pairs $f=(f_1, f_2)$ of C^∞ functions on $\mathcal{H}_{r_1, r_2}$ with the following properties: f_i is a non-constant function with a single variable $x_i (i=1, 2)$; $(f_1+f_2)(dx_1^2+dx_2^2)$ defines a complete Riemannian metric on $\mathcal{H}_{r_1, r_2}$.

We say that two elements (f_1, f_2) and (h_1, h_2) of $\tilde{\mathcal{D}}_{r_1, r_2}$ are equivalent if there is a conformal transformation φ of $\mathcal{H}_{r_1, r_2}$ such that

$$\varphi^* h_i = f_i + (-1)^i a \ (i = 1, 2) \qquad \text{if } r_1 \leqq r_2,$$
$$\varphi^* h_1 = f_2 + a, \quad \varphi^* h_2 = f_1 - a, \quad \text{if } r_1 = r_2$$

for some constant $a \in \boldsymbol{R}$. Let $\mathcal{D}_{r_1, r_2}$ denote the set of all equivalence classes in $\tilde{\mathcal{D}}_{r_1, r_2}$.

For each $(f_1, f_2) \in \tilde{\mathcal{D}}_{r_1, r_2}$ we assign a Liouville surface

$$\left(\mathcal{H}_{r_1, r_2}, (f_1+f_2)(dx_1^2+dx_2^2), \quad \frac{1}{f_1+f_2}(f_2\xi_1^2 - f_1\xi_2^2)\right).$$

It is easy to see that mutually equivalent elements give mutually equivalent Liouville surfaces and we have

THEOREM 3.13. *The assignment above gives the one-to-one correspondence between the set*

$$\mathcal{D}_{0,0} \cup \mathcal{D}_{0,1} \cup \mathcal{D}_{1,1} \cup \bigcup_{1<r_2 \leqq \infty} \mathcal{D}_{1, r_2} \cup \mathcal{D}_{\infty, \infty}$$

and the set of equivalence classes of simply connected complete Liouville surfaces with $\#\mathfrak{N}=0$.

4. Quadratic surfaces in the hyperbolic 3-space.

In this section we show that quadratic surfaces in the hyperbolic 3-space $\boldsymbol{H}^3(-1)$ of constant sectional curvature -1 are also Liouville surfaces as Euclidean ones. We restrict ourselves to hyperboloids of two sheets and elliptic paraboloids, but the similar computation shows that all such surfaces are Liouville surfaces.

Let M denote a hyperboloid of two sheets in $\boldsymbol{R}^3$ defined by a quadratic equation

$$\frac{x_0^2}{a_0}+\frac{x_1^2}{a_1}+\frac{x_2^2}{a_2}=1$$

with $a_0<a_1<0<a_2$. Then elliptic coordinates $(u_1, u_2)\in(-\infty, a_0]\times[a_0, a_1]$ are given by

$$x_0^2=\frac{a_0(u_1-a_0)(u_2-a_0)}{(a_1-a_0)(a_2-a_0)}$$

$$x_1^2=\frac{a_1(u_1-a_1)(u_2-a_1)}{(a_0-a_1)(a_2-a_1)}$$

$$x_2^2=\frac{a_2(u_1-a_2)(u_2-a_2)}{(a_0-a_2)(a_1-a_2)}.$$

We adopt a half-space $\{(x_0, x_1, x_2);\ x_1>0\}$ with the Riemannian metric $g_{-1}=(dx_0^2+dx_1^2+dx_2^2)/x_1^2$ as the hyperbolic 3-space $\boldsymbol{H}^3(-1)$. Let S be the connected component of $M\cap\boldsymbol{H}^3(-1)$ with $0<x_2$. Then we have

PROPOSITION 4.1. *The first fundamental form of $S\subset\boldsymbol{H}^3(-1)$ in the elliptic coordinates is given by*

$$ds^2=\frac{(a_0-a_1)(a_2-a_1)}{a_1}\left(\frac{-1}{a_1-u_1}+\frac{1}{a_1-u_2}\right)(U_1du_1^2+U_2du_2^2)$$

where $U_i=U_i(u_i)=(-1)^iu_i/f(u_i)$; $f(u_i)=4(a_0-u_i)(a_1-u_i)(a_2-u_i)$.

THEOREM 4.2. *In the elliptic coordinates (u_1, u_2) on S, the geodesics are characterized by*

$$\sqrt{U_1}\dot{u}_1\bigg/\sqrt{\frac{-1}{a_1-u_1}+\frac{1}{a_1-\gamma}}\mp\sqrt{U_2}\dot{u}_2\bigg/\sqrt{\frac{1}{a_1-u_2}-\frac{1}{a_1-\gamma}}=0,$$

together with the condition $ds^2(u, \dot{u})=\mathrm{const}$. Here γ is a constant with value in $(-\infty, a_0)$ or (a_0, a_1).

The constant γ is called the parameter of the geodesic.

PROOF. Choose $\gamma\in(-\infty, a_0)$ or (a_0, a_1). On the subdomain of those $(u_1, u_2)\in(-\infty, a_0)\times(a_0, a_1)$ which satisfy $u_1<\gamma<u_2$, we introduce new coordinates u_1', u_2' by

$$du_1'=\sqrt{\frac{-1}{a_1-u_1}+\frac{1}{a_1-\gamma}}\sqrt{U_1}\,du_1\pm\sqrt{\frac{1}{a_1-u_2}-\frac{1}{a_1-\gamma}}\sqrt{U_2}\,du_2,$$

$$du_2' = \sqrt{U_1}\,du_1 \Big/ \sqrt{\frac{-1}{a_1-u_1}+\frac{1}{a_1-\gamma}} \mp \sqrt{U_2}\,du_2 \Big/ \sqrt{\frac{1}{a_1-u_2}-\frac{1}{a_1-\gamma}}\,.$$

In these coordinates, the line element is given by

$$ds^2 = \frac{(a_0-a_1)(a_2-a_1)}{a_1}\left\{du_1'^2+\left(\frac{-1}{a_1-u_1}+\frac{1}{a_1-\gamma}\right)\left(\frac{1}{a_1-u_2}-\frac{1}{a_1-\gamma}\right)du_2'^2\right\}.$$

Hence u_1'-curves are geodesics. Our equation is equivalent to $du_2'/dt=0$. ■

COROLLARY 4.3. *Define*

$$F(u,\dot{u}) = \frac{(a_0-a_1)(a_2-a_1)}{a_1}\left(\frac{-1}{a_1-u_1}+\frac{1}{a_1-u_2}\right)\left(\frac{U_1\dot{u}_1^2}{a_1-u_2}+\frac{U_2\dot{u}_2^2}{a_1-u_1}\right).$$

Then $F(u,\dot{u})=\text{const}=1/(a_1-\gamma)$ *if* $u(t)=(u_1(t), u_2(t))$ *is a geodesic with parameter* γ *parametrized by arc length, i.e.,* (S, ds^2, F) *is a Liouville surface with*

$$\mathfrak{N} = \{(0, \sqrt{a_1(a_0-a_1)/(a_2-a_1)}, \sqrt{a_2(a_0-a_2)/(a_1-a_2)})\}.$$

Next we shall study the elliptic paraboloid in $\boldsymbol{H}^3(-1)$. Let M denote a surface in $\boldsymbol{R}^3$ defined by a quadratic equation

$$\frac{x_0^2}{a_0}+\frac{x_1^2}{a_1}=2x_2$$

with $0<a_0<a_1$. Then elliptic coordinates $(u_1, u_2)\in[a_0, a_1]\times[a_1, \infty)$ are given by

$$x_0^2 = \frac{a_0(u_1-a_0)(u_2-a_0)}{a_1-a_0}$$

$$x_1^2 = \frac{a_1(u_1-a_1)(u_2-a_1)}{a_0-a_1}$$

$$x_2^2 = \frac{u_1+u_2-a_0-a_1}{2}.$$

We adopt a half-space $\{(x_0, x_1, x_2);\ x_0>0\}$ with the Riemannian metric $g_{-1}=(dx_0^2+dx_1^2+dx_2^2)/x_0^2$ as the hyperbolic 3-space $\boldsymbol{H}^3(-1)$. Put $S=M\cap\boldsymbol{H}^3(-1)$. Then we have

PROPOSITION 4.4. *The first fundamental form of* $S\subset\boldsymbol{H}^3(-1)$ *in the elliptic coordinates is given by*

$$ds^2 = \frac{a_1-a_0}{a_0}\left(\frac{1}{u_1-a_0}+\frac{-1}{u_2-a_0}\right)(U_1du_1^2+U_2du_2^2),$$

where $U_i=U_i(u_i)=(-1)^iu_i/f(u_i)$; $f(u_i)=4(a_0-u_i)(a_1-u_i)$.

THEOREM 4.5. *In the elliptic coordinates* (u_1, u_2) *on* S, *the geodesics are characterized by*

$$\sqrt{U_1}\,\dot{u}_1\Big/\sqrt{\frac{1}{u_1-a_0}-\frac{1}{\gamma-a_0}}\mp\sqrt{U_2}\,\dot{u}_2\Big/\sqrt{\frac{-1}{u_2-a_0}+\frac{1}{\gamma-a_0}}=0$$

together with the condition $ds^2(u, \dot{u})=\mathrm{const}$. *Here* γ *is a constant with value in* (a_0, a_1) *or* (a_1, ∞).

The constant γ is called the parameter of the geodesic.

PROOF. Choose $\gamma\in(a_0, a_1)$ or (a_1, ∞). On the subdomain of those $(u_1, u_2)\in(a_0, a_1)\times(a_1, \infty)$ which satisfy $u_1<\gamma<u_2$, we introduce new coordinates u_1', u_2' by

$$du_1'=\sqrt{\frac{1}{u_1-a_0}-\frac{1}{\gamma-a_0}}\sqrt{U_1}\,du_1\pm\sqrt{\frac{-1}{u_2-a_0}+\frac{1}{\gamma-a_0}}\sqrt{U_2}\,du_2,$$

$$du_2'=\sqrt{U_1}\,du_1\Big/\sqrt{\frac{1}{u_1-a_0}-\frac{1}{\gamma-a_0}}\mp\sqrt{U_2}\,du_2\Big/\sqrt{\frac{-1}{u_2-a_0}+\frac{1}{\gamma-a_0}}.$$

In these coordinates, the line element is given by

$$ds^2=\frac{a_1-a_0}{a_0}\left\{du_1'^2+\left(\frac{-1}{u_2-a_0}+\frac{1}{\gamma-a_0}\right)\left(\frac{1}{u_1-a_0}-\frac{1}{\gamma-a_0}\right)du_2'^2\right\}.$$

Hence u_1'-curves are geodesics. Our equation is equivalent to $du_2'/dt=0$. ∎

COROLLARY 4.6. *Define*

$$F(u, \dot{u})=\frac{a_1-a_0}{a_0}\left(\frac{-1}{u_2-a_0}+\frac{1}{u_1-a_0}\right)\left(\frac{U_1\dot{u}_1^2}{u_2-a_0}+\frac{U_2\dot{u}_2^2}{u_1-a_0}\right).$$

Then $F(u, \dot{u})=\mathrm{const}=1/(\gamma-a_0)$, *if* $u(t)=(u_1(t), u_2(t))$ *is a geodesic with parameter* γ *parametrized by arc length, i.e.,* (S, ds^2, F) *is a Liouville surface with* $\mathfrak{N}=\{(\sqrt{a_0(a_1-a_0)}, 0, (a_1-a_0)/2)\}$.

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