

## Existence of positive entire solutions for higher order quasilinear elliptic equations

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### 1. Introduction.

This paper is concerned with the problem of existence of positive entire solutions for the  $2M$ -th order quasilinear elliptic equation

$$(1.1) \quad (-\Delta)^M u = f(x, u, -\Delta u, \dots, (-\Delta)^{M-1} u), \quad x \in \mathbf{R}^N,$$

where  $M \geq 2$ ,  $N \geq 2$ ,  $\Delta$  is the  $N$ -dimensional Laplace operator and  $f \in C_{loc}^\theta(\mathbf{R}^N \times \mathbf{R}^M)$ ,  $0 < \theta < 1$ . An entire solution of (1.1) is defined to be a function  $u$  which is of class  $C^{2M}(\mathbf{R}^N)$  and satisfies (1.1) at every point of  $\mathbf{R}^N$ .

Beginning with Kusano and Swanson [10], several authors have developed existence theory of *radial* entire solutions for higher order elliptic equations of the type (1.1) with radial symmetry; see e. g. the papers [1, 2, 6-10]. A natural question then arises: Is it possible to construct *non-radial* entire solutions of the equation (1.1) without radial symmetry? An answer to this question has been given by Edelson and Vakilian [3] and Kusano and Swanson [11], who have examined the equation  $(-\Delta)^M u = f(x, u)$  by employing entirely different methods. A principal tool used in the paper [11] is an extension of the supersolution-subsolution method (super-subsolution method for short) which has proved to be very powerful in establishing the existence of entire solutions for second order elliptic equations of the form  $-\Delta u = f(x, u)$ . Such an extension in [11] relies on the derivation of a super-subsolution principle holding for second order elliptic systems of the form

$$(1.2) \quad -\Delta u_i = f_i(x, u_1, \dots, u_M), \quad x \in \mathbf{R}^N, \quad i = 1, 2, \dots, M.$$

It will be natural to expect that the super-subsolution principle for (1.2) given in [11] could be generalized so as to give rise to a new super-subsolution method for constructing non-radial entire solution of (1.1), thereby generalizing considerably the results of [3] and [11]. The purpose of this paper is to verify

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the truth of this expectation by showing that the desired generalization can be made possible with the aid of a super-subsolution principle for (1.2) recently established by Furusho [5]. The statement of our super-subsolution principle for (1.1) is given in Section 2. The main existence results for (1.1) are stated and proved in Sections 3 and 4; Section 3 concerns entire solutions which are bounded from above and below by positive constants, while Section 4 deals with positive entire solutions which decay to zero as  $x$  tends to infinity. Section 5 examines the equation

$$(1.3) \quad (-\Delta)^M u = \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i u]^{\gamma_i}, \quad x \in \mathbf{R}^N, \quad N \geq 2,$$

which is an important special case of (1.1), where the  $\gamma_i$  are constants and the  $p_i(x)$  are functions of class  $C_{loc}^\theta(\mathbf{R}^N)$  with  $\theta \in (0, 1)$ . The existence theorems of Sections 3 and 4 are applicable to the case of positive  $\gamma_i$  but not to the case of negative  $\gamma_i$ . It is shown that the latter case of (1.3) can also be handled in the framework of the existence theory of Section 2.

**2. The supersolution-subsolution principle.**

The purpose of this section is to develop a general principle, called the super-subsolution method, by means of which the existence of non-radial entire solutions can be assured for the higher order elliptic equation (1.1). The derivation of the desired principle is based on the following simple observation: If  $u \in C_{loc}^{2M+\theta}(\mathbf{R}^N)$  is an entire solution of (1.1), then the vector function  $(u_1, \dots, u_M)$  with  $u_i = (-\Delta)^{i-1} u$ ,  $i=1, 2, \dots, M$ , is of class  $C_{loc}^{2+\theta}(\mathbf{R}^N, \mathbf{R}^M)$  and satisfies the elliptic system

$$(2.1) \quad \begin{cases} -\Delta u_i = u_{i+1}, & i=1, \dots, M-1, \\ -\Delta u_M = f(x, u_1, \dots, u_M), & x \in \mathbf{R}^N. \end{cases}$$

Conversely, if  $(u_1, u_2, \dots, u_M) \in C_{loc}^{2+\theta}(\mathbf{R}^N, \mathbf{R}^M)$  is a solution of (2.1), then the first component  $u=u_1$  is an entire solution of (1.1).

Very recently, Furusho [5, Theorem 3.1] has obtained a general super-subsolution principle for second order elliptic systems which, when specialized to (1.2), implies the following statement.

**THEOREM 2.0.** *Let  $f_i(x, \xi_1, \dots, \xi_M)$ ,  $i=1, 2, \dots, M$ , be functions of class  $C_{loc}^\theta(\mathbf{R}^N, \mathbf{R}^M)$ ,  $0 < \theta < 1$ , and suppose that there exist vector functions  $(v_1, \dots, v_M)$  and  $(w_1, \dots, w_M)$  of class  $C_{loc}^{2+\theta}(\mathbf{R}^N, \mathbf{R}^M)$  such that  $w_i \leq v_i$  in  $\mathbf{R}^N$ ,  $i=1, \dots, M$ ,*

$$-\Delta v_i(x) \geq f_i(x, \sigma_1, \dots, \sigma_M), \quad x \in \mathbf{R}^N, \quad i=1, \dots, M,$$

*for any  $(\sigma_1, \dots, \sigma_M) \in \mathbf{R}^M$  satisfying  $w_j(x) \leq \sigma_j \leq v_j(x)$ ,  $j \neq i$ ,  $\sigma_i = v_i(x)$ , and*

$$-\Delta w_i(x) \leq f_i(x, \tau_1, \dots, \tau_M), \quad x \in \mathbf{R}^N, \quad i=1, \dots, M,$$

for any  $(\tau_1, \dots, \tau_M) \in \mathbf{R}^M$  satisfying  $w_j(x) \leq \tau_j \leq v_j(x)$ ,  $j \neq i$ ,  $\tau_i = w_i(x)$ . Then, the system (1.2) has an entire solution  $(u_1, \dots, u_M) \in C_{\text{loc}}^{2M+\theta}(\mathbf{R}^N, \mathbf{R}^M)$  such that  $w_i \leq u_i \leq v_i$  in  $\mathbf{R}^N$ ,  $i=1, \dots, M$ .

Applying this theorem to the system (2.1), we have a super-subsolution principle for the equation (1.1) on which most of the development of this paper depends.

**THEOREM 2.1.** *If there exists a pair of functions  $v$  and  $w$  of class  $C_{\text{loc}}^{2M+\theta}(\mathbf{R}^N)$  such that*

$$(2.2) \quad (-\Delta)^i w(x) \leq (-\Delta)^i v(x), \quad x \in \mathbf{R}^N, \quad i=0, 1, \dots, M-1,$$

and

$$(2.3) \quad (-\Delta)^M v(x) \geq f(x, \sigma_0, \sigma_1, \dots, \sigma_{M-1}) \geq (-\Delta)^M w(x),$$

for any  $(\sigma_0, \sigma_1, \dots, \sigma_{M-1}) \in \mathbf{R}^M$  with  $(-\Delta)^i w(x) \leq \sigma_i \leq (-\Delta)^i v(x)$ ,  $i=0, 1, \dots, M-1$ , at every fixed point  $x \in \mathbf{R}^N$ , then the equation (1.1) has an entire solution  $u \in C_{\text{loc}}^{2M+\theta}(\mathbf{R}^N)$  satisfying

$$(2.4) \quad (-\Delta)^i w(x) \leq (-\Delta)^i u(x) \leq (-\Delta)^i v(x), \quad x \in \mathbf{R}^N, \quad i=0, 1, \dots, M-1.$$

The functions  $v$  and  $w$  in Theorem 2.1 are said to be a *supersolution* and a *subsolution* of (1.1), respectively. Theorem 2.1 asserts that the existence of a supersolution and a subsolution of (1.1) guarantees the existence of an entire solution of the equation under consideration.

As is easily seen, the statement of Theorem 2.1 becomes much simpler when the function  $f(x, \xi_1, \dots, \xi_M)$  in (1.1) is monotone in the variables  $\xi_j$ ,  $j=1, 2, \dots, M$ .

**COROLLARY 2.1.** *Let  $f(x, \xi_1, \dots, \xi_M)$  be nondecreasing in  $\xi_j$ ,  $j=1, 2, \dots, M$ . If there exists a pair of functions  $v$  and  $w$  in  $C_{\text{loc}}^{2M+\theta}(\mathbf{R}^N)$  which satisfy (2.2),*

$$(-\Delta)^M v(x) \geq f(x, v(x), -\Delta v(x), \dots, (-\Delta)^{M-1} v(x)), \quad x \in \mathbf{R}^N$$

and

$$(-\Delta)^M w(x) \leq f(x, w(x), -\Delta w(x), \dots, (-\Delta)^{M-1} w(x)), \quad x \in \mathbf{R}^N,$$

then the equation (1.1) has an entire solution  $u$  satisfying (2.4).

**COROLLARY 2.2.** *Let  $f(x, \xi_1, \dots, \xi_M)$  be nonincreasing in  $\xi_j$ ,  $j=1, 2, \dots, M$ . If there exists a pair of functions  $v$  and  $w$  in  $C_{\text{loc}}^{2M+\theta}(\mathbf{R}^N)$  which satisfy (2.2),*

$$(-\Delta)^M v(x) \geq f(x, w(x), -\Delta w(x), \dots, (-\Delta)^{M-1} w(x)), \quad x \in \mathbf{R}^N$$

and

$$(-\Delta)^M w(x) \leq f(x, v(x), -\Delta v(x), \dots, (-\Delta)^{M-1} v(x)), \quad x \in \mathbf{R}^N,$$

then the equation (1.1) has an entire solution  $u$  satisfying (2.4).

Corollary 2.1 was first proved by Kusano and Swanson [11, Theorem 2.1] under slightly more restrictive assumptions on  $f(x, \xi_1, \dots, \xi_M)$ . Corollary 2.2 seems to be new. An example of equations to which Corollary 2.2 applies is the equation (1.3) in which all the functions  $p_i(x)$  are nonnegative and all the exponents  $\gamma_i$  are nonpositive.

**COROLLARY 2.3.** *Let  $p_i \in C_{loc}^\theta(\mathbf{R}^N)$ ,  $p_i \geq 0$ ,  $\gamma_i \geq 0$ ,  $i=0, 1, \dots, M-1$ . If there exists a pair of functions  $v$  and  $w$  of class  $C_{loc}^{2M+\theta}(\mathbf{R}^N)$  such that*

$$0 < (-\Delta)^i w(x) \leq (-\Delta)^i v(x), \quad x \in \mathbf{R}^N, \quad i=0, 1, \dots, M-1,$$

$$(-\Delta)^M v(x) \geq \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i w(x)]^{-\gamma_i}, \quad x \in \mathbf{R}^N$$

and

$$(-\Delta)^M w(x) \leq \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i v(x)]^{-\gamma_i}, \quad x \in \mathbf{R}^N,$$

then the equation

$$(-\Delta)^M u = \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i u]^{-\gamma_i}, \quad x \in \mathbf{R}^N$$

has an entire solution  $u$  satisfying (2.4).

A crucial step in applying the above-mentioned principles is the detection or construction of supersolutions and subsolutions of the equations under study. In what follows super- and subsolutions will always be sought in the form of radial functions  $y(|x|)$  with  $y(t)$  satisfying ordinary differential equations of the type

$$(2.5) \quad (-\Delta)^M y = g(t, y, -\Delta y, \dots, (-\Delta)^{M-1} y), \quad t \geq 0,$$

where the functions  $g$  are chosen to dominate  $f$  or to be dominated by  $f$  in a certain sense, and  $\Delta$  is understood to mean its one-dimensional polar form  $t^{1-N} \cdot d/dt(t^{N-1}d/dt)$ . In solving (2.5) a central role will be played by the integral operator  $\Psi$  defined by

$$(2.6) \quad (\Psi h)(t) = \begin{cases} t^{2-N} \int_0^t (s^{N-3} \int_s^\infty r h(r) dr) ds, & \text{for } t > 0, \\ \frac{1}{N-2} \int_0^\infty s h(s) ds, & \text{for } t = 0. \end{cases}$$

It is known [4] that  $\Psi$  has the properties:

$$(2.7) \quad -\Delta(\Psi h)(t) = h(t), \quad t \geq 0 \quad \text{and} \quad \lim_{t \rightarrow \infty} \Psi h(t) = 0$$

for every functions  $h \in C(\bar{\mathbf{R}}_+)$ ,  $\bar{\mathbf{R}}_+ = [0, \infty)$ , such that  $\int_0^\infty s |h(s)| ds < \infty$ . The first relation of (2.7) says that  $-\Psi$  is a kind of inverse of the one-dimensional polar

form of the Laplace operator. Additional important properties of  $\Psi$  which will be needed in the subsequent sections are summarized in the following lemma.

Let us denote by  $L^1_\lambda(\bar{\mathbf{R}}_+)$ ,  $\lambda > 0$ , the set of all real valued measurable functions  $h$  on  $\bar{\mathbf{R}}_+$  such that

$$\int_0^\infty s^\lambda |h(s)| ds < \infty.$$

LEMMA 2.1. *Suppose that  $N \geq 2M + 1$  and let  $j \in \{1, \dots, M\}$ . Then, the  $j$ -th iterate  $\Psi^j$  of  $\Psi$  maps  $C(\bar{\mathbf{R}}_+) \cap L^1_{2j-1}(\bar{\mathbf{R}}_+)$  into  $C^{2j}(\bar{\mathbf{R}}_+)$  and has the following properties.*

(i) *If  $h \in C(\bar{\mathbf{R}}_+) \cap L^1_{2j-1}(\bar{\mathbf{R}}_+)$  is nonnegative on  $\bar{\mathbf{R}}_+$ , then*

$$\begin{aligned} I_1(N, j) \int_0^\infty \min\{s, s^{N-1}\} h(s) ds \cdot q_{N,j}(t) &\leq \Psi^j h(t) \\ &\leq I_2(N, j) \int_0^\infty s^{2j-1} h(s) ds, \quad t \geq 0, \end{aligned}$$

where  $q_{N,j}(t) = \min\{1, t^{2j-N}\}$ ,  $I_1(N, j) = [(N-2)^j(N-4) \cdots (N-2j)]^{-1}$  and  $I_2(N, j) = [2^j(j-1)!(N-2)(N-4) \cdots (N-2j)]^{-1}$ .

(ii) *If  $h \in C(\bar{\mathbf{R}}_+) \cap L^1_{N-1}(\bar{\mathbf{R}}_+)$  is nonnegative on  $\bar{\mathbf{R}}_+$ , then*

$$\begin{aligned} I_1(N, j) \int_0^\infty \min\{s, s^{N-1}\} h(s) ds \cdot q_{N,j}(t) &\leq \Psi^j h(t) \\ &\leq J_2(N, j) \int_0^\infty \max\{s, s^{N-1}\} h(s) ds \cdot q_{N,j}(t), \quad t \geq 0, \end{aligned}$$

where  $I_1(N, j)$  is as above and  $J_2(N, j) = [2^j(N-2)(N-4) \cdots (N-2j)]^{-1}$ .

For the proof of this lemma see Kusano, Naito and Swanson [9].

### 3. Uniformly positive bounded entire solutions.

Our purpose here is to apply the super-subsolution principle mentioned in the preceding section to construct positive entire solutions of the equation (1.1) which tend to positive constants as  $x$  tends to infinity. The structure hypotheses required for (1.1) are as follows.

(F<sub>1</sub>)  $f \in C^0_{\text{loc}}(\mathbf{R}^N \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{M-1})$ ,  $0 < \theta < 1$ , and there is a nonnegative function  $F(t, \xi_1, \dots, \xi_M)$  in  $C^0_{\text{loc}}(\bar{\mathbf{R}}_+^{M+1})$  which is nondecreasing in  $\xi_i$ ,  $i=1, \dots, M$ , and satisfies  $|f(x, \xi_1, \xi_2, \dots, \xi_M)| \leq F(|x|, \xi_1, |\xi_2|, \dots, |\xi_M|)$  for  $(x, \xi_1, \xi_2, \dots, \xi_M) \in \mathbf{R}^N \times \bar{\mathbf{R}}_+ \times \mathbf{R}^{M-1}$ .

(F<sub>2</sub>) (Superlinearity.) For any fixed  $(t, \xi_1, \xi_2, \dots, \xi_M) \in \bar{\mathbf{R}}_+^{M+1}$ ,  $F(t, \lambda\xi_1, \lambda\xi_2, \dots, \lambda\xi_M)/\lambda$  is nondecreasing in  $\lambda > 0$  and

$$\lim_{\lambda \rightarrow +0} \frac{F(t, \lambda\xi_1, \lambda\xi_2, \dots, \lambda\xi_M)}{\lambda} = 0.$$

(F<sub>3</sub>) (Sublinearity.) For any fixed  $(t, \xi_1, \xi_2, \dots, \xi_M) \in \bar{\mathbf{R}}_+^{M+1}$ ,  $F(t, \lambda\xi_1, \lambda\xi_2, \dots, \lambda\xi_M)/\lambda$  is nonincreasing in  $\lambda > 0$  and

$$\lim_{\lambda \rightarrow \infty} \frac{F(t, \lambda\xi_1, \lambda\xi_2, \dots, \lambda\xi_M)}{\lambda} = 0.$$

THEOREM 3.1. Let  $N \geq 2M+1$  and suppose that either  $\{(F_1), (F_2)\}$  or  $\{(F_1), (F_3)\}$  is satisfied. If

$$(3.1) \quad \int_0^\infty t^{2M-1} F(t, c, \dots, c) dt < \infty$$

for some constant  $c > 0$ , then there exist infinitely many positive entire solutions  $u(x)$  of (1.1) such that

$$(3.2) \quad \begin{cases} \lim_{|x| \rightarrow \infty} u(x) = \text{constant} > 0, \\ \lim_{|x| \rightarrow \infty} (-\Delta)^j u(x) = 0, \quad j = 1, 2, \dots, M-1. \end{cases}$$

PROOF. The conclusion follows from Theorem 2.1 if infinitely many values of  $k > 0$  are found such that, for every such  $k$ , there exist a supersolution  $v_k$  and a subsolution  $w_k$  of (1.1) satisfying

$$(3.3) \quad \lim_{|x| \rightarrow \infty} v_k(x) = \lim_{|x| \rightarrow \infty} w_k(x) = k.$$

Since the condition (3.1) implies, via the Lebesgue dominated convergence theorem, that

$$\lim_{k \rightarrow * } k^{-1} \int_0^\infty t^{2j-1} F(t, 2k, \dots, 2k) dt = 0, \quad j = 1, \dots, M,$$

where  $* = 0$  or  $\infty$  according as (F<sub>2</sub>) or (F<sub>3</sub>) holds, there exists an interval  $I$ , which is of the form  $(0, k_0)$  in case (F<sub>2</sub>) holds and is of the form  $(k_0, \infty)$  in case (F<sub>3</sub>) holds, such that

$$(3.4) \quad \begin{cases} I_2(N, M) \int_0^\infty t^{2M-1} F(t, 2k, \dots, 2k) dt < k, \\ I_2(N, j) \int_0^\infty t^{2j-1} F(t, 2k, \dots, 2k) dt < 2k, \quad j = 1, 2, \dots, M-1. \end{cases}$$

for every  $k \in I$ , where  $I_2(N, j)$  is as in (i) of Lemma 2.1.

Let  $\mathcal{C} = C^{2M-2}(\bar{\mathbf{R}}_+)$  be the Fréchet space with the topology induced by the semi-norms

$$\|y\|_n = \sum_{i=0}^{M-1} \max_{0 \leq t \leq n} |(-\Delta)^i y(t)|, \quad n = 1, 2, \dots.$$

where

$$-\Delta y(t) = -t^{1-N} \left( \frac{d}{dt} t^{N-1} \frac{d}{dt} y \right)(t),$$

and consider a closed convex subset  $Y_k$ ,  $k \in I$ , of  $C$  defined by

$$Y_k = \{y \in C : k \leq y(t) \leq 2k, 0 \leq (-\Delta)^i y(t) \leq 2k, t \geq 0, i = 1, 2, \dots, M-1\}.$$

It can be shown that the mapping  $\mathcal{F}_k$  defined by

$$\mathcal{F}_k y(t) = k + [\Psi^M F(\cdot, y, -\Delta y, \dots, (-\Delta)^{M-1} y)](t), \quad t \geq 0$$

has a fixed point in  $Y_k$  with the aid of the Schauder-Tychonoff fixed point theorem. To do this it suffices to verify that (i)  $\mathcal{F}_k(Y_k) \subset Y_k$ ; (ii)  $\mathcal{F}_k$  is continuous in the  $C$ -topology; and (iii)  $\mathcal{F}_k(Y_k)$  is relatively compact in  $C$ .

(i) If  $y \in Y_k$ , then from (F<sub>1</sub>), (3.4) and (i) of Lemma 2.1 it follows that

$$k \leq \mathcal{F}_k y(t) \leq k + [\Psi^M F(\cdot, 2k, \dots, 2k)](t) < 2k, \quad t \geq 0,$$

and

$$\begin{aligned} 0 &\leq (-\Delta)^i (\mathcal{F}_k y)(t) = [\Psi^{M-i} F(\cdot, y, -\Delta y, \dots, (-\Delta)^{M-1} y)](t) \\ &\leq [\Psi^{M-i} F(\cdot, 2k, \dots, 2k)](t) < 2k, \quad t \geq 0, i = 1, 2, \dots, M-1. \end{aligned}$$

This implies that  $\mathcal{F}_k(Y_k) \subset Y_k$ .

(ii) Let  $\{y_\nu\}$  be a sequence of elements of  $Y_k$  converging to  $y \in Y_k$  in  $C$  and introduce the abbreviations

$$\begin{aligned} g_\nu(t) &= F(t, y_\nu(t), -\Delta y_\nu(t), \dots, (-\Delta)^{M-1} y_\nu(t)), \quad \nu = 1, 2, \dots, \\ g(t) &= F(t, y(t), -\Delta y(t), \dots, (-\Delta)^{M-1} y(t)). \end{aligned}$$

Then,  $\{g_\nu(t)\}$  converges to  $g(t)$  locally uniformly in  $\bar{R}_+$ , and

$$0 \leq g_\nu(t), g(t) \leq F(t, 2k, \dots, 2k), \quad t \geq 0, \nu = 1, 2, \dots.$$

In view of (3.1) the Lebesgue dominated convergence theorem implies that the sequence  $\{\Psi g_\nu(t)\}$  converges to  $\Psi g(t)$  locally uniformly in  $\bar{R}_+$ . Since

$$0 \leq \Psi g_\nu(t), \Psi g(t) \leq [\Psi F(\cdot, 2k, \dots, 2k)](t), \quad t \geq 0, \nu = 1, 2, \dots,$$

a similar argument shows that  $\{\Psi^2 g_\nu(t)\}$  converges to  $\Psi^2 g(t)$  locally uniformly in  $\bar{R}_+$ . Repetition of this procedure leads to the conclusion that, for every  $j=0, 1, \dots, M-1$ , the sequence  $\{\Psi^j g_\nu(t)\}$  converges to  $\Psi^j g(t)$  uniformly on any compact subinterval of  $\bar{R}_+$ . Since

$$(-\Delta)^i \mathcal{F}_k y_\nu(t) - (-\Delta)^i \mathcal{F}_k y(t) = \Psi^{M-i} g_\nu(t) - \Psi^{M-i} g(t), \quad i = 0, 1, \dots, M-1,$$

it follows that  $\{\mathcal{F}_k y_\nu\}$  converges to  $\mathcal{F}_k y$  in the topology of  $C$ . This proves the continuity of the mapping  $\mathcal{F}_k$ .

(iii) Define the sets

$$(-\Delta)^i \mathcal{F}_k(Y_k) = \{(-\Delta)^i \mathcal{F}_k y : y \in Y_k\}, \quad i = 0, 1, \dots, M-1.$$

All of these sets are uniformly bounded in  $\bar{\mathbf{R}}_+$ , since the relation  $\mathcal{F}_k(Y_k) \subset Y_k$  implies  $0 \leq [(-\Delta)^i \mathcal{F}_k y](t) \leq 2k, t \geq 0, i = 0, 1, \dots, M-1$ , for  $y \in Y_k$ . These sets are locally equicontinuous in  $\bar{\mathbf{R}}_+$ , since if  $y \in Y_k$ , then for any fixed  $T > 0$

$$\begin{aligned} | [(-\Delta)^i \mathcal{F}_k y]'(t) | &= | (\Psi^{M-i} g)'(t) | = \left| \frac{1}{t^{N-1}} \int_0^t s^{N-1} (\Psi^{M-i-1} g)(s) ds \right| \\ &\leq K_i T, \quad t \in [0, T], i = 0, 1, \dots, M-1, \end{aligned}$$

where  $' = d/dt$  and

$$K_i = 2k \quad \text{for } i = 0, 1, \dots, M-2; K_{M-1} = \max_{0 \leq t \leq T} F(t, 2k, \dots, 2k).$$

The relative compactness of  $\mathcal{F}_k(Y_k)$  in  $\mathcal{C}$  then follows from the Ascoli-Arzelà theorem.

Therefore there exists an element  $y_k \in Y_k$  such that  $y_k = \mathcal{F}_k y_k$ . It is easy to see that  $y_k$  is a solution of the differential equation

$$(3.5) \quad (-\Delta)^M y(t) = F(t, y(t), -\Delta y(t), \dots, (-\Delta)^{M-1} y(t)), \quad t > 0,$$

and satisfies

$$(3.6) \quad \begin{cases} k \leq y_k(t) < 2k, 0 \leq (-\Delta)^i y_k(t) < 2k, & t \geq 0, i = 1, 2, \dots, M-1, \\ \lim_{t \rightarrow \infty} y_k(t) = k, \lim_{t \rightarrow \infty} (-\Delta)^i y_k(t) = 0, & i = 1, 2, \dots, M-1. \end{cases}$$

Now define the functions  $v_k(x)$  and  $w_k(x)$  by

$$(3.7) \quad v_k(x) = y_k(|x|), w_k(x) = 2k - v_k(x), \quad x \in \mathbf{R}^N.$$

Then,  $v_k(x)$  and  $w_k(x)$  are of class  $C_{loc}^{2M+\theta}(\mathbf{R}^N)$  by the standard elliptic regularity and satisfy (3.3) and

$$(3.8) \quad 0 < w_k(x) < v_k(x), \quad (-\Delta)^i w_k(x) \leq 0 \leq (-\Delta)^i v_k(x), \\ x \in \mathbf{R}^N, i = 1, 2, \dots, M-1.$$

Furthermore, if  $x \in \mathbf{R}^N$  and if  $(\sigma_0, \dots, \sigma_{M-1}) \in \mathbf{R}^M$  is any vector such that  $(-\Delta)^i w_k(x) \leq \sigma_i \leq (-\Delta)^i v_k(x), i = 0, 1, \dots, M-1$ , then by (3.5) and (F<sub>1</sub>)

$$\begin{aligned} (-\Delta)^M v_k(x) &= F(|x|, v_k(x), -\Delta v_k(x), \dots, (-\Delta)^{M-1} v_k(x)) \\ &\geq F(|x|, |\sigma_0|, |\sigma_1|, \dots, |\sigma_{M-1}|) \geq f(x, \sigma_0, \sigma_1, \dots, \sigma_{M-1}) \end{aligned}$$

and

$$\begin{aligned} (-\Delta)^M w_k(x) &= -F(|x|, v_k(x), -\Delta v_k(x), \dots, (-\Delta)^{M-1} v_k(x)) \\ &\leq -F(|x|, |\sigma_0|, |\sigma_1|, \dots, |\sigma_{M-1}|) \leq f(x, \sigma_0, \sigma_1, \dots, \sigma_{M-1}), \end{aligned}$$



which implies that  $v_k(x)$  and  $w_k(x)$  are, respectively, a supersolution and a sub-solution of (1.1). From Theorem 2.1 it follows that (1.1) has an entire solution  $u(x)$  such that

$$(-\Delta)^i w_k(x) \leq (-\Delta)^i u(x) \leq (-\Delta)^i v_k(x), \quad x \in \mathbf{R}^N, i = 0, 1, \dots, M-1.$$

From (3.6) it is obvious that  $u(x)$  has the desired property (3.2). This completes the proof.

**4. Decaying positive entire solutions.**

This section is devoted to the construction of positive decaying entire solutions of (1.1) by means of the super-subsolution method. The following hypothesis is needed for this purpose.

(F<sub>4</sub>)  $f \in C_{loc}^0(\mathbf{R}^N \times \bar{\mathbf{R}}_+^M)$ ,  $\theta \in (0, 1)$ ;  $f$  is nonnegative and satisfies

$$(4.1) \quad \varphi(|x|, \xi_1) \leq f(x, \xi_1, \dots, \xi_M) \leq F(|x|, \xi_1, \dots, \xi_M) \\ \text{for } (x, \xi_1, \dots, \xi_M) \in \mathbf{R}^N \times \bar{\mathbf{R}}_+^M,$$

where  $\varphi \in C_{loc}^0(\bar{\mathbf{R}}_+^2)$  and  $F \in C_{loc}^0(\bar{\mathbf{R}}_+^{M+1})$  are nonnegative functions such that  $\varphi(t, \xi_1)$  is nondecreasing in  $\xi_1$  and  $F(t, \xi_1, \dots, \xi_M)$  is nondecreasing in  $\xi_j, j=1, \dots, M$ . In addition  $F$  satisfies (F<sub>3</sub>) and  $\varphi$  satisfies

$$(4.2) \quad \lim_{\lambda \rightarrow +0} \frac{\varphi(t, \lambda \xi_1)}{\lambda} = \infty$$

for any  $t$  in some subinterval of  $\mathbf{R}_+$  and for any  $\xi_1 > 0$ .

**THEOREM 4.1.** *Let  $N \geq 2M+1$  and (F<sub>4</sub>) hold. If  $F$  satisfies (3.1) for some  $c > 0$ , then there exists an entire solution  $u(x)$  of (1.1) such that*

$$(4.3) \quad \begin{cases} (-\Delta)^i u(x) > 0, & x \in \mathbf{R}^N, \\ \lim_{|x| \rightarrow \infty} (-\Delta)^i u(x) = 0, & i = 0, \dots, M-1. \end{cases}$$

**PROOF.** Choose a positive constant  $k > 1$  so large that

$$(4.4) \quad I_2(N, j) \int_0^\infty \max\{s, s^{2j-1}\} F(s, k, \dots, k) ds \leq k, \quad j = 1, \dots, M,$$

$$(4.5) \quad I_1(N, j) \int_0^\infty \min\{s, s^{N-1}\} \varphi(s, k^{-1} q_{N, M}(s)) ds \geq k^{-1}, \quad j = 1, \dots, M,$$

where  $I_1(N, j)$ ,  $I_2(N, j)$  and  $q_{N, M}(t)$  are as in Lemma 2.1. Such a choice of  $k$  is indeed possible, since (4.4) follows from (3.1), (F<sub>3</sub>) and the Lebesgue convergence theorem, and (4.5) follows from (4.2) and Fatou's lemma.

Let  $\mathcal{C} = C^{2M-2}(\bar{\mathbf{R}}_+)$  be as in the proof of Theorem 3.1 and consider the mapping

$$\mathcal{G}z(t) = [\Psi^M F(\cdot, z, -\Delta z, \dots, (-\Delta)^{M-1}z)](t), \quad t \geq 0,$$

on the set

$$Z_1 = \{z \in \mathcal{C} : k^{-1}q_{N, M-i}(t) \leq (-\Delta)^i z(t) \leq k, t \geq 0, i = 0, 1, \dots, M-1\}$$

which is a closed convex subset of  $\mathcal{C}$ . If  $z \in Z_1$ , then as in the proof of Theorem 3.1 one sees by (4.4) that

$$0 \leq (-\Delta)^i(\mathcal{G}z)(t) \leq k, \quad t \geq 0, i = 0, 1, \dots, M-1,$$

and obtains, by using (F<sub>4</sub>), (ii) of Lemma 2.1 and (4.5),

$$\begin{aligned} (-\Delta)^i(\mathcal{G}z)(t) &\geq [\Psi^{M-i}\varphi(\cdot, k^{-1}q_{N, M})](t) \\ &\geq I_1(N, M-i) \int_0^\infty \min\{s, s^{N-1}\} \varphi(s, k^{-1}q_{N, M}(s)) ds \cdot q_{N, M-i}(t) \\ &\geq k^{-1}q_{N, M-i}(t), \quad t \geq 0, i = 0, 1, \dots, M-1. \end{aligned}$$

This shows that  $\mathcal{G}(Z_1) \subset Z_1$ . The continuity of  $\mathcal{G}$  and the relative compactness of  $\mathcal{G}(Z_1)$  can also be proved without difficulty, and so  $\mathcal{G}$  has a fixed element  $z_0$  in  $Z_1 : z_0 = \mathcal{G}z_0$ .

Let us define the functions  $v(x)$  and  $w(x)$  by

$$v(x) = z_0(|x|), \quad w(x) = [\Psi^M \varphi(\cdot, k^{-1}q_{N, M})](|x|), \quad x \in \mathbf{R}^N,$$

and check that these are a supersolution and a subsolution of (1.1) generating the desired decaying entire solution. From (F<sub>4</sub>) and the fact that  $z_0 \in Z_1$  it follows that

$$\begin{aligned} (-\Delta)^i v(x) &= [\Psi^{M-i} F(\cdot, z_0, \dots, (-\Delta)^{M-1}z_0)](|x|) \\ &\geq [\Psi^{M-i} \varphi(\cdot, z_0)](|x|) \geq [\Psi^{M-i} \varphi(\cdot, k^{-1}q_{N, M})](|x|) \\ &= (-\Delta)^i w(x), \quad x \in \mathbf{R}^N, i = 0, 1, \dots, M-1 \end{aligned}$$

which implies that

$$(4.6) \quad \lim_{|x| \rightarrow \infty} (-\Delta)^i v(x) = \lim_{|x| \rightarrow \infty} (-\Delta)^i w(x) = 0, \quad i = 0, 1, \dots, M-1.$$

Furthermore, if  $x \in \mathbf{R}^N$  and  $(-\Delta)^i w(x) \leq \sigma_i \leq (-\Delta)^i v(x), i = 0, 1, \dots, M-1$ , then

$$\begin{aligned} (-\Delta)^M v(x) &= F(|x|, v(x), -\Delta v(x), \dots, (-\Delta)^{M-1}v(x)) \\ &\geq F(|x|, \sigma_0, \sigma_1, \dots, \sigma_{M-1}) \geq f(x, \sigma_0, \sigma_1, \dots, \sigma_{M-1}) \end{aligned}$$

and

$$\begin{aligned} (-\Delta)^M w(x) &= \varphi(|x|, k^{-1}q_{N, M}(|x|)) \leq \varphi(|x|, w(x)) \\ &\leq \varphi(|x|, \sigma_0) \leq f(x, \sigma_0, \sigma_1, \dots, \sigma_{M-1}). \end{aligned}$$

Consequently, by Theorem 2.1, the equation (1.1) possesses an entire solution

$u(x)$  such that  $(-\Delta)^i w(x) \leq (-\Delta)^i u(x) \leq (-\Delta)^i v(x)$  in  $\mathbf{R}^N$ ,  $i=0, 1, \dots, M-1$ . The solution  $u(x)$  clearly enjoys the property (4.3), and the proof is complete.

No information is available from Theorem 4.1 about precise order of decay of the entire solution obtained therein. One can indicate a condition (stronger than (3.1)) which allows (1.1) to have a decaying entire solution with specific order of decay at infinity as the following theorem shows.

**THEOREM 4.2.** *Let  $N \geq 2M+1$  and  $(F_4)$  hold. If*

$$(4.7) \quad \int_0^\infty t^{N-1} F(t, q_{N,M}(t), q_{N,M-1}(t), \dots, q_{N,1}(t)) dt < \infty,$$

then there exists an entire solution  $u(x)$  of (1.1) such that

$$(4.8) \quad k_i^{-1} q_{N,M-i}(|x|) \leq (-\Delta)^i u(x) \leq k_i q_{N,M-i}(|x|), \quad x \in \mathbf{R}^N$$

for some positive constants  $k_i$ ,  $i=0, 1, \dots, M-1$ .

**PROOF.** Take a positive constant  $k > 1$  large enough so that

$$(4.9) \quad I_1(N, j) \int_0^\infty \min\{s, s^{N-1}\} \varphi(s, k^{-1} q_{N,M}(s)) ds \geq k^{-1}, \quad j=1, \dots, M,$$

$$(4.10) \quad J_2(N, j) \int_0^\infty \max\{s, s^{N-1}\} F(s, k q_{N,M}(s), \dots, k q_{N,1}(s)) ds \leq k, \quad j=1, \dots, M,$$

where  $J_2(N, j)$  is as in (ii) of Lemma 2.1, and define the set

$$Z_2 = \{z \in C : k^{-1} q_{N,M-i}(t) \leq (-\Delta)^i z(t) \leq k q_{N,M-i}(t), t \geq 0, i=0, \dots, M-1\}.$$

Then, it can be shown, with the use of the results of Lemma 2.1, that the same mapping  $\mathcal{G}$  as in the proof of Theorem 4.1 maps  $Z_2$  continuously into a relatively compact subset of  $Z_2$ . Therefore there exists a fixed element  $z_0 \in Z_2$  of  $\mathcal{G}$ . Now it is not difficult to verify that the functions  $v(x) = z_0(|x|)$  and  $w(x) = [\Psi^M \varphi(\cdot, k^{-1} q_{N,M})](|x|)$  become, respectively, a supersolution and a subsolution of (1.1). Thus Theorem 2.1 ensures the existence of an entire solution  $u(x)$  of (1.1) satisfying  $(-\Delta)^i w(x) \leq (-\Delta)^i u(x) \leq (-\Delta)^i v(x)$  in  $\mathbf{R}^N$ ,  $i=0, 1, \dots, M-1$ . Because of (4.7) the second statement of Lemma 2.1 shows that, for each  $i \in \{0, 1, \dots, M-1\}$ , the functions  $(-\Delta)^i v(x)$  and  $(-\Delta)^i w(x)$  behave like positive constant multiples of  $q_{N,M-i}(|x|)$  as  $|x| \rightarrow \infty$ , and hence the solution  $u(x)$  has the required property (4.8). This completes the proof.

## 5. Special equations.

An example of equations to which the results of §§3-4 apply is the following :

$$(5.1) \quad (-\Delta)^M u = \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i u]^{r_i}, \quad x \in \mathbf{R}^N, N \geq 2,$$

where each  $p_i \in C_{loc}^\theta(\mathbf{R}^N)$ ,  $\theta \in (0, 1)$ , and each  $\gamma_i$  is either zero or the ratio of positive odd integers,  $i=0, \dots, M-1$ . This equation is a special case of (1.1) with

$$f(x, \xi_1, \dots, \xi_M) = \sum_{i=0}^{M-1} p_i(x) \xi_{i+1}^{\gamma_i}, \quad x \in \mathbf{R}^N, (\xi_1, \dots, \xi_M) \in \mathbf{R}^M.$$

The condition (F<sub>1</sub>) is satisfied with the choice

$$F(t, \xi_1, \dots, \xi_M) = \sum_{i=0}^{M-1} p_i^*(t) \xi_{i+1}^{\gamma_i}, \quad t \in \bar{\mathbf{R}}_+, (\xi_1, \dots, \xi_M) \in \bar{\mathbf{R}}_+^M,$$

where  $p_i^*(t) = \max_{|x|=t} |p_i(x)|$ ,  $i=0, 1, \dots, M-1$ . The condition (F<sub>2</sub>) or (F<sub>3</sub>) holds according to whether  $\gamma_i > 1$ ,  $i=0, 1, \dots, M-1$ , or  $0 \leq \gamma_i < 1$ ,  $i=0, 1, \dots, M-1$ . If  $0 \leq \gamma_i < 1$ ,  $p_i(x) \geq 0$  in  $\mathbf{R}^N$ ,  $i=0, 1, \dots, M-1$ , and  $p_0(0) > 0$ , then the condition (F<sub>4</sub>) holds with the above  $F$  and  $\varphi(t, \xi_1) = p_{0*}(t) \xi_1^{\gamma_0}$ , where  $p_{0*}(t) = \min_{|x|=t} p_0(x)$ .

The results that follow from Theorems 3.1, 4.1 and 4.2 applied to (5.1) are listed below.

**THEOREM 5.1.** *Let  $N \geq 2M+1$ .*

(i) *If either  $\gamma_i > 1$ ,  $i=0, 1, \dots, M-1$  or  $0 \leq \gamma_i < 1$ ,  $i=0, 1, \dots, M-1$ , and*

$$(5.2) \quad \int_0^\infty t^{2M-1} p_i^*(t) dt < \infty, \quad i=0, 1, \dots, M-1,$$

*then there exist infinitely many positive entire solutions  $u(x)$  of (5.1) such that*

$$(5.3) \quad \begin{cases} \lim_{|x| \rightarrow \infty} u(x) = \text{constant} > 0, \\ \lim_{|x| \rightarrow \infty} (-\Delta)^i u(x) = 0, \quad i=1, \dots, M-1. \end{cases}$$

(ii) *If  $0 \leq \gamma_i < 1$ ,  $p_i(x) \geq 0$ ,  $x \in \mathbf{R}^N$ ,  $i=0, 1, \dots, M-1$ ,  $p_0(0) > 0$ , and (5.2) is satisfied, then there exists an entire solution  $u(x)$  of (5.1) such that*

$$(5.4) \quad \begin{cases} (-\Delta)^i u(x) > 0, \quad x \in \mathbf{R}^N, \\ \lim_{|x| \rightarrow \infty} (-\Delta)^i u(x) = 0, \quad i=0, 1, \dots, M-1. \end{cases}$$

(iii) *Let  $\gamma_i$  and  $p_i$  be as in (ii). If*

$$(5.5) \quad \int_0^\infty t^{N-1-\gamma_i(N-2M+2i)} p_i^*(t) dt < \infty, \quad i=0, 1, \dots, M-1,$$

*then there exists an entire solution  $u(x)$  of (5.1) such that*

$$(5.6) \quad k_i^{-1} q_{N, M-i}(|x|) \leq (-\Delta)^i u(x) \leq k_i q_{N, M-i}(|x|), \quad x \in \mathbf{R}^N$$

*for some positive constants  $k_i$ ,  $i=0, 1, \dots, M-1$ .*

Let us now consider the singular equation

$$(5.7) \quad (-\Delta)^M u = \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i u]^{-\gamma_i}, \quad x \in \mathbf{R}^N, N \geq 2,$$

where  $p_i \in C_{\text{loc}}^{\theta}(\mathbf{R}^N)$  is nonnegative and each  $\gamma_i$  is a nonnegative constant,  $i=0, 1, \dots, M-1$ . It is clear that none of the theorems of §§3-4 is applicable to this equation. It will be shown that the existence of bounded positive entire solutions can be established for (5.7) on the basis of Corollary 2.2 or 2.3.

**THEOREM 5.2.** *Let  $N \geq 2M+1$  and suppose that  $0 \leq \gamma_0 \gamma_i < 1$ ,  $i=0, 1, \dots, M-1$ , and  $p_0(0) > 0$ . If*

$$(5.8) \quad \int_0^{\infty} t^{2M-1} p_0^*(t) dt < \infty,$$

$$(5.9) \quad \int_0^{\infty} t^{2M-1+\gamma_i(N-2M+2i)} p_i^*(t) dt < \infty, \quad i=1, \dots, M-1,$$

then the equation (5.7) has infinitely many positive entire solutions  $u(x)$  satisfying (5.3).

**PROOF.** Put  $z_0(t) = [\Psi^M p_{0*}](t)$  for  $t \geq 0$ , where  $p_{0*}(t) = \min_{|x|=t} p_0(x)$ . Since  $0 \neq p_{0*}(t) \geq 0$  for  $t \geq 0$ , by (i) of Lemma 2.1 there are positive constants  $\alpha_i$  such that

$$(5.10) \quad (-\Delta)^i z_0(t) \geq \alpha_i q_{N, M-i}(t), \quad t \geq 0, i=0, 1, \dots, M-1.$$

Put

$$c_0 = \sup_{t \geq 0} [\Psi^M p_0^*](t) \quad \text{and} \quad c_i = \sup_{t \geq 0} [\Psi^M p_i^* \{(-\Delta)^i z_0\}^{-\gamma_i}](t), \quad i=1, \dots, M-1,$$

take a positive constant  $k$  small enough so that

$$(5.11) \quad k^{1+\gamma_0} \leq (1+c_0)^{-\gamma_0},$$

$$(5.12) \quad \begin{cases} k^{1+\gamma_0} \leq (M+1)^{-\gamma_0}, \\ k^{1-\gamma_0 \gamma_i} c_i^{\gamma_0} \leq (M+1)^{-\gamma_0}, \quad i=0, 1, \dots, M-1, \end{cases}$$

and define

$$z(t) = k(1+z_0(t)), \quad t \geq 0.$$

The choice of such  $k$  is possible by the condition  $0 \leq \gamma_0 \gamma_i < 1$ . It is easy to check that

$$(5.13) \quad (-\Delta)^M z(t) = k p_{0*}(t), \quad t > 0,$$

$$(5.14) \quad \lim_{t \rightarrow \infty} z(t) = k$$

and

$$(5.15) \quad \begin{cases} z(t) \geq k(1+\alpha_0 q_{N, M}(t)), \quad t \geq 0, \\ (-\Delta)^i z(t) \geq k \alpha_i q_{N, M-i}(t), \quad t \geq 0, i=1, \dots, M-1. \end{cases}$$

Consider the function

$$y(t) = k + \sum_{i=0}^{M-1} [\Psi^M p_i^* \{(-\Delta)^i z\}^{-r_i}](t), \quad t \geq 0.$$

In view of (5.8), (5.9), (5.15) and Lemma 2.1  $y(t)$  is well-defined and satisfies

$$(5.16) \quad (-\Delta)^M y(t) = \sum_{i=0}^{M-1} p_i^*(t) [(-\Delta)^i z]^{-r_i}(t), \quad t > 0,$$

$$(5.17) \quad \lim_{t \rightarrow \infty} y(t) = k$$

and

$$(5.18) \quad k \leq y(t) \leq k + \sum_{i=0}^{M-1} c_i k^{-r_i}, \quad t \geq 0.$$

Now define  $v(x)$  and  $w(x)$  by  $v(x)=y(|x|)$  and  $w(x)=z(|x|)$ , respectively. Then (5.11) and the relation  $\sup_{t \geq 0} z(t) \leq k(1+c_0)$  lead to the inequalities

$$\begin{aligned} [\Psi^{M-i}(p_0^* z^{-r_0})](t) &\geq k^{-r_0}(1+c_0)^{-r_0}(\Psi^{M-i} p_0^*)(t) \\ &\geq k(\Psi^{M-i} p_0^*)(t) \geq k(\Psi^{M-i} p_{0*})(t), \quad t \geq 0, i = 0, 1, \dots, M-1, \end{aligned}$$

which gives

$$(5.19) \quad (-\Delta)^i w(x) \leq (-\Delta)^i v(x), \quad x \in \mathbf{R}^N, i = 0, 1, \dots, M-1.$$

Furthermore (5.16) implies

$$\begin{aligned} (-\Delta)^M v(x) &= \sum_{i=0}^{M-1} p_i^*(|x|) [(-\Delta)^i w(x)]^{-r_i} \\ &\geq \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i w(x)]^{-r_i}, \quad x \in \mathbf{R}^N. \end{aligned}$$

On the other hand,  $v(x)$  satisfies  $v(x)^{-r_0} \geq k, x \in \mathbf{R}^N$ . In fact, in case  $\gamma_0=0$  this is obvious since  $k \leq 1$ , and in case  $\gamma_0 > 0$ , this follows from (5.12) and (5.18) as the computation below shows:

$$\begin{aligned} v(x)^{-r_0} &= y(|x|)^{-r_0} \geq \left(k + \sum_{i=0}^{M-1} c_i k^{-r_i}\right)^{-r_0} \\ &\geq \left((M+1)^{-1} k^{-1/r_0} + \sum_{i=0}^{M-1} (M+1)^{-1} k^{-1/r_0}\right)^{-r_0} = (k^{-1/r_0})^{-r_0} = k. \end{aligned}$$

From this it follows that

$$\begin{aligned} (-\Delta)^M w(x) &= k p_{0*}(|x|) \leq p_{0*}(|x|) v(x)^{-r_0} \leq p_0(x) v(x)^{-r_0} \\ &\leq \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i v(x)]^{-r_i}, \quad x \in \mathbf{R}^N. \end{aligned}$$

The above observation shows that the functions  $v(x)$  and  $w(x)$  are, respectively, a supersolution and a subsolution of (5.7) in the sense of Corollary 2.3, and so

there exists a positive entire solution  $u(x)$  of (5.7) lying between  $v(x)$  and  $w(x)$ . This finishes the proof of the theorem.

Our final theorem concerns the existence of decaying positive entire solutions of (5.7).

**THEOREM 5.3.** *Let  $N \geq 2M+1$  and suppose that  $0 \leq \gamma_0 \gamma_i < 1$ ,  $i=0, 1, \dots, M-1$ , and  $p_0(0) > 0$ .*

(i) *If*

$$(5.20) \quad \int_0^\infty t^{2M-1+\gamma_i(N-2M+2i)} p_i^*(t) dt < \infty, \quad i=0, 1, \dots, M-1,$$

*then the equation (5.7) has an entire solution  $u(x)$  satisfying (5.4).*

(ii) *If*

$$(5.21) \quad \int_0^\infty t^{N-1+\gamma_i(N-2M+2i)} p_i^*(t) dt < \infty, \quad i=0, 1, \dots, M-1,$$

*then the equation (5.7) has an entire solution  $u(x)$  satisfying (5.6).*

**PROOF.** (i) Let  $z_0$  be as in the proof of Theorem 5.2 and put

$$c_i = \sup_{t \geq 0} [\Psi^M p_i^* \{(-\Delta)^i z_0\}^{-\gamma_i}](t), \quad i=0, 1, \dots, M-1.$$

Choose a constant  $k > 0$  so that

$$(5.22) \quad k^{1+\gamma_0} \leq c_0^{-\gamma_0},$$

$$(5.23) \quad k^{1-\gamma_0 \gamma_i} c_i^{\gamma_0} \leq M^{-\gamma_0}, \quad i=0, 1, \dots, M-1,$$

and put

$$z(t) = k z_0(t), \quad y(t) = \sum_{i=0}^{M-1} [\Psi^M p_i^* \{(-\Delta)^i z\}^{-\gamma_i}](t), \quad t \geq 0.$$

Then the functions  $v(x) = y(|x|)$  and  $w(x) = z(|x|)$  satisfy

$$\begin{aligned} (-\Delta)^j v(x) &= \sum_{i=0}^{M-1} [\Psi^{M-j} p_i^* \{(-\Delta)^i z\}^{-\gamma_i}](|x|) \\ &\geq [\Psi^{M-j} p_0^* z^{-\gamma_0}](|x|) \geq (k c_0)^{-\gamma_0} (\Psi^{M-j} p_0^*)(|x|) \\ &\geq k^{-1-\gamma_0} c_0^{-\gamma_0} (-\Delta)^j w(x) \geq (-\Delta)^j w(x) > 0, \quad x \in \mathbf{R}^N, j=0, \dots, M-1, \end{aligned}$$

where (5.22) and the fact that  $\sup_{x \in \mathbf{R}^N} w(x) = k c_0$  have been used.

Furthermore, from the definition of  $v(x)$  and  $w(x)$  one obtains

$$\begin{aligned} (-\Delta)^M v(x) &= \sum_{i=0}^{M-1} p_i^*(|x|) [(-\Delta)^i w(x)]^{-\gamma_i} \\ &\geq \sum_{i=0}^{M-1} p_i(x) [(-\Delta)^i w(x)]^{-\gamma_i}, \quad x \in \mathbf{R}^N, \end{aligned}$$

and one sees with the use of (5.22) and (5.23) that

$$\begin{aligned} v(x)^{-\gamma_0} &= y(|x|)^{-\gamma_0} = \left( \sum_{i=0}^{M-1} [\Psi^M p_i^* \{(-\Delta)^i z\}^{-r_i}] (|x|) \right)^{-\gamma_0} \\ &= \left( \sum_{i=0}^{M-1} k^{-r_i} [\Psi^M p_i^* \{(-\Delta)^i z_0\}^{-r_i}] (|x|) \right)^{-\gamma_0} \geq \left( \sum_{i=0}^{M-1} c_i k^{-r_i} \right)^{-\gamma_0} \geq (k^{-1/r_0})^{-\gamma_0} = k, \end{aligned}$$

so that

$$\begin{aligned} (-\Delta)^M w(x) &= k p_{0*}(|x|) \leq p_{0*}(|x|) v(x)^{-\gamma_0} \\ &\leq \sum_{i=0}^{M-1} p(x) [(-\Delta)^i v(x)]^{-r_i}, \quad x \in \mathbf{R}^N. \end{aligned}$$

The conclusion then follows from Corollary 2.3.

(ii) Let  $y$  and  $z$  be as in (i). Then, as was shown in (i), (5.7) has a positive entire solution  $u(x)$  satisfying  $(-\Delta)^i z(|x|) \leq (-\Delta)^i u(x) \leq (-\Delta)^i y(|x|)$ ,  $x \in \mathbf{R}^N$ ,  $i=0, 1, \dots, M-1$ . Since  $p_{0*} \in L^1_{N-1}(\mathbf{R}_+)$ , and

$$(-\Delta)^i z(t) = k [\Psi^{M-i} p_{0*}](t), \quad t \geq 0, i=0, 1, \dots, M-1,$$

from (ii) of Lemma 2.1 it follows that

$$(5.24) \quad \alpha_i^{-1} q_{N, M-i}(t) \leq (-\Delta)^i z(t) \leq \alpha_i q_{N, M-i}(t), \quad t \geq 0,$$

for some constants  $\alpha_i > 1$ ,  $i=0, 1, \dots, M-1$ . Combining (5.20) with (5.24) gives

$$\int_0^\infty t^{N-1} p_i^*(t) [(-\Delta)^i z(t)]^{-r_i} dt < \infty, \quad i=0, 1, \dots, M-1,$$

and (ii) of Lemma 2.1 then implies that

$$(5.25) \quad \beta_i^{-1} q_{N, M-i}(t) \leq (-\Delta)^i y(t) \leq \beta_i q_{N, M-i}(t), \quad t \geq 0$$

for some constants  $\beta_i > 1$ ,  $i=0, 1, \dots, M-1$ . From (5.24) and (5.25) it follows that the solution  $u(x)$  satisfies (5.6). This completes the proof.

REMARK. The condition  $0 \leq \gamma_0 \gamma_i < 1$ ,  $i=0, 1, \dots, M-1$ , in Theorems 5.2 and 5.3 seems to be stronger than necessary. It is our conjecture that the Theorems 5.2 and 5.3 hold without this condition, that is, for any nonnegative values of  $\gamma_i$ ,  $i=0, 1, \dots, M-1$ .

## References

- [1] R. Dalmasso, Solutions d'équations elliptiques semi-linéaires d'ordre  $2m$ , Funkcial. Ekvac., **33** (1990), 277-289.
- [2] R. Dalmasso, Positive solutions of superlinear biharmonic equations, Funkcial. Ekvac., **34** (1991), 403-422.
- [3] A.L. Edelson and R. Vakilian, Asymptotic behaviour of higher order semilinear equations, Nonlinear Anal., **15** (1990), 17-26.



- [4] N. Fukagai, On decaying entire solutions of second order sublinear elliptic equations, *Hiroshima Math. J.*, 14 (1984), 551-562.
- [5] Y. Furusho, Existence of positive entire solutions of weakly coupled semilinear elliptic systems, *Proc. Royal Soc. Edinburgh Sect. A*, 120 (1992), 79-91.
- [6] T. Kusano and M. Naito, Kiguradze class for radial entire solutions of higher order quasilinear elliptic equations, *Hiroshima Math. J.*, 22 (1992), 301-363.
- [7] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions to even order semilinear elliptic equations in the plane, *Proc. Royal Soc. Edinburgh Sect. A*, 105 (1987), 275-287.
- [8] T. Kusano, M. Naito and C. A. Swanson, Entire solutions of a class of even order quasilinear elliptic equations, *Math. Z.*, 195 (1987), 151-163.
- [9] T. Kusano, M. Naito and C. A. Swanson, Radial entire solutions of even order semilinear elliptic equations, *Canad. J. Math.*, 40 (1988), 1281-1300.
- [10] T. Kusano and C. A. Swanson, Positive entire solutions of semilinear biharmonic equations, *Hiroshima Math. J.*, 17 (1987), 13-28.
- [11] T. Kusano and C. A. Swanson, A general method for quasilinear elliptic problems in  $\mathbf{R}^N$ , *J. Math. Anal. Appl.*, 167 (1992), 414-428.

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