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# Asymptotic profiles of blow-up solutions of the nonlinear Schrödinger equation with critical power nonlinearity

Dedicated to Professor R. Iino on his 70th birthday

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## 1. Introduction.

We shall consider the blow-up problem for the nonlinear Schrödinger equation:

$$\mathrm{C(p)}\qquad \begin{cases} \text{(NS)} & 2i\dfrac{\partial u}{\partial t}+\Delta u+|u|^{p-1}u=0 \qquad (t, x)\in \boldsymbol{R}_{+}\times \boldsymbol{R}^{N}, \\ \text{(IV)} & u(0, x)=u_{0}(x), \qquad x\in \boldsymbol{R}^{N}. \end{cases}$$

Here $i=\sqrt{-1}$, $u_{0}\in H^{1}(\boldsymbol{R}^{N})$ and $\Delta$ is the Laplace operator on $\boldsymbol{R}^{N}$. The nonlinear Schrödinger equation of the form (NS) arises in various domains of physics, *e.g.*, fluids, plasmas and optics. The equation (NS) also derived from a field equation for a quantum mechanical nonrelativistic many body system in the semi-classical limit.

The unique local existence of solutions of C(p) is well known for $1<p<2^{*}-1$ ($2^{*}=2N/(N-2)$ if $N\geqq 3$, $=\infty$ if $N=1, 2$): For any $u_{0}\in H^{1}(\boldsymbol{R}^{N})$, there exists a unique solution $u(t, x)$ of C(p) in $C([0, T_{m}); H^{1}(\boldsymbol{R}^{N}))$ for some $T_{m}\in(0, \infty]$ (maximal existence time), and $u(t)$ satisfies the following two conservation laws of $L^{2}$ and the energy:

$$\|u(t)\|=\|u_{0}\|, \tag{1.1}$$

$$E_{p+1}(u(t))\equiv\|\nabla u(t)\|^{2}-\frac{2}{p+1}\|u(t)\|_{p+1}^{p+1}=E_{p+1}(u_{0}), \tag{1.2}$$

for $t\in[0, T_{m})$, where $\|\cdot\|$ and $\|\cdot\|_{p+1}$ denotes the $L^{2}$ norm and $L^{p+1}$ norm respectively. Furthermore $T_{m}=\infty$ or $T_{m}<\infty$ and $\lim_{t\to T_{m}}\|\nabla u(t)\|=\infty$. For details, see *e.g.*, [**11, 12, 14**].

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As for the existence and non-existence of global solutions of C(p), the following is well known (see [**11, 12, 13, 14, 30, 31**]).

(i) If $1<p<1+4/N$, there exists a global solution $u\in C_b(\boldsymbol{R}; H^1(\boldsymbol{R}^N))$, for any $u_0\in H^1(\boldsymbol{R}^N)$, where $C_b(\boldsymbol{R}; H^1(\boldsymbol{R}^N))=C(\boldsymbol{R}; H^1(\boldsymbol{R}^N))\cap L^\infty(\boldsymbol{R}; H^1(\boldsymbol{R}^N))$.

(ii) If $1+4/N\leqq p<2^*-1$, there is a subset $\mathcal{B}\in H^1(\boldsymbol{R}^N)$ such that for any $u_0\in\mathcal{B}$ the solution of C(p) blows up, *i.e.* the $L^2$ norm of its gradient explodes in finite time $T_m$.

As we have seen above, the number $p=1+4/N$ is the critical number for the existence of blow-up solutions of C(p). In what follows, we refer to (NS) with $p=1+4/N$ as (NSC), *i.e.*,

$$\text{(NSC)} \qquad 2i\frac{\partial u}{\partial t}+\Delta u+|u|^{4/N}u=0\,,$$

and we shall use the notations

$$E(v)=E_\sigma(v)\,, \qquad \sigma=2+\frac{4}{N}\,.$$

The nonlinear Schrödinger equation of the form (NSC) is of physical interest, because (NSC) with $N=2$ arises in a theory of the stationary self-focusing of a laser beam propagating along the $t$-axis in a nonlinear medium (see *e.g.*, [**1, 2, 15, 40**]). We may say that the blow-up of solution corresponds to the focusing of the laser beam.

Recently, many mathematicians have studied the formation of singularities in blow-up solutions of (NSC) near blow-up time (*e.g.*, [**7, 19, 20, 21, 23, 24, 25, 27, 29, 35, 37, 38**]). Here, it is worth while to note that (NSC) has a remarkable property that it is invariant under the pseudo-conformal transformations (see *e.g.*, [**7, 29**] and (1.7) below). This symmetry seems to be closely related to the structure of solutions of (NSC) (see *e.g.*, Weinstein [**37, 38**], Nawa and M. Tsutsumi [**29**] and Cazenave and Weissler [**7**]): in the super critical case ($p>1+4/N$), Merle [**19**] suggested that every blow-up solution of (NS) has a strong limit in $L^2$ at blow-up times; in the critical case ($p=1+4/N$), Nawa [**24, 25**] and Weinstein [**38**] showed that every blow-up solution of (NSC) loses its $L^2$ continuity at blow-up time because of the concentration of its $L^2$ mass (see also Merle [**20**], Merle and Y. Tsutsumi [**21**], and Y. Tsutsumi [**35**]). Moreover we know how amount the blow-up solution of (NSC) concentrate their $L^2$ mass. Precisely, if the solution $u(t)$ of (NSC) blow up at time $T_m>0$, then we have

$$(1.3) \qquad \sup_{R>0}\Big(\liminf_{t\uparrow T_m}\Big(\sup_{y\in\boldsymbol{R}^N}\int_{|x-y|\leqq R(T_m-t)^{1/2}}|u(t,x)|^2dx\Big)\Big)\geqq\|Q\|^2,$$

where $Q$ is a nontrivial solution of the elliptic equation

$$\Delta Q - Q + |Q|^{4/N} Q = 0 \tag{1.4}$$

such that

$$\frac{2}{\sigma}\|Q\|^{4/N} = \inf_{\substack{v\in H^1(\mathbf{R}^N)\\ v\neq 0}} \frac{\|v\|^{4/N}\|\nabla v\|^2}{\|v\|_\sigma^\sigma} \tag{1.5}$$
$$= \inf_{\substack{v\in H^1(\mathbf{R}^N)\\ v\neq 0}} \left\{\frac{2}{\sigma}\|v\|^{4/N}\,;\ \|\nabla v\|^2 - \frac{2}{\sigma}\|v\|_\sigma^\sigma \leqq 0\right\}.$$

The focusing of a laser beam could be understood mathematically as "mass concentration" phenomena of blow-up solutions of (NSC).

REMARK 1.1. The left hand side of (1.3) measures the "size" of the "largest" singularity, since the blow-up solution, in general, has several $L^2$-concentration points (see Merle [**20**] and Nawa [**25**]).

REMARK 1.2. (1) The equation (1.4) is a time-independent version of (NSC) and arises in various domain of physics. See [**3, 6, 32, 36**] and Proposition 2.5 of this paper for the existence of positive solutions of (1.4) and for the associated minimization problems. The standard argument shows that $Q\in\mathcal{S}$ (the space of $C^\infty$ functions of rapid decreasing). We can also prove that $E(Q)=0$.

(2) By the first equality in (1.5) and the conservation law (1.2), we see that if $\|u_0\|<\|Q\|$, the corresponding solution exists globally in time. For this, see Weinstein [**36, 37**]. In this sense, the estimate (1.3) is optimal.

However the profiles of blow-up solutions have not been investigated so well. Concerning this problem, the following results are known.

(I) Let $u(t)$ be a solution of (NSC) such that $\|u(t)\|=\|Q\|$ and $\|\nabla u(t)\|\to\infty$ as $t\to T_m$ for some $T_m\in(0,\infty]$. Then we have, for $\lambda(t)=\|\nabla u(t)\|^{-1}$,

$$\|\lambda(t)^{N/2}u(t, \lambda(t)(\cdot-\gamma(t)))e^{i\theta(t)} - Q(\cdot)\| \longrightarrow 0 \quad \text{as } t\to T_m \tag{1.6}$$

for some $\gamma(t)\in\mathbf{R}^N$ and $\theta(t)\in\mathbf{R}$ (Weinstein [**37**]).

(II) Let $u(t)$ be a solution of (NSC) such that $xu(t)\in L^2(\mathbf{R}^N)$ and $\|\nabla u(t)\|\to\infty$ as $t\to T_m$ for some $T_m\in(0,\infty)$. If $u(t)$ satisfies $\|(x-a)u(t)\|\to 0$ $(t\to T_m)$, then $u(t)$ must be of the form:

$$(T_m-t)^{-N/2}\exp\left(\frac{-i|x_t(a,v)|^2}{2(T_m-t)}\right)V\left(\frac{t}{T_m(T_m-t)}, \frac{x_t(a,v)}{T_m-t}\right)e^{iv/T_m(x-(v/2T_m)t)}, \tag{1.7}$$

where $V(t,x)$ is also a solution of (NSC) in $C(\mathbf{R}_+; H^1(\mathbf{R}^N))\cap L^2(\mathbf{R}^N; |x|^2dx)$ such that $E(V(t))=0$, and where

$$x_t(a,v) = x-a+v-\frac{v}{T_m}t$$

for an appropriate $v\in\mathbf{R}^N$ (Nawa and M. Tsutsumi [**29**]).

(III) For given $L$ points $\{a^1, a^2, \cdots, a^L\} \subset \boldsymbol{R}^N$, there exists a blow-up solution $u(t)$ of (NSC) such that

$$\left\| u(t) - \sum_{j=1}^{L} Q^j(t) \right\|_\sigma \longrightarrow 0 \quad \text{as } t \to T_m, \tag{1.8}$$

where

$$Q^j(t, x) = (T_m - t)^{-N/2} \exp\left(\frac{-i|x-a^j|^2}{2(T_m-t)}\right) Q\left(\frac{x-a^j}{T_m-t}\right) e^{it/2T_m(T_m-t)} \tag{1.9}$$

for $T_m \in (0, \infty)$ (Merle [20]).

REMARK 1.3. $Q(x)e^{it/2}$, which is a standing wave solution of (NSC), is transformed into $Q^j$ by the space-time transformation appearing in the left hand side of (1.7) with $a = a^j$ and $v = 0$. We call this transformation pseudo-conformal transformation. Since we have $E(Q(\cdot)e^{it/2}) = 0$, $Q^j$ is a blow-up solution of (NSC) such that $\|(x - a^j)u(t)\| \to 0$ $(t \to T_m)$ by virtue of (II).

These results require additional conditions on initial data (or solutions): $\|u_0\| = \|Q\|$ for (I); $|x| u_0 \in L^2(\boldsymbol{R}^N)$ for (II) and (III). Our purpose here is to investigate the asymptotic profile of generic $H^1$-blow-up solution of (NSC).

We have

THEOREM 1. *Let $u(t)$ be a singular solution of (NSC) such that*

$$\limsup_{t \to T_m} \|\nabla u(t)\| = \limsup_{t \to T_m} \|u(t)\|_\sigma = \infty \tag{1.10}$$

*for some $T_m \in (0, \infty]$. Let $\{t_n\}$ be any sequence such that*

$$\sup_{t \in [0, t_n)} \|u(t)\|_\sigma = \|u(t_n)\|_\sigma. \tag{1.11}$$

*For this $\{t_n\}$, we put*

$$\lambda_n = \frac{1}{\|u(t_n)\|_\sigma^{\sigma/2}} \tag{1.12}$$

*and, we consider the scaled functions*

$$u_n(t, x) = \lambda_n^{N/2} \overline{u(t_n - \lambda_n^2 t, \lambda_n x)} \tag{1.13}$$

*for $t \in [0, t_n/\lambda_n^2)$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exist*

(i) *a finite number of nontrivial solutions $u^1, u^2, \cdots, u^L$ of (NSC) in $C_b(\boldsymbol{R}_+; H^1(\boldsymbol{R}^N))$ with $E(u^j) = 0$, and*

(ii) *sequences $\{\gamma_n^1\}, \{\gamma_n^2\}, \cdots, \{\gamma_n^L\}$ in $\boldsymbol{R}^N$,*

*such that, for any $T > 0$,*

$$\lim_{n \to \infty} \sup_{t \in [0, T]} \left\| u_n(t, \cdot) - \sum_{j=1}^{L} u^j(t, \cdot - \gamma_n^j) \right\|_\sigma = 0, \tag{1.14}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|\nabla u_n(t,\ \cdot)-\sum_{j=1}^{L}\nabla u^j(t,\ \cdot-\gamma_n^j)\right\|=0\,, \tag{1.15}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\left\|u_n(t,\ \cdot)-\sum_{j=1}^{L}u^j(t,\ \cdot-\gamma_n^j)-\phi_n(t,\ \cdot)\right\|=0\,, \tag{1.16}$$

*where*

$$\phi_n(t,\ \cdot)=\exp\left(\frac{it}{2}\Delta\right)*\left(u_n(0,\ \cdot)-\sum_{j=1}^{L}u^j(0,\ \cdot-\gamma_n^j)\right). \tag{1.17}$$

*Furthermore we have*

$$\|u_0\|^2\geqq\sum_{j=1}^{L}\|u^j(t)\|^2\geqq L\|Q\|^2\,, \tag{1.18}$$

*where $Q$ is a nontrivial solution of* (1.4) *and* (1.5).

REMARK 1.4. (1) If the solution satisfies $\limsup_{t\to T_m}\|\nabla u(t)\|=\infty$, then we have, by the energy conservation law $\limsup_{t\to T_m}\|u(t)\|_\sigma=\infty$. So, (1.10) is always assured. If $T_m<\infty$, we have (1.10) with lim sup replaced by lim.

(2) We can choose a sequence as in (1.11), since we have (1.10).

(3) The scaled function $u_n$ in (1.13) also solves (NSC), and satisfies $\|u_n(t)\|=\|u(t)\|$ and $E(u_n(t))=\lambda_n^2E(u(t))$. This is a special feature of (NSC).

Each $u^j$ can be considered to correspond to the "strong" singularity in blow-up solution, since one has, by (1.16),

$$\lim_{n\to\infty}\sup_{t\in[t_n-\lambda_n^2T,\,t_n]}\left\|\overline{u(t,\ \cdot)}-\sum_{j=1}^{L}u_n^j(t,\ \cdot)-\tilde{\phi}_n(t,\ \cdot)\right\|=0\,, \tag{1.19}$$

where

$$u_n^j(t,\ x)=\frac{1}{\lambda_n^{N/2}}u^j\left(\frac{t_n-t}{\lambda_n^2},\ \frac{x-\gamma_n^j\lambda_n}{\lambda_n}\right), \tag{1.20}$$

$$\tilde{\phi}_n(t,\ x)=\frac{1}{\lambda_n^{N/2}}\phi_n\left(\frac{t_n-t}{\lambda_n^2},\ \frac{x}{\lambda_n}\right). \tag{1.21}$$

We note that there is a possibility that $\tilde{\phi}_n$ produces "weak" singularities, around which the rate of blow-up is lower than $\|u(t)\|_\sigma$. If $N\geqq 2$, "weak" singularities may form a $N-1$ dimensional manifold as in the case of semilinear heat equations (Giga and Kohn [**10**]). If $\tilde{\phi}_n$ produces no singularity, we can safely say that the blow-up set consists of finite number of points as in the case of one-dimensional semilinear heat equations (Chen and Matano [**8**]). However, there still remains a possibility of ergodic behavior of singularities, *i.e.*, even in the case of $L=1$, $\{\lambda_n\gamma_n^1\}$ may perform an ergodic behavior (in full sequence).

Theorem 1 seems to be closely related to a phenomenon which has been observed in various nonlinear problems by the name of bubble theorem or concentrated compactness theorem (for example, see [**4, 16, 17, 18, 22, 33, 34**]). In fact, the proof of this theorem is inspired by Brézis and Coron [**4**]. One may find that the underlying idea being the method of concentrated compactness due to Lions [**17, 18**]. However, we do not use the general method of it. Our basic tool is the compactness device as in Lieb [**16**] (see also Brézis and Lieb [**6**] and Fröhlich, Lieb and Loss [**9**]). We extend Lieb's compactness lemma to space-time one, with which the Ascoli-Arzela theorem plays a crucial role in our analysis working with the scaled solutions of (NSC) defined by (1.13). Proposition 3.1 in Sect. 3 of this paper is the heart of the matter. The use of the general method of concentrated compactness in the study of blow-up problem for the nonlinear Schrödinger equation can be traced back to Weinstein [**37**].

We can safely say that our analysis investigates, by means of Proposition 3.1, how the "dichotomy" (in the terminology of concentrated compactness) occurs in the sequence $\{u_n\}$. Theorem 1 asserts that $u_n$ behaves like a finite superposition of *zero energy* time global solutions of (NSC) (see (1.13)-(1.16)). It is worth while to note again that the scaled function $u_n$ also solves (NSC), and satisfies $\|u_n(t)\|=\|u(t)\|$ and $E(u_n(t))=\lambda_n^2 E(u(t))$ ($\to 0$ as $n\to\infty$). We iteratively use Proposition 3.1 to construct $u^j$'s (see Sect. 4). Here the important thing is the finiteness of $u^j$'s. This follows from $E(u^j)=0$ (for any $j$), since in this case we have $\|u^j\|\geqq\|Q\|$ for any $j$ (see (1.5) and Proposition 2.5 in Sect. 2 of this paper). If the iteration was not terminated at some finite index, we would have by the construction of $u^j$'s that $\limsup_{k\to\infty}\sum_{j=1}^k E(u^j(t))\leqq 0$ (see the argument in Sect. 4 bellow (4.43)). Hence we can conclude $E(u^j)=0$ for any $j$, if we know that every bounded global solution must have nonnegative energy, *i.e.*;

THEOREM 2. *If the solution $u(t)$ of (NSC) belongs to $C_b([0,\infty);H^1(\boldsymbol{R}^N))$, then its energy must be nonnegative, i.e.,*

$$E(u(0))\geqq 0\,.$$

*In other words, if $E(u(0))<0$, then there exists $T_m\in(0,\infty]$ such that the corresponding solution $u(t)$ of (NSC) satisfies*

$$\limsup_{t\to T_m}\|\nabla u(t)\|=\limsup_{t\to T_m}\|u(t)\|_\sigma=\infty\,,$$

*i.e., $u(t)$ belows up (in finite time) or grows up (at infinity). If $T_m<\infty$, we can replace* lim sup *by* lim.

This is an improvement of previous results concerning the existence of blow-up solution of (NSC) in the sence that we do not require additional conditions on initial data except $E(u(0))<0$. However Theorem 2 does not assert that every negative energy initial datum leads to the blow-up solution of

$C(1+4/N)$. There remains a possibility that $T_m=\infty$. The blow-up of negative energy solutions has been proved under some conditions: $|x|u_0\in L^2(\boldsymbol{R}^N)$ (Glassey [**13**]) while this is an important class of initial data and quite reasonable physically; $N\geqq 2$ and $u_0$ is radially symmetric (Ogawa-Y. Tsutsumi [**30**]); $N=1$ and $p=1+4/N$ (Ogawa-Y. Tsutsumi [**31**]). Hence the results of Ogawa-Tsutsumi [**30, 31**] ensure that Theorem 2 with $T_m<\infty$ holds true for the case of $N=1$, and of $N\geqq 2$ and $u(0)$ being radially symmetric. We refer [**28**] for the proof of Theorem 2 ($N\geqq 2$), although we do not prove it in this paper (see Remark (Added on Revision) below Corollary 2 of this section). We may say that Theorem 2 is a weak version of a theorem which has long been speculated. It is worth while to note here that the essential results of this paper are valid for all dimensions $N\geqq 1$.

Next two corollaries give sufficiently conditions that we have $L=1$ in Theorem 1. We, however, do not need Theorem 2 to prove $L=1$ in the proofs of both corollaries.

COROLLARY 1. *Let $u(t)$, $\{t_n\}$, $u_n(t)$ and $Q(x)$ be as in Theorem* 1. *Assume in addition $u(0)$ is radially symmetric. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exists a nontrivial solution $u^1$ of (NSC) in $C_b(\boldsymbol{R}_+;H^1(\boldsymbol{R}^N))$ with $E(u^1)=0$, and such that, for any $T>0$,*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u_n(t,\cdot)-u^1(t,\cdot)\|_\sigma=0\,, \tag{1.22}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|\nabla u_n(t,\cdot)-\nabla u^1(t,\cdot)\|=0\,, \tag{1.23}$$

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u_n(t,\cdot)-u^1(t,\cdot)-\phi_n(t,\cdot)\|=0\,, \tag{1.24}$$

*where*

$$\phi_n(t,\cdot)=\exp\left(\frac{it}{2}\Delta\right)*(u_n(0,\cdot)-u^1(0,\cdot))\,. \tag{1.25}$$

COROLLARY 2. *Let $u(t)$, $\{t_n\}$, $u_n(t)$ and $Q(x)$ be as in Theorem* 1. *Assume in addition $\|u(0)\|=\|Q\|$. Then there exists a subsequence of $\{u_n\}$ (still denoted by $\{u_n\}$), which satisfies the following properties: there exist*

(i) *a nontrivial solution $u^1$ of (NSC) in $C_b(\boldsymbol{R}_+;H^1(\boldsymbol{R}^N))$ with $E(u^1)=0$, and*

(ii) *a sequence $\{\gamma_n^1\}$, in $\boldsymbol{R}^N$,*

*such that, for any $T>0$,*

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|u_n(t,\cdot)-u^1(t,\cdot-\gamma_n^1)\|=0\,. \tag{1.26}$$

REMARK (Added on Revision). When this paper was submitted for publication, the title of the paper was "*Formation of singularities in solutions of the one dimensional nonlinear Schrödinger equation with critical power nonlinearity*". The referee kindly suggested the author to rephrase the title and introduction, since the essential results of this paper are valid for all dimensions $N \geqq 1$, and since Theorem 2 ("Theorem 0" in the first manuscript) does hold for data in $H^1(\boldsymbol{R}^N)$ with $|x|u_0 \in L^2(\boldsymbol{R}^N)$ which is an important class of initial data and quite reasonable physically. The author is grateful to the referee for this and his valuable comments. The author [**28**], however, proved Theorem 2 for $N \geqq 2$ after first submission of this paper. Its proof is also relevant to a phenomenon which has been observed in various nonlinear problems by the name of bubble theorem or concentrated compactness theorem. The proof, which is long and rather technical, proceeds combining the results (or methods) of Nawa [**24, 27**], Ogawa-Y. Tsutsumi [**30, 31**] and Proposition 3.1 of this paper.

The paper is organized as follows.

In Section 2, we give a lemma concerning the evolution operator for the free Schrödinger equation, and give one proposition which concerns (1.4), (1.5) and Remark 1.2.

In Section 3, we prepare a key proposition to prove Theorem 1.

In Section 4, we prove Theorem 1.

In Section 5, we prove Corollaries 1 and 2.

Throughout this paper we will use the following notations:

NOTATIONS. $\partial_t = \partial/\partial t$, $\partial_k = \partial/\partial x_k$, $\nabla = (\partial_1, \cdots, \partial_n)$; $\langle \cdot, \cdot \rangle$ denotes the scalar product in $L^2$ and various pairing of dual spaces of functions; $\mu$ denotes the Lebesgue measure on $\boldsymbol{R}^N$; The set $\{x \in \boldsymbol{R}^N; f(x) > \varepsilon\}$ is simply represented as $[f > \varepsilon]$, which is also used to denote the characteristic function of this set; $B(y; R) = \{x \in \boldsymbol{R}^N; |x-y| \leqq R\}$; Let $\Omega \subset \boldsymbol{R}^N$ be open. The symbol $\omega \Subset \Omega$ means that $\bar{\omega}$ (the closure of $\omega$) is compact and $\bar{\omega} \subset \Omega$; $C(I; F)$ denotes the space of strongly continuous function from an interval $I \subset \boldsymbol{R}$ to a Fréchet space $F$; $L^\theta(I; B)$ denotes the space of measurable functions $v$ from an interval $I \subset \boldsymbol{R}$ to a Banach space B such that $\|v(\cdot)\|_B \in L^\theta(I)$; $C_b(I; F) = L^\infty(I; F) \cap C(I; F)$; $U = U(t) = \exp((it/2)\Delta)$.

## 2. Preliminaries

We recall Proposition D in [**25**] as Proposition 2.1 below. Although we have already proved this in [**25**] to study the "mass concentration" phenomena in blow-up solutions of (NSC) to obtain the formula (1.3), we shall give the proof of Proposition 2.1, since the proof of Theorem 1 in Sect. 4 proceeds after the model of it. Using this, we also consider (1.4) and (1.5) in the previous section (see Proposition 2.5 below).

PROPOSITION 2.1. *Let $\{f_n(x)\}$ be a bounded sequence of functions in $H^1(\boldsymbol{R}^N)$ such that, for some positive constants $C_\sigma$,*

$$\limsup_{n\to\infty} \|f_n\|_\sigma^\sigma \geqq C_\sigma > 0\,, \tag{2.1}$$

$$\limsup_{n\to\infty} E(f_n) = \limsup_{n\to\infty}\left(\|\nabla f_n\|^2 - \frac{2}{\sigma}\|f_n\|_\sigma^\sigma\right) = E_0\,. \tag{2.2}$$

*Then there exist*

(i) *a family of functions in $H^1(\boldsymbol{R}^N)$: $\mathfrak{A} = \{f^1, f^2, \cdots\}$, and*

(ii) *a family of sequences in $\boldsymbol{R}^N$: $\mathfrak{B} = \{\{y_n^1\}, \{y_n^2\}, \cdots\}$*

*such that we have*

$$\lim_{n\to\infty}\left|\sum_{k=2}^{j} y_n^k\right| = \infty \qquad (j \geqq 2), \tag{2.3}$$

*and, for some subsequence (still denoted by the same letter), we have*

$$f_n^1 \equiv f_n(\cdot + y_n^1) \longrightarrow f^1 \not\equiv 0\,, \tag{2.4}$$

$$f_n^j \equiv (f_n^{j-1} - f^{j-1})(\cdot + y_n^j) \longrightarrow f^j \not\equiv 0 \qquad (j \geqq 2)\,, \tag{2.5}$$

*weakly in $H^1(\boldsymbol{R}^N)$ and strongly in $L^q(\Omega)$ for any $\Omega \Subset \boldsymbol{R}^N$ and $q \in [2, 2^*)$, and*

$$\lim_{n\to\infty}\int_{\boldsymbol{R}^N} \big|\,|f_n^j|^q - |f_n^j - f^j|^q - |f^j|^q\,\big|\,dx = 0\,, \qquad q \in [2, 2^*)\,, \tag{2.6}$$

$$\lim_{n\to\infty}\{E(f_n^j) - E(f_n^j - f^j) - E(f^j)\} = 0\,, \tag{2.7}$$

$$E_0 - \lim_{n\to\infty} E(f_n^j - f^j) = \sum_{k=1}^{j} E(f^k)\,. \tag{2.8}$$

*Furthermore, we have: If $L \equiv \#\mathfrak{A} < \infty$,*

$$\lim_{n\to\infty} \|f_n^L - f^L\|_{L^\sigma(\boldsymbol{R}^N)} = 0\,, \tag{2.9}$$

$$\lim_{n\to\infty}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{|x-y|<R} |f_n^L(x) - f^L(x)|^2\,dx\right\} = 0\,; \tag{2.10}$$

*if $\#\mathfrak{A} = \infty$,*

$$\lim_{j\to\infty}\lim_{n\to\infty} \|f_n^j - f^j\|_{L^\sigma(\boldsymbol{R}^N)} = 0\,, \tag{2.11}$$

$$\lim_{j\to\infty}\lim_{n\to\infty}\left\{\sup_{y\in\boldsymbol{R}^N}\int_{|x-y|<R} |f_n^j(x) - f^j(x)|^2\,dx\right\} = 0\,, \tag{2.12}$$

*for any $R > 0$.*

Proposition 2.1 asserts that $f_n$ behaves like a superposition of several functions of the form $f^j(t, x-\sum_{k=1}^{j} y_n^k)$, $(j=1, 2, \cdots, )$ as $n\to\infty$.

The proof of Proposition 2.1 is inspired by the method of concentrated compactness due to Lions [**17**, **18**]. However, we do not use the general method of it. Our basic tool is the same compactness device as in Lieb [**16**]. We repeatedly use Lieb's lemma to decompose $\{f_n\}$ iteratively into several parts with the help of Brézis-Lieb's lemma [**5**].

We collect here these results needed for the proof of Proposition 2.1.

LEMMA 2.2 (Fröhlich, Lieb and Loss [**9**]). *Let* $1<\alpha<\beta<\gamma$ *and let* $g(x)$ *be a measurable function on* $\boldsymbol{R}^N$ *such that, for some positive constants* $C_\alpha$, $C_\beta$ *and* $C_\gamma$,

$$\|g\|_\alpha \leqq C_\alpha, \qquad \|g\|_\beta \geqq C_\beta>0, \qquad \|g\|_\gamma \leqq C_\gamma . \tag{2.13}$$

*Then we have*

$$\mu([|g|>\eta]) > C_1 \tag{2.14}$$

*for some* $\eta$, $C_1>0$ *depending on* $\alpha$, $\beta$, $\gamma$, $C_\alpha$, $C_\beta$, $C_\gamma$ *but not on* $g$.

LEMMA 2.3 (Lieb [**16**]). (1) *In addition to the assumption of Lemma* 2.2, *we assume*

$$\|\nabla g\|_\alpha \leqq C_2 , \tag{2.15}$$

*for some positive constants* $C_2$. *Then there exists a shift* $T_y g(x)=g(x+y)$ *such that, for some constant* $\delta=\delta(C_1, C_2, \eta)$,

$$\mu\Big(B(0;1)\cap\Big[|T_y g| > \frac{\eta}{2}\Big]\Big) > \delta . \tag{2.16}$$

(2) *Let* $1<\alpha<\infty$ *and let* $\{f_n\}$ *be a uniformly bounded sequence of functions in* $W^{1,\alpha}(\boldsymbol{R}^N)$ *such that* $\mu([|f_n|>\eta])\geqq C$ *for some positive constants* $\eta$ *and* $C$. *Then there exists a sequence* $\{y_n\}$ *in* $\boldsymbol{R}^N$ *such that, for some subsequence (still denoted by the same letter)*,

$$f_n(\cdot+y_n) \rightharpoonup f \not\equiv 0 , \tag{2.17}$$

*weakly in* $W^{1,\alpha}(\boldsymbol{R}^N)$.

LEMMA 2.4 (Brézis and Lieb [**5**]). *Let* $\{f_n(x)\}$ *be an bounded family in* $L^\alpha(\boldsymbol{R}^N)$ *for* $\alpha\in(0, \infty)$. *Suppose that* $f_n\to f$ *a.e. in* $\boldsymbol{R}^N$. *Then we have*

$$\lim_{n\to\infty}\int_{\boldsymbol{R}^N} \big||f_n|^\alpha-|f_n-f|^\alpha-|f|^\alpha\big|\, dx = 0 . \tag{2.18}$$

REMARK 2.1. One may find proofs of these results in the next section, since we extend these results to functions of space-time variables in $C_b(I; W^{1,\alpha}(\boldsymbol{R}^N))$ for $I\subset\boldsymbol{R}$ in Sect. 3. So, Lemmas 2.2–2.4 are considered to be stationary versions of lemmas in Sect. 3.

PROOF OF PROPOSITION 2.1. In what follows, we shall often extract subsequences with explicitly mentioning this fact. By the assumption (2.1), Lemma 2.2 and Lemma 2.3 (2), we can shift each $f_n$ so that

$$f_n^1 \equiv f_n(\cdot + y_n^1) \longrightarrow f^1 \not\equiv 0, \tag{2.19}$$

weakly in $H^1(\boldsymbol{R}^N)$, strongly in $L^q(\Omega)$ for any $\Omega \Subset \boldsymbol{R}^N$ and $q \in [1, 2^*)$, and $a.e.$ in $\boldsymbol{R}^N$. Hence we have, by Lemma 2.4,

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^N} ||f_n^1|^2 - |f_n^1 - f^1|^q - |f^1|^q| dx = 0, \qquad q \in [2, 2^*). \tag{2.20}$$

Moreover we have

$$\lim_{n\to\infty} \|\nabla f_n^1\|^2 - \|\nabla f_n^1 - \nabla f^1\|^2 - \|\nabla f^1\|^2 = 0, \tag{2.21}$$

since $\nabla f_n^1 \rightharpoonup \nabla f^1$ weakly in $L^2$. Combining (2.20) with $q=\sigma$ and (2.21), we have

$$\lim_{n\to\infty} \{E(f_n^1) - E(f_n^1 - f^1) - E(f^1)\} = 0. \tag{2.22}$$

Suppose that $\limsup_{n\to\infty} \|f_n^1 - f^1\|_\sigma \neq 0$. Then there exists a sequence $\{y_n^2\}$ in $\boldsymbol{R}^N$ such that

$$f_n^2 \equiv (f_n^1 - f^1)(\cdot + y_n^2) \longrightarrow f^2 \not\equiv 0, \tag{2.23}$$

weakly in $H^1(\boldsymbol{R}^N)$, strongly in $L^q(\Omega)$ for any $\Omega \Subset \boldsymbol{R}^N$ and $q \in [2, 2^*)$, and $a.e.$ in $\boldsymbol{R}^N$ by Lemma 2.2 and Lemma 2.3 again. Moreover we have, by Lemma 2.4,

$$\lim_{n\to\infty} \int_{\boldsymbol{R}^N} ||f_n^2|^q - |f_n^2 - f^2|^q - |f^2|^q| dx = 0, \qquad q \in [2, 2^*), \tag{2.24}$$

$$\lim_{n\to\infty} \{E(f_n^2) - E(f_n^2 - f^2) - E(f^2)\} = 0, \tag{2.25}$$

$$E_0 - \lim_{n\to\infty} E(f_n^2 - f^2) = \sum_{k=1}^{2} E(f^k), \tag{2.26}$$

since we have

$$\lim_{n\to\infty} \|f_n^1 - f^1\|_q = \lim_{n\to\infty} \|f_n^2\|_q, \qquad q \in [2, 2^*), \tag{2.27}$$

$$\lim_{n\to\infty} E(f_n^1 - f^1) = \lim_{n\to\infty} E(f_n^2) \tag{2.28}$$

by the translation invariance of $\|\cdot\|_q$ and $E(\cdot)$. The local $L^2$ convergence of (2.24) and the nontriviality of $f^2$ yield that $\lim_{n\to\infty} |y_n^2| = \infty$: if not, we have that, for any $K \Subset \boldsymbol{R}^N$,

$$\lim_{n\to\infty} \int_K |f_n(x + y_n^1) - f^1(x) - f^2(x - y_n^2)|^2 dx = 0, \tag{2.29}$$

from which we conclude $f^2 \equiv 0$ by the local $L^2$ convergence of (2.19).

Repeating this procedure until the quantity $\limsup_{n\to\infty} \|f_n^j - f^j\|_\sigma$ becomes 0, we obtain desired families $\mathfrak{A}$ and $\mathfrak{B}$. It remains to prove (2.11). Suppose the contrary that, for some positive constant $\varepsilon_0$ and a subsequence $\{j(k)\}$ of $\{j\}$, one has

$$\liminf_{n\to\infty} \|f_n^{j(k)} - f^{j(k)}\|_\sigma^\sigma > \varepsilon_0 .$$

Hence there is a constant $C_0$ essentially depends on $\varepsilon_0$ such that

$$\|f^{j(k)+1}\|_\sigma^\sigma > C_0 , \tag{2.30}$$

since the size of $\|f^j\|_\sigma$ essentially depends on the lower bound of $\|f_n^{j-1} - f^{j-1}\|_\sigma^\sigma$ by Lemma 2.2, Lemma 2.3 (2) and the construction of $f^j$. We choose $k \in \boldsymbol{N}$ large enough (specified latter). Using the formula (2.6) with $j=1, 2, \cdots, j(k)$, we have by (2.30) that, for enough subsequence with respect to $n$,

$$\begin{aligned} \limsup_{n\to\infty} \|f_n\|_\sigma^\sigma &> \limsup_{n\to\infty} (\|f_n\|_\sigma^\sigma - \|f_n^{j(k)} - f^{j(k)}\|_\sigma^\sigma) \\ &= \sum_{j=1}^{j(k)} \|f^j\|_\sigma^\sigma > \sum_{l=1}^{k} \|f^{j(l)}\|_\sigma^\sigma \\ &> kC_0 . \end{aligned} \tag{2.31}$$

Thus we reach a contradiction, if we take $k$ as $kC_0 \geqq \limsup_{n\to\infty} \|f_n\|_\sigma^\sigma$.

We next consider (1.4) and (1.5) in the previous section.

PROPOSITION 2.5. *Let*

$$m = \inf_{\substack{v \in H^1(\boldsymbol{R}^N) \\ v \neq 0}} \left\{ \|v\| \,;\, E(v) \equiv \|\nabla v\|^2 - \frac{2}{\sigma} \|v\|_\sigma^\sigma \leqq 0 \right\} , \tag{2.32}$$

$$\frac{1}{C_N} = \inf_{\substack{v \in H^1(\boldsymbol{R}^N) \\ v \neq 0}} \frac{\|v\|^{4/N} \|\nabla v\|^2}{\|v\|_\sigma^\sigma} \equiv \inf_{\substack{v \in H^1(\boldsymbol{R}^N) \\ v \neq 0}} J(v) . \tag{2.33}$$

*There is a function $Q \in H^1(\boldsymbol{R}^N) - \{0\}$ such that*

$$\|Q\| = m , \tag{2.34}$$

$$\Delta Q - Q + |Q|^{4/N} Q = 0 , \tag{2.35}$$

$$\frac{2}{\sigma} \|Q\|^{4/N} = \frac{1}{C_N} , \tag{2.36}$$

$$E(Q) = 0 . \tag{2.37}$$

REMARK 2.1. (1) The constant $C_N$ in (2.22) is the best constant for the Gagliardo-Nirenberg inequality, so that

$$\|v\|_\sigma^\sigma \leq C_N \|v\|^{4/N} \|\nabla v\|^2 \tag{G-N}$$

holds true for any $v \in H^1(\boldsymbol{R}^N)$.

(2) The estimate of the best constant $C_N$ in terms of the solution of (2.35) can be traced back to Weinstein [**36**]. He consider the minimizing sequence for (2.33), and employ the radial rearrangement and radial compactness lemma to obtain the suitable minimizer. We shall give an another proof of Proposition 2.5. Our proof concerns the $L^2$ minimizing sequence of (2.32) instead of (2.33), and apply Proposition 2.1 to it. Thus our analysis do not rely on the radial rearrangement and radial compactness lemma.

PROOF OF PROPOSITION 2.5. First we note that $m>0$, more precisely

$$\frac{2}{\sigma} m^{4/N} \geqq \frac{1}{C^N} \tag{2.38}$$

by the Gagliardo-Nirenberg inequality (G-N).

Let $\{v_n\} \subset H^1(\boldsymbol{R}^N)$ be a minimizing sequence for (2.32), *i.e.*,

$$\lim_{n\to\infty} \|v_n\| = m\,, \tag{2.39}$$

$$E(v_n) \leqq 0 \qquad \text{for any } n \in \boldsymbol{N}. \tag{2.40}$$

It is worth while to note that the boundedness of $\{v_n\}$ in $H^1(\boldsymbol{R}^N)$ is not known. So we consider the following scaled function:

$$Q_n(x) = \nu_n^{N/2} v_n(\nu_n x)\,, \qquad \nu_n = \frac{1}{\|v_n\|_\sigma^{\sigma/2}}\,, \tag{2.41}$$

so that we have

$$\begin{aligned} &\|Q_n\| = \|v_n\| \longrightarrow m \qquad \text{as } n\to\infty\,, \\ &\|Q_n\|_\sigma = 1\,, \\ &E(Q_n) = \nu_n^2 E(v_n)\,. \end{aligned} \tag{2.42}$$

Hence we get an $H^1$-bounded minimizing sequence $\{Q_n\}$ for (2.32). We apply Proposition 2.1 to this $\{Q_n\}$ to obtain a subsequence of $\{Q_n\}$ (we still denote it by $\{Q_n\}$) which satisfies

$$Q_n^1 \equiv Q_n(\cdot + y_n^1) \longrightarrow Q^1 \not\equiv 0 \qquad \text{weakly in } H^1(\boldsymbol{R}^N)\,, \tag{2.43}$$

$$\lim_{n\to\infty} \{E(Q_n^1) - E(Q_n^1 - Q^1) - E(Q^1)\} = 0\,, \tag{2.44}$$

$$\lim_{n\to\infty} (\|Q_n^1\|^2 - \|Q_n^1 - Q^1\|^2 - \|Q^1\|^2) = 0\,, \tag{2.45}$$

for some $\{y_n^1\} \subset \boldsymbol{R}^N$. We note that $Q_n^1$ is also a $H^1$-bounded minimizing sequence of (2.32). Now we suppose that $E(Q_1)>0$, so that (2.44) and the fact $E(Q_n^1) \leqq 0$ yield that $E(Q_n^1 - Q^1) \leqq 0$ for sufficiently large $n$. Thus we have $\|Q_n^1 - Q^1\| \geqq m$ for large $n$ by the definition of $m$. Since $\lim_{n\to\infty} \|Q_n^1\| = m$, we get from (2.45) that $\|Q^1\| \leqq 0$, which is a contradiction. Thus we obtain

$$E(Q^1) \leqq 0 . \tag{2.46}$$

It follows from (2.46) and the definition of $m$ that $\|Q^1\| \geqq m$, so that we have

$$\|Q^1\| = m , \tag{2.47}$$

since $Q_n^1 \rightharpoonup Q^1$ weakly in $L^2(\boldsymbol{R}^N)$. Thus we get $\lim_{n\to\infty} \|Q_n^1 - Q^1\| = 0$. (So we have $L=1$ in the terminology of Proposition 2.1)

Let $s = \sqrt{\sigma\|\nabla Q^1\|^2/2\|Q^1\|_\sigma^\sigma}$ ($\leqq 1$), and put $Q_s = Q^1\left(\frac{\cdot}{s}\right)$. Then we have

$$E(Q_s) = 0 , \qquad \|Q_s\| = s^{N/2}\|Q^1\| \leqq \|Q^1\| . \tag{2.48}$$

Hence $s$ must be 1. Thus we obtain

$$E(Q_1) = 0 , \tag{2.49}$$

and we have $\lim_{n\to\infty} \|Q_n^1 - Q^1\|_{H^1(\boldsymbol{R}^N)} = 0$.

Let $\{w_n\} \subset H^1(\boldsymbol{R}^N)$ be a minimizing sequence for (2.33). We rescale $w_n$ as follows:

$$W_n(x) = w_n\left(\frac{x}{\tilde{\nu}_n}\right), \qquad \tilde{\nu}_n = \sqrt{\frac{\sigma\|\nabla w_n\|^2}{2\|w_n\|_\sigma^\sigma}} . \tag{2.50}$$

Then one has

$$J(W_n) = J(w_n), \tag{2.51}$$

$$E(W_n) = \tilde{\nu}_n^{N-2}\left(\|\nabla w_n\|^2 - \tilde{\nu}_n^2 \frac{2}{\sigma}\|w_n\|_\sigma^\sigma\right) = 0 , \tag{2.52}$$

so that

$$\frac{1}{C_N} = \lim_{n\to\infty} \frac{2}{\sigma}\|W_n\|^{4/N} , \qquad E(W_n) = 0 . \tag{2.53}$$

Thus by the definition of $m$, we have $(2/\sigma)m^{4/N} \leqq 1/C_N$. Hence we obtain, by (2.38),

$$\frac{2}{\sigma} m^{4/N} = \frac{1}{C_N} . \tag{2.54}$$

Thus $Q^1$ is a critical point of $J(\cdot)$. Since $|\nabla|Q^1|| \leqq |\nabla Q^1|$, we many assume $Q^1 \geqq 0$. So we have

$$\frac{d}{dt} J(Q^1 + t\varphi)\Big|_{t=0} = 0 \tag{2.55}$$

for any $\varphi \in C_0^\infty(\boldsymbol{R}^N)$. Hence $Q^1$ satisfies

$$\Delta Q^1 - \left(\frac{2\|\nabla Q^1\|^2}{N\|Q^1\|^2}\right)Q^1 + |Q^1|^{4/N} Q^1 = 0 \tag{2.56}$$

in the sense of distribution.

Taking

$$Q(x)=\hat{v}^{N/2}Q^1(\hat{v}x), \qquad \hat{v}=\sqrt{\frac{N\|Q^1\|^2}{2\|\nabla Q^1\|^2}}, \tag{2.57}$$

one can easily verify that this $Q$ satisfies (2.35) and $\|Q\|=\|Q^1\|=m$.

We conclude this section with the estimates of the evolution operator of the free Schrödinger equation. We say that a pair $(\nu, \rho)$ of indices is *admissible* if

$$\frac{1}{2}-\frac{1}{n}<\frac{1}{\rho}\leqq\frac{1}{2}$$

and

$$\frac{2}{\nu}=\frac{n}{2}-\frac{n}{\rho}\equiv\delta(\rho).$$

LEMMA 2.6. (1) *For every $\phi\in L^2$ and for every admissible pair $(\nu, \rho)$, the function $t\mapsto U(t)\phi$ belongs to $C(\boldsymbol{R}; L^2)\cap L^\nu(\boldsymbol{R}; L^\rho)$ and satisfies*

$$\|U(\cdot)\phi\|_{L^\nu(\boldsymbol{R};L^\rho)}\leqq C\|\phi\|_2,$$

*where $C$ is independent of $\phi\in L^2$.*

(2) *Let $I$ be an interval $I\subset\boldsymbol{R}$ and let $t_0\in\bar{I}$. Let $(\kappa, \theta)$ be an admissible pair and let $v\in L^{\kappa'}(I; L^{\theta'})$, where $1/\kappa'+1/\kappa=1/\theta'+1/\theta=1$. Then, for every admissible pair $(\nu, \rho)$, the function $t\mapsto\int_{t_0}^t U(t-\tau)v(\tau)d\tau$ belongs to $C(\bar{I}; L^2)\cap L^\nu(I; L^\rho)$ and satisfies*

$$\left\|\int_{t_0} U(\cdot-\tau)v(\tau)d\tau\right\|_{L^\nu(I;L^\rho)}\leqq C\|v\|_{L^{\kappa'}(I;L^{\theta'})},$$

*where $C$ is independent of $v\in L^{\kappa'}(I; L^{\theta'})$.*

For Lemma 2.6, see [**11, 14, 39**].

## 3. A Convergence result.

The aim of this section is to prove the following proposition, which is crucial for the proof of Theorem 1.

PROPOSITION 3.1. *Let $\{v_n\}$ be an equibounded family in $C([0, T]; H^1(\boldsymbol{R}^N))$ such that*

$$2i\frac{\partial v_n}{\partial t}+\Delta v_n+|v_n|^{4/N}v_n=g_n, \tag{3.1}$$

$$\sup_{t\in[0,T]}\|v_n(t)\|_\sigma\neq 0. \tag{3.2}$$

*Here* $\{g_n\}$ *is an equibounded family in* $C([0, T]; L^{\sigma'}(\boldsymbol{R}^N))$ *such that, for any* $R>0$,

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|g_n(t, \cdot)\|_{\sigma'} = 0\,, \tag{3.3}$$

*where* $1/\sigma+1/\sigma'=1$. *Then there exist*

(i) *a nontrivial solution* $v$ *of (NSC) in* $C([0, T]; H^1(\boldsymbol{R}^N))$ *and*

(ii) *a sequence* $\{\gamma_n\}\subset\boldsymbol{R}^N$

*such that for* $\Omega\Subset\boldsymbol{R}^N$ *and for some subsequence (still denoted by the same letter),*

$$\tilde{v}_n \equiv v_n(\cdot, \cdot+\gamma_n) \xrightarrow{\;*\;} v \qquad \textit{weakly* in } L^\infty([0, T]; H^1(\boldsymbol{R}^N)), \tag{3.4}$$

$$\tilde{v}_n \longrightarrow v \qquad \textit{strongly in } C([0, T]; L^\alpha(\Omega)) \textit{ for } \alpha\in[2, 2^*) \quad \textit{as } n\to\infty. \tag{3.5}$$

*Furthermore we have*

$$|\tilde{v}_n|^{4/N}\tilde{v}_n-|\tilde{v}_n-v|^{4/N}(\tilde{v}_n-v)-|v|^{4/N}v \longrightarrow 0 \tag{3.6}$$

$$\textit{strongly in} \quad C([0, T]; L^{\sigma'}(\boldsymbol{R}^N)),$$

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \int_{\boldsymbol{R}^N} \big||\tilde{v}_n|^\alpha-|\tilde{v}_n-v|^\alpha-|v|^\alpha\big| dx = 0\,, \qquad \alpha\in[2, 2^*)\,, \tag{3.7}$$

$$\lim_{n\to\infty} \int_0^T \{E(\tilde{v}_n)-E(\tilde{v}_n-v)-E(v)\} dt = 0\,, \tag{3.8}$$

*and for any* $t\in[0, T]$

$$\lim_{n\to\infty} \{E(\tilde{v}_n(t))-E((\tilde{v}_n-v)(t))-E(v(t))\} = 0\,. \tag{3.9}$$

To prove Proposition 3.1, we prepare the following lemmas.

LEMMA 3.2. *Let* $1<\alpha<\beta<\gamma$ *and* $I\subset\boldsymbol{R}$. *Let* $g(t, x)$ *be a measurable function on* $I\times\boldsymbol{R}^N$ *such that, for some positive constants* $C_\alpha$, $C_\beta$, *and* $C_\gamma$,

$$\operatorname*{ess.sup}_{t\in I} \|g(t)\|_\alpha^\alpha \leqq C_\alpha, \quad \operatorname*{ess.sup}_{t\in I} \|g(t)\|_\beta^\beta \geqq C_\beta>0, \quad \operatorname*{ess.sup}_{t\in I} \|g(t)\|_\gamma^\gamma \geqq C_\gamma. \tag{3.10}$$

*Then we have*

$$\operatorname*{ess.sup}_{t\in I} \mu([|g(t, \cdot)|>\eta]) > C_1 \tag{3.11}$$

*for some* $\eta$, $C_1>0$ *depending on* $\alpha$, $\beta$, $\gamma$, $C_\alpha$, $C_\beta$, $C_\gamma$, *but not on* $g$.

PROOF. Simple calculation with (3.10) implies that, for sufficiently small $\eta>0$,

$$\int_{R^N}|g(t, x)|^\beta dx$$

$$=\int_{[|g(t,\cdot)|<\eta]}|g(t, x)|^\beta dx+\int_{[\eta<|g(t,\cdot)|<1/\eta]}|g(t, x)|^\beta dx+\int_{[|g(t,\cdot)|>1/\eta]}|g(t, x)|^\beta dx$$

$$\leqq\frac{C_\beta}{4C_\alpha}\int_{[|g(t,\cdot)|<\eta]}|g(t, x)|^\alpha dx+\int_{[\eta<|g(t,\cdot)|<1/\eta]}|g(t, x)|^\beta dx$$

$$+\frac{C_\beta}{4C_\gamma}\int_{[|g(t,\cdot)|>1/\eta]}|g(t, x)|^\gamma dx$$

$$\leqq\frac{C_\beta}{4C_\alpha}\operatorname*{ess.sup}_{t\in I}\|g(t)\|_\alpha^\alpha+\int_{[\eta<|g(t,\cdot)|<1/\eta]}|g(t, x)|^\beta dx+\frac{C_\beta}{4C_\gamma}\operatorname*{ess.sup}_{t\in I}\|g(t)\|_\gamma^\gamma$$

$$\leqq\frac{C_\beta}{2}+\mu([|g(t, \cdot)|>\eta])\Big(\frac{1}{\eta}\Big)^\beta.$$

Thus we have (3.11) with $C_1=(C_\beta/2)\eta^\beta$.

LEMMA 3.3. *Let $1<\alpha<\infty$ and $I\Subset R$. Let $g(t, x)\in L^\infty(I; W^{1,\alpha}(R^N))$ such that*

$$\operatorname*{ess.sup}_{t\in I}\|\nabla g(t)\|_\alpha\leqq C_2\,, \tag{3.12}$$

$$\operatorname*{ess.sup}_{t\in I}\mu([|g(t)|>\eta])\geqq C_3 \tag{3.13}$$

*for some positive constants $C_2$ and $C_3$. Then there exists a shift $(T_y g)(t, x)=g(t, x+y)$ such that, for some constant $\delta=\delta(C_1, C_2, \eta)$,*

$$\operatorname*{ess.sup}_{t\in I}\mu\Big(B(0;1)\cap\Big[|T_y g(t)|>\frac{\eta}{2}\Big]\Big)>\delta\,. \tag{3.14}$$

PROOF. For simplicity, we suppose that $g\in C_b(I; W^{1,\alpha}(R^N))$. In general case, we employ Lusin's theorem.

We borrow the idea of Brézis in Lieb [16]. Let $f$ be a non trivial function such that $f(\cdot)\in C_b(I; W^{1,\alpha}_{\mathrm{loc}}(R^N))$, $\sup_{t\in I}\|\nabla f(t, \cdot)\|_\alpha\leqq 1$. Let

$$K=1+\frac{1}{\sup_{t\in R}\|f(t)\|_\alpha^\alpha},$$

$$C_y=\text{cube in } R^N \text{ with center } y \text{ and the side length } \frac{2}{\sqrt{N}}.$$

First we claim that there exists a point $(s, y)\in I\times R^N$ such that

$$\int_{C_y}|\nabla f(s, x)|^\alpha dx<K\int_{C_y}|f(s, x)|^\alpha dx\,. \tag{3.15}$$

Indeed suppose the contrary that, for any $(s, y)\in I\times R^N$,

$$(3.16)\qquad \int_{C_y} |\nabla f(s, x)|^{\alpha} dx \geqq K\int_{C_y} |f(s, x)|^{\alpha} dx\,.$$

We choose $\{y_k\}$ in $\boldsymbol{R}^N$ such that $\mathring{C}_{y_k} \cap \mathring{C}_{y_j} = \emptyset$ for $j \neq k$ and $\bigcup_{k=1}^{\infty} C_{y_k} = \boldsymbol{R}^N$. For this $\{y_k\}$, we have

$$(3.17)\qquad \int_{C_{y_k}} |\nabla f(s, x)|^{\alpha} dx \geqq K\int_{C_{y_k}} |f(s, x)|^{\alpha} dx$$

for any $s \in \boldsymbol{R}$. Summing (3.17) from $k$ equals 1 to $\infty$, we have

$$(3.18)\qquad 1 \geqq \int_{\boldsymbol{R}^N} |\nabla f(s, x)|^{\alpha} dx \geqq K\int_{\boldsymbol{R}^N} |f(s, x)|^{\alpha} dx$$

for any $s \in \boldsymbol{R}$. Now we take the supremum of the right hand side of (3.18) with respect to $s \in I$, so that we obtain, by the definition of $K$,

$$(3.19)\qquad 1 \geqq 1 + \sup_{t \in \boldsymbol{R}} \|f(t)\|_{\alpha}^{\alpha} > 1\,,$$

which is a contradiction.

By (3.15) we have

$$(3.20)\qquad \int_{C_y} |\nabla f(s, x)|^{\alpha} + |f(s, x)|^{\alpha} dx < (K+1)\int_{C_y} |f(s, x)|^{\alpha} dx\,.$$

On the other hand, by the Sobolev inequality, we have

$$(3.21)\qquad \int_{C_y} |\nabla f(s, x)|^{\alpha} + |f(s, x)|^{\alpha} dx \geqq S\Big(\int_{C_y} |f(s, x)|^{\alpha^*} dx\Big)^{\alpha/\alpha^*}\,,$$

where $1/\alpha^* + 1/N = 1/\alpha$ if $\alpha < N$ and, if $\alpha \geqq N$, $\alpha^*$ is arbitrary with $\alpha < \alpha^* < \infty$. $S$ depends only on $\alpha$, $\alpha^*$. Combining (3.20) and (3.21), we have that, for some open, small interval $I_s$ containing $s$,

$$(3.22)\qquad S\sup_{t \in I_s}\Big(\int_{C_y} |f(t, x)|^{\alpha^*} dx\Big)^{\alpha/\alpha^*} < (K+1)\int_{C_y} |f(s, x)|^{\alpha} dx\,.$$

Here we have used the fact that $f(t)$ is continuous in $t$ with values in $W^{1,\alpha}_{\mathrm{loc}}(\boldsymbol{R}^N)$. From (3.22), we have by the Hölder inequality that, for $\tau \in I_s$ (if necessary, taking smaller interval),

$$(3.23)\qquad S\sup_{t \in I_s}\Big(\int_{C_y} |f(t, x)|^{\alpha^*} dx\Big)^{\alpha/\alpha^*}$$
$$< (K+1)\mu(C_y \cap \operatorname{supp} f(\tau, \cdot))^{1-\alpha/\alpha^*}\Big(\int_{C_y} |f(\tau, x)|^{\alpha^*} dx\Big)^{\alpha/\alpha^*}\,.$$

Hence it follows that

$$(3.24)\qquad S < (K+1)\operatorname*{ess.\,sup}_{t \in I} \mu(C_y \cap \operatorname{supp} f(t, \cdot))^{1-\alpha/\alpha^*}\,.$$

Now we put $f(t, x) = \max(|g(t, x)| - \eta/2, 0)$. For simplicity we assume that

$\|\nabla g(t)\|_{\alpha}\leqq 1$ so that $\sup_{t\in \boldsymbol{R}}\|\nabla f(t,\ \cdot)\|_{\alpha}\leqq 1$. From (3.13), we have

$$\sup_{t\in \boldsymbol{R}}\|f(t)\|_{\alpha}^{\alpha}\geqq\left(\frac{\eta}{2}\right)^{\alpha}\sup_{t\in \boldsymbol{R}}\mu\left(\left[\,|g(t,\ \cdot)|>\frac{\eta}{2}\right]\right)\geqq\left(\frac{\eta}{2}\right)^{\alpha}C_{3}\,, \tag{3.25}$$

and thus $K\leqq 1+2^{\alpha}/\eta^{\alpha}C_{3}$. From (3.24) we deduce (3.14) for some point $y\in \boldsymbol{R}^{N}$ and some constant $\delta$ depending only on $N$, $\alpha$, $\eta$, $C_{2}$ and $C_{1}$.

REMARK 3.1. Since we have Lemma 3.2, the condition (3.13) is always assured if $g$ satisfies

$$\operatorname*{ess.sup}_{t\in I}\|g(t)\|_{\alpha}^{\alpha}\leqq C_{\alpha},\quad \operatorname*{ess.sup}_{t\in I}\|g(t)\|_{\beta}^{\beta}\geqq C_{\beta}>0,\quad \operatorname*{ess.sup}_{t\in I}\|g(t)\|_{\alpha*}^{\alpha*}\leqq C_{\gamma},$$

where $\beta=\alpha(1+\alpha/N)$.

This lemma and Corollary 3.4 below are closely related to the compactness device as in Lieb [**16**].

COROLLARY 3.4. *Let* $1<\alpha<\infty$ *and* $I\Subset \boldsymbol{R}$. *Let* $\{v_{n}(t,\ x)\}$ *be a uniformly bounded sequence of functions in* $C_{b}(I\,;W^{1,\alpha}(\boldsymbol{R}^{N}))$ *such that* $\mathrm{ess.sup}_{t\in I}\,\mu([\,|v_{n}(t)|>\eta])\geqq C_{3}$ *for some positive constant* $\eta$ *and* $C_{3}$. *Furthermore we suppose that* $\{v_{n}(t,\ x)\}$ *is an equi-continuous family in* $C_{b}(I\,;\,L^{\alpha}(\boldsymbol{R}^{N}))$. *Then there exist a sequence* $\{y_{n}\}$ *in* $\boldsymbol{R}^{N}$ *and a nontrivial function* $v\in L^{\infty}(I\,;W^{1,\alpha}(\boldsymbol{R}^{N}))$ *such that for* $\Omega\Subset \boldsymbol{R}^{N}$ *and for some subsequence (still denoted by the same letter),*

$$\tilde{v}_{n}\equiv v_{n}(\cdot,\ \cdot+y_{n})\xrightarrow{\ *\ }v\qquad \textit{weakly* in } L^{\infty}(I\,;W^{1,\alpha}(\boldsymbol{R}^{N}))\,, \tag{3.26}$$

$$\tilde{v}_{n}\longrightarrow v\qquad \textit{strongly in } C(I\,;\,L^{\alpha}(\Omega)) \tag{3.27}$$

*as* $n\rightarrow\infty$.

PROOF. We note that, for any $\{x_{n}\}\subset \boldsymbol{R}^{N}$, $\{v_{n}(t,\ x+x_{n})\}$ is also an equi-continuous family in $C_{b}(I;\,L^{\alpha}(\boldsymbol{R}^{N}))$. Thus this corollary is a direct consequence of Lemma 3.3 and the Ascoli-Arzelà theorem.

To treat the nonlinear term in (3.1), we need

LEMMA 3.5. *Let* $1<\alpha<\infty$. *Let* $\{f_{n}(t,\ x)\}$ *be an bounded family in* $L^{\alpha}(I\times\Omega)$ *where* $I\times\Omega\subset \boldsymbol{R}\times \boldsymbol{R}^{N}$. *Suppose that* $f_{n}\rightarrow f$ *a.e. in* $I\times\Omega$. *Then we have*

$$|f_{n}|^{\alpha-2}f_{n}-|f_{n}-f|^{\alpha-2}(f_{n}-f)-|f|^{\alpha-2}f\longrightarrow 0\qquad \textit{in } L^{\alpha'}(I\times\Omega)\,, \tag{3.28}$$

*where* $1/\alpha+1/\alpha'=1$, *and we have*

$$\lim_{n\rightarrow\infty}\iint_{I\times\Omega}\big|\,|f_{n}|^{\alpha}-|f_{n}-f|^{\alpha}-|f|^{\alpha}\big|\,dtdx=0\,. \tag{3.29}$$

*Furthermore if* $I$ *is a compact interval, and if* $\{f_{n}(t,\ x)\}$ *is an equi-continuous family in* $C(I\,;\,L^{\alpha}(\Omega))$, *then we have*

$$|f_{n}|^{\alpha-2}f_{n}-|f_{n}-f|^{\alpha-2}(f_{n}-f)-|f|^{\alpha-2}f\longrightarrow 0\qquad \textit{in } L^{\infty}(I\,;\,L^{\alpha'}(\Omega))\,, \tag{3.30}$$

*and*

$$\limsup_{n\to\infty}\sup_{t\in I}\int_{\Omega}||f_n|^{\alpha}-|f_n-f|^{\alpha}-|f|^{\alpha}|dx=0. \tag{3.31}$$

PROOF. If we establish the assertions of (3.28) and (3.29), one can easily verify (3.30) and (3.31).

(3.29) is purely a consequence of the Brézis-Lieb lemma [**5**]. The convergence (3.28) also follows from the same argument performed in [**5**]. One can easily observed that for any $\varepsilon>0$, there exists a positive constant $C_{\varepsilon}$ such that

$$||f_n|^{\alpha-2}f_n-|f_n-f|^{\alpha-2}(f_n-f)-|f|^{\alpha-2}f|\leqq\varepsilon|f_n-f|^{\alpha-1}+C_{\varepsilon}|f|^{\alpha-1}. \tag{3.32}$$

We note that for $I\Subset\boldsymbol{R}$,

$$\varepsilon|f_n-f|^{\alpha-1}+C_{\varepsilon}|f|^{\alpha-1}\in L^{\alpha'}(I\times\boldsymbol{R}^N). \tag{3.33}$$

Now set

$$W_{\varepsilon,n}=[||f_n|^{\alpha-2}f_n-|f_n-v|^{\alpha-2}(f_n-f)-|f|^{\alpha-2}f|-\varepsilon|f_n-f|^{\alpha-1}]_+ \tag{3.34}$$

where $[a]_+=\max(a, 0)$, so that $W_{\varepsilon,n}(t, x)\to0$ *a.e.* as $n\to\infty$ and by (3.32), $W_{\varepsilon,n}\leqq C_{\varepsilon}|f|^{\alpha-1}$. Thus the dominated convergence theorem implies

$$\lim_{n\to\infty}\iint_{I\times\boldsymbol{R}^N}W_{\varepsilon,n}(t, x)^{\alpha'}dt\,dx=0. \tag{3.35}$$

However we have

$$||f_n|^{\alpha-2}f_n-|f_n-v|^{\alpha-2}(f_n-f)-|f|^{\alpha-2}f|\leqq W_{\varepsilon,n}+\varepsilon|f_n-f|^{\alpha-1} \tag{3.36}$$

and, thus

$$\begin{aligned}\limsup_{n\to\infty}\iint_{I\times\boldsymbol{R}^N}&||f_n|^{\alpha-2}f_n-|f_n-v|^{\alpha-2}(f_n-f)-|f|^{\alpha-2}f|^{\alpha'}dtdx\\&\leqq 2^{\alpha'-1}\varepsilon\limsup_{n\to\infty}\iint_{I\times\boldsymbol{R}^N}|f_n-f|^{\alpha}dtdx.\end{aligned} \tag{3.37}$$

Hence we obtain (3.28).

Now we are in a position to prove Proposition 3.1.

PROOF OF PROPOSITION 3.1. Since $\{v_n\}$ and $\{g_n\}$ are equi-bounded families in $L^{\infty}(I; H^1(\boldsymbol{R}^N))$ and $L^{\infty}(I; L^{\sigma'}(\boldsymbol{R}^N))$, respectively, it follows from (3.1) that $\{\partial v_n/\partial t\}$ is an equibounded family in $L^{\infty}(\boldsymbol{R}; H^{-1})$, where $H^{-1}$ is the dual space of $H^1(\boldsymbol{R}^N)$. From the identity

$$\|v_n(t)-v_n(s)\|^2=2\int_s^t\Re\left\langle\frac{\partial}{\partial\tau}v_n(\tau), v_n(\tau)-v_n(s)\right\rangle d\tau, \tag{3.38}$$

we have

$$(3.39)\qquad \|v_n(t)-v_n(s)\|^2 \leqq 4|t-s|\,\|v_n\|_{L^\infty(R,H^1)}\left\|\frac{\partial}{\partial\tau}v_n\right\|_{L^\infty(R,H^{-1})},$$

where $t,\ s\in I$ and $H^1=H^1(\boldsymbol{R}^N)$. By the Gagliardo-Nirenberg inequality, we also obtain, for $\alpha\in(2,\ 2^*)$,

$$(3.40)\qquad \|v_n(t)-v_n(s)\|_\alpha^\alpha \leqq C|t-s|^\theta\|v_n\|_{L^\infty(R;H^1)}^{\theta+N(\alpha-2)/2}\left\|\frac{\partial}{\partial\tau}v_n\right\|_{L^\infty(R,H^{-1})}^\theta,$$

where $\theta=\alpha/2-N(\alpha-2)/4$. Hence, for any $\{x_n\}\subset\boldsymbol{R}^N$, $\{v_n(t,\ x+x_n)\}$ is an equi-continuous family in $C_b(\boldsymbol{R}\,;L^\alpha(\boldsymbol{R}^N))$ for $\alpha\in[2,\ 2^*)$. On the other hand, we have, by (3.2) and Lemma 3.2, that, for some positive constant $C$ independent of $n$,

$$(3.41)\qquad \operatorname*{ess.sup}_{t\in I}\mu([|v_n(t)|>\eta])\geqq C\,.$$

Thus, by virtue of Corollary 3.4, there exist a nontrivial function $v$ in $C([0,\ T]\,;H^1(\boldsymbol{R}^N))$ and a sequence $\{\gamma_n\}\subset\boldsymbol{R}^N$ such that, for $\Omega\Subset\boldsymbol{R}^N$ and for some subsequence (still denoted by the same letter),

$$(3.42)\qquad \tilde{v}_n\equiv v_n(\cdot,\ \cdot+\gamma_n)\xrightarrow{\ *\ } v\qquad \text{weakly* in } L^\infty([0,\ T]\,;H^1(\boldsymbol{R}^N)),$$

$$(3.43)\qquad \tilde{v}_n\to v\qquad \text{strongly in } C([0,\ T]\,;L^\alpha(\Omega)) \text{ for } \alpha\in[2,\ 2^*)\ \text{ and}$$

as $n\to\infty$.

We claim that the limit function $v$ solves (NSC). Since $v_n$ solves (3.1), it holds that, for any $\chi\in C_0^\infty(\boldsymbol{R})$ and $\varphi\in C_0^\infty(\boldsymbol{R}^N)$,

$$(3.45)\qquad -\int_R\langle 2i\tilde{v}_n,\ \varphi\rangle\dot{\chi}\,dt+\int_R\langle\Delta\tilde{v}_n,\ \varphi\rangle\chi\,dt+\int_R\langle|\tilde{v}_n|^{4/N}\tilde{v}_n,\ \varphi\rangle\lambda\,dt=\int_R\langle\tilde{g}_n,\ \varphi\rangle\chi\,dt,$$

where $\tilde{g}_n\equiv g_n(t,\ x+\gamma_n)$ and $\dot{}=d/dt$. Hence, by (3.3), (3.42) and (3.43), we have

$$(3.46)\qquad 2i\frac{d}{dt}\langle v,\ \varphi\rangle+\langle\Delta v,\ \varphi\rangle+\langle|v|^{4/N}v,\ \varphi\rangle=0$$

in $\mathcal{D}'(\boldsymbol{R})$ (the dual of $C_0^\infty(\boldsymbol{R})$), so that the standard argument shows

$$(3.47)\qquad 2i\frac{\partial v}{\partial t}+\Delta v+|v|^{4/N}v=0\,,\qquad \text{in } H^{-1}.$$

By (3.39), we have $v\in C_b(I\,;L^2(\boldsymbol{R}^N))$. This fact together with $v\in L^\infty(I\,;H^1(\boldsymbol{R}^N))$ implies $v\in C_w(I\,;H^1(\boldsymbol{R}^N))$. Hence $v$ has the definite initial value $v(0)\in H^1(\boldsymbol{R}^N)$. Thus the uniqueness theorem of Kato [**14**] yields $v\in C_b(\boldsymbol{R}\,;H^1(\boldsymbol{R}^N))$.

Since $\tilde{v}_n$ converges to $v$ in $C(I\,;L^2(\Omega))$ and *a fortiori* in $L^2(I\times\Omega)$ for any $\Omega\Subset\boldsymbol{R}^N$, we can extract a subsequence from $\{\tilde{v}_n\}$ (still denoted by $\{\tilde{v}_n\}$) such that $\tilde{v}_n\to v$ *a.e.* $I\times\Omega$. Thus (3.6) and (3.7) follow from Lemma 3.5 and the equi-continuity of $\{\tilde{v}_n\}$. (3.8) follows from (3.6) and the weak convergence of $\{\nabla b_n\}$ in $L^2(I\times\boldsymbol{R}^N)$. (3.9) follows from (3.6) and the weak convergence of $\{\nabla\tilde{v}_n(t)\}$ in $L^2(\boldsymbol{R}^N)$.

## 4. Proof of Theorem 1.

We shall prove Theorem 1 by using Proposition 3.1 in the previous section.

Let $u(t)$ be the singular solution of (NSC) with initial datum $u(0)=u_0\in H^1(\boldsymbol{R}^N)$ such that $u(\cdot)\in C([0, T_m); H^1(\boldsymbol{R}^N))$ and

$$\limsup_{t\to T_m}\|\nabla u(t)\| = \limsup_{t\to T_m}\|u(t)\|_\sigma = \infty \tag{4.1}$$

for some $T_m\in(0, \infty]$. Let $\{t_n\}$ be any sequence such that

$$\sup_{t\in[0, t_n)}\|u(t)\|_\sigma = \|u(t_n)\|_\sigma . \tag{4.2}$$

For this $\{t_n\}$, we put

$$\lambda_n = \frac{1}{\|u(t_n)\|_\sigma^{\sigma/2}} \tag{4.3}$$

and, we consider the scaled functions

$$u_n(t, x) = \lambda_n^{N/2}\overline{u(t_n-\lambda_n^2 t, \lambda_n x)} \tag{4.4}$$

for $t\in[0, t_n/\lambda_n^2)$. One can easily see that

$$2i\frac{\partial u_n}{\partial t}+\Delta u_n+|u_n|^{4/N}u_n = 0 , \tag{4.5}$$

$$\|u_n(t)\| = \|u_0\| , \tag{4.6}$$

$$\sup_{t\in[0, T]}\|u_n(t)\|_\sigma = 1 \qquad \text{for any } T>0, \tag{4.7}$$

$$E(u_n(t)) = \lambda_n^2 E(u_0)(\to 0) \qquad \text{as } n\to\infty . \tag{4.8}$$

From (4.5), (4.6) and (4.7), it follows that $\{u_n\}$ is an equi-bounded family in $L^\infty(\boldsymbol{R}_+; H^1(\boldsymbol{R}^N))$. Thus, by Proposition 3.1, there exist

(i) a nontrivial solution $u^1$ of (NSC) in $C([0, \infty); H^1(\boldsymbol{R}^N))$

(ii) a sequence $\{y_n^1\}\subset\boldsymbol{R}^N$

such that for $\Omega\Subset\boldsymbol{R}^N$ and for some subsequence (still denoted by the same letter),

$$u_n^1 \equiv u_n(\cdot, \cdot+y_n^1) \xrightarrow{\ *\ } u^1 \qquad \text{weakly* in } L^\infty([0, \infty); H^1(\boldsymbol{R}^N)), \tag{4.10}$$

$$u_n^1 \to u^1 \qquad \text{strongly in } C([0, T]; L^\alpha(\Omega)) \tag{4.11}$$

for $\alpha\in[2, 2^*)$ as $n\to\infty$. Furthermore we have

$$|u_n^1|^{4/N}u_n^1-|u_n^1-u^1|^{4/N}(u_n^1-u^1)-|u^1|^{4/N}u^1 \longrightarrow 0 \tag{4.12}$$

strongly in $C([0, T]; L^{\sigma'}(\boldsymbol{R}^N))$,

$$\lim_{n\to\infty}\sup_{t\in[0, T]}\int_{\boldsymbol{R}^N}\left||u_n^1|^\alpha-|u_n^1-u^1|^\alpha-|u^1|^\alpha\right|dx = 0 , \qquad \alpha\in[2, 2^*), \tag{4.13}$$

$$\lim_{n\to\infty}\int_0^T\{E(u_n^1)-E(u_n^1-u^1)-E(u^1)\}\,dt=0\,, \tag{4.14}$$

and, for any $t\in \boldsymbol{R}_+$

$$\lim_{n\to\infty}\{E(u_n^1(t))-E((u_n^1-u^1)(t))-E(u^1(t))\}=0\,. \tag{4.15}$$

Here we note that $u^1(t)$ is defined on $\boldsymbol{R}_+$, and that $u^1\in L^\infty(\boldsymbol{R}_+;H^1(\boldsymbol{R}^N))$. Hence we obtain, by Theorem 2, that

$$E(u^1(t))=E(u^1(0))\geqq 0 \qquad t\in[0,\infty)\,. \tag{4.16}$$

Suppose that $\limsup_{n\to\infty}\|u_n^1-u^1\|_\sigma\neq 0$. $u_n^1-u^1$ satisfies

$$2i\frac{\partial(u_n^1-u^1)}{\partial t}+\Delta(u_n^1-u^1)+|u_n^1-u^1|^{4/N}(u_n^1-u^1)=g_n^1\,, \tag{4.17}$$

where

$$g_n^1(t,x)=-(|u_n^1|^{4/N}u_n^1-|u_n^1-u^1|^{4/N}(u_n^1-u^1)-|u^1|^{4/N}u^1)(t,x)\,. \tag{4.18}$$

We note here that (4.12) implies that for any $\{x_n\}\subset\boldsymbol{R}^N$ and for any $T>0$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|g_n^1(t,\cdot+x_n)\|_\sigma=0\,. \tag{4.19}$$

Clearly $\{u_n^1-u^1\}$ is an equi-bounded family in $L^\infty([0,\infty);H^1(\boldsymbol{R}^N))$. Thus we apply Proposition 3.1 to $\{u_n^1-u^1\}$ to obtain

(i) a nontrivial solution $u^2$ of (NSC) in $C_b([0,\infty);H^1(\boldsymbol{R}^N))$ with $E(u^2(t))=E(u^2(0))\geqq 0$, $t\in[0,\infty)$ (by Theorem 2) and

(ii) a sequence $\{y_n^2\}\subset\boldsymbol{R}^N$

such that for $\Omega\Subset\boldsymbol{R}^N$ and for some subsequence (still denoted by the same letter),

$$u_n^2\equiv(u_n^1-u^1)(\cdot,\cdot+y_n^2)\xrightarrow{*}u^2 \tag{4.20}$$

weakly* in $L^\infty([0,\infty);H^1(\boldsymbol{R}^N))$, and

$$u_n^2\longrightarrow u^2 \qquad \text{strongly in } C([0,T];L^\alpha(\Omega)) \tag{4.21}$$

for $\alpha\in[2,2^*)$ as $n\to\infty$. Furthermore we have

$$|u_n^2|^{4/N}u_n^2-|u_n^2-u^2|^{4/N}(u_n^2-u^2)-|u^2|^{4/N}u^2\longrightarrow 0 \tag{4.22}$$

strongly in $C([0,T];L^{\sigma'}(\boldsymbol{R}^N))$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\int_{\boldsymbol{R}^N}||u_n^2|^\alpha-|u_n^2-u^2|^\alpha-|u^2|^\alpha|\,dx=0\,, \qquad \alpha\in[2,2^*)\,, \tag{4.23}$$

$$\lim_{n\to\infty}\int_0^T\{E(u_n^2)-E(u_n^2-u^2)-E(u^2)\}\,dt=0\,, \tag{4.24}$$

and, for any $t\in\boldsymbol{R}_+$,

$$\lim_{n\to\infty}\{E(u_n^2(t))-E((u_n^2-u^2)(t))-E(u^2(t))\}=0\,. \tag{4.25}$$

We also have by (4.10) and (4.20) that

$$\lim_{n\to\infty}|y_n^2|=\infty\,, \tag{4.26}$$

and we can see that $(u_n^2-u^2)$ satisfies

$$2i\frac{\partial(u_n^2-u^2)}{\partial t}+\Delta(u_n^2-u^2)+|u_n^2-u^2|^{4/N}(u_n^2-u^2)=g_n^2\,, \tag{4.27}$$

where

$$g_n^2(t,\ x)=g_n^1(t,\ x+y_n^1)-(|u_n^2|^{4/N}u_n^2-|u_n^2-u^2|^{4/N}(u_n^2-u^2)-|u^2|^{4/N}u^2)(t,\ x)\,. \tag{4.28}$$

(4.12) and (4.22) imply that for any $\{x_n\}\subset \boldsymbol{R}^N$ and for any $T>0$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|g_n^2(t,\ \cdot+x_n)\|_\sigma=0\,. \tag{4.29}$$

It is worthwhile to note that it holds from (4.13), (4.15), (4.23) and (4.25) that, for any $t\in\boldsymbol{R}_+$ and $\alpha\in[2,\ 2^*)$,

$$\lim_{n\to\infty}(\|u_n(t)\|_\alpha^\alpha-\|(u_n^2-u^2)(t)\|_\alpha^\alpha)=\sum_{k=1}^{2}\|u^k(t)\|_\alpha^\alpha\,, \tag{4.30}$$

$$\lim_{n\to\infty}\{E(u_n(t))-E((u_n^2-u^2)(t))\}=\sum_{k=1}^{2}E(u^k(t))\,, \tag{4.31}$$

since $\|\cdot\|_\alpha$ and $E(\cdot)$ are invariant under the action of space-translations.

The proof of Theorem 1 consists of iterating the construction of Proposition 3.1. In what follows, we freely take enough subsequences. Repeating the procedure above, we inductively obtain: $(j\geqq 2)$

(i) nontrivial solutions $u^j$ of (NSC) in $C_b([0,\ \infty);H^1(\boldsymbol{R}^N))$ with $E(u^j(t))=E(u^j(0))\geqq 0$ $t\in[0,\ \infty)$ (by Theorem 2) and

(ii) a sequence $\{y_n^j\}\subset\boldsymbol{R}^N$ with $\lim_{n\to\infty}|\sum_{k=2}^{j}y_n^k|=\infty$

such that for $\Omega\Subset\boldsymbol{R}^N$ and for any $T>0$,

$$u_n^j\equiv(u_n^{j-1}-u^{j-1})(\cdot,\ \cdot+y_n^j)\xrightarrow{\ *\ }u^j \tag{4.32}$$

weakly* in $L^\infty([0,\ \infty);H^1(\boldsymbol{R}^N))$, and

$$u_n^j\longrightarrow u^j \qquad \text{strongly in } C([0,\ T];L^\alpha(\Omega)) \tag{4.33}$$

for $\alpha\in[2,\ 2^*)$ as $n\to\infty$; and furthermore we have

$$|u_n^j|^{4/N}u_n^j-|u_n^j-u^j|^{4/N}(u_n^j-u^j)-|u^j|^{4/N}u^j\longrightarrow 0 \tag{4.34}$$

strongly in $C([0,\ T];L^{\sigma'}(\boldsymbol{R}^N))$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\int_{\boldsymbol{R}^N}||u_n^j|^\alpha-|u_n^j-u^j|^\alpha-|u^j|^\alpha|\,dx=0\,, \tag{4.35}$$

$$\lim_{n\to\infty}\int_0^T \{E(u_n^j)-E(u_n^j-u^j)-E(u^j)\}\,dt = 0\,, \tag{4.36}$$

and, for any $t\in\boldsymbol{R}_+$,

$$\lim_{n\to\infty}\{E(u_n^j(t))-E((u_n^j-u^j)(t))-E(u^j(t))\} = 0\,, \tag{4.37}$$

$$\lim_{n\to\infty}(\|u_n(t)\|_\alpha^\alpha-\|(u_n^j-u^j)(t)\|_\alpha^\alpha) = \sum_{k=1}^{j}\|u^k(t)\|_\alpha^\alpha\,, \tag{4.38}$$

$$\lim_{n\to\infty}\{E(u_n(t))-E((u_n^j-u^j)(t))\} = \sum_{k=1}^{j}E(u^k(t))\,; \tag{4.39}$$

besides, $(u_n^j-u^j)$ satisfies

$$2i\frac{\partial(u_n^j-u^j)}{\partial t}+\Delta(u_n^j-u^j)+|u_n^j-u^j|^{4/N}(u_n^j-u^j) = g_n^j\,, \tag{4.40}$$

where

$$g_n^j(t,x)=g_n^{j-1}(t,x+y_n^{j-1})-(|u_n^j|^{4/N}u_n^j-|u_n^j-u^j|^{4/N}(u_n^j-u^j)-|u^j|^{4/N}u^j)(t,x)\,. \tag{4.41}$$

(4.34) implies that for any $\{x_n\}\subset\boldsymbol{R}^N$ and for any $T>0$,

$$\lim_{n\to\infty}\sup_{t\in[0,T]}\|g_n^j(t,\cdot+x_n)\|_\sigma = 0\,. \tag{4.42}$$

We claim that the iteration must terminated at some index $L\in\boldsymbol{N}$. Suppose the contrary that $L=\infty$. In this case, we have

$$\lim_{j\to\infty}\lim_{n\to\infty}\sup_{t\in[0,T]}\|(u_n^j-u^j)(t)\|_\sigma = 0\,. \tag{4.43}$$

Indeed; suppose the contrary that, for some positive constant $\varepsilon_0$ and a subsequence $\{j(k)\}$ of $\{j\}$, one has

$$\liminf_{k\to\infty}\lim_{n\to\infty}\sup_{t\in[0,T]}\|(u_n^{j(k)}-u^{j(k)})(t))\|_\sigma > \varepsilon_0\,. \tag{4.44}$$

Hence there is a constant $C_0$ essentially depends on $\varepsilon_0$ such that

$$\sup_{t\in[0,T]}\|u^{j(k)+1}(t)\|_\sigma^\sigma > C_0\,, \tag{4.45}$$

since the size of $\sup_{t\in[0,T]}\|u^j(t)\|_\sigma$ essentially depends on the lower bound of $\sup_{t\in[0,T]}\|(u_n^{j-1}-u^{j-1})(t)\|_\sigma$ by Lemma 3.2, Lemma 3.3 and the construction of $u^j$. We choose $k\in\boldsymbol{N}$ large enough (specified latter). Using the formula (4.38) with $\alpha=\sigma$ and $j=j(k)$, we have by (4.45) that, for enough subsequence with respect to $n$,

$$\begin{aligned} 1 &> \limsup_{n\to\infty}\sup_{t\in[0,T]}(\|u^{j(k)+1}(t)\|_\sigma^\sigma-\|(u_n^{j(k)}-u^{j(k)})(t)\|_\sigma^\sigma) \\ &= \sup_{t\in[0,T]}\sum_{j=1}^{j(k)}\|u^j(t)\|_\sigma^\sigma > \sup_{t\in[0,T]}\sum_{l=1}^{k}\|u^{j(l)}(t)\|_\sigma^\sigma \\ &> kC_0\,. \end{aligned} \tag{4.46}$$

Thus we reach a contradiction, if we take $k$ as $kC_0 \geqq 1$. Hence we get (4.43). (4.39) together with (4.8) and (4.43) yields that

$$0 \leqq \lim_{j\to\infty} \sum_{k=1}^{j} E(u^k(t)) \leqq \lim_{j\to\infty} \lim_{n\to\infty} \sup_{t\in[0,T]} \|(u_n^j - u^j)(t)\|_\sigma^\sigma = 0, \tag{4.47}$$

so that we have

$$E(u^k(t)) = 0, \qquad j \in \boldsymbol{N}. \tag{4.48}$$

Then, by Proposition 2.5, we have

$$\|u^j(t)\| = \|u^j(0)\| \geqq \|Q\|, \qquad j \in \boldsymbol{N}. \tag{4.49}$$

This together with (4.38) with $\alpha=2$ implies

$$j\|Q\|^2 \leqq \sum_{k=1}^{j} \|u^k(t)\|^2 \leqq \|u_0\|^2, \tag{4.50}$$

which is a contradiction. Thus we obtain $L<\infty$, so that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|(u_n^L - u^L)(t)\|_\sigma = 0 \tag{4.51}$$

which implies that (1.14) with $\gamma_n^j = \sum_{k=1}^{j} y_n^k$, since we have

$$(u_n^L - u^L)(t, x) = u_n\Big(t, x+\sum_{k=1}^{L} y_n^k\Big) - \sum_{j=1}^{L} u^j\Big(t, x+\sum_{k=j+1}^{L} y_n^k\Big). \tag{4.52}$$

(1.15) follows from (4.39), (4.51) and (4.52). It remains to prove (1.16). Noting that

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \||u_n^L - u^L|^{4/N}(u_n^L - u^L)\|_{\sigma'} = 0 \tag{4.53}$$

we have, by (4.34) with $j=L$ and (4.52), that

$$|u_n|^{4/N} u_n(t, \cdot) - \sum_{j=1}^{L} |u^j|^{4/N} u^j(t, \cdot - \gamma_n^j) \longrightarrow 0 \tag{4.54}$$

strongly in $C([0, T]; L^{\sigma'}(\boldsymbol{R}^N))$. Since $u_n$ satisfies

$$u_n(t) = U(t)u_n(0) + \frac{i}{2}\int_0^t U(t-\tau)(|u_n|^{4/N} u_n)(\tau)d\tau, \tag{4.55}$$

and $\tilde{u}^j(t, x) \equiv u^j(t, x-\gamma_n^j)$ satisfies

$$\tilde{u}^j(t) = U(t)\tilde{u}^j(0) + \frac{i}{2}\int_0^t U(t-\tau)(|\tilde{u}^j|^{4/N}\tilde{u}^j)(\tau)d\tau, \tag{4.56}$$

we have

$$u_n(t) - \sum_{j=1}^{L} \tilde{u}^j(t) - \phi^u(t) = \frac{i}{2}\int_0^t U(t-\tau)\Big(|u_n|^{4/N} u_n - \sum_{j=1}^{L} |\tilde{u}^j|^{4/N}\tilde{u}^j\Big)d\tau, \tag{4.57}$$

where

$$\phi_n(t,\ \cdot) = U(t)\Big(u_n(0) - \sum_{j=1}^{L} \tilde{u}^j(0)\Big). \tag{4.58}$$

Estimating (4.57) with the help of Lemma 2.6 (2) ($\nu=\infty$, $\rho=2$ and $\kappa=\theta=\sigma$), we have (1.16) by virtue of (4.54).

## 5. Proofs of Corollaries.

Let $u(t)$, $\{t_n\}$, $u_n(t)$ and $Q(x)$ be as in the previous section. We shall prove Corollaries 1 and 2 which give sufficiently conditions that we have $L=1$ in Theorem 1. In the proofs of both corollaries we, however, do not need Theorem 2 to prove $L=1$.

PROOF OF COROLLARY 1. Let $H_r^1(\boldsymbol{R}^N)=\{v\in H^1(\boldsymbol{R}^N)\,;\, v(x)=v(|x|),\ x\in\boldsymbol{R}^N\}$. Assume $u(0)$ is radially symmetric and $N\geqq 2$. Then the corresponding solution $u(t)$ is also radially symmetric in space variables. Precisely $u(\cdot)\in C([0,\ T_m)\,;\, H_r^1(\boldsymbol{R}^N))$. We note the following fact: if, in Proposition 3.1, we assume in addition that $\{v_n\}\subset C_b([0,\ T]\,;\, H_r^1(\boldsymbol{R}^N))$, then we have that $\gamma_n\equiv 0$, and that

$$v_n \longrightarrow v \qquad \text{strongly in } C([0,\ T]\,;\, L^\alpha(\boldsymbol{R}^N)) \text{ for } \alpha\in(2,\ 2^*) \tag{5.1}$$

as $n\to\infty$, since the embedding $L^\alpha(\boldsymbol{R}^N)\hookrightarrow H_r^1(\boldsymbol{R}^N)$ for $\alpha\in(2,\ 2^*)$ is compact. Thus we have in the same way as in the proof of Theorem 1 that, for radially symmetric family $\{u_n\}$, there exists a nontrivial solution $u^1$ of (NSC) in $C([0,\ \infty)\,;\, H_r^1(\boldsymbol{R}^N))$ such that for some subsequence (still denoted by the same letter),

$$|u_n|^{4/N}u^u - |u^1|^{4/N}u^1 \longrightarrow 0 \qquad \text{strongly in } C([0,\ T]\,;\, L^{\sigma'}(\boldsymbol{R}^N)), \tag{5.2}$$

$$\lim_{n\to\infty}\ \sup_{t\in[0,T]} \|(u_n-u^1)(t)\|_\sigma = 0 \qquad \alpha\in(2,\ 2^*)\,, \tag{5.3}$$

$$\lim_{n\to\infty}\int_0^T \{E(u_n)-E(u_n-u^1)-E(u^1)\}\,dt = 0\,, \tag{5.4}$$

and for any $t\in[0,\ T]$

$$\lim_{n\to\infty}\{E(u_n(t))-E((u_n-u^1)(t))-E(u^1(t))\} = 0\,. \tag{5.5}$$

The strong convergence (5.3) implies $L-1$ in the terminology in Theorem 1. From (5.3) and (5.4), we have

$$E(u^1(t)) = E(u^1(0)) \leqq 0\,. \tag{5.6}$$

Theorem 2 is only used to prove $E(u(t))=0$.

PROOF OF COROLLARY 2. Assume $\|u_0\|=\|Q\|$. In the same way as in the proof of Proposition 2.5, we can prove that (in the proof of Theorem 1) we have

$$|u_n^1|^{4/N}u_n^1 - |u^1|^{4/N}u^1 \longrightarrow 0 \tag{5.7}$$

strongly in $C([0, T]; L^{\sigma'}(\boldsymbol{R}^N))$, and we have

$$\lim_{n\to\infty} \sup_{t\in[0,T]} \|(u_n^1 - u^1)(t)\|_\alpha = 0 \qquad \alpha\in[2, 2^*), \tag{5.8}$$

$$E(u^1) = 0. \tag{5.9}$$

Thus we have $L=1$ in the terminology in Theorem 1. We note that we do not use Theorem 2.

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