J. Math. Soc. Japan
Vol. 46, No. 4, 1994

Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and hyperbolic fibre spaces

By Makoto SUZUKI

(Received Oct. 26, 1992)
(Revised April 21, 1993)

1. Introduction.

Let Y be a complete hyperbolic complex space. We assume that Y is hyperbolically imbedded into an irreducible compact complex space $\bar{Y}$ as its Zariski open subset. Let X be a Zariski open subset of an irreducible compact complex space. We denote by $Hol(X, Y)$ (resp. $Mer_{dom}(X, Y)$) the set of all holomorphic (resp. dominant meromorphic) mappings of X into Y, where a mapping is said to be dominant if its image contains a nonempty open subset. In this paper, by making use of the theory developed by Noguchi [**12**, **13**, **16**] we study the structure of $Hol(X, Y)$. We first prove the following finiteness theorem for mappings in noncompact case, which was conjectured by Noguchi (cf. [**16**], Conjecture (5.5)),

FINITENESS THEOREM (cf. Theorem 2.3). *Let X and Y be as above. Then $Mer_{dom}(X, Y)$ is a finite set.*

This is regarded as the splitting case of the finiteness theorem of the sections of hyperbolic fibre spaces, and plays an essential role in considering the structure of hyperbolic fibre spaces in more general setting below. In the case of a noncompact quotient D/Γ of a bounded symmetric domain D in the complex vector space by a torsion free arithmetic discrete subgroup Γ of the identity component of the holomorphic automorphism group of D, D/Γ is complete hyperbolic and hyperbolically imbedded into its Satake compactification (cf. [**6**]). Thus applying the theorem to this case, we see the finiteness of $Mer_{dom}(X, D/\Gamma)$. Tsushima [**20**] obtained this result by showing the finiteness of dominant strictly rational maps into a smooth algebraic variety of log-general type. In the case where Y is a Riemann surface of finite type (g, n) with $2g-2+n>0$, Imayoshi [**2**] proved the above finiteness theorem. The compact version of the above theorem (a Lang's conjecture in [8]) was recently solved by Noguchi [**16**] (see § 2 for precise statement). Using this, Noguchi [**12**, **11**, **16**] proved the finiteness theorem of non-constant holomorphic sections and of trivial fibre subspaces of hyperbolic fibre spaces (see § 3 for definitions and precise statement), which gave a proof of a higher dimensional analogue of Mordell's conjecture over function fields (see Noguchi [**14**, **11**, **16**] and their references for related Diophantine geometry). We consider a non-compact version of finiteness theorem of non-constant holomorphic sections and of trivial fibre subspaces of hyperbolic fibre spaces. Let Y be as in the above Finiteness Theorem and X be a nonsingular Zariski open subset of an irreducible compact complex space $\bar{X}$. Let $(Y\times X, P, X)$ be the trivial hyperbolic fibre space with the natural projection $P: Y\times X\to X$. We obtain

THEOREM 4.1. *Let X and Y be as above. Then $(Y\times X, P, X)$ contains only finitely many meromorphically trivial fibre subspaces, and carries only finitely many holomorphic sections except for constant ones in those bimeromorphic trivializations.*

In the case of nonsingular fibre spaces whose fibres are Riemann surfaces of fixed finite type (g, n) with $2g-2+n>0$ and whose base spaces are Riemann surfaces of finite type, Imayoshi and Shiga [**4**] showed the finiteness of non-constant sections by making use of the theory of Teichmüller spaces and Kleinian groups. In the case of nontrivial normal fibre spaces, which has a compactification, such that the fibres are noncompact curves and that its generic fibres are hyperbolic, Zaidenberg [**22**] gave two proofs of the finiteness theorem. One followed the approach used in [**12**], which is the same one as we take. The other was based on the idea, which was suggested by Parshin, that the case was reduced to the case of compact fibres, i. e., Manin's theorem (an analogue of Mordell's conjecture for curves over function fields) (cf. [**10**]). When the fibre dimension is equal to one, our theorem says nothing but the finiteness of non-constant holomorphic mappings. But when the fibre dimension is greater than one, we can see a detailed structure of the moduli space $Hol(X, Y)$.

In § 2, we give a proof of the Finiteness Theorem and obtain an estimate of dimensions of the moduli space $Hol(X, Y)$. In § 3, we recall Noguchi's finiteness theorem of nonconstant sections and of the trivial fibre subspaces, and prove it for the sake of convenience. In § 4, Theorem 4.1 is proved.

The author is grateful to express his sincere gratitude to Professor J. Noguchi for valuable discussion and useful suggestion.

General Remark. Throughout this paper, we assume that all complex spaces are paracompact and reduced and that all complex manifolds are connected. The term "hyperbolic" is always used in the sense of Kobayashi. For complex hyperbolic geometry, refer to Kobayashi [**5**], Lang [**9**], Noguchi and Ochiai [**17**] and Noguchi [**14**].

2. Finiteness of mappings in noncompact case.

Let X be a Zariski open subset of an irreducible compact complex space $\bar{X}$. Let Y be an irreducible complex space. A mapping f of X into Y is said to be *dominant* if the image of X by f contains a non-empty open subset of Y. We denote by $Mer_{dom}(X, Y)$ (resp. $Hol_{dom}(X, Y)$) the set of all dominant meromorphic (resp. holomorphic) mappings from X into Y. Recently, Noguchi [**16**] proved the following finiteness theorem, which was conjectured by Lang [8] from Diophantine geometric point of view;

NOGUCHI'S FINITENESS THEOREM (cf. [**16**], Theorem A). *If Y is compact hyperbolic, then $Mer_{dom}(X, Y)$ is a finite set.*

This is a higher dimensional version of classical de Franchis theorem, which asserts that there are only a finite number of surjective holomorphic mappings of a fixed compact Riemann surface onto another fixed compact Riemann surface of genus greater than one. We consider a noncompact version of Noguchi's finiteness theorem.

Let X be a Zariski open subset of a compact complex manifold $\tilde{X}$ such that the boundary $\partial X := \tilde{X} - X$ of X is a hypersurface with only normal crossings. Let Y be a connected Zariski open subset of a compact complex space $\bar{Y}$. Assume that Y is complete hyperbolic and hyperbolically imbedded into $\bar{Y}$. The spaces $Hol(X, Y)$ and $Hol(\tilde{X}, \bar{Y})$ are equipped with compact-open topology. Since $\tilde{X}$ is compact, according to Douady's theory (cf. [1]) the latter space carries the universal structure of complex space such that its underlying topology coincides with compact-open topology and the evaluation mapping

$$Hol(\tilde{X}, \bar{Y}) \times \tilde{X} \ni (f, x) \longmapsto f(x) \in \bar{Y}$$

is holomorphic. The extension and convergence theorem of Noguchi (cf. [**13**], Theorem 1.19) implies that the extension mapping of $Hol(X, Y)$ into $Hol(\tilde{X}, \bar{Y})$ is homeomorphic onto the image of the mapping and by the natural identification the space $Hol(X, Y)$ is regarded as the topological subspace of $Hol(\tilde{X}, \bar{Y})$. In fact the following structure theorem due to Noguchi holds;

NOGUCHI'S STRUCTURE THEOREM (cf. [**13**], Theorem 2.8). i) *The space $Hol(X, Y)$ is a Zariski open subset of the compact analytic subspace $\overline{Hol(X, Y)}$ of $Hol(\tilde{X}, \bar{Y})$ where $\overline{Hol(X, Y)}$ is the closure of $Hol(X, Y)$ in $Hol(\tilde{X}, \bar{Y})$ and the evaluation mapping*

$$\Phi : Hol(X, Y) \times X \ni (f, x) \longmapsto f(x) \in Y$$

is holomorphic and extends to a holomorphic mapping

$$\bar{\Phi} : Hol(X, Y)\times\tilde{X} \longrightarrow \bar{Y}\ .$$

ii) (*universality*) *For a complex space* T *and a holomorphic mapping* $\phi: T\times X\to Y$, *the natural mapping*

$$T \ni t \longmapsto \phi(t, \cdot) \in Hol(X, Y)$$

is holomorphic.

We set

$$Hol(k\,; X, Y) := \{f\in Hol(X, Y)\,;\ \mathrm{rank}\, f=k\},$$

where k is a nonnegative integer. Then we know the following

PROPOSITION 2.1 (cf. Noguchi [**13**], Proposition 2.18, Theorem 3.3, i)). *$Hol(k\,; X, Y)$ is open and closed in $Hol(X, Y)$, hence it carries a structure of complex space. In particular, $Hol(n\,; X, Y)$ where $n=\dim Y$ is a compact complex space.*

In this section, for any element $g\in Hol(X, Y)$, we denote the extension of g to $\tilde{X}$ by the same letter g. Put $\partial Y := \bar{Y}-Y$. The next assertion essentially follows from the proof of Noguchi's structure theorem, i) (cf. the proof of Theorem 2.8, i) in [**13**], pp. 23-24). For the sake of convenience and the later use, we state it in the following

LEMMA 2.2. *Let Z be a connected component of $Hol(X, Y)$. Take an element $g_0\in Z$ and put $\partial X_0 := \mathrm{Supp}\, g_0^{-1}(\partial Y)$ and $Z_0 := \{g\in Z\,;\ \mathrm{Supp}\, g^{-1}(\partial Y)=\partial X_0\}$. Then $Z=Z_0$.*

PROOF. Since Y is complete hyperbolic, $g^{-1}(\partial Y)$ must be a hypersurface in $\tilde{X}$ for $g\in Z$ and is contained in ∂X. Note that the convergence in compact-open topology of $Hol(X, Y)$ implies one in compact-open topology of $Hol(\tilde{X}, \bar{Y})$ by the extention and convergence theorem of Noguchi (cf. [**13**], Theorem 1.19). Then Z_0 is closed since for any sequence $\{g_\nu\}_{\nu=1}^{\infty}$ of Z_0 converging to $g\in Z$

$$\mathrm{Supp}\, g^{-1}(\partial Y) = \lim_{\nu\to\infty} \mathrm{Supp}\, g_\nu^{-1}(\partial Y)$$

where the limit set in the right is used in the following sense: A point $x\in\tilde{X}$ is in the set $\lim \mathrm{Supp}\, g_\nu^{-1}(\partial Y)$ if and only if for every neighborhood N of x there is a positive integer ν_0 such that $N\cap \mathrm{Supp}\, g_\nu^{-1}(\partial Y)\neq\phi$ for all $\nu\geqq\nu_0$ (see [**13**], the proof of Theorem (2.8), p. 23). We see that Z_0 is open in Z. In fact, if Z_0 is not open, there exist an element $g\in Z_0$ and a sequence $\{g_\nu\}\subset Z$ such that $\lim_\nu g_\nu=g$ and $\mathrm{Supp}\, g_\nu^{-1}(\partial Y)\neq\partial X_0$. Since ∂X is compact, the number of the distinct supports of the hypersurfaces in $\tilde{X}$ which are contained in ∂X is only finite. Thus by taking a subsequence if necessary, we see that all the inverse images $g_\nu^{-1}(\partial Y)$ coincide and are different from ∂X_0. It follows from the first half of the proof that $\mathrm{Supp}\, g^{-1}(\partial Y)=\partial X_0$. This contradicts that g is in Z_0. ■

THEOREM 2.3. *Let Y be a complete hyperbolic complex space which is hyperbolically imbedded into an irreducible compact complex space $\bar{Y}$ and is a Zariski open subset of $\bar{Y}$. Then for any Zariski open subset X of any irreducible compact complex space $\bar{X}$, $Mer_{dom}(X, Y)$ is a finite set.*

PROOF. Assume that $Mer_{dom}(X, Y)$ is not finite. Let $\bar{X}^* \xrightarrow{\alpha} \bar{X}$ be a resolution of singularities such that $\bar{X}^* - \alpha^{-1}(X)$ is a hypersurface with only normal crossings and put $X^* := \alpha^{-1}(X)$. Then we have $f \circ (\alpha|_{X^*}) \in Mer(X^*, Y)$ for $f \in Mer(X, Y)$. Since X^* is nonsingular and Y is complete hyperbolic, $f \circ (\alpha|_{X^*})$ is, in fact, a holomorphic mapping of X^* into Y (cf. [5], Chapter VIII, Theorem 1.2). Then replacing X^* by X and putting $\tilde{X} := \bar{X}^*$ and $\partial X := \tilde{X} - X$, we may assume that $\tilde{X}$ is a compact complex manifold, X a Zariski open subset of $\tilde{X}$ and ∂X a hypersurface with only normal crossings. Assume that $Hol_{dom}(X, Y)$ is not finite. It follows from Proposition 2.1 that $Hol(n; X, Y)$ is a compact complex space with positive dimension where $n = \dim Y$. Take an irreducible component Z of $Hol(n; X, Y)$ with $\dim Z > 0$. The extension mapping $g_0 : \tilde{X} \to \bar{Y}$ is surjective by the irreducibility of $\bar{Y}$. If we put $\partial X_0 := \operatorname{Supp} g_0^{-1}(\partial Y)$, we see from Lemma 2.2 that $\operatorname{Supp} g^{-1}(\partial Y) = \partial X_0$ for any $g \in Z$. Consider the evaluation mapping

$$\Phi : Z \times \tilde{X} \ni (z, x) \longmapsto z(x) \in \bar{Y},$$

which is holomorphic. Put $X_0 := \tilde{X} - \partial X_0$ and take a point $x_0 \in X_0$. Then the subset $Z(x_0) := \{\Phi(z, x_0) \in \bar{Y} ; z \in Z\}$ of $\bar{Y}$ is a compact hyperbolic complex subspace of Y. Let Y_0 be an irreducible compact hyperbolic complex subspace of Y containing $Z(x_0)$ with the maximum dimension among those subspaces. Take an element $z_0 \in Z$ at which Z is nonsingular. Since the mapping $z_0|_{X_0} : X_0 \to Y$ is proper holomorphic, $(z_0|_{X_0})^{-1}(Y_0)$ is a compact subvariety in X_0. Let X_0' be the irreducible component of $(z_0|_{X_0})^{-1}(Y_0)$ containing x_0. Then the mapping $z_0|_{X_0'} : X_0' \to Y$ is surjective. Moreover, the subset $\Phi(Z \times X_0')$ of Y is an irreducible compact hyperbolic complex subspace containing Y_0. Thus we see that $\Phi(Z \times X_0') = Y_0$. Because of the finiteness of holomorphic mappings in Z which map a given point to a given point (cf. [21], Theorem 1), together with $\dim Z > 0$, it holds that the mapping $Z \to Hol_{dom}(X_0', Y_0)$ is non-constant. This contradicts Noguchi's finiteness theorem, and we complete the proof. ■

REMARK. The use of Lemma 2.2 was pointed out by Professor J. Noguchi and makes the proof of Theorem 2.3 simpler than the original one.

If we put $X = Y$ in Theorem 2.3, we obtain the following, which was proved in the case where Y is nonsingular by Miyano and Noguchi (see [11], Theorem (2.4)).

Corollary 2.4. *Let Y be as in Theorem* 2.3. *Then the bimeromorphic automorphism group of Y is finite.*

Remark 2.5. Note that the finiteness of the biholomorphic automorphism group in Theorem 2.4 of Miyano and Noguchi [**11**] implies the finiteness of the bimeromorphic one. In fact, if Y is nonsingular, the bimermorphic automorphism is really biholomorphic (cf. Kobayashi [**5**], Theorem 1.2). If Y has singularity, it is not always the case. For example, let Y be a compact Riemann surface with genus greater than one and the nontrivial biholomorphic automorphism group. Take a nontrivial biholomorphic automorphism g and two distinct points y_1, y_2 in Y such that $g(y_1)\neq y_2$. Identifying y_1 with y_2, we obtain a singular irreducible curve Y'. Y is the normalization of Y'. It is clear that g induces a bimeromorphic automorphism of Y', which is not biholomorphic. Kodama constructed an example of a normal irreducible complete hyperbolic complex space which has a bimeromorphic and non-biholomorphic automorphism (cf. Kodama [**7**]).

Now, we can obtain some information about the moduli spaces of holomorphic mappings in our situation. Let X be a Zariski open subset of an irreducible compact complex space $\tilde{X}$. We assume that $\partial X := \tilde{X}-X$ is a hypersurface with only normal crossings and X is nonsingular. Let Y be as in Theorem 2.3 and Z be a connected component of $Hol(X, Y)$. Note that the closure $\bar{Z}$ of Z in $Hol(\tilde{X}, \bar{Y})$ is a compact complex subspace of $Hol(\tilde{X}, \bar{Y})$ and Z is a Zariski open subset of $\bar{Z}$ by Noguchi's structure theorem (see the theorem before Proposition 2.1). We see that some of the nature of the target space transfer themselves to their moduli spaces of holomorphic mappings.

Proposition 2.6 (cf. [**13**], Remark (2.16) and [**11**], Lemma (2.13)).

i) *Z is complete hyperbolic and hyperbolically imbedded into $\bar{Z}$.*

ii) *If Y is quasi-projective algebraic and carries a projective compactification $\bar{Y}$ such that Y is hyperbolically imbedded into $\bar{Y}$, then Z is quasi-projective.*

See for the proof the papers cited above.

Proposition 2.7. *The space $Hol(k\,; X, Y)$ is compact for $k>\dim\partial Y$ where $\partial Y := \bar{Y}-Y$.*

The proof is same as in Noguchi [**13**], Theorem (3.3), i). We obtain an estimate of the complex dimension of the moduli spaces.

Theorem 2.8. *Let Z be an irreducible component of $Hol(X, Y)$. If Z contains a non-constant holomorphic mapping, then* $\dim Z\leqq \dim Y-1$.

Proof. According to Noguchi's structure theorem (cf. the theorem before

Proposition 2.1), the evaluation mapping

$$\Phi : Z \times X \ni (z, x) \longmapsto z(x) \in Y$$

is holomorphic and is holomorphically extended to the mapping

$$\bar{\Phi} : \bar{Z} \times \tilde{X} \longrightarrow \bar{Y} .$$

Since $\bar{Z}$ is compact, for any $x \in X$ the mapping

$$\bar{\Phi}(\cdot, x) : \bar{Z} \ni z \longmapsto z(x) \in \bar{Y}$$

is finite. Thus $\dim \bar{Z} \leqq \dim \bar{Y}$. Assume that $\dim Z = \dim Y$. Then $\Phi(\cdot, x) \in Hol_{dom}(Z, Y)$ for any $x \in X$. It follows from Theorem 2.3 that $\Phi(\cdot, x) : Z \rightarrow Y$ is a holomorphic mapping independent of $x \in X$. Thus each $z \in Z$ is a constant mapping. ∎

REMARK 2.9. In the case where Y is a noncompact quotient of a bounded symmetric domain in the complex vector space by a torsion free arithmetic discrete subgroup of the identity component of its holomorphic automorphism group, more precise estimates were obtained in Sunada [**19**], Theorem B, and Noguchi [**13**], Theorem (4.7), (4.10) (see for the compact quotient case Noguchi and Sunada [**18**], and Imayoshi [**2**, **3**]). In fact, in that case, $\dim Z$ is not greater than the maximum value of dimension of the proper boundary components of the bounded symmetric domain.

PROPOSITION 2.10. *Suppose that* $\operatorname{codim} \partial Y \geqq 2$. *Let* z_0 *be an element in* $Hol(n-1; X, Y)$ *such that* $z_0(X)$ *is relatively compact in* Y ($n = \dim Y$). *Then* $\dim_{z_0} Hol(n-1; X, Y) = 0$.

PROOF. By Proposition 2.7 it holds that $Hol(n-1; X, Y)$ is compact. Let Z be an irreducible component of $Hol(n-1; X, Y)$ containing z_0. Then for any element $z \in Z$ the extension of z maps $\tilde{X}$ into Y. Thus the image Y' of $Z \times \tilde{X}$ under the evaluation mapping $\bar{\Phi} : \overline{Hol(X, Y)} \times \tilde{X} \rightarrow \bar{Y}$ is an irreducible compact complex subspace of Y, hence is hyperbolic. Since $\dim Y' = n-1$, we see that $Z \subset Hol_{surj}(\tilde{X}, Y')$. The result follows from Noguchi's finiteness theorem (cf. the theorem before Proposition 2.1). ∎

3. Finiteness of nontrivial sections and of trivial fibre subspaces in compact case.

Let $\bar{R}$ and $\bar{W}$ be irreducible compact complex spaces and $\bar{\Pi} : \bar{W} \rightarrow \bar{R}$ a surjective holomorphic mapping with connected fibres. Let R be a nonsingular Zariski open subset of $\bar{R}$ and $\partial R := \bar{R} - R$. Put

$$W := \overline{W}|_R = \overline{\Pi}^{-1}(R), \qquad \Pi := \overline{\Pi}|_W .$$

Suppose that each fibre $W_t := \Pi^{-1}(t)$ is irreducible for $t \in R$. We denote by Γ the set of all holomorphic sections of the fiber space (W, Π, R).

DEFINITION 3.1 (cf. [**12**], § 1). We call a fibre space (W, Π, R) a *hyperbolic fibre space* if all the fibres W_t for $t \in R$ are hyperbolic. We say that the fibre space (W, Π, R) is *hyperbolically imbedded into* $(\overline{W}, \overline{\Pi}, \overline{R})$ *along* ∂R if for any $t \in \partial R$ there are neighborhoods U and V of t in $\overline{R}$ such that U is relatively compact in V and $W|_{U-\partial R}$ is hyperbolically imbedded into $\overline{W}|_V$.

Noguchi proved the following global triviality for normal hyperbolic fibre spaces (cf. [**12**], Main Theorem (3.2) and [**16**], Theorem A).

NOGUCHI'S TRIVIALITY THEOREM FOR HYPERBOLIC FIBRE SPACES. *Let (W, Π, R) be a hyperbolic fibre space. Suppose that (W, Π, R) is hyperbolically imbedded into a compact fibre space $(\overline{W}, \overline{\Pi}, \overline{R})$ along ∂R and that W is normal. If there exists a point $t \in R$ such that $\Gamma(t) := \{s(t) \in W_t : s \in \Gamma\}$ is Zariski dense in W_t, then (W, Π, R) is holomorphically trivial, i.e., there is a biholomorphic mapping $F: W_t \times R \to W$ such that $P = \Pi \circ F$ where $P: W_t \times R \to R$ is the natural projection.*

Noguchi [**16**] considered hyperbolic fibre spaces in a more general setting and obtained the following finiteness theorem for sections and for trivial fibre subspaces of a hyperbolic fibre space, which gave a partial answer to the higher dimensional analogue of Mordell's conjecture over function fields posed by Lang (cf. [8], p. 781 and Remark 3.5 in this section). For the sake of convenience and the later use, we recall it with the sketch of the proof.

DEFINITION 3.2. We say that a fibre space (W, Π, R) is *meromorphically trivial* if (W, Π, R) is bimeromorphically isomorphic to some trivial fibre space over R, i.e., there exist a trivial fibre space $(W_0 \times R, P, R)$ and a bimeromorphic mapping $G: W_0 \times R \to W$ with $P = \Pi \circ G$.

THEOREM 3.3 (cf. [**16**], Theorem B and its correction). *Let (W, Π, R) be a hyperbolic fibre space. Assume that (W, Π, R) is hyperbolically imbedded into some compact fibre space $(\overline{W}, \overline{\Pi}, \overline{R})$ along ∂R. Then (W, Π, R) contains only finitely many meromorphically trivial fibre subspaces with positive dimensional fibres, and carries only finitely many holomorphic sections except for constant ones in those bimeromorphic trivializations.*

PROOF. We have after a finite succession of proper modification, a resolution $\alpha: \overline{R}_0 \to \overline{R}$ such that $\overline{R}_0$ is smooth and $\overline{R}_0 - \alpha^{-1}(R)$ is a hypersurface with only normal crossings. Putting $R_0 := \alpha^{-1}(R)$, the mapping $\alpha | R_0 : R_0 \to R$ is biholomorphic. Put

$$\overline{W}_0 := \overline{R}_0 \times_{\overline{R}} \overline{W}\,, \qquad W_0 := R_0 \times_R W\,.$$

Let $\overline{\Pi}_0 : \overline{W}_0 \to \overline{R}_0$ and $\Pi_0 : W_0 \to R_0$ be the natural projections. Then the fibre space (W_0, Π_0, R_0) is hyperbolic and is hyperbolically imbedded into $(\overline{W}_0, \overline{\Pi}_0, \overline{R}_0)$ along $\partial R_0 := \overline{R}_0 - R_0$. Thus we may assume that $\overline{R}$ is smooth and that $\partial R := \overline{R} - R$ is a hypersurface with only normal crossings. Let Γ be the set of all holomorphic sections of (W, Π, R). Then every $s \in \Gamma$ extends holomorphically to a section of $(\overline{W}, \overline{\Pi}, \overline{R})$ (cf. [**12**], Lemma (2.1)). Γ is identified with the set of all holomorphic sections of $(\overline{W}, \overline{\Pi}, \overline{R})$. Then Γ, endowed with compact-open topology, is compact and carries a complex structure with universal property such that the mapping

$$\Psi : \Gamma \times \overline{R} \ni (s, t) \longmapsto s(t) \in \overline{W}$$

is holomorphic (cf. [**12**], the proof of Main Theorem 3.2). Let

$$\Gamma = \Gamma_1 \cup \cdots \cup \Gamma_l \qquad (l < \infty)$$

be the decomposition into irreducible components. Take any Γ_i with $\dim \Gamma_i > 0$. We denote Γ_i by Γ'. Put

$$\overline{W}' := \Psi(\Gamma' \times \overline{R}), \quad \overline{\Pi}' := \overline{\Pi}|_{\overline{W}'}, \quad W' := \overline{W}'|_R \quad \text{and} \quad \Pi' := \overline{\Pi}'|_{W'}.$$

Then $(\overline{W}', \overline{\Pi}', \overline{R})$ is a compact fibre subspace of $(\overline{W}, \overline{\Pi}, \overline{R})$ with fibres of positive dimension. (W', Π', R) is a hyperbolic fibre space and is hyperbolically imbedded into $(\overline{W}', \overline{\Pi}', \overline{R})$ along ∂R. Let $\overline{W}'_N \xrightarrow{\beta} \overline{W}'$ be the normalization of $\overline{W}'$. Then $\overline{W}'_N$ is an irreducible normal compact complex space and the mapping β from $\overline{W}'_N$ onto $\overline{W}'$ is a proper finite map. Hence $\overline{W}'_{N,t} := (\overline{\Pi}' \circ \beta)^{-1}(t)$ $(= \beta^{-1}(\overline{W}'_t))$ is compact hyperbolic for each $t \in R$. Put

$$W'_N := \beta^{-1}(W'), \quad \overline{\Pi}'_N := \overline{\Pi}' \circ \beta : \overline{W}'_N \to \overline{R} \quad \text{and} \quad \Pi'_N := \overline{\Pi}'_N|_{W'_N}.$$

Then we see that (W'_N, Π'_N, R) is a hyperbolic fibre space and is hyperbolically imbedded into the compact fibre space $(\overline{W}'_N, \overline{\Pi}'_N, \overline{R})$ along ∂R. Put

$$A := \{s \in \Gamma' : s(\overline{R}) \subset \overline{W}'_{sing}\}$$

where $\overline{W}'_{sing}$ denotes the set of all singular points of $\overline{W}'$. Then A is a proper analytic subset of Γ'. For any $s \in \Gamma' - A$, there exists a unique holomorphic mapping s_N from $\overline{R}$ into $\overline{W}'_N$ such that $\beta \circ s_N = s$ on $\overline{R}$ and then $s_N \in \Gamma(\overline{R}, \overline{W}'_N)$. On the other hand, since (W'_N, Π'_N, R) is hyperbolically imbedded into the compact fibre space $(\overline{W}'_N, \overline{\Pi}'_N, \overline{R})$ along ∂R, the set of holomorphic sections of (W'_N, Π'_N, R) forms a normal family (cf. [**12**], Theorem 2.2) and is identified with the set of holomorphic sections $\Gamma(\overline{R}, \overline{W}'_N)$ of $(\overline{W}'_N, \overline{\Pi}'_N, \overline{R})$, which is compact. Thus the mapping

$$\beta_* : \Gamma(\bar{R}, \overline{W}'_N) \ni s_N \longmapsto \beta \circ s_N \in \Gamma(\bar{R}, \overline{W})$$

is proper holomorphic. Therefore we see that there exists an irreducible component Γ'_N of $\beta_*^{-1}(\Gamma')$ such that $\beta_*(\Gamma'_N)=\Gamma'$. It follows that the evaluation mapping

$$\Psi_N : \Gamma_N \times \bar{R} \ni (s_N, t) \longmapsto s_N(t) \in \overline{W}'_N$$

is a proper surjective holomorphic mapping. Then we see that for $t \in \bar{R}$

$$\overline{W}'_{N,t} := \bar{\Pi}'^{-1}_N(t) = \Gamma'_N(t) := \{s_N(t) \in \overline{W}'_N \,;\, s_N \in \Gamma'_N\},$$

which is compact and irreducible. It follows from Noguchi's triviality theorem for hyperbolic fibre spaces (cf. the theorem after Definition 3.1) that (W'_N, Π'_N, R) is a trivial fibre space, i.e., if we denote the general fibre by $W'_{N,0}$, then

$$W'_N \cong W'_{N,0} \times R$$

as fibre spaces. Note that since W'_N is normal and R is nonsingular, the general fibre $W'_{N,0}$ is normal. We see that each element of Γ'_N is a constant section of the trivial fibre space (W'_N, Π'_N, R) and that $W'_{N,0}$ is biholomorphic to Γ'_N. In fact, we can write through the global trivialization

$$\Psi_N|_{\Gamma'_N \times R} : \Gamma'_N \times R \ni (s, t) \longmapsto (\phi_N(s, t), t) \in W'_{N,0} \times R\,.$$

The mapping $\phi_N(\cdot, t) : \Gamma'_N \to W'_{N,0}$ is a surjective holomorphic mapping for each $t \in R$. It follows from Noguchi's finiteness theorem for mappings (cf. § 2) that $\phi_N(\cdot, t)$ is independent of $t \in R$. Thus each element of Γ'_N corresponds to a constant section of the trivial fibre space (W'_N, Π'_N, R'). Since Γ'_N is the space of sections, $\phi_N(\cdot, t) : \Gamma'_N \to W'_{N,0}$ is injective, hence is bijective. Since $W'_{N,0}$ is normal, Γ'_N is biholomorphic to $W'_{N,0}$ by Zariski's main theorem. Since Γ is compact, there are only finitely many zero dimensional irreducible components of Γ, which are finite holomorphic sections except for constant ones of the above meromorphically trivial fibre subspaces.

Let $(V, \Pi|_V, R)$ be a hyperbolic fibre subspace with compact fibres of (W, Π, R) which is meromorphically trivial. We show that $(V, \Pi|_V, R)$ is a fibre subspace of one of the above meromorphically trivial fibre subspaces. Let $V_0 \times R$ be the bimeromorphic trivialization of V where V_0 is a compact complex space. Taking the desingularization $V'_0 \xrightarrow{\alpha'} V_0$ due to Hironaka, we may assume that

$$V'_0 \times R \xrightarrow{\tau} V$$

gives a bimeromorphic trivialization. Since $V'_0 \times R$ is nonsingular and $(V, \Pi|_V, R)$

is a hyperbolic fibre space with compact hyperbolic fibres, we see that the mapping $\tau: V_0' \times R \to V$ becomes holomorphic. Thus the mapping

$$V_0' \ni v \longmapsto \tau(v, \cdot) \in \Gamma$$

is holomorphic. Then we can take the irreducible component which contains the image of V_0' by the above mapping. ■

REMARK 3.4. In the above theorem, since the mapping

$$\beta_* |_{\Gamma_N'} : \Gamma_N' \longmapsto \Gamma'$$

is a proper finite modification, the space Γ_N' is the normalization of Γ'. Then Γ_N' gives the normalization of almost all (except for a proper subvariety of R) fibres W_t' of the trivial fibre space (W_N', Π_N', R). Moreover if $\Pi'(\overline{W}_{sing}') \neq \bar{R}$, then

$$\Gamma' \times (R - \Pi'(\overline{W}_{sing}')) \xrightarrow{\text{onto}} W'|_{R-\Pi'(\overline{W}_{sing}')} \cong \Gamma_N' \times (R - \Pi'(\overline{W}_{sing}')) .$$

Thus it follows that Γ' is biholomorphic to Γ_N', so is normal. In this case Γ' is imbedded into W.

REMARK 3.5. In the above theorem, the assumption of hyperbolically imbeddedness is essentially used in the proof. This condition is automatically satisfied when both the fiber dimension and the base dimension are equal to one, i.e., if (W, Π, R) is a smooth fibre space of curves with genus greater than one and $\dim R = 1$, then there is a compactification $(\overline{W}, \overline{\Pi}, \bar{R})$ of (W, Π, R) such that (W, Π, R) is hyperbolically imbedded into $(\overline{W}, \overline{\Pi}, \bar{R})$ along ∂R (cf. [**12**], §5, [**11**], Theorem 3.11 and [**22**], Theorem on domination). Thus, Noguchi's triviality theorem for hyperbolic fibre spaces (cf. the theorem after Definition 3.1) combined with this gave an another proof of Manin's theorem (an analogue of the Mordell's conjecture over function fields) (cf. [**10**]). In the higher dimensional case, however, every hyperbolic fibre space does not have such a nice imbedding even after a finite Galois extension of the base space. Noguchi gave such an example (cf. [**15**]).

COROLLARY 3.6. *Let* (W, Π, R) *be as in Theorem* 3.3. *If there is a point* $t \in R$ *such that* $\Gamma(t)$ *is Zariski dense in* W_t, *then* (W, Π, R) *is bimeromorphically trivial, more precisely, the fibre space* (W_N, Π_N, R) *obtained by taking the normalization of* W *as in Theorem* 3.3 *is a holomorphically trivial fibre space.*

PROOF. Since $\Gamma(t)$ is Zariski dense in the fibre of t, we can take an irreducible component Γ_0 of Γ such that $\Psi(\bar{R} \times \Gamma_0) = \overline{W}$. In fact, if there is no such Γ_0, the image of t by any irreducible component of Γ is a proper analytic subset of the fibre of t. ■

A fibre space (W, Π, R) is locally trivial if for each point t in R, there exists a neighborhood U of t in R such that $(\Pi^{-1}(U), \Pi|_{\Pi^{-1}(U)}, U)$ is a holomorphically trivial fibre space. In Corollary 3.6, if (W, Π, R) is locally trivial, under the assumption of Corollary 3.6, it is globally trivial.

COROLLARY 3.7. *Let (W, Π, R) be as in Theorem 3.3. Suppose that W is quasiprojective and that the set of all holomorphic sections Γ of (W, Π, R) is infinite. Then there exists a hyperbolic fibre subspace $(W', \Pi|_{W'}, R)$ of (W, Π, R) with irreducible curves as fibres, which is either generically trivial, i.e., there exists a proper subvariety V (which may be empty) of $\bar{R}$ such that the hyperbolic fibre space $(W'|_{R-V}, \Pi|_{(W'|R-V)}, R-V)$ is holomorphically trivial, or locally non-trivial.*

PROOF. Since Γ is infinite, there is an irreducible component Γ' of Γ with $\dim \Gamma'>0$. If $\dim \Gamma'=1$, we put

$$W' := \Psi(\Gamma' \times R) \subset W$$

where Ψ is as in the proof of Theorem 3.3. If $\dim \Gamma' \neq 1$, we can construct a hyperbolic fibre subspace with one dimensional fibres as follows. From the proof of Theorem 3.3 we see that

$$\overline{W}'_N \cong \Gamma'_N \times \bar{R}$$

as fibre spaces where $\overline{W}'_N$ is the normalization of the closure $\overline{W}'$ of W' in $\overline{W}$ and Γ'_N is the normalization of Γ' (see also Remark 3.4). Take $t \in R$ such that $\Gamma'(t) \not\subset W'_{sing}$. Then since $\Gamma'(t)$ is projective algebraic, so is its normalization Γ'_N. Therefore one can find a compact irreducible curve $C' \subset \Gamma'$ so that the dimension of the image of the holomorphic mapping

$$\Psi|_{C' \times \bar{R}} : C' \times \bar{R} \longrightarrow \overline{W}'$$

is equal to $\dim R+1$. Replacing C' with Γ', we put

$$W' := \Psi(\Gamma' \times R) \quad \text{and} \quad \overline{W}' := \Psi(\Gamma' \times \bar{R}) .$$

Now let $\mathfrak{N}(\overline{W}')$ be the non-normal locus:

$$\mathfrak{N}(\overline{W}') := \{x \in \overline{W}' : \overline{W}' \text{ is not normal at } x\}.$$

If $\mathfrak{N}(\overline{W}')=\phi$ (i.e., $\overline{W}$ is normal), then $(W', \Pi|_{W'}, R)$ is globally trivial by Noguchi's triviality theorem (see the theorem after Definition 3.1). Assume that $\mathfrak{N}(\overline{W}') \neq \phi$. Taking the normalization $\overline{W}'_N \xrightarrow{\beta} \overline{W}'$ of $\overline{W}'$ and putting

$$\bar{\Pi}'_N := (\bar{\Pi}|_{\overline{W}'}) \circ \beta, \quad W'_N := \beta^{-1}(W') \quad \text{and} \quad \Pi'_N := \bar{\Pi}'_N|_{W'_N} ,$$

(W'_N, Π'_N, R) is the globally trivial hyperbolic fibre space by Corollary 3.6. If

$\bar{\Pi}(\mathfrak{N}(\overline{W}'))$ is a proper analytic subset of $\bar{R}$, the hyperbolic fibre space

$$(W'|_{R-\bar{\Pi}(\mathfrak{N}(\overline{W}'))}, \text{ the restriction of } \Pi,\ R-\bar{\Pi}(\mathfrak{N}(\overline{W}')))$$

is globally trivial. Suppose that $\bar{\Pi}(\mathfrak{N}(\overline{W}'))=\bar{R}$. Let $p:\overline{W}'_N\to\Gamma'_N$ be the natural projection through the identification of $\overline{W}'_N$ with $\Gamma'_N\times\bar{R}$. Put $S_N:=p(\beta^{-1}(\mathfrak{N}(\overline{W}')))$, $S:=\beta_*(S_N)$ and

$$A:=\{s\in\Gamma'\,;\,s(\bar{R})\subset\mathfrak{N}(\overline{W}')\}.$$

Then clearly $A\subset S$. Since the restriction $\beta_*|_{\Gamma'_N-\beta_*^{-1}(A)}:\Gamma'_N-\beta_*^{-1}(A)\to\Gamma'-A$ is biholomorphic and $\beta_*^{-1}(S)=S_N$, the mapping

$$\beta_*|_{\Gamma'_N-S_N}:\Gamma'_N-S_N\longrightarrow\Gamma'-S$$

is biholomorphic. In general, we have $S_N(\bar{R})\supset\beta^{-1}(\mathfrak{N}(\overline{W}'))$. If S_N is a proper analytic subset of Γ'_N, then S_N is a finite set. Hence $S_N(\bar{R})=\beta^{-1}(\mathfrak{N}(\overline{W}'))$. Thus we see that $S(\bar{R})=\mathfrak{N}(\overline{W}')$ and that the restriction

$$\Psi|_{(\Gamma'-S)\times\bar{R}}:(\Gamma'-S)\times\bar{R}\longrightarrow\overline{W}'-\mathfrak{N}(\overline{W}')$$

is a biholomorphism. The former equality implies that $A=S$ and therefore $A(\bar{R})=\mathfrak{N}(\overline{W}')$. Since A is finite, it follows that

$$\Psi:\Gamma'\times\bar{R}\longrightarrow\overline{W}'$$

is bijective, hence it induces the isomorphism as fibre spaces. Suppose that $S_N=\Gamma'_N$. Let

$$\beta^{-1}(\mathfrak{N}(\overline{W}'))=\bigcup_{j=0}^{\nu}V_j\qquad(\nu<\infty)$$

be the irreducible decomposition. If there exists a point t at which $(\overline{W}',\bar{\Pi}|_{\overline{W}'},\bar{R})$ is locally trivial, i.e., $(\bar{\Pi}|_{\overline{W}'})^{-1}(U)\cong\Gamma'(t)\times U$ where U is some neighborhood of t in $\bar{R}$, then $\Gamma'(t)$ is a finite set of non-normal points. Since the pull-back of those points by β_* is a finite subset of Γ'_N, the projection p is locally constant on $\bar{\Pi}'^{-1}_N(U)\cap\beta^{-1}(\mathfrak{N}(\overline{W}'))$. Thus p is constant on each irreducible component which intersects $\bar{\Pi}'^{-1}_N(U)$, in particular, on each V_j with $\bar{\Pi}'_N(V_j)=\bar{R}$. We see that the image of sum of the other components by $\bar{\Pi}'_N$ becomes a proper analytic subset of $\bar{R}$. We complete the proof. ∎

REMARK. Corollary 3.7 is a slightly more detailed version of Corollary 3.6 in Noguchi [**12**].

It is necessary to take the normalization of the total space in the proof of Theorem 3.3. We give an example of the non-normal hyperbolic fibre spaces with infinitely many sections which is locally nontrivial. The author wishes to thank Professor Tetsuo Ueda for his help in constructing this example.

EXAMPLE 3.8. Let R be a compact Riemann surface of genus greater than one. Let σ be a holomorphic automorphism of R which is not the identity mapping and ι be the identity mapping of R. Put

$$\hat{\sigma}(t) = (\sigma(t), t) \in R\times R \qquad \text{for } t\in R$$

and

$$\hat{\iota}(t) = (t, t) \in R\times R \qquad \text{for } t\in R .$$

We define an equivalence relation on $R\times R$ as follows: for $y_1, y_2\in R\times R$, $y_1\sim y_2$ if and only if there exists a point $t\in R$ such that $y_1=\hat{\sigma}(t)$ and $y_2=\hat{\iota}(t)$. Put $W:= R\times R/\sim$. Then we see that W is a complex space and that the projection $\beta: R\times R\to W$ is holomorphic. Let Π be the projection such that $\Pi\circ\beta=P_2$ on $R\times R$ where $P_2: R\times R\to R$ is the second projection. Then (W, Π, R) is a hyperbolic space with compact hyperbolic fibres and carries infinitely many sections which come from the trivial fibre space $(R\times R, P_2, R)$ through the projection β. Since $R\times R$ is the normalization of W, the trivial fibre space $(R\times R, P_2, R)$ gives the meromorphic trivialization of (W, Π, R). The fibre space (W, Π, R) is locally nontrivial. In fact, suppose that there exists a local trivialization $\varphi: W|_U \xrightarrow{\cong} W_0\times U$ where U is an open set in R and W_0 is an irreducible curve. We take the normalizations of the domain and the image of the localization and consider the lifting $\tilde{\varphi}$ of the mapping φ between the two normalizations. Then we see that $\tilde{\varphi}$ generates infinitely many holomorphic automorphisms of R. This is absurd since R is compact hyperbolic.

4. Finiteness of nontrivial sections and of trivial fibre subspace in noncompact case.

We consider about finiteness of trivial fibre subspaces in the case where all the fibres are noncompact. We treat only a special case, trivial fibre spaces, but the result supplies information about the moduli of holomorphic mappings. Let $\bar{X}$ be an irreducible compact complex space and X be a nonsingular Zariski open subset of $\bar{X}$.

THEOREM 4.1. *Let X be as above. Let Y be an irreducible complete hyperbolic complex space. Suppose that Y is hyperbolically imbedded into some compact complex space $\bar{Y}$ and Y is Zariski open in $\bar{Y}$. Then the trivial fibre space $(Y\times X, P, X)$ contains only finitely many meromorphically trivial fibre subspaces where P is the natural projection (i.e., any meromorphically trivial fibre subspace of $(\bar{Y}\times\bar{X}, P, \bar{X})$ is a trivial fibre subspace of one of them) and carries only finitely many holomorphic sections except for constant ones in those bimeromorphic trivializations.*

PROOF. Let $(V, P|_V, X)$ be a meromorphically trivial fibre subspace of $(Y\times X, P, X)$ such that the closure $\bar{V}$ of V in $\bar{Y}\times\bar{X}$ is a subvariety of $\bar{Y}\times\bar{X}$. Let $V\sim V_0\times X$ be the bimeromorphic trivialization. Then there exists a fibre V_t of $(V, P|_V, X)$ such that V_t is bimeromorphic to V_0. Let $\bar{V}_t$ be the closure of V_t in $\bar{Y}$, so that $\bar{V}_t$ is a compact subvariety of $\bar{Y}$. Let $\bar{V}_r\xrightarrow{\alpha}\bar{V}_t$ be the resolution of singularities such that $\partial V_r:=\bar{V}_r-\alpha^{-1}(V_t)$ is a hypersurface with only normal crossings and put $V_r:=\alpha^{-1}(V_t)$. We denote by f the bimeromorphic mapping of $V_r\times X$ to V. Since Y is complete hyperbolic and $V_r\times X$ is nonsingular, then the meromorphic mapping $P_Y\circ f$ of $V_r\times X$ to Y is holomorphic where $P_Y: Y\times X\to Y$ is the first projection (cf. [5], Chapter VIII, Theorem 1.2). We may assume that $\bar{X}$ is smooth and $\partial X:=\bar{X}-X$ is a hypersurface with only normal crossings. $Hol(X, Y)$ is a Zariski open subset of a compact analytic subspace $\overline{Hol(X, Y)}$ of $Hol(\bar{X}, \bar{Y})$ by Noguchi's structure theorem (cf. the theorem before Proposition 2.1). Let $\bigcup_{j=1}^{l}\bar{Z}_j$ $(l<\infty)$ be the irreducible decomposition of $\overline{Hol(X, Y)}$ and put $Z_j:=\bar{Z}_j\cap Hol(X, Y)$ for each j. Note that all the mappings in Z_j have the same rank (see Proposition 2.1). The mapping

$$\Psi: \overline{Hol(X, Y)}\times\bar{X}\ni(s, x)\longmapsto(s(x), x)\in\bar{Y}\times\bar{X}$$

is an injective holomorphic mapping and through this correspondence $Hol(X, Y)$ is regarded as the space of sections of the fibre space $(Y\times X, P, X)$. We denote the constant holomorphic section $X\ni x\mapsto(v, x)\in V_r\times X$ for $v\in V_r$ by the same letter v. Put $V_r':=\{v\in V_r; v(X)\not\subset \text{the indeterminacy locus of } f\}$. Then V_r' is a nonempty Zariski dense open subset of $\bar{V}_r$. The mapping

$$\sigma: V_r'\ni v\longmapsto P_Y\circ f\circ v\in Hol(X, Y)$$

is holomorphic. Let $\bar{V}_r''\xrightarrow{\alpha'}\bar{V}_r$ be the modification along $\bar{V}_r-V_r'$ such that the mapping $\alpha'|_{V_r''}: V_r''\to V_r'$ is biholomorphic and $\partial V_r'':=\bar{V}_r''-V_r''$ is a hypersurface with only normal crossings where $V_r'':=\alpha'^{-1}(V_r')$. Since $Hol(X, Y)$ is complete hyperbolic and hyperbolically imbedded into $\overline{Hol(X, Y)}$ (cf. Proposition 2.6, i)), the holomorphic mapping $\sigma\circ(\alpha'|_{V_r''}): V_r''\to Hol(X, Y)$ is holomorphically extended to the mapping $\overline{\sigma\circ(\alpha'|_{V_r''})}$ of $\bar{V}_r''$ to $\overline{Hol(X, Y)}$. Let $\bar{Z}_j$ be the component which contains $\overline{\sigma\circ(\alpha'|_{V_r''})}(\bar{V}_r'')$ and put $Z:=Z_j$. Then $\bar{V}_r$ is meromorphically imbedded into $\bar{Z}$ so that $\dim\bar{Z}\geqq\dim\bar{V}_r=\dim V_0$.

Case (a). Let Z be compact. Put $W:=\Psi(Z\times X)$. Then $\bar{W}=\Psi(Z\times\bar{X})$ is a compact complex subspace of $\bar{Y}\times\bar{X}$. Put $\bar{\Pi}:=P|_{\bar{W}}$ and $\Pi:=\bar{\Pi}|_W$. Since for any $x\in X$

$$W_x:=\Pi^{-1}(x)=\{(s(x), x)\in Y\times X; s\in Z\}\subset Y\times\{x\},$$

the fibre space (W, Π, X) is a hyperbolic fibre space whose fibres are compact. It is clear that (W, Π, X) is hyperbolically imbedded into $(\bar{W}, \bar{\Pi}, \bar{X})$ along ∂X. Thus by Corollary 3.6 if we take the normalizations W_N and Z_N of W and Z

respectively, then W_N is isomorphic to $Z_N \times X$ as fibre space. In particular, (W, Π, X) is bimeromorphically equivalent to the trivial fibre space $(Z_N \times X, P, X)$.

Case (b). Let Z be noncompact. In this case Z is a Zariski open subset of the compact complex space $\bar{Z}$ by Noguchi's structure theorem (cf. the theorem before Proposition 2.1 in § 2). It follows from the complete hyperbolicity of Y that for each $s \in \partial Z := \bar{Z} - Z$, the image of $\bar{X}$ under the extension of s is contained in ∂Y. Let W and W_x be as in case (a). Then for $x \in X$

$$\overline{W}_x = \{(s(x), x) \in \bar{Y} \times X ; s \in \bar{Z}\} \subset \bar{Y} \times \{x\}$$

where $\overline{W}_x$ is the closure of W_x in $\bar{Y} \times X$. Since Y is complete hyperbolic and hyperbolically imbedded into $\bar{Y}$, W_x is also complete hyperbolic and hyperbolically imbedded into $\overline{W}_x$. $\overline{W} = \Psi(\bar{Z} \times \bar{X})$ is a compact complex subspace of $\bar{Y} \times \bar{X}$. Put

$$\bar{\Pi} := P|_{\overline{W}} \quad \text{and} \quad \Pi := \bar{\Pi}|_W .$$

Then (W, Π, X) is a hyperbolic fibre space with complete hyperbolic fibres which are hyperbolically imbedded into compact complex spaces as Zariski open subsets. $\bar{Z}$ is identified via the mapping $\bar{Z} \ni s \mapsto \Psi(s, \cdot) \in Hol(\bar{X}, \overline{W})$ with some compact irreducible component $\bar{\Gamma}$ of the complex space $\Gamma(\bar{X}, \overline{W})$ of all section of the fibre space $(\overline{W}, \bar{\Pi}, \bar{X})$. Since ∂Z is a proper analytic subset of $\bar{Z}$, Z corresponds to a Zariski open dense subset Γ of $\bar{\Gamma}$. Put $\partial\Gamma := \bar{\Gamma} - \Gamma$. Note that $\gamma(\bar{X}) \subset \partial W := \overline{W} - W$ for $\gamma \in \partial\Gamma$. Take the normalization $\overline{W}_N \xrightarrow{\beta} \overline{W}$ of $\overline{W}$. Then $\overline{W}_N$ is an irreducible compact normal complex space. Put

$$W_N := \beta^{-1}(W), \quad \bar{\Pi}_N := \bar{\Pi} \circ \beta, \quad \Pi_N := \bar{\Pi}_N|_{W_N}$$

and for $x \in X$,

$$W_{N,x} := (\Pi \circ \beta)^{-1}(x) = \beta^{-1}(W_x) .$$

Then for $x \in X$,

$$\overline{W}_{N,x} = (\bar{\Pi} \circ \beta)^{-1}(x) = \beta^{-1}(\overline{W}_x) ,$$

and $W_{N,x}$ is complete hyperbolic and is hyperbolically imbedded into $\overline{W}_{N,x}$. Thus (W_N, Π_N, X) is a hyperbolic fibre space whose fibres have the above properties. We can find an irreducible compact complex subspace $\bar{\Gamma}_N$ of $\Gamma(\bar{X}, \overline{W}_N)$ such that $\beta_*(\bar{\Gamma}_N) = \bar{\Gamma}$ where β_* is the pull-back of $\Gamma(\bar{X}, \overline{W}_N)$ to $\Gamma(\bar{X}, \overline{W})$. Put $\Gamma_N := \beta_*^{-1}(\Gamma) \cap \bar{\Gamma}_N$. Then $\Gamma_N = \bar{\Gamma}_N - \beta_*^{-1}(\partial\Gamma)$ is a Zariski open subset of $\bar{\Gamma}_N$. Let Ψ_N be the holomorphic mapping of $\bar{\Gamma}_N \times \bar{X}$ into $\overline{W}_N$ which is the restriction of the evaluation mapping of $Hol(\bar{X}, \overline{W}_N) \times \bar{X}$ into $\overline{W}_N$. Then we see that

$$\Psi_N(\bar{\Gamma}_N \times \bar{X}) = \overline{W}_N, \quad \Psi_N(\Gamma_N \times X) = W_N, \quad \Psi_N(\bar{\Gamma}_N \times \{x\}) = \overline{W}_{N,x},$$

and

$$\Psi_N(\Gamma_N \times \{x\}) = W_{N,x} \qquad \text{for } x \in \bar{X} .$$

Note that since $\bar{\Gamma}_N$ is irreducible, $\overline{W}_{N,x}$ is irreducible and so is $W_{N,x}$. Applying the argument in the proof of Main Theorem 3.2 of Noguchi [**12**] (cf. [**12**], p. 37), we can construct the horizontal direction field η in the fibre space $(\overline{W}_N, \overline{\Pi}_N, \overline{X})$. Then we see that the fibre space $(\overline{W}_N, \overline{\Pi}_N, \overline{X})$ is a holomorphic fibre bundle with typical fibre $\overline{W}_{N,0}$. If we put $\widetilde{W}_N := \overline{\Psi}_N(\Gamma_N \times \overline{X})$ and $\widetilde{\Pi}_N := \overline{\Pi}_N|_{\widetilde{W}_N}$, it follows from the construction of η that the fibre subspace $(\widetilde{W}_N, \widetilde{\Pi}_N, \overline{X})$ is a hyperbolic fibre subbundle of $(\overline{W}_N, \overline{\Pi}_N, \overline{X})$ with typical fibre $W_{N,0}$, whose structure group is the holomorphic automorphism group of $W_{N,0}$, i.e., for a sufficiently small neighborhood U of a point $x \in \overline{X}$, there is a trivialization $\overline{W}_N|_U \cong \overline{W}_{N,0} \times U$ and $\widetilde{W}_N|_U \cong W_{N,0} \times U$. Since $\overline{W}_N|_U$ is normal, so is $\overline{W}_{N,0}$ and $W_{N,0}$ also. Through this local trivialization, consider the following mapping:

$$\overline{\Psi}_N : \Gamma_N \times U \ni (\gamma, x) \longmapsto (\psi(\gamma, x), x) \in W_{N,0} \times U .$$

Since the mappings $\psi(\cdot, x) : \Gamma_N \to W_{N,0}$ are surjective for all $x \in U$, it follows from Theorem 2.3 that $\psi(\cdot, x)$ does not depend on $x \in U$. This implies that for two distinct elements $\gamma_1, \gamma_2 \in \Gamma_N$, $\gamma_1(x) \neq \gamma_2(x)$ for any $x \in \overline{X}$ since Γ_N is the space of sections. Thus the holomorphic mapping $\psi(\cdot, x)$ of Γ_N to $W_{N,0}$ is bijective. Since $W_{N,0}$ is normal, it becomes the biholomorphism between them. Then we see that the mapping $\overline{\Psi}_N|_{\Gamma_N \times \overline{X}} : \Gamma_N \times \overline{X} \to \widetilde{W}_N$ is biholomorphic. Since $\beta(\widetilde{W}_N|_X) = W$, we see that (W, Π, X) is meromorphically trivial. ■

From the above proof we obtain an estimate of the dimension of moduli spaces of the holomorphic mappings with intermediate rank.

COROLLARY 4.2. *Let X and Y be as in Theorem* 4.1 *and $f \in Hol(X, Y)$. If $f(X)$ is not relatively compact in Y, then the dimension of the irreducible component of $Hol(X, Y)$ which contains f is not greater than the dimension of ∂Y.*

REMARK 4.3. In the case where Y is the quotient space of a bounded symmetric domain by a torsion free arithmetic discrete subgroup of the identity component of the holomorphic automorphism group, Corollary 4.2 was obtained in Noguchi [**13**], Theorem 4.7 (iv).

References

[1] A. Douady, Le problème des modules pour les sous-espaces analytiques compacts d'un espace analytique donné, Ann. Inst. Fourier (Grenoble), **16** (1966), 1-95.

[2] Y. Imayoshi, Generalization of de Franchis theorem, Duke Math. J., **50** (1983), 393-408.

[3] Y. Imayoshi, Holomorphic maps of projective algebraic manifolds into C-hyperbolic manifolds, J. Math. Soc. Japan, **46** (1994), 289-307.

[4] Y. Imayoshi and H. Shiga, A finiteness theorem for holomorphic families of Riemann surfaces, in Proc. Holomorphic Functions and Moduli, MSRI Berkeley, 1986.

[5] S. Kobayashi, Hyperbolic Manifolds and Holomorphic Mappings, Marcel Dekker, New York, 1970.

[6] S. Kobayashi and T. Ochiai, Satake compactification and the great Picard theorem, J. Math. Soc. Japan, **23** (1971), 340-350.
[7] A. Kodama, On bimeromorphic automorphisms of hyperbolic complex spaces, Nagoya Math. J., **73** (1979), 1-5.
[8] S. Lang, Higher dimensional Diophantine problems, Bull. Amer. Math. Soc., **80** (1974), 779-787.
[9] S. Lang, Introduction to Complex Hyperbolic Spaces, Springer-Verlag, New York-Berlin-Heidelberg, 1987.
[10] Ju. I. Manin, Rational points of algebraic curves over function fields, Izv. Akad. Nauk SSSR Ser. Mat., **27** (1963), 1395-1440.
[11] T. Miyano and J. Noguchi, Moduli spaces of harmonic and holomorphic mappings and Diophantine geometry, in Prospects in Complex Geometry, Proc. 25th Taniguchi International Symposium, Katata/Kyoto, 1989, Lecture Notes in Math., **1468**, Springer-Verlag, Heidelberg, 1991.
[12] J. Noguchi, Hyperbolic fibre spaces and Mordell's conjecture over function fields, Publ. Res. Inst. Math. Sci. Kyoto University, **21** (1985), 27-46.
[13] J. Noguchi, Moduli spaces of holomorphic mappings into hyperbolically imbedded complex spaces and locally symmetric spaces, Invent. Math., **93** (1988), 15-34.
[14] J. Noguchi, Hyperbolic manifolds and Diophantine geometry, Sugaku Expositions, Amer. Math. Soc., Providence, Rhode Island, 1991. (Translation of Japanese version, Sugaku, **41** (1989), 320-334.)
[15] J. Noguchi, An example of a hyperbolic fiber space without hyperbolic embedding into compactification, Proceedings of Osaka International Conference, Osaka 1990.
[16] J. Noguchi, Meromorphic mappinngs into compact hyperbolic complex spaces and geometric Diophantine problems, Internat. J. Math., **3** (1992), 277-289, and its correction (to appear).
[17] J. Noguchi and T. Ochiai, Geometric Function Theory in Several Complex Variables, Transl. Math. Monographs., **80**, Amer. Math. Soc., Providence, Rhode Island, 1990.
[18] J. Noguchi and T. Sunada, Finiteness of the family of rational and meromorphic mappings into algebraic varieties, Amer. J. Math., **104** (1982), 887-900.
[19] T. Sunada, Holomorphic mappings into a compact quotient of symmetric bounded domain, Nagoya Math. J., **64** (1976), 159-175.
[20] R. Tsushima, Rational maps to varieties of hyperbolic type, Proc. Japan Acad. Ser. A, **55** (1979), 95-100.
[21] T. Urata, Holomorphic mappings into taut complex analytic spaces, Tohoku Math. J., **31** (1979), 349-353.
[22] M. G. Zaidenberg, A function-field analog of the Mordell conjecture: A noncompact version, Math. USSR-Izv., **35** (1990), 61-81.

Makoto SUZUKI

Department of Mathematics
Faculty of Science
Hiroshima University
Higashi-Hiroshima 724
Japan

Current Address
Department of Civil Engineering
Faculty of Engineering
Hiroshima Institute of Technology
Miyake, Saeki-ku, Hiroshima 731-51
Japan