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# $G$-$s$-cobordant manifolds are not necessarily $G$-homeomorphic for arbitrary compact Lie groups $G$

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## § 1. Introduction.

The classical $h$-cobordism theorem and the $s$-cobordism theorem have played an important role in numerous aspects of geometric topology, including the classification of manifolds by surgery [35], [22], [3], [28], [21], [36], [34], [43], [5], [9], [23], [42].

In [1], we discussed equivariant versions of these theorems.

Let $G$ be a compact Lie group and $X$ a finite $G$-CW complex. S. Illman [14] defined the equivariant Whitehead group $\mathrm{Wh}_G(X)$ of $X$ and the equivariant Whitehead torsion $\tau_G(f)$ for a $G$-homotopy equivalence $f: X \to Y$ between finite $G$-CW-complexes $X$, $Y$ as an element of $\mathrm{Wh}_G(X)$. When $\tau_G(f)=0$, $f$ is called a *simple $G$-homotopy equivalence*.

Let $W$ be a compact smooth $G$-manifold whose boundary $\partial W$ is the disjoint union $X \amalg Y$ of two closed $G$-invariant submanifolds. If the inclusion maps

$$i_X : X \longrightarrow W \quad \text{and} \quad i_Y : Y \longrightarrow W$$

are $G$-homotopy equivalences, then the triad $(W; X, Y)$ is called a *$G$-$h$-cobordism*.

When $G$ is a finite group, $W$ admits a unique smooth $G$-triangulation [15]. Accordingly the equivariant Whitehead torsion $\tau_G(i_X)$ is well-defined. On the other hand the investigation of T. Matumoto and M. Shiota [26] enables us to define the equivariant Whitehead torsion $\tau_G(i_X)$ even when $G$ is a compact Lie group. Notice that $\tau_G(i_X)$ is often written as $\tau_G(W, X)$.

A $G$-$h$-cobordism $(W; X, Y)$ is called a *$G$-$s$-cobordism* when $\tau_G(i_X)$ vanishes. The two $G$-manifolds $X$ and $Y$ are then called *$G$-$s$-cobordant*.

We say that the $G$-$s$-cobordism theorem holds for a $G$-$s$-cobordism $(W; X, Y)$ if $W$ is $G$-diffeomorphic to the product $X \times I$ rel $X$ where $I$ is the interval

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[0, 1] with trivial $G$-action.

Let $H, K$ be isotropy groups appearing in $W$ and

$$W^H = \coprod_\lambda W^H_\lambda, \qquad W^K = \coprod_\mu W^K_\mu$$

be the decompositions to connected components of respective fixed point sets. We now consider two conditions.

(*1) (Gap hypothesis) If $W^K_\mu \supsetneqq W^H_\lambda$, then $\dim W^K_\mu - \dim W^H_\lambda \geqq \dim G+3$ for any pair of components $W^K_\mu$ and $W^H_\lambda$.

(*2) If $H$ is a maximal isotropy group, then $\dim W^H_\lambda \geqq \dim G+6$ for any components $W^H_\lambda$.

Then we have

THEOREM 1.1 [**1**]. *Let G be a compact Lie group and* $(W; X, Y)$ *a G-s-cobordism. If W satisfies the conditions* (*1) *and* (*2) *above, then we have a G-diffeomorphism*

$$W \cong X\times I \qquad \mathit{rel}\ X.$$

*In particular, X is G-diffeomorphic to Y.*

On the other hand, we have shown in [**20**] that $G$-$s$-cobordism theorems do not hold in general for many compact Lie groups $G$ if the condition (*1) is not satisfied. The $G$-$s$-cobordisms $(W; X, Y)$ provided there as counterexamples are such that $X$ is $G$-diffeomorphic to $Y$, but $W$ is not $G$-homeomorphic to $X\times I$.

In the present paper, we show that $G$-$s$-cobordant manifolds are not necessarily $G$-homeomorphic. Namely we have

THEOREM 1.2. *Let G be an arbitrary non-trivial compact Lie group. Then there exists a G-s-cobordism* $(W; X, Y)$ *such that X is not G-homeomorphic to Y. In particular, W is not G-homeomorphic to* $X\times I$.

REMARK 1.3. Similar results related with Theorem 1.1 were also obtained in [**7**], [**30**], [**6**], [**2**], [**39**], [**38**], [**19**].

REMARK 1.4. In the non-equivariant case, Milnor has given examples of $h$-cobordant manifolds which are not diffeomorphic [**29**]. Moreover F.T. Farrell and W.C. Hsiang have shown that $h$-cobordant manifolds are not necessarily homeomorphic [**10**]. It is needless to say that these $h$-cobordant manifolds are not $s$-cobordant.

REMARK 1.5. In the equivariant case, W. Browder and F. Quinn have shown that there is a $Z_2$-$h$-cobordism $(W; S^n_1, S^n_2)$ such that $W$ is not $Z_2$-homeomorphic to $S^n_1\times I$ [**7**]. But $S^n_1$ and $S^n_2$ are $Z_2$-homeomorphic in this case.

REMARK 1.6. By combining the results of S. Illman [**15**], T. Matumoto [**25**], I. M. James and G. B. Segal [**17**], C. H. Giffen [**11**] and D. W. Sumners [**41**], we get a $Z_p$-$h$-cobordism $(W_1; X_1, Y_1)$ such that $X_1$ and $Y_1$ are not $Z_p$-homeomorphic (see § 4). Unfortunately, however, we do not know whether the $Z_p$-$h$-cobordism is a $Z_p$-$s$-cobordism or not. Therefore the consideration in § 4 is indispensable even for $G=Z_p$.

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## § 2. Induced transformation groups.

We first introduce basic notations. Let $G$ be a compact Lie group. Whenever $H$ is a closed subgroup of $G$, $(H)$ denotes the conjugacy class of $H$ in $G$. Let $X$ be a $G$-space. We shall denote the isotropy group at $x\in X$ by $G_x$, namely $G_x=\{g\in G \mid gx=x\}$, and the $G$ orbit of $x$ by $G(x)$, namely $G(x)=\{gx\in X \mid g\in G\}$. A $G$-space $X$ is called a *semi-free* $G$-space when $G_x$ is either $G$ or the unit group $\{e\}$ for every $x\in X$. The orbit space of a $G$-space $X$ is denoted by $X/G$. For a subgroup $H$ of $G$, we shall put $X^H=\{x\in X \mid G_x\supset H\}$, $X(H)=\{x\in X \mid (G_x)=(H)\}$.

In the following, we introduce the notion of induced transformation groups.

Let $G$ be a compact Lie group and $H$ a closed subgroup of $G$. Let $X$ be an $H$-space. Consider the space $G\times X$ and define an $H$-action $\phi: H\times(G\times X)\to G\times X$ by

$$\phi(h, (g, x))=(gh^{-1}, hx) \qquad \text{for } h\in H,\ g\in G,\ x\in X.$$

We define $G\times_H X$ to be the orbit space of $G\times X$ under this $H$-action. Let $\pi: G\times X\to G\times_H X$ be the natural projection and denote $\pi(g, x)=[g, x]$. Now define a $G$-action $\psi: G\times(G\times_H X)\to G\times_H X$ by $\psi(g', [g, x])=[g'g, x]$.

The space $G\times_H X$ together with this $G$-action is called an *induced transformation group*.

LEMMA 2.1. *For a closed subgroup $K$ of $H$, we have*

$$(G\underset{H}{\times} X)(K)=G\underset{H}{\times}\{\bigcup_{\substack{K'<H \\ (K')=(K)}} X(K')\}$$

*where the union is taken over all the closed subgroups $K'$ of $H$ such that $K'$ is conjugate to $K$ in $G$.*

PROOF. Lemma 2.1 follows immediately from the following relation;

$$G_{[g,x]} = gH_x g^{-1}.$$

PROPOSITION 2.2. *If $X$ is an $H$-space, then there is a canonical homeomorphism*

$$f : (G \underset{H}{\times} X)/G \longrightarrow X/H$$

*with* $f((G \times_H X)(H)/G) = X^H$.

PROOF. Set $f(G([g, x])) = H(x)$. Then it is easy to see that $f$ is a well-defined continuous map. Conversely define a map

$$f' : X/H \longrightarrow (G \underset{H}{\times} X)/G$$

by setting

$$f'(H(x)) = G([e, x]).$$

Then one verifies easily that $f'$ is also a well-defined continuous map and that

$$f \cdot f' = f' \cdot f = \text{identity}.$$

Hence both $f$ and $f'$ are homeomorphisms.

In view of Lemma 2.1, we have

$$(G \underset{H}{\times} X)(H) = G \underset{H}{\times} \{ \bigcup_{\substack{K' < H \\ (K') = (H)}} X(K') \}.$$

Since there exists an element $g$ of $G$ such that $gK'g^{-1} = H$, we have

$$g^{-1}Hg = K' \subset H.$$

Then it is shown in [4] that $g$ belongs to the normalizer $N(H)$ of $H$ in $G$. Hence $K'$ exactly coincides with $H$. Thus we have

$$\bigcup_{\substack{K' < H \\ (K') = (H)}} X(K') = X(H) = X^H.$$

It follows that

$$(G \underset{H}{\times} X)(H)/G = (G \underset{H}{\times} X^H)/G.$$

Obviously we have

$$f((G \underset{H}{\times} X^H)/G) = X^H.$$

This makes the proof of Proposition 2.2 complete.

## § 3. Equivariant Whitehead torsions of induced transformation groups.

Denote by $\mathring{D}^n$ the $n$-dimensional open disk with trivial $H$-action. Each $H$-cell of an $H$-CW complex has the form $H/K\times\mathring{D}^n$ where $K$ is a closed subgroup of $H$. Making use of the canonical $G$-homeomorphism

$$G\underset{H}{\times}(H/K\times\mathring{D}^n)=G/K\times\mathring{D}^n\,,$$

we have

LEMMA 3.1 [14]. *If $X$ is a finite $H$-CW complex, then $G\underset{H}{\times}X$ is a finite $G$-CW complex.*

Each element of $\mathrm{Wh}_H(X)$ is represented by a finite $H$-CW pair $(V, X)$ such that $X$ is an $H$-deformation retract of $V$. The element represented by such a pair $(V, X)$ is denoted by $\tau_H(V, X)$ and is called the equivariant Whitehead torsion of $(V, X)$. Then it is easy to see that $G\times_H X$ is a $G$-deformation retract of $G\times_H V$. Hence the $G$-CW pair $(G\times_H V, G\times_H X)$ represents an element of $\mathrm{Wh}_G(G\times_H X)$ and we have

LEMMA 3.2. [14]. *The assignment $\tau_H(V, X)\rightarrow\tau_G(G\times_H V, G\times_H X)$ gives a well-defined homomorphism*

$$i_*:\mathrm{Wh}_H(X)\longrightarrow\mathrm{Wh}_G(G\underset{H}{\times}X)\,.$$

Suppose hereafter that $H$ is a finite subgroup of a compact Lie group $G$.

Let $(W;X, Y)$ be a smooth $H$-$h$-cobordism. Namely $W$ is a compact $H$-manifold with boundary $\partial W=X\amalg Y$ (disjoint union) and the inclusions

$$i_X:X\longrightarrow W\quad\text{and}\quad i_Y:Y\longrightarrow W$$

are $H$-homotopy equivalences.

According to [15], $W$ and $X$ admit unique smooth $H$-triangulations and hence the equivariant Whitehead torsion $\tau_H(W, X)$ is well-defined. Consider the induced transformation groups $G\times_H W$ and $G\times_H X$. Then it follows from Lemma 3.1 that $G\times_H W$ and $G\times_H X$ have the induced $G$-CW complex structures. Hence we have a homomorphism

$$i_*:\mathrm{Wh}_H(X)\longrightarrow\mathrm{Wh}_G(G\underset{H}{\times}X)$$

by Lemma 3.2.

On the other hand, $G\times_H X$ has the induced smooth $G$-manifold structure as follows. Since the map

$$\phi:H\times(G\times X)\longrightarrow G\times X$$

defined by $\phi(h,(g,x))=(gh^{-1}, hx)$ for $h\in H$, $g\in G$, $x\in X$ gives a smooth free $H$-action, the orbit space $G\times_H X$ of $G\times X$ under this action is naturally given a smooth structure so that the $G$-action on $G\times_H X$ is smooth. Similar for $G\times_H W$. Hence the equivariant Whitehead torsion $\tau_G(G\times_H W, G\times_H X)_{\mathrm{MS}}$ in the sense of Matumoto and Shiota [26] is defined.

We now claim the following

LEMMA 3.3. *The $G$-CW complex structure in Lemma 3.1 coincides with that of Matumoto and Shiota, and we have*

$$i_*\tau_H(W, X)=\tau_G(G\underset{H}{\times}W, G\underset{H}{\times}X)_{\mathrm{MS}}.$$

PROOF. In the following, the reader is referred to [12], [13], [26], [27], [32], [33]. T. Matumoto and M. Shiota defined the equivariant Whitehead torsion $\tau_G(G\times_H W, G\times_H X)_{\mathrm{MS}}$ by using a subanalytic triangulations of the orbit spaces $(G\times_H W)/G$ and $(G\times_H X)/G$. Notice that the orbit space $X/H$ is endowed with a canonical triangulation [15]. Concerning the induced $G$-CW complex structure on $G\times_H X$, the orbit space $(G\times_H X)/G$ is endowed with a canonical triangulation and we have a canonical isomorphism of simplicial complexes:

$$X/H\cong(G\underset{H}{\times}X)/G.$$

Hereafter we identify $X/H$ with $(G\times_H X)/G$ by this canonical isomorphism.

Recall that every Lie group $G$ has a unique real analytic structure. Moreover every smooth $G$-manifold $X$ is equivariantly diffeomorphic to a real analytic $G$-manifold. When both $G$ and $X$ are compact, such a real analytic $G$-manifold structure is unique.

According to [26], the smooth $G$-manifold $G\times_H X$ admits a $G$-CW complex structure which induces a subanalytic triangulation on the orbit space $(G\times_H X)/G$.

Consider the following commutative diagram

$$\begin{tikzcd} G\times X \arrow[r, "\pi_2"] \arrow[d, "\pi_1"] & X \arrow[d, "\pi_3"] \\ G\underset{H}{\times}X \arrow[r, "\pi_4"] & (G\underset{H}{\times}X)/G=X/H \end{tikzcd}$$

where $\pi_2$ is the projection to the second factor and the other $\pi_i$ are orbit maps.

Since $\pi_1: G\times X\to G\times_H X$ is a finite covering, there is a local analytic section for the projection $\pi_1$. Obviously $\pi_2$ is an analytic map. Moreover it is easy to see that the orbit space $X/H$ is a subanalytic set and the projection

$\pi_3: X \to X/H$ is a subanalytic map.

Putting all this together, we have that $\pi_4$ is locally a composite of two analytic maps and a subanalytic map. Notice that a map $f: A \to B$ between compact subanalytic sets $A$, $B$ is subanalytic if $f$ is locally subanalytic.

Since all the spaces in the diagram above are compact, we can conclude that the map $\pi_4$ is subanalytic.

Namely the triangulation of the orbit space $(G \times_H X)/G$ is nothing but the subanalytic triangulation of [**26**].

Thus we have shown that the induced $G$-CW complex structure on $G \times_H X$ gives the $G$-CW complex structure in the sense of [**26**].

Similar for $G \times_H W$ and the induced $G$-CW pair $(G \times_H W, G \times_H X)$ represents the equivariant Whitehead torsion $\tau_G(G \times_H W, G \times_H X)_{\mathrm{MS}}$ in the sense of Matumoto and Shiota [**26**].

This makes the proof of Lemma 3.3 complete.

COROLLARY 3.4. *If* $\tau_H(W, X)=0$, *then we have* $\tau_G(G \times_H W, G \times_H X)_{\mathrm{MS}}=0$.

PROOF. This is an immediate consequence of Lemmas 3.2 and 3.3.

## § 4. Construction of counterexamples.

We start by recalling a theorem of Sumners. Let $S^n$ and $B^n$ denote the $n$-sphere and the $n$-ball respectively. For a ball pair $(B^{n+3}, kB^{n+1})$, we denote by $\partial(B^{n+3}, kB^{n+1})$ the boundary sphere pair. Denote by $Z_p$ the cyclic group of order $p$. A manifold pair $(M, N)$ is said to *admit* a $Z_p$*-action* if there exists a semi-free $Z_p$-action on $M$ such that the fixed point set is $N$.

THEOREM OF SUMNERS [**41**]. *For each pair* $(n, p)$ *with* $n \geqq 2$ *and* $p \geqq 2$, *there are infinitely many knots* $(S^{n+2}, kS^n)$ *and ball pairs* $(B^{n+3}, kB^{n+1})$ *satisfying the following conditions*:

(i) $(S^{n+2}, kS^n)=\partial(B^{n+3}, kB^{n+1})$

(ii) $(B^{n+3}, kB^{n+1})$ *admit* $Z_p$*-actions.*

Let $(S^{n+2}, kS^n)=\partial(B^{n+3}, kB^{n+1})$ be one of the non-trivial knots in Theorem of Sumners (see also [**11**], [**8**]). Choose an arbitrary point $x$ from the interior of $kB^{n+1}$. Let $D(x)$ be a $Z_p$-invariant closed tubular neighbourhood of $x$ in $B^{n+3}$ satisfying

$$D(x) \subset \operatorname{Int} B^{n+3}$$

where $\operatorname{Int} B^{n+3}$ denotes the interior of $B^{n+3}$. Then we put

$$W_1 = B^{n+3} - \operatorname{Int} D(x), \qquad X_1 = \partial B^{n+3} = S^{n+2}, \qquad Y_1 = \partial D(x)$$

where $\partial B^{n+3}$ and $\partial D(x)$ denote the boundaries of $B^{n+3}$ and $D(x)$ respectively.

It follows from the uniqueness of tubular neighbourhoods that $W_1$ is diffeomorphic to $S^{n+2}\times I$ and the fixed point set $W_1^{Z_p}$ is diffeomorphic to $X_1^{Z_p}\times I=S^n\times I$. Hence the following inclusion maps

$$X_1 \longrightarrow W_1, \quad Y_1 \longrightarrow W_1, \quad X_1^{Z_p} \longrightarrow W_1^{Z_p}, \quad Y_1^{Z_p} \longrightarrow W_1^{Z_p}$$

are homotopy equivalences. Since $W_1$, $X_1$ and $Y_1$ have $Z_p$-triangulations [**15**], $X_1$ and $Y_1$ are $Z_p$-deformation retracts of $W_1$ by [**25**] and [**17**]. Namely the triad $(W_1; X_1, Y_1)$ is a $Z_p$-$h$-cobordism.

Next we consider the following triad

$$(W_2; X_2, Y_2)=(W_1; X_1, Y_1)\times S^{2k+1}=(W_1\times S^{2k+1}; X_1\times S^{2k+1}, Y_1\times S^{2k+1})$$

where $S^{2k+1}$ is the $(2k+1)$-sphere with trivial $Z_p$-action.

Let $G$ be an arbitrary compact Lie group including $Z_p$ as a subgroup. Finally we consider the following triad consisting of induced transformation groups

$$(W; X, Y)=G\underset{Z_p}{\times}(W_2; X_2, Y_2)=(G\underset{Z_p}{\times}W_2; G\underset{Z_p}{\times}X_2, G\underset{Z_p}{\times}Y_2).$$

Then we have

THEOREM 4.1. *The triad $(W; X, Y)$ is a $G$-$s$-cobordism such that $X$ is not $G$-homeomorphic to $Y$.*

PROOF. It follows from the product formula for equivariant Whitehead torsion [**16**] that the inclusion map $X_2\to W_2$ is a simple $Z_p$-homotopy equivalence. In another word, the triad $(W_2; X_2, Y_2)$ is a $Z_p$-$s$-cobordism. By virtue of Corollary 3.4, it follows that the triad $(W; X, Y)$ is a $G$-$s$-cobordism.

In the following we shall show that $X$ is not $G$-homeomorphic to $Y$. To see this, we suppose tentatively that there exists a $G$-homeomorphism $f: X\to Y$. Since $f$ is a $G$-homeomorphism, $f$ induces a homeomorphism

$$\bar{f}: X_2/Z_p \longrightarrow Y_2/Z_p$$

with

$$\bar{f}(X_2^{Z_p})=Y_2^{Z_p}$$

by Proposition 2.2. Consequently, we have a homeomorphism

$$\bar{f}_0: X_2/Z_p - X_2^{Z_p} \longrightarrow Y_2/Z_p - Y_2^{Z_p}.$$

Since $Z_p$ acts trivially on $S^{2k+1}$, there are canonical homeomorphisms

$$h_1: X_2/Z_p - X_2^{Z_p} \longrightarrow (X_1/Z_p - X_1^{Z_p})\times S^{2k+1}$$

$$h_2: Y_2/Z_p - Y_2^{Z_p} \longrightarrow (Y_1/Z_p - Y_1^{Z_p})\times S^{2k+1}.$$

Thus we get a homeomorphism

$$\varphi = h_2 \cdot \bar{f}_0 \cdot h_1^{-1} : (X_1/Z_p - X_1^{Z_p}) \times S^{2k+1} \longrightarrow (Y_1/Z_p - Y_1^{Z_p}) \times S^{2k+1} .$$

As a consequence, $\varphi$ induces an isomorphism

$$\varphi_* : \pi_i(X_1/Z_p - X_1^{Z_p}) \oplus \pi_i(S^{2k+1}) \longrightarrow \pi_i(Y_1/Z_p - Y_1^{Z_p}) \oplus \pi_i(S^{2k+1})$$

of homotopy groups. Since the $Z_p$-action on $Y_1$ is linear, one verifies easily that $Y_1/Z_p - Y_1^{Z_p}$ is homotopy equivalent to the circle $S^1$.

We now consider two cases.

In case $k=0$: When $i=1$, the isomorphism $\varphi_*$ above has the form

$$\varphi_* : \pi_1(X_1/Z_p - X_1^{Z_p}) \oplus Z \longrightarrow Z \oplus Z$$

where $Z$ denotes the group of integers. It follows from the fundamental theorem of abelian groups that $\pi_1(X_1/Z_p - X_1^{Z_p})$ is isomorphic to $Z$. When $i \geqq 2$, we have $\pi_i(Y_1/Z_p - Y_1^{Z_p}) \cong \pi_i(S^1) \cong 0$. Therefore we have $\pi_i(X_1/Z_p - X_1^{Z_p}) \cong 0$ for $i \geqq 2$. Note that $Z_p$ acts freely and smoothly on $X_1 - X_1^{Z_p}$. Hence we have the principal fiber bundle:

$$Z_p \longrightarrow X_1 - X_1^{Z_p} \longrightarrow X_1/Z_p - X_1^{Z_p} ,$$

which yields the following homotopy exact sequence

$$\cdots \longrightarrow \pi_i(Z_p) \longrightarrow \pi_i(X_1 - X_1^{Z_p}) \longrightarrow \pi_i(X_1/Z_p - X_1^{Z_p}) \longrightarrow \cdots .$$

As a consequence, we have isomorphisms

$$\pi_i(X_1 - X_1^{Z_p}) \cong \begin{cases} Z & \text{for } i=1 \\ 0 & \text{for } i \geqq 2 . \end{cases}$$

Obviously $X_1 - X_1^{Z_p}$ has the homotopy type of a finite CW complex. Thus we can conclude that $X_1 - X_1^{Z_p}$ is homotopy equivalent to the circle $S^1$ by the theorem of J. H. C. Whitehead [**44**]. But this contradicts the choice of the knot $(X_1, X_1^{Z_p}) = (S^{n+2}, kS^n)$ [**41**] (see also [**40**], [**24**], [**37**]).

In case $k \geqq 1$: When $i=1$, the isomorphism $\varphi_*$ above has the form

$$\varphi_* : \pi_1(X_1/Z_p - X_1^{Z_p}) \longrightarrow \pi_1(Y_1/Z_p - Y_1^{Z_p}) \cong Z .$$

Since $\pi_i(Y_1/Z_p - Y_1^{Z_p}) \cong \pi_i(S^1) \cong 0$ for $i \geqq 2$, we have an isomorphism

$$\varphi_* : \pi_i(X_1/Z_p - X_1^{Z_p}) \oplus \pi_i(S^{2k+1}) \longrightarrow \pi_i(S^{2k+1}) \qquad \text{for } i \geqq 2 .$$

It follows from Serre [**31**] that $\pi_i(S^{2k+1})$ is finitely generated. Therefore by the isomorphism $\varphi_*$ above $\pi_i(X_1/Z_p - X_1^{Z_p})$ is a subgroup of a finitely generated abelian group. It is well-known that a subgroup of a finitely generated abelian group is also a finitely generated abelian group. Hence we can apply the fundamental theorem of abelian groups and conclude that

$$\pi_i(X_1/Z_p - X_1^{Z_p}) \cong 0 \qquad \text{for } i \geqq 2\,.$$

By making use of the homotopy exact sequence above, we have again isomorphisms

$$\pi_i(X_1 - X_1^{Z_p}) \cong \begin{cases} Z & \text{for } i=1 \\ 0 & \text{for } i \geqq 2\,. \end{cases}$$

Thus we can conclude that $X_1 - X_1^{Z_p}$ is homotopy equivalent to the circle $S^1$ in this case too. But this contradicts also the choice of the knot $(X_1, X_1^{Z_p}) = (S^{n+2}, kS^n)$ [**41**].

This makes the proof of Theorem 4.1 complete.

PROOF OF THEOREM 1.2. Let $G$ be an arbitrary non-trivial compact Lie group. When $G$ is of positive dimension, there is a maximal torus $T^i$ of positive dimension. Hence an arbitrary cyclic group $Z_p$ is a subgroup of $G$. When $G$ is a finite group, there is a cyclic subgroup $Z_p$ of $G$ with $p \geqq 2$. Thus for an arbitrary non-trivial compact Lie group $G$, there is a cyclic subgroup $Z_p$ of $G$ with $p \geqq 2$. Therefore Theorem 4.1 yields Theorem 1.2.

## § 5. Concluding remarks.

Stable equivalence of $G$-manifolds is discussed in [**18**], [**1**]. If we stabilize a $G$-$s$-cobordism with respect to spheres or disks of suitable $G$-representation spaces, then the conditions (*1) and (*2) are automatically satisfied and we have a stable $G$-$s$-cobordism theorem.

On the other hand, it follows from the product formula for equivariant Whitehead torsion [**16**] that any $G$-$h$-cobordism can be altered into a $G$-$s$-cobordism by multiplying it by the unit sphere $S(V)$ of an arbitrary unitary complex representation space $V$ of $G$ in the case where $G$ is finite. It turns out that if we make use of the unit sphere $S(V)$ of a suitable unitary representation space $V$ of $Z_p$ instead of the sphere $S^{2k+1}$ with trivial action in § 4, then the $G$-$s$-cobordism theorem holds.

In the following we shall give such an example. Let $(W_1; X_1, Y_1)$ be the $Z_p$-$h$-cobordism in § 4 with $\dim W_1 \geqq 5$. Denote by $V$ a unitary representation space of $Z_p$ such that $Z_p$ acts freely on the unit sphere $S(V)$. Let $G$ be a compact Lie group including $Z_p$ as a subgroup. Then the $G$-$s$-cobordism theorem holds for the triad $G \times_{Z_p} (W_1; X_1, Y_1) \times S(V)$ by Theorem 1.1. In particular $G \times_{Z_p} X_1 \times S(V)$ is $G$-diffeomorphic to $G \times_{Z_p} Y_1 \times S(V)$.

This example shows that it is essential to show in the proof of Theorem 4.1 that $G \times_{Z_p} X_1 \times S^{2k+1}$ is not $G$-homeomorphic to $G \times_{Z_p} Y_1 \times S^{2k+1}$, even if they have factors of $S^{2k+1}$.

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