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An exact sequence related to Adams-Novikov E_2-terms of a cofibering

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§1. Introduction.

Let E be a ring spectrum. The E-Adams spectral sequence with computable E_2-term is a useful tool for computing stable homotopy groups of spectra. We concern here about the BP-Adams spectral sequence $E_2=\mathrm{Ext}_{BP_*BP}^{s,*}(BP_*, BP_*(X))\Rightarrow\pi_{*-s}(X)$, in which BP is the Brown-Peterson spectrum at a prime p with the coefficient ring $BP_*=\pi_*(BP)=\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots]$. For a spectrum X, we especially concern for the sphere spectrum S^0 and the Toda-Smith spectrum $V(n)$. In their paper [8], Miller, Ravenel, and Wilson have introduced the chromatic spectral sequence to compute the E_2-term of the above spectral sequence. Let $I_i=(p, v_1, \cdots, v_{i-1})$ be the invariant prime ideal of BP_*. They have defined BP_*BP-comodules A_i^j and LA_i^j by taking $A_i^0=\pi_*(BP)/I_i$, $LA_i^j=v_{i+j}^{-1}A_i^j$ and the short exact sequences $0\to A_i^j\to LA_i^j\to A_i^{j+1}\to 0$, inductively. The chromatic spectral sequence converging to $\mathrm{Ext}_{BP_*BP}^s(BP_*, BP_*/I_i)$ is given by the chromatic resolution

$$0\longrightarrow BP_*/I_i\longrightarrow LA_i^0\longrightarrow LA_i^1\longrightarrow\cdots$$

associated to the short exact sequence above. Then they have defined "the universal Greek letter map" $\eta:\mathrm{Ext}_{BP_*BP}^s(BP_*, A_i^j)\to\mathrm{Ext}_{BP_*BP}^{s+j}(BP_*, BP_*/I_i)$ by taking the composition of the coboundary homomorphisms, and the Greek letter elements to be the image of η.

In this paper we also consider the Johnson-Wilson spectrum $E(n)$ for $n\geqq 0$ with $E(n)_*=\boldsymbol{Z}_{(p)}[v_1, v_2, \cdots, v_n, v_n^{-1}]$. Define $E(n)_*E(n)$-comodules B_i^j and LB_i^j in the same way as the case of BP by replacing the first step of the induction by $B_i^0=E(n)_*/I_i$ (see §3). Throughout the paper, we use the following notation for the Hopf algebroids appeared above:

$$(A, \Gamma)=(BP_*, BP_*BP)$$

and

$$(B, \Sigma)=(E(n)_*, E(n)_*E(n)=E(n)_*\otimes_A\Gamma\otimes_A E(n)_*).$$

Let L_nX (resp. C_nX) for a spectrum X denote the Bousfield localization (resp. acyclization) with respect to the spectrum $E(n)$ for $n>0$. Then we have the cofibration

$$V(i-1) \longrightarrow L_n V(i-1) \longrightarrow C_n V(i-1)$$

for the Toda-Smith spectrum $V(i-1)$, which is known to exist if $i-1<4$ and the prime $p \geqq 2i-1$ (cf. [**10**]). Applying the homotopy $\pi_*(-)$ gives rise to the exact sequence $\pi_*(V(i-1)) \to \pi_*(L_n V(i-1)) \to \pi_*(C_n V(i-1))$ and we have the Adams-Novikov spectral sequences computing these homotopy groups. The maps in the cofiber sequence above induce the maps of these E_2-terms

$$\mathrm{Ext}^s_\Gamma(A, A^0_i) \longrightarrow \mathrm{Ext}^s_\Sigma(B, B^0_i),$$

$$\mathrm{Ext}^s_\Sigma(B, B^0_i) \longrightarrow \mathrm{Ext}^{s-t}_\Gamma(A, A^{t+1}_i)$$

and

$$\mathrm{Ext}^{s-t}_\Gamma(A, A^{t+1}_i) \xrightarrow{\eta} \mathrm{Ext}^{s+1}_\Gamma(A, A^0_i).$$

Our main result of this paper is to show that these maps give the exact sequence in the following results:

THEOREM A. *For $i+j \leqq n$, we denote $t=n-i-j$. Then there exists an exact sequence*

$$\cdots \xrightarrow{\eta} \mathrm{Ext}^s_\Gamma(A, A^j_i) \to \mathrm{Ext}^s_\Sigma(B, B^j_i) \to \mathrm{Ext}^{s-t}_\Gamma(A, A^{n-i+1}_i) \xrightarrow{\eta} \mathrm{Ext}^{s+1}_\Gamma(A, A^j_i) \to \cdots,$$

in which $\mathrm{Ext}^{s-t}_\Gamma=0$ *for* $s-t<0$.

Here we note that if $j=0$, then every map in the sequence is induced from the corresponding one in the cofibering $V(i-1) \to L_n V(i-1) \to C_n V(i-1)$ if the spectrum $V(i-1)$ exists. Therefore this is the exact sequence of E_2-terms of the Adams-Novikov spectral sequences computing homotopy groups of these spectra and so we obtain some information on the homotopy from the E_2-terms that are generally studied algebraically (see § 6).

We obtain this exact sequence by the Mahowald spectral sequence converging to $\mathrm{Ext}^t_\Sigma(B, B^j_i)$ whose differential d_r is related to the one of the BP-Adams spectral sequence (cf. [**3**], [**6**]). Its E_2-term is given by $\mathrm{Ext}^s_\Gamma(A, \mathrm{Ext}^t_\Sigma(B, B\Gamma^j_i))$, where $B\Gamma^j_i$ is a Σ-comodule defined by $B\Gamma^j_i = B^j_i \otimes_A \Gamma$. The E_2-term now is read off from

THEOREM B. *For $i+j<n$,*

$$\mathrm{Ext}^t_\Sigma(B, B\Gamma^j_i) = \begin{cases} A^j_i & \text{for } t=0, \\ A^{n-i+1}_i & \text{for } t=n-i-j, \\ 0 & \text{otherwise.} \end{cases}$$

Furthermore, we have the following theorem for the Mahowald differential d_r and the universal Greek letter map η.

THEOREM C. *For* $t=n-i-j$,

$$d_{t+1}=(-1)^s\eta : \mathrm{Ext}_\Gamma^s(A, A_i^{n-i+1}) \longrightarrow \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j).$$

Then the exact sequence of Theorem A follows from the standard argument of the spectral sequence (see Th. 3.20).

These results are generalized in the following sense: Suppose that a spectrum X has the BP_*-homology $BP_*(X)=A_j^i\otimes_A F_*$ for a connected locally finite free A-module F_*. Then the E_2-terms for the homotopy groups of X, L_nX and C_nX are given by

$$\mathrm{Ext}_\Gamma^*(A, A_j^i\otimes_A F_*),$$

$$\mathrm{Ext}_\Sigma^*(B, B_j^i\otimes_A F_*)$$

and

$$\mathrm{Ext}_\Gamma^*(A, A_j^{n-j+1}\otimes_A F_*),$$

respectively. We now have the following theorem relating with the cofibering

$$X \longrightarrow L_nX \longrightarrow C_nX.$$

THEOREM D. *There exists an exact sequence*

$$\cdots \xrightarrow{\eta} \mathrm{Ext}_\Gamma^s(A, A_j^i\otimes_A F_*) \to \mathrm{Ext}_\Sigma^s(B, B_j^i\otimes_A F_*) \to \mathrm{Ext}_\Gamma^{s-t}(A, A_j^{n-j+1}\otimes_A F_*)$$

$$\xrightarrow{\eta} \mathrm{Ext}_\Gamma^{s+1}(A, A_j^i\otimes_A F_*) \to \cdots,$$

for the integers i and j with $i+j\leqq n$, in which $t=n-i-j$ and $\mathrm{Ext}_\Gamma^{s-t}=0$ *for* $s-t<0$.

As an example of X, we have $W_k(i)$, which is defined to be a spectrum that has BP_*-homology

$$BP_*(W_k(i))=BP_*/I_{i+1}[t_1, \cdots, t_k]=A_{i+1}^0\otimes_A A[t_1, \cdots, t_k]$$

as a $BP_*(BP)$-subcomodule algebra of $BP_*(BP)/I_{i+1}=\boldsymbol{Z}_{(p)}[t_1, t_2, \cdots]$. Then $W_k(-1)$ is Ravenel's spectrum $T(k)$ and $W_k(i)$ would be $T(k)\wedge V(i)$ if $V(i)$ exists. We have the existence of $W_k(4)$ for $k>0$, though we have no idea about $V(4)=W_0(4)$ ([**14**], [**13**]). We also know the existence of $W_k(i)$ for $k\geqq i$ ([**13**]).

This paper is organized as follows. In § 2 we study about the Mahowald and the Cartan-Eilenberg spectral sequences and then chromatic spectral sequence in § 3. We prove Theorem C in § 4 by studying the definitions of these spectral sequences. We then give a lemma in § 5 to give more general results. In the last section we give some remarks on these results.

The authors would like to thank the referee for suggesting them that the results are applied not only to Toda-Smith spectra but to other spectra.

§2. Mahowald spectral sequence.

In this section we prepare some notations in the homological algebra of the Hopf algebroids (cf. [2], [7], [10, Appendix 1]), and prove that the Mahowald spectral sequence is the same as the Cartan-Eilenberg one.

Let (A, Γ) be a Hopf algebroid with a coproduct $\Delta_\Gamma: \Gamma \to \Gamma \otimes_A \Gamma$ and N a right Γ-comodule with a coaction $\psi_\Gamma: N \to N \otimes_A \Gamma$. Then we have the cobar complex $C_\Gamma^*(N)$ consisting of Γ-comodules

$$C_\Gamma^s(N) = N \otimes_A \Gamma^s \ (\Gamma^s = \Gamma \otimes_A \cdots \otimes_A \Gamma \text{ with } s \text{ factors of } \Gamma)$$

and differentials δ_Γ given by

$$\begin{aligned}\delta_\Gamma(m \otimes \gamma_s \otimes \cdots \otimes \gamma_1) = {} & m \otimes \gamma_s \otimes \cdots \otimes \gamma_1 \otimes 1 \\ & + \textstyle\sum_{i=1}^s (-1)^i m \otimes \gamma_s \otimes \cdots \otimes \Delta_\Gamma(\gamma_i) \otimes \cdots \otimes \gamma_1 \\ & + (-1)^{s+1} \psi_\Gamma(m) \otimes \gamma_s \otimes \cdots \otimes \gamma_1\end{aligned}$$

for $m \in N$ and $\gamma_i \in \Gamma$ with $1 \leqq i \leqq s$. We define $\mathrm{Ext}_\Gamma^s(A, N)$ for a Γ-comodule N as the cohomology $H^s(C_\Gamma^*(N); \delta_\Gamma)$ of the cobar complex $C_\Gamma^*(N)$. By the definition of the coboundary homomorphism, we have the following

LEMMA 2.1. *Let* $\alpha: 0 \to N \xrightarrow{i} M \xrightarrow{j} L \to 0$ *be a short exact sequence of right* Γ-*comodules and* $\delta^\alpha: \mathrm{Ext}_\Gamma^s(A, L) \to \mathrm{Ext}_\Gamma^{s+1}(A, N)$ *the coboundary homomorphisms. If there exist elements* $x \in C_\Gamma^s(L)$, $y \in C_\Gamma^s(M)$ *and* $z \in C_\Gamma^{s+1}(N)$ *such that* $j_*(y) = x$ *and* $\delta_\Gamma(y) = i_*(z)$, *then*

$$\delta^\alpha([x]) = [z],$$

for $[x] \in H^s(C_\Gamma^*(L))$ *and* $[z] \in H^{s+1}(C_\Gamma^*(N))$. *Here* $[x]$ *denotes the cohomology class represented by* x *and* i_* *and* j_* *are the induced maps on the cohomologies.*

Next, let (B, Σ) be another Hopf algebroid with a coproduct $\Delta_\Sigma: \Sigma \to \Sigma \otimes_B \Sigma$ and N a left Σ- and right Γ-comodule with coactions $\psi_\Sigma: N \to \Sigma \otimes_B N$ and $\psi_\Gamma: N \to N \otimes_A \Gamma$ such that $(1 \otimes \psi_\Gamma)\psi_\Sigma = (\psi_\Sigma \otimes 1)\psi_\Gamma$. Then there exists a cobar double complex $C_{\Sigma\text{-}\Gamma}^{s,t} = \Sigma^t \otimes_B C_\Gamma^s(N)$ with differentials δ_Γ and δ_Σ defined by

$$\delta_\Gamma = 1 \otimes \delta_\Gamma: C_{\Sigma\text{-}\Gamma}^{s,t}(N) \longrightarrow C_{\Sigma\text{-}\Gamma}^{s+1,t}(N)$$

and

$$\begin{aligned}\delta_\Sigma(\sigma_1 \otimes \sigma_2 \otimes \cdots \otimes \sigma_t \otimes m \otimes \gamma) = {} & 1 \otimes \sigma_1 \otimes \cdots \otimes \sigma_t \otimes m \otimes \gamma \\ & + \sum_{i=1}^t (-1)^i \sigma_1 \otimes \cdots \otimes \Delta_\Sigma(\sigma_i) \otimes \cdots \otimes \sigma_t \otimes m \otimes \gamma \\ & + (-1)^{t+1} \sigma_1 \otimes \cdots \otimes \psi_\Sigma(m) \otimes \gamma\end{aligned}$$

for $\sigma_i \in \Sigma$, $m \in N$ and $\gamma \in \Gamma^s$. We note that $\delta_\Sigma \delta_\Gamma = \delta_\Gamma \delta_\Sigma$. Consider a total complex $T^m = \bigoplus_{s+t=n} C^{s,t}_{\Sigma-\Gamma}(N)$ with a differential $\delta_T(x) = \delta_\Gamma(x) + (-1)^{s+1}\delta_\Sigma(x)$ for $x \in C^{s,t}_{\Sigma-\Gamma}(N)$. Then we obtain two spectral sequences $\{E^{s,t}_r, d_r\}$ and $\{\bar{E}^{s,t}_r, d_r\}$ converging to $H^{s+t}(T^*; \delta_T)$ by filtering

$$T^{s+t} = F^{0,s+t} \supset \cdots \supset F^{s,t} = \bigoplus_{a \geqq s} C^{a,s+t-a}_{\Sigma-\Gamma}(N) \supset F^{s+1,t-1} \supset \cdots \supset F^{s+t,0} \supset 0$$

and

$$T^{s+t} = \bar{F}^{s+t,0} \supset \cdots \supset \bar{F}^{s,t} = \bigoplus_{b \geqq t} C^{s+t-b,b}_{\Sigma-\Gamma}(N) \supset \bar{F}^{s-1,t+1} \supset \cdots \supset \bar{F}^{0,s+t} \supset 0.$$

Consider the following commutative diagrams

$$\begin{array}{ccccccccc} 0 & \longrightarrow & F^{s+1,t-1} & \subset & F^{s,t} & \longrightarrow & C^{s,t}_{\Sigma-\Gamma}(N) & \longrightarrow & 0 \\ & & \downarrow \delta_T & & \downarrow \delta_T & & \downarrow (-1)^{s+1}\delta_\Sigma & & \\ 0 & \longrightarrow & F^{s+1,t} & \subset & F^{s,t+1} & \longrightarrow & C^{s,t+1}_{\Sigma-\Gamma}(N) & \longrightarrow & 0 \end{array}$$

and

$$\begin{array}{ccccccccc} 0 & \longrightarrow & \bar{F}^{s-1,t+1} & \subset & \bar{F}^{s,t} & \longrightarrow & C^{s,t}_{\Sigma-\Gamma}(N) & \longrightarrow & 0 \\ & & \downarrow \delta_T & & \downarrow \delta_T & & \downarrow \delta_\Gamma & & \\ 0 & \longrightarrow & \bar{F}^{s,t+1} & \subset & \bar{F}^{s+1,t} & \longrightarrow & C^{s+1,t}_{\Sigma-\Gamma}(N) & \longrightarrow & 0, \end{array}$$

and we have $E^{s,t}_1 = H^t(C^{s,*}_{\Sigma-\Gamma}(N); (-1)^{s+1}\delta_\Sigma)$, $\bar{E}^{s,t}_1 = H^s(C^{*,t}_{\Sigma-\Gamma}(N); \delta_\Gamma)$, $d_1 = \delta_{\Gamma*}$: $H^t(C^{s,*}_{\Sigma-\Gamma}(N)) \to H^t(C^{s+1,*}_{\Sigma-\Gamma}(N))$ and $\bar{d}_1 = (-1)^{s+1}\delta_{\Sigma*}$: $H^s(C^{*,t}_{\Sigma-\Gamma}(N)) \to H^s(C^{*,t+1}_{\Sigma-\Gamma}(N))$ by definition. Then we have the following

LEMMA 2.2. i) *If Γ is A-flat, then*

$$E^{s,t}_2 = H^s(H^t(C^{*,*}_{\Sigma-\Gamma}(N); (-1)^{s+1}\delta_\Sigma); \delta_{\Gamma*})$$
$$= \mathrm{Ext}^s_\Gamma(A, \mathrm{Ext}^t_\Sigma(B, N)),$$

and if Σ is B-flat, then

$$\bar{E}^{s,t}_2 = H^t(H^s(C^{*,*}_{\Sigma-\Gamma}(N); \delta_\Gamma); (-1)^{s+1}\delta_{\Sigma*})$$
$$= \mathrm{Ext}^t_\Sigma(B, \mathrm{Ext}^s_\Gamma(A, N)).$$

ii) *If* $\mathrm{Ext}^s_\Gamma(A, N)=0$ *for* $s>0$ *and* Σ *is B-flat, then the sequence* $\{E^{s,t}_r\}$ *converges to* $H^t(T^*; \delta_T) = \bar{E}^{0,t}_2 = \mathrm{Ext}^t_\Sigma(B, \mathrm{Ext}^0_\Gamma(A, N))$.

iii) *For* $x \in C^{s,t}_{\Sigma-\Gamma}(N)$, *if* $\delta_\Sigma(x)=0$ *and* $d_i([x])=0$ *for* $1 \leqq i \leqq r$ ($[x] \in E^{s,t}_{i+1}$), *then there exist elements* $x_i \in C^{s+i,t-i}_{\Sigma-\Gamma}(N)$ *for* $0 \leqq i \leqq r$ *such that*

$$x_0 = x,\ \delta_\Gamma(x_i) = (-1)^{s+i+1}\delta_\Sigma(x_{i+1}),\quad \textit{and} \tag{2.3}$$
$$d_{r+1}([x]) = [\delta_\Gamma(x_r)] \in E^{s+r+1,t-r}_{r+1}.$$

PROOF. i) and ii) are trivial.

iii) In the case $r=0$, this is trivial. We proceed by induction on r. Assume first that there exist r and x_i above and $d_{r+1}([x]) = [\delta_\Gamma(x_r)] = 0 \in$

$E_{r+1}^{s+r+1,t-r}$. This implies that $[\delta_\Gamma(x_r)] = d_r([y]) \in E_r^{s+r+1,t-r}$ for some $[y] \in E_r^{s+1,t-1}$, and so there exist $y_j \in C_{\Sigma-\Gamma}^{s+j+1,t-j-1}(N)$ for $0 \leqq j \leqq r-1$ such that $\delta_\Sigma(y_0) = 0$, $[y] = [y_0] \in E_r^{s+1,t-1}$, $\delta_\Sigma(y_j) = (-1)^{s+j+2}\delta_\Sigma(y_{j+1})$ and $[\delta_\Gamma(y_{r-1})] = [\delta_\Gamma(x_r)] \in E_r^{s+r+1,t-r}$. Then, for elements $x_0' = x_0$ and $x_i' = x_i - y_{i-1}$ $(1 \leqq i \leqq r)$, (2.3) holds and $[\delta_\Gamma(x_r')] = 0 \in E_r^{s+r+1,t-r}$. Repeating this process, we may assume that $[\delta_\Gamma(x_r)] = 0 \in E_1^{s+r+1,t-r} = H^{t-r}(C_{\Sigma-\Gamma}^{s+r+1,*}(N))$ for elements of (2.3). Hence there exists $x_{r+1} \in C_{\Sigma-\Gamma}^{s+r+1,t-r-1}(N)$ such that $\delta_\Gamma(x_r) = (-1)^{s+r+1}\delta_\Sigma(x_{r+1})$. Moreover, $\delta_T(x_{i+1}) = \delta_\Gamma(x_{i+1}) + (-1)^{s+i+2}\delta_\Sigma(x_{r+1}) = \delta_\Gamma(x_{i+1}) - \delta_\Gamma(x_i)$, and so $\delta_\Gamma(x_{r+1}) = \delta_\Gamma(x_r) + \delta_T(x_{r+1}) = \delta_\Gamma(x_{r-1}) + \delta_T(x_r + x_{r+1}) = \cdots = \delta_\Gamma(x_0) + \delta_T(\sum_{i=1}^{r+1} x_i)$ and $\sum_{i=1}^{r+1} x_i \in F^{s+1,t-1}$. This implies the desired equation $d_{r+2}([x_0]) = [\delta_\Gamma(x_{r+1})]$ by the definition of d_r.

q.e.d.

Now let E and F be ring spectra and $\lambda: E \to F$ a ring map. We assume that $E_*(E)$ and $F_*(F)$ are flat over $\pi_*(E)$ and $\pi_*(F)$, respectively. We apply the argument above for the case of $(A, \Gamma) = (\pi_*(E), E_*(E))$, $(B, \Sigma) = (\pi_*(F), F_*(F))$ and $N = F_*(X \wedge E)$ for any spectrum X. The ring map λ induces a map $\lambda_*: (A, \Gamma) \to (B, \Sigma)$ of Hopf algebroids, and we see that $\mathrm{Ext}_\Gamma^s(A, N) = 0$ for $s > 0$, and $= F_*(X)$ for $s = 0$ by the same way as the proof of [**3**, Lemma 2.2]. Hence we have a spectral sequence of Lemma 2.2.

PROPOSITION 2.4. *There exists a spectral sequence* $\{E_r^{s,t}, d_r\}$ *converging to* $\mathrm{Ext}_\Sigma^{s+t}(B, F_*(X))$ *with* E_2*-term* $E_2^{s,t} = \mathrm{Ext}_\Gamma^s(A, \mathrm{Ext}_\Sigma^t(B, F_*(X \wedge E)))$.

On the other hand, by [**3**] and [**6**], we have a Mahowald spectral sequence $\{E(Mah)_r^{s,t}, d_r^{Mah}\}$ converging to $\mathrm{Ext}_\Sigma^{s+t}(B, F_*(X))$ as follows:

Consider the cofibering $\alpha: S^0 \xrightarrow{\iota} E \to \bar{E}$ associated to the unit map $\iota: S^0 \to E$, and the spectra $\bar{E}^s \wedge X = \bar{E} \wedge \cdots \wedge \bar{E} \wedge X$ (s-times smash products). Then it gives rise to the long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_\Sigma^t(B, F_*(\bar{E}^s \wedge X)) \longrightarrow \mathrm{Ext}_\Sigma^t(B, F_*(E \wedge \bar{E}^s \wedge X))$$
$$\longrightarrow \mathrm{Ext}_\Sigma^t(B, F_*(\bar{E}^{s+1} \wedge X)) \longrightarrow \mathrm{Ext}_\Sigma^{t+1}(B, F_*(\bar{E}^s \wedge X)) \longrightarrow \cdots,$$

which defines an exact couple and the associated spectral sequence is the Mahowald one. By definition, $E(Mah)_1^{s,t} = \mathrm{Ext}_\Sigma^t(B, F_*(E \wedge \bar{E}^s \wedge X))$ and $E(Mah)_2^{s,t} = \mathrm{Ext}_\Gamma^s(A, \mathrm{Ext}_\Sigma^t(B, F_*(X \wedge E)))$ (cf. [**3**, Th. 4.7]).

THEOREM 2.5. *Let* $\{E_r^{s,t}, d_r\}$ *be the spectral sequence of Proposition* 2.4. *For* $r=2$ *and* $x \in C_{\Sigma-\Gamma}^{s,t}(F_*(X \wedge E))$, $E(Mah)_2^{s,t} = E_2^{s,t}$ *and* $d_r^{Mah}([x]) = (-1)^{s+r+1} d_r([x])$ $([x] \in E(Mah)_r^{s,t} = E_r^{s,t})$, *and so* $E(Mah)_{r+1}^{s,t} = E_{r+1}^{s,t}$ *for* $r \geqq 1$.

PROOF. If there exist elements x_i of Lemma 2.2 (iii), then $d_{r+1}^{Mah}([x]) = (-1)^{s+r+1}[\delta_\Gamma(x_r)]$ by [**3**, Lemma 4.10]. This implies the above theorem.

q.e.d.

§ 3. Chromatic spectral sequences.

The chromatic spectral sequence is introduced by [8] to compute the E_2-term of the BP-Adams spectral sequence for the sphere spectrum and for the Toda-Smith spectrum. In the same way we can construct similar one for another ring spectrum as follows:

Let $E(n)$ be the Johnson-Wilson ring spectrum with the coefficient ring $\pi_*(E(n)) = \boldsymbol{Z}_{(p)}[v_1, \cdots, v_n, v_n^{-1}]$. We denote that $(A, \Gamma) = (\pi_*(BP), BP_*(BP))$, $(B, \Sigma)=(\pi_*(E(n)), E(n)_*(E(n)))=(B, B\otimes_A\Gamma\otimes_A B)$, and $B\Gamma=E(n)_*(BP)=B\otimes_A\Gamma$. For $N=A$, B and $B\Gamma$, we define N_i^j and $LN_i^j=v_{i+j}^{-1}N_i^j$ inductively by taking $N_i^0=N/I_i$ and short exact sequences

$$0 \longrightarrow N_i^j \xrightarrow{i} LN_i^j \xrightarrow{\pi} N_i^{j+1} \longrightarrow 0 \tag{3.1}$$

of Γ-comodules (or Σ-comodules). Then we obtain a long exact sequence

$$0 \longrightarrow N_i^j \longrightarrow LN_i^j \longrightarrow LN_i^{j+1} \longrightarrow \cdots \tag{3.2}$$

by splicing (3.1).

In the first place, consider the case of $N=A$. In this case, (3.1) gives us a long exact sequence

$$\cdots \longrightarrow \mathrm{Ext}_\Gamma^s(A, A_i^j) \xrightarrow{i_*} \mathrm{Ext}_\Gamma^s(A, LA_i^j) \xrightarrow{\pi_*} \mathrm{Ext}_\Gamma^s(A, A_i^{j+1}) \xrightarrow{\delta} \cdots. \tag{3.3}$$

Now this sequence gives us the exact couple and the chromatic spectral sequence converging to $\mathrm{Ext}_\Gamma^{s+j}(A, A_i^0)$ with E_1-term $\mathrm{Ext}_\Gamma^s(A, LA_i^j)$ (cf. [8], [**10**, § 5]). Then (3.2) for $j=0$ is the chromatic resolution

$$0 \longrightarrow BP_*/I_i \longrightarrow LA_i^0 \longrightarrow LA_i^1 \longrightarrow \cdots. \tag{3.4}$$

"The universal Greek letter map"

$$\eta^{t+1} = \delta \cdots \delta : \mathrm{Ext}_\Gamma^s(A, A_i^{j+t+1}) \longrightarrow \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j), \tag{3.5}$$

is introduced in [8] and defines related Greek letter elements. By Lemma 2.1, we have the following

LEMMA 3.6. *For any cocycle x of $C_\Gamma^s(A_i^{j+t+1})$, we can find a set of elements $y\in C_\Gamma^{s+t+1}(A_i^j)$ and $x_k\in C_\Gamma^{s+k}(LA_i^{j+t-k})$ for $0\leqq k\leqq t$ such that $\pi_*(x_0)=x$, $\delta_\Gamma(x_k)=i_*\pi_*(x_{k+1})$ for $0\leqq k\leqq t-1$ and $\delta_\Gamma(x_t)=i_*(y)$. Then $\eta^{t+1}([x])=[y]$.*

For the E_1-term of the chromatic spectral sequence, we know the Morava vanishing Theorem (cf. [**10**, § 6]). Let (K_i, Σ_i) denote the Hopf algebroid $(K(i)_*, K(i)_*K(i))$ studied in [**10**, § 6]. Here $K_i=K(i)_*=\boldsymbol{Z}/(p)[v_i, v_i^{-1}]$ and $\Sigma_i=K(i)_*K(i)=K(i)_*\otimes_A\Gamma\otimes_A K(i)_*$.

THEOREM 3.7. *If* $(p-1)\nmid i$ *then* $\mathrm{Ext}^s_{\Sigma_i}(K_i, K_i)=0$ *for* $s>i^2$.

Since we have the isomorphism $\mathrm{Ext}^s_\Gamma(A, LA^0_i)=\mathrm{Ext}^s_{\Sigma_i}(K_i, K_i)$ of [7, Th. 2.10], we see immediately the following

COROLLARY 3.8. *If* $(p-1)\nmid i$ *then* $\mathrm{Ext}^s_\Gamma(A, LA^0_i)=0$ *for* $s>i^2$.

COROLLARY 3.9. i) ([8, Cor. 3.17]). *If* $(p-1)\nmid(i+j)$, *then* $\mathrm{Ext}^s_\Gamma(A, LA^j_i)=0$ *for* $s>(i+j)^2$.

ii) *If* $p-1$ *does not divide integers* $i+j+k$ *for* $0\leqq k\leqq t$, *then the universal Greek letter map* η^{t+1} *of* (3.5) *is an isomorphism for* $s>(i+j+t)^2$, *and an epimorphism for* $s=(i+j+t)^2$.

Next consider the case of $N=B$. Then (3.1) and (3.2) give us a long exact sequence

$$(3.10)\qquad \cdots\longrightarrow \mathrm{Ext}^t_\Sigma(B, B^j_i)\longrightarrow \mathrm{Ext}^t_\Sigma(B, LB^j_i)\longrightarrow \mathrm{Ext}^t_\Sigma(B, B^{j+1}_i)\longrightarrow\cdots$$

and a resolution

$$(3.11)\qquad 0\longrightarrow B^j_i\longrightarrow LB^j_i\longrightarrow LB^{j+1}_i\longrightarrow\cdots\longrightarrow LB^{n-i}_i\longrightarrow 0.$$

We notice that

$$(3.12)\qquad B^{n-i}_i=LB^{n-i}_i \quad\text{and}\quad B^j_i=LB^j_i=0 \quad\text{for } i+j>n.$$

By the exact couple, we have a spectral sequence such that it converging to $\mathrm{Ext}^{s+t}_\Sigma(B, B^j_i)$ and its E_1-term is $\mathrm{Ext}^t_\Sigma(B, LB^{j+s}_i)$. For this E_1-term, we have the following

PROPOSITION 3.13. *If* $(p-1)\nmid(i+j)$, *then*

$$\mathrm{Ext}^t_\Sigma(B, LB^j_i)=0 \qquad \textit{for}\quad t>(i+j)^2.$$

PROOF. Noticing that $\Sigma=B\otimes_A\Gamma\otimes_A B$, we have $\Sigma_i=K_i\otimes_B\Sigma\otimes_B K_i$, which induces the map $(B, \Sigma)\to(K_i, \Sigma_i)$ of Hopf algebroids. We also obtain an isomorphism $\Sigma\otimes_B K_i\cong LB^0_i\otimes_{K_i}\Sigma_i$ from the one $\Gamma\otimes_A K_i\cong LA^0_i\otimes_{K_i}\Sigma_i$ shown in [7, Prop. 2.4]. This gives another isomorphism $(\Sigma\otimes_B K_i)\square_{\Sigma_i}K_i\cong LB^0_i$. Apply now Shapiro's Lemma [7, Prop. 1.4], and obtain an isomorphism

$$\mathrm{Ext}^t_\Sigma(B, LB^0_i)\cong\mathrm{Ext}^t_{\Sigma_i}(K_i, K_i),$$

which is zero if $(p-1)\nmid i$ and $t>i^2$ by Theorem 3.7. We now proceed by induction on j. Consider the long exact sequence associated to the short exact sequence

$$0\longrightarrow LB^{j-1}_{i+1}\longrightarrow LB^j_i\xrightarrow{v_i} LB^{j}_{i}{}'\longrightarrow 0,$$

and we see that the induced map $v_{i*}: \mathrm{Ext}^t_\Sigma(B, LB^j_i) \to \mathrm{Ext}^t_\Sigma(B, LB^j_i)$ is an isomorphism by the assumption of induction and so $\mathrm{Ext}^t_\Sigma(B, LB^j_i)=0$ for $t>(i+j)^2$ and $(p-1) \nmid (i+j)$ since this Ext-group is v_i-torsion. q.e.d.

If $p-1$ does not divide any $i+j+k$ for $0 \leqq k \leqq n-i-j$, then $\mathrm{Ext}^{t-k}_\Sigma(B, LB^{j+k}_i)=0$ for $t>n^2+n-i-j$ by the above proposition and $t-k>n^2+n-i-j-k \geqq n^2 \geqq (i+j+k)^2$. Moreover, $LB^{j+k}_i=0$ for $k>n-i-j$, and so the spectral sequence associated to (3.10) implies immediately the following

COROLLARY 3.14. *If $p-1$ does not divide any of $i+j, i+j+1, \cdots, n$, then* $\mathrm{Ext}^t_\Sigma(B, B^j_i)=0$ *for* $t>n^2+n-i-j$.

In the last place, consider the case of $N=B\Gamma$. We note that $B\Gamma^j_i=LB\Gamma^j_i$ for $i+j=n$ and $B\Gamma^j_i=LB\Gamma^j_i=0$ for $i+j>n$.

LEMMA 3.15. *For $i \leqq n$,* $\mathrm{Ext}^t_\Sigma(B, LB\Gamma^0_i)=LA^0_i$ *for $s=0$, and $=0$ for $s>0$.*

PROOF. This is similarly shown as the first step of the proof of Proposition 3.13. By [7, Prop. 2.8], we have $(\Gamma \otimes_A K_i)\square_{\Sigma_i}(K_i \otimes_A \Gamma) \cong LA^0_i \otimes_A \Gamma$, and so $B \otimes_A LA^0_i \otimes_A \Gamma \cong (B \otimes_A \Gamma \otimes_A K_i)\square_{\Sigma_i}(K_i \otimes_A \Gamma)=(\Sigma \otimes_B K_i)\square_{\Sigma_i}(K_i \otimes_A \Gamma)$. Noticing that $B \otimes_A LA^0_i \otimes_A \Gamma = LB\Gamma^0_i$, we have $\mathrm{Ext}^t_\Sigma(B, LB\Gamma^0_i)=\mathrm{Ext}^t_\Sigma(B, (\Sigma \otimes_B K_i)\square_{\Sigma_i}(K_i \otimes_A \Gamma)) \cong \mathrm{Ext}^t_{\Sigma_i}(K_i, K_i \otimes_A \Gamma)$ by the Shapiro's lemma [7, Prop. 1.4]. Moreover we have

$$\mathrm{Ext}^t_{\Sigma_i}(K_i, K_i \otimes_A \Gamma) = \mathrm{Ext}^t_{\Sigma_i}(K_i, \Sigma_i \otimes_{K_i} LA^0_i) = \begin{cases} LA^0_i & \text{for } t=0, \\ 0 & \text{for } t>0, \end{cases}$$

by [7, Prop. 2.4]. q.e.d.

A short exact sequence $0 \to LB\Gamma^{j-1}_{i+1} \to LB\Gamma^j_i \xrightarrow{v_i} LB\Gamma^j_i \to 0$ induces a commutative diagram

$$\begin{array}{ccccccccc}
0 \longrightarrow & LA^{j-1}_{i+1} & \longrightarrow & LA^j_i & \xrightarrow{v_i} & LA^j_i & \longrightarrow & 0 \\
& \downarrow h & & \downarrow h & & \downarrow h & & \\
0 \to & \mathrm{Ext}^0_\Sigma(B, LB\Gamma^{j-1}_{i+1}) & \to & \mathrm{Ext}^0_\Sigma(B, LB\Gamma^j_i) & \xrightarrow{v_i} & \mathrm{Ext}^0_\Sigma(B, LB\Gamma^j_i) & \to & \mathrm{Ext}^1_\Sigma(B, LB\Gamma^{j-1}_{i+1}).
\end{array}$$

By induction on j with the lemma above, we see that $\mathrm{Ext}^t_\Sigma(B, LB\Gamma^j_i)=0$ for $t>0$ and h is an isomorphism since LA^j_i and $LB\Gamma^j_i$ are v_i-torsion as long as $j>0$.

THEOREM 3.16. *For $i+j \leqq n$,* $\mathrm{Ext}^t_\Sigma(B, LB\Gamma^j_i)=LA^j_i$ *for $t=0$, and $=0$ for $t>0$.*

This theorem shows that the sequence (3.2) for $N=B\Gamma$

$$0 \longrightarrow B\Gamma^j_i \longrightarrow LB\Gamma^j_i \longrightarrow LB\Gamma^{j+1}_i \longrightarrow \cdots \longrightarrow LB\Gamma^{n-i}_i \longrightarrow 0 \tag{3.17}$$

is a Σ-resolution of $B\Gamma_i^j$ with $\mathrm{Ext}_\Sigma^t(B, LB\Gamma_i^{j+a})=0$ for all $t>0$. By [7, Lemma 1.1], this fact and (3.2) for $N=A$ imply the following

THEOREM 3.18.

$$\mathrm{Ext}_\Sigma^t(B, B\Gamma_i^j) = \begin{cases} A_i^j & t=0, \\ A_i^{n-i+1} & t=n-i-j, \\ 0 & \text{otherwise}, \end{cases} \quad \text{for } i+j<n,$$
$$= \begin{cases} LA_i^j & t=0, \\ 0 & t>0, \end{cases} \quad \text{for } i+j=n, \text{ and}$$
$$=0 \quad \text{for } i+j>n.$$

Now, consider the spectral sequence $\{E_r^{s,t}, d_r\}$ of Lemma 2.2 for $N=B\Gamma_i^j$. Since $B\Gamma_i^j=B_i^j\otimes_A\Gamma$, we see that $\mathrm{Ext}_\Gamma^s(A, B\Gamma_i^j)=0$ for $s>0$ and $=B_i^j$ for $s=0$. Then Lemma 2.2 shows that this spectral sequence converges to $\mathrm{Ext}_\Sigma^{s+t}(B, B_i^j)$ and has E_2-term $E_2^{s,t}=\mathrm{Ext}_\Gamma^s(A, \mathrm{Ext}_\Sigma^t(B, B\Gamma_i^j))$. By the above theorem, $d_r=0$ for $r\neq n-i-j+1$, and so we have two exact sequences

$$\text{(3.19)} \quad \begin{aligned} &0 \longrightarrow E_\infty^{s,0} \longrightarrow \mathrm{Ext}_\Sigma^s(B, B_i^j) \longrightarrow E_\infty^{s-t,t} \longrightarrow 0, \quad \text{and} \\ &0 \longrightarrow E_\infty^{s-t,t} \longrightarrow E_2^{s-t,t} \xrightarrow{d_{i+1}} E_2^{s+1,0} \longrightarrow E_\infty^{s+1,0} \longrightarrow 0, \end{aligned}$$

in which $t=n-i-j$. By splicing (3.19), we have the exact sequence of Theorem A.

THEOREM 3.20. i) $\mathrm{Ext}_\Gamma^s(A, LA_i^{n-i})=\mathrm{Ext}_\Sigma^s(B, B_i^{n-i})$.

ii) $\mathrm{Ext}_\Sigma^s(B, B_i^j)=\mathrm{Ext}_\Gamma^s(A, LA_i^j)$ *for* $s<n-i-j$.

iii) *The following is a long exact sequence*

$$\cdots \xrightarrow{d_{i+1}} \mathrm{Ext}_\Gamma^s(A, A_i^j) \xrightarrow{\lambda_*} \mathrm{Ext}_\Sigma^s(B, B_i^j) \longrightarrow \mathrm{Ext}_\Gamma^{s-t}(A, A_i^{n-i+1}) \xrightarrow{d_{i+1}} \mathrm{Ext}_\Gamma^{s+1}(A, A_i^j) \xrightarrow{\lambda_*} \cdots,$$

in which $t=n-i-j$ *and* λ_* *denotes the induced map from the canonical map* λ: $\Gamma\to\Sigma$ *of Hopf algebroids, and note that* $\mathrm{Ext}_\Gamma^{s-t}=0$ *for* $s-t<0$.

We notice that since λ_* is induced from the Thom map $BP\to E(n)$, it is also obtained from the localization map $X\to L_nX$. In fact, the Thom map induces the map between the Adams-Novikov spectral sequences based on BP and $E(n)$ which converges to the homotopy groups of X and L_nX, respectively. We further notice that Corollary 3.9(ii) follows from the exact sequence above and Corollary 3.14 for $n=i+j+t$.

§ 4. Mahowald's differential and the Greek letter map.

In this section, we use the following notation. The short exact sequences (3.1) for $N=A$ and $B\Gamma$ induce the exact sequences

$$0 \longrightarrow C_{\Sigma-\Gamma}^{s,t}(B\Gamma_i^j) \xrightarrow{i_{BP}} C_{\Sigma-\Gamma}^{s,t}(LB\Gamma_i^j) \xrightarrow{\pi_{BP}} C_{\Sigma-\Gamma}^{s,t}(B\Gamma_i^{j+1}) \longrightarrow 0, \text{ and}$$

$$0 \longrightarrow C_\Gamma^s(A_i^j) \xrightarrow{i_A} C_\Gamma^s(LA_i^j) \xrightarrow{\pi_A} C_\Gamma^s(A_i^{j+1}) \longrightarrow 0.$$

The Hurewitz map $h: A=\pi_*(BP)\to B\Gamma=E(n)_*(BP)$ induces two homomorphisms $h_A: C_\Gamma^s(A_i^j)\to C_{\Sigma-\Gamma}^{s,0}(B\Gamma_i^j)$ and $h_{LA}: C_\Gamma^s(LA_i^j)\to C_{\Sigma-\Gamma}^{s,0}(LB\Gamma_i^j)$. Then we take $\{C^{s,t,u}, \delta_\Gamma, \delta_\Sigma, \delta_L\}$ as follows:

$$C^{s,t,u}=\begin{cases} C_{\Sigma-\Gamma}^{s,t}(LB\Gamma_i^{j+u}) & \text{for } s\geqq 0,\ t\geqq 0,\ u\geqq 0,\\ C_{\Sigma-\Gamma}^{s,t}(B\Gamma_i^j) & \text{for } s\geqq 0,\ t\geqq 0,\ u=-1,\\ C_\Gamma^s(LA_i^{j+u}) & \text{for } s\geqq 0,\ t=-1,\ u\geqq 0,\\ C_\Gamma^s(A_i^j) & \text{for } s\geqq 0,\ t=-1,\ u=-1, \text{ and}\\ 0 & \text{otherwise,}\end{cases}$$

$\delta_\Gamma: C^{s,t,u}\to C^{s+1,t,u}$ for $s\geqq 0$ and $\delta_\Sigma: C^{s,t,u}\to C^{s,t+1,u}$ for $t\geqq 0$ are the same as mentioned before in § 2, δ_Σ for $t=-1$ is h_{LA} for $u\geqq 0$ and h_A for $u=-1$, δ_L: $C^{s,t,u}\to C^{s,t,u+1}$ for $u\geqq 0$ is $\pi_{B\Gamma}i_{B\Gamma}$ for $t\geqq 0$ and $\pi_A i_A$ for $t=-1$, and δ_L for $u=-1$ is $i_{B\Gamma}$ for $t\geqq 0$ and i_A for $t=-1$.

PROPOSITION 4.1. i) *If $t=n-i-j\geqq 0$ and there exists an element $x(s,t,-1)\in C^{s,t,-1}=C_{\Sigma-\Gamma}^{s,t}(B\Gamma_i^j)$ with $\delta_\Sigma(x(s,t,-1))=0$, then we have elements $x(a,b,c)\in C^{a,b,c}$ for $a+b+c=s+t-1$ such that*

$$(4.2)\qquad \begin{aligned} &x(a,b,c)=0 \qquad \textit{for}\quad a<s \textit{ or } c<-1, \textit{ and}\\ &\delta_\Sigma(x(a,b,c))=\delta_\Gamma(x(a-1,b+1,c))+(-1)^a\delta_L(x(a,b+1,c-1)).\end{aligned}$$

ii) *For the case of $c=-1$,*

$$d_{t+1}([x(s,t,-1)])=(-1)^\varepsilon[\delta_\Gamma(x(s+t,0,-1))],$$

in which $\varepsilon=t(2s+t+1)/2$, $[x(s,t,-1)]\in E_2^{s,t}=H^s(H^t(C_{\Sigma-\Gamma}^{*,*}(B\Gamma_i^j)))$ and $[\delta_\Gamma(x(s+t,0,-1))]\in E_2^{s+t+1,0}=H^{s+t+1}(H^0(C_{\Sigma-\Gamma}^{*,*}(B\Gamma_i^j)))$ in the spectral sequence of Lemma 2.2.

iii) *For the case of $b=-1$,*

$$\eta^{t+1}([\pi_A(x(s,-1,t))])=(-1)^{\varepsilon'}[x(s+t+1,-1,-1)],$$

in which $\varepsilon'=(t+1)(2s+t)/2$, $[\pi_A(x(s,-1,t))]\in \mathrm{Ext}_\Gamma^s(A, A_i^{j+t+1})=H^s(C_\Gamma^*(A_i^{j+t+1}))$, $[x(s+t+1,-1,-1)]\in \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j)=H^{s+t+1}(C_\Gamma^*(A_i^j))$, and η^{t+1} is the Greek *letter map of* (3.5).

iv) *A correspondence of* $x(s, t, -1)$ *to* $\pi_A(x(s, -1, t))$ *induces an isomorphism* $f: \mathrm{Ext}^t_\Sigma(B, B\Gamma^j_i)\otimes_A\Gamma^s = H^t(C^{s,*}_{\Sigma-\Gamma}(B\Gamma^j_i)) \to A^{j+t+1}_i\otimes\Gamma^s = C^s_\Gamma(A^{j+t+1}_i)$.

PROOF. i) We shall adopt the induction on $k = a+c-s+1$. For $k \leqq 0$, (4.2) is equivalent to

$$x(a, b, c) = 0 \quad \text{for} \quad a < s \text{ or } c < -1, \quad \text{and}$$
$$\delta_\Sigma(x(s, t, -1)) = 0.$$

Assume that there exist elements $x(a, b, c)$ with (4.2) for $0 \leqq a+c-s+1 < k$. We see that $n=i+j+t=i+j+c+(a-s)+(b+1)\geqq i+j+c$ and $n-i-j=t=k-1+b+1>b+1$ if we put $k=a+c-s+1$, $a\geqq s\geqq 0$, $b\geqq -1$ and $c\geqq -1$, which shows $H^{b+1}(C^{a,*,c};\delta_\Sigma)=0$ by Theorems 3.16 and 3.18. We now compute

$$\begin{aligned}
&\delta_\Sigma(\delta_\Gamma(x(a-1, b+1, c))+(-1)^a\delta_L(x(a, b+1, c-1)))\\
&\quad = \delta_\Gamma\delta_\Sigma(x(a-1, b+1, c))+(-1)^a\delta_L\delta_\Sigma(x(a, b+1, c-1))\\
&\quad = \delta_\Gamma(\delta_\Gamma(x(a-2, b+2, c))+(-1)^{a-1}\delta_L(x(a-1, b+2, c-1)))\\
&\qquad +(-1)^a\delta_L(\delta_\Gamma(x(a-1, b+2, c-1))+(-1)^a\delta_L(x(a, b+2, c-2)))\\
&\quad = 0,
\end{aligned}$$

and so we have $x(a, b, c)$ with $\delta_\Sigma(x(a, b, c))=\delta_\Gamma(x(a-1, b+1, c))+(-1)^a\delta_L(x(a, b+1, c-1))$.

ii) For $c=-1$, (4.2) implies $\delta_\Sigma(x(a, b, -1))=\delta_\Gamma(x(a-1, b+1, -1))$. Then we can take x_i in Lemma 2.2 iii) by $(-1)^{\varepsilon(i)}x(s+i, t-i, -1)$, where $\varepsilon(i) = i(2s+i+1)/2$, and so

$$d_{t+1}[x(s, t, -1)] = (-1)^{\varepsilon(t)}[\delta_\Gamma(x(s+t, 0, -1))].$$

iii) By (4.2), $\delta_\Gamma(x(a, -1, c))+(-1)^{a+1}\delta_L(x(a+1, -1, c-1)) = \delta_\Sigma(x(a+1, -2, c))=0$. Then we can take x_k and y in Lemma 3.6 by

$$x_k = (-1)^{\varepsilon'(k)}x(s+k, -1, t-k) \quad \text{and} \quad y = (-1)^{\varepsilon'(t+1)}x(s+t+1, -1, -1),$$

where $\varepsilon'(k)=k(2s+k-1)/2$, and so $\eta^{t+1}([\pi_*(x(s, -1, t))]) = (-1)^{\varepsilon'(t+1)}[x(s+t+1, -1, -1)]$.

iv) By (4.2), $x(s-1, b+1, c)=0$, and so $\delta_\Sigma(x(s, b, c)) = (-1)^s\delta_L(x(s, b+1, c-1))$. We take x_k by $x(s, t-k, k-1)$. Then $x_0=x(s, t, -1)$, $x_{t+1}=x(s, -1, t)$ and

$$\delta_\Sigma(x_k) = (-1)^s\delta_L(x_{k+1}) \quad \text{for} \quad 0\leqq k\leqq t. \tag{4.3}$$

For another $x'_k(0\leqq k\leqq t+1)$ satisfying (4.3) and $[x'_0]=[x_0]\in H^t(C^{s,*}_{\Sigma-\Gamma}(B\Gamma^j_i))$, we see that $\pi_A(x'_{t+1})=\pi_A(x_{t+1})\in A^{j+t+1}_i\otimes\Gamma^s$ by Theorem 3.16 and (3.2) for $N=A$,

and so f is well defined. Moreover, f is isomorphic by the diagram chasing.
q.e.d.

Now, $\delta_\Sigma = h_A : C^{s+t+1,-1,-1} = C_\Gamma^{s+t+1}(A_i^j) \to C^{s+t+1,0,-1} = C_{\Sigma-\Gamma}^{s+t+1,0}(B\Gamma_i^j)$ induces an isomorphism $h' : C_\Gamma^{s+t+1}(A_i^j) \overset{\cong}{\to} \mathrm{Ext}_\Sigma^0(B, B\Gamma_i^j) \otimes \Gamma^{s+t+1} = H^0(C_{\Sigma-\Gamma}^{s+t+1,*}(B\Gamma_i^j); \delta_\Sigma)$ by (3.17) and Theorem 3.18, and so an isomorphism $h'_* : \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j) = H^{s+t+1}(C_\Gamma^*(A_i^j)) \overset{\cong}{\to} E_2^{s+t+1,0} = \mathrm{Ext}_\Gamma^{s+t+1}(A, \mathrm{Ext}_\Sigma^0(B, B\Gamma_i^j)) = H^{s+t+1}(H^0(C_{\Sigma-\Gamma}^{*,*}(B\Gamma_i^j)))$. By the above proposition, we have $h'_*\eta^{t+1}f([x(s, t, -1)]) = h'_*\eta^{t+1}([\pi_A(x(s, -1, t))]) = (-1)^{\varepsilon'} h'_*([x(s+t+1, -1, -1)]) = (-1)^{\varepsilon'}[\delta_\Gamma(x(s+t, 0, -1))] = (-1)^s d_{t+1}([x(s, t, -1)])$. This completes the proof of Theorem C in the introduction.

§ 5. Generalization.

In this section we consider more general cases. For a graded A-module M_*, M_r stands for the subset of M_* with degree r, and Λ_r, for the subset of a set Λ_* of the generators of M_*. A graded A-module M_* is said to be *connected* if it has an integer r_0 such that $M_r = 0$ for any $r < r_0$, and *locally finite* if $\Lambda_r = 0$ for each $r \in \boldsymbol{Z}$.

Let F_* be a Γ-comodule which is a connected locally finite free A-module and Λ_* the set of its generators. We may assume that $\Lambda_r = 0$ for $r < 0$. Denote $F(r)_*$ the A-submodule of F_* generated by the elements of $\cup_{k \le r} \Lambda_k$, and we have the exact sequence

$$(5.1) \qquad 0 \longrightarrow F(r)_* \overset{\subset}{\longrightarrow} F(r+1)_* \longrightarrow G(r+1)_* \longrightarrow 0,$$

in which $G(r)_*$ is the trivial Γ-comodule generated by the elements of Λ_r. For a Γ-comodule M, we use here the notation

$$\mathrm{Ext}^{s,t}(M),$$

which stands for $\mathrm{Ext}_\Gamma^{s,t}(A, M)$ or $\mathrm{Ext}_\Sigma^{s,t}(B, B \otimes_A M)$. Now applying $\mathrm{Ext}^{*,*}(-)$ to the short exact sequence (5.1), we have the exact couple, which also gives the Cartan-Eilenberg type spectral sequence

$$E_2^s = \mathrm{Ext}^s(M) \otimes \boldsymbol{Z}_{(p)} \otimes_A F_* \Rightarrow \mathrm{Ext}^s(M \otimes_A F_*)$$

for some Γ-comodule M. Here $\boldsymbol{Z}_{(p)}$ has the Γ-comodule structure as a subcomodule of A. Then we have

LEMMA 5.2. *If* $\mathrm{Ext}^s(M) = 0$ *for some* s *and* Γ*-comodule, then so is* $\mathrm{Ext}^s(M \otimes_A F_*)$.

Applying the lemma to Theorem 3.20, we have the similar result.

THEOREM 5.3. *Let F_* be a Γ-comodule which is a connected locally finite free A-module. Then we have the following*:

i) $\mathrm{Ext}_\Gamma^s(A, LA_i^{n-i}\otimes_A F_*)=\mathrm{Ext}_\Sigma^s(B, B_i^{n-i}\otimes_A F_*)$.

ii) $\mathrm{Ext}_\Sigma^s(B, B_i^j\otimes_A F_*)=\mathrm{Ext}_\Gamma^s(A, LA_i^j\otimes_A F_*)$ *for* $s<n-i-j$.

iii) *The following is a long exact sequence*

$$\cdots \xrightarrow{d_{t+1}} \mathrm{Ext}_\Gamma^s(A, A_i^j\otimes_A F_*) \xrightarrow{\lambda_*} \mathrm{Ext}_\Sigma^s(B, B_i^j\otimes_A F_*) \longrightarrow \mathrm{Ext}_\Gamma^{s-t}(A, A_i^{n-i+1}\otimes_A F_*)$$
$$\xrightarrow{d_{t+1}} \mathrm{Ext}_\Gamma^{s+1}(A, A_i^j\otimes_A F_*) \xrightarrow{\lambda_*} \cdots,$$

in which $t=n-i-j$ and λ_* denotes the induced map from the canonical map λ: $\Gamma\to\Sigma$ of Hopf algebroids, and note that $\mathrm{Ext}_\Gamma^{s-t}=0$ *for* $s-t<0$.

In the previous section we only use the fact whether or not the Ext-groups are zero. So we can replace N in the previous sections by $N\otimes_A F_*$, since tensoring with F_* preserves exactness. Immitating the definition of the universal Greek letter map, we have the map

$$\eta = \eta\otimes id : \mathrm{Ext}_\Gamma^s(A, A_i^{n-i+1}\otimes_A F_*) \longrightarrow \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j\otimes_A F_*).$$

Here η on the right is the universal Greek letter map. We thus apply Lemma 5.2 to Theorem 5.3 and get

THEOREM 5.4. *For $t=n-i-j$,*

$$d_{t+1} = (-1)^s\eta : \mathrm{Ext}_\Gamma^s(A, A_i^{n-i+1}\otimes_A F_*) \longrightarrow \mathrm{Ext}_\Gamma^{s+t+1}(A, A_i^j\otimes_A F_*).$$

§6. Some remarks.

In [**9**, Th. 5.10], Ravenel gives the following cofibration:

$$\Sigma^{-n-1}N_{n+1}X \longrightarrow X \xrightarrow{\eta X} L_nX, \tag{6.1}$$

in which BP_*-homology of $N_{n+1}X$ is A_j^{n+1} for $X=V(j-1)$, the Toda-Smith spectrum by [**11**]. The E_2-terms of the Adams-Novikov spectral sequence converging to $\pi_*(N_{n+1}X)$, $\pi_*(X)$ and $\pi_*(L_nX)$ are

$$\mathrm{Ext}_\Gamma(A, A_j^{n+1}), \quad \mathrm{Ext}_\Gamma(A, A/I_j) \quad \text{and} \quad \mathrm{Ext}_\Sigma(B, B/I_j),$$

respectively, for the Hopf algebroids $(A, \Gamma)=(BP_*, BP_*BP)$ and $(B, \Sigma)=(E(n)_*, E(n)_*E(n))$ as before. Theorems 3.20 and C shows

PROPOSITON 6.2. *We have the long exact sequence*

$$\cdots \longrightarrow \mathrm{Ext}_\Gamma^t(A, A_j^{n+1}) \xrightarrow{\eta} \mathrm{Ext}_\Gamma^{t+n+1}(A, A/I_j) \xrightarrow{(\eta V)_*} \mathrm{Ext}_\Sigma^{t+n+1}(B, B/I_j)$$
$$\longrightarrow \mathrm{Ext}_\Gamma^{t+1}(A, A_j^{n+1}) \longrightarrow \cdots,$$

where η denotes the universal Greek letter map, $(\eta V)_*$ the induced map by the localization one for $V(j-1)$ and $(B, \Sigma)=(E(n+j)_*, E(n+j)_*E(n+j))$.

In fact, the map $\mathrm{Ext}_\Gamma^{t+n+1}(A, A/I_j)\to\mathrm{Ext}_\Sigma^{t+n+1}(B, B/I_j)$ is induced from the canonical map $\Gamma\to\Sigma$ and so it is induced from the localization map ηV. We generalize this a little more. Let $W_k(j)$ denote a spectrum such that

$$BP_*(W_k(j)) = (A/I_{j+1})[t_1, \cdots, t_k] = (BP_*/I_{j+1})[t_1, \cdots, t_k]$$

as a $BP_*(BP)$-subcomodule algebra of $\Gamma/I_{j+1}=BP_*(BP)/I_{j+1}$. Then we know the existence of the spectrum for $k\geqq j$, or for $j\leqq 4$ and the prime $p>2j$. (cf. [**10**], [**13**], [**14**]). For $j=-1$, it is known as Ravenel's spectrum and denoted by $T(k)$. Then the cofiber sequence (6.1) turns into

$$\Sigma^{-n-1}N_{n+1}W_k(j-1) \longrightarrow W_k(j-1) \xrightarrow{\eta W} L_nW_k(j-1),$$

and the BP_*-homology of each spectrum is $A_j^{n+1}[t_1, \cdots, t_k]$, $(A/I_j)[t_1, \cdots, t_k]$ and $(v_n^{-1}A/I_j)[t_1, \cdots, t_k]$, respectively. Now our corresponding result on their E_2-terms is

PROPOSITION 6.3. *We have the long exact sequence*

$$\cdots \longrightarrow \mathrm{Ext}_\Gamma^t(A, A_j^{n+1}[t_1, \cdots, t_k]) \xrightarrow{\eta} \mathrm{Ext}_\Gamma^{t+n+1}(A, A/I_j[t_1, \cdots, t_k])$$
$$\xrightarrow{(\eta W)_*} \mathrm{Ext}_\Sigma^{t+n+1}(B, B/I_j[t_1, \cdots, t_k]) \longrightarrow \mathrm{Ext}_\Gamma^{t+1}(A, A_j^{n+1}[t_1, \cdots, t_k]) \longrightarrow \cdots,$$

where η denotes the map induced from the universal Greek letter map, $(\eta W)_*$ the induced map by the localization one for $W_k(j-1)$ and $(B, \Sigma)=(E(n+j)_*, E(n+j)_*E(n+j))$.

These are the results concerning the cofibration (6.1). As an immediate consequence of Theorem C, the definition of Greek letter elements shows the following

PROPOSITION 6.4. *The homotopy groups $\pi_*(L_nS)$ of the localized sphere L_nS do not contain the $(n+1)$-st Greek letter elements.*

In the same way as defining A_j^i, we can define $E(n)_j^i$ inductively as follows:

$$E(n)_j^0 = E(n)_*/I_j, \qquad LE(n)_j^i = v_{i+j}^{-1}E(n)_j^i$$

and the cofibering

$$0 \longrightarrow E(n)_j^i \longrightarrow LE(n)_j^i \longrightarrow E(n)_j^{i+1} \longrightarrow 0.$$

Note that if we fix n then it is same as B_j^i used above. In this case, $LE(n)_j^{n-j}=E(n)_j^{n-j}$ and so $E(n)_j^i=0$ for $i+j>n$. Then we can proceed our argument for $(A, \Gamma)=(E(n)_*, E(n)_*E(n))$ and $(B, \Sigma)=(E(m)_*, E(m)_*E(m))$ as long as $m<n$. For $m=n-1$, we have the long exact sequence as Proposition 6.3 in the commutative diagram

$$\begin{array}{ccccccc}
\longrightarrow \mathrm{Ex}^s(A_0^{n+1}) & \xrightarrow{\eta} & \mathrm{Ex}^{s+n+1}(A) & \longrightarrow & \mathrm{Ex}_{(n)}^{s+n+1}(E(n)_*) & \longrightarrow & \mathrm{Ex}^{s+1}(A_0^n) \\
\| & \delta & \uparrow \eta & & \uparrow \eta_n & & \\
\longrightarrow \mathrm{Ex}^s(A_0^{n+1}) & \longrightarrow & \mathrm{Ex}^{s+1}(A_0^n) & \longrightarrow & \mathrm{Ex}^{s+1}(LA_0^n) & \longrightarrow & \mathrm{Ex}^{s+1}(A_0^n) \\
 & & \uparrow & & \uparrow & & \\
 & & \mathrm{Ex}_{(n-1)}^{s+n}(E(n-1)_*) & = & \mathrm{Ex}_{(n-1)}^{s+n}(E(n-1)_*). & &
\end{array}$$

Here we use the notation: $A=BP_*$,

$$\mathrm{Ex}^s(M) = \mathrm{Ext}^s_{BP_*BP}(BP_*, M), \qquad \mathrm{Ex}^s_{(m)}(M) = \mathrm{Ext}^s_{E(m)_*E(m)}(E(m)_*, M)$$

and η_n is the universal Greek letter map based on $E(n)_*$. In particular, we have

PROPOSITION 6.5. *If* $\mathrm{Ex}^{s+n}_{(n-1)}(E(n-1)_*)=0$, *then*

$$\mathrm{Ex}^{s+1}(LA_0^n) \cong \mathrm{Ex}^{s+n+1}_{(n)}(E(n)_*).$$

This result relates to the conjecture

$$\mathrm{Ex}^s(LA_0^n) = 0 \quad \text{if } s > n^2-n \quad \text{and} \quad n < p-1,$$

which is arisen from a result of [**12**]. Furthermore, we know the triviality $\mathrm{Ex}^s_{(n-1)}(E(n-1)_*)=0$ for $s>(n-1)^2+(n-1)=n^2-n$ and $n<p$ by [**9**] and so this proposition is applied to this conjecture to be

QUESTION. Is $\mathrm{Ext}^s_{E(n)_*E(n)}(E(n)_*, E(n)_*)=0$ for $s>n^2$?

For the case $n=2$, the result of [**9**, (10, 10)] says that $\mathrm{Ext}^s_{E(2)_*E(2)}(E(2)_*, E(2)_*)=0$ for $s>6$ and the results of [**12**] indicates that it is trivial for $s>5$. Recently it is proved in [**15**] that the Ext group is trivial for $s=5$.

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