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On energy decay-nondecay problems for wave equations with nonlinear dissipative term in $\boldsymbol{R}^N$

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1. Introduction.

In this paper we are concerned with the energy decay and nondecay problems of solutions to the wave equation

$$w_{tt}-\Delta w+\lambda w+\beta(x, t, w_t)w_t = 0, \qquad (x, t) \in \boldsymbol{R}^N\times(0, \infty) \tag{1.1}$$

with initial data

$$w(x, 0) = w_1(x) \quad \text{and} \quad w_t(x, 0) = w_2(x), \quad x\in\boldsymbol{R}^N . \tag{1.2}$$

Here $w_t=\partial w/\partial t$, $w_{tt}=\partial^2 w/\partial t^2$, Δ is the N-dimensional Laplacian, $\lambda\geqq 0$ and $\beta(x, t, w_t)w_t$ represents a dissipative term. In the following it is restricted to the power nonlinearity

$$\beta(x, t, w_t(x, t)) = b(x, t)|w_t(x, t)|^{\rho-1} \tag{1.3}$$

with $b(x, t)\geqq 0$ and $\rho>1$, or to the cubic convolution

$$\beta(x, t, w_t(x, t)) = (V_\gamma * w_t(t)^2)(x) = \int_{\boldsymbol{R}^N} V_\gamma(x-y)w_t(y, t)^2 dy \tag{1.4}$$

with $V_\gamma(x)=|x|^{-\gamma}$ $(0<\gamma<N)$. In order to guarantee regularities of solutions, we require in (1.3)

$$|b_t(x, t)|+|\nabla b(x, t)| \leqq Cb(x, t) \tag{1.5}$$

for some $C>0$, where $\nabla f=(\partial_1 f, \cdots, \partial_N f)$, $\partial_j=\partial/\partial x_j$.

We use the following notation: L^p $(1\leqq p\leqq\infty)$ is the usual space of all L^p-functions in $\boldsymbol{R}^N$; If X is a Banach space and $I\subset\boldsymbol{R}$ is an interval, then by $C(I; X)$ and $L^p(I; X)$ we mean the space of all X-valued continuous and L^p-functions on I, respectively; H^k $(k=1, 2, \cdots)$ is the Sobolev space with norm

$$\|f\|_{H^k} = \left\{\sum_{|\alpha|\leqq k}\int_{\boldsymbol{R}^N}|\nabla^\alpha f(x)|^2 dx\right\}^{1/2} < \infty,$$

where α are the multi-indices; E is the space of pairs $f=\{f_1, f_2\}$ of functions such that

$$\|f\|_E = \left\{\frac{1}{2}\int_{R^N}[|\nabla f_1(x)|^2+\lambda f_1(x)^2+f_2(x)^2]dx\right\}^{1/2} < \infty .$$

Now, assume

$$\{w_1, w_2\} \in H^2\times(H^1\cap L^q), \tag{1.6}$$

where $q=2\rho$ for power nonlinearity (1.3) and $q=6N/(3N-2\gamma)$ for cubic convolution (1.4). Then it is known (see e.g., Lions-Strauss [**4**], Strauss [**15**], Motai [**11**] and Mochizuki and Motai [**10**]) that the above initial value problem has a global solution satisfying the following properties:

(i) $w(t)\in C([0, \infty); E)$ and satisfies the energy equation

$$\|w(t)\|_E^2+\int_0^t\int_{R^N}\beta(x, \tau, w_t(x, \tau))w_t(x, \tau)^2dxd\tau = \|w(0)\|_E^2 \tag{1.7}$$

for any $t>0$.

(ii) $w(t)\in C([0, T]; L^2)$ for any $T>0$.

(iii) $w_{tt}(t)$, $\nabla w_t(t)$, $\Delta w(t)$, $\beta(\cdot, t, w_t(t))w_t(t)\in L^\infty([0, T]; L^2)$ for any $T>0$.

We see from (1.7) that the energy $\|w(t)\|_E^2$ of solution $w(t)$ is decreasing in $t>0$. Thus, a question naturally rises whether it decays or not as t goes to infinity. Our purpose of the present paper is to give a partial answer to this question under some additional conditions on the initial data.

We shall show that the energy decay occurs for a dense class of initial data in $H^2\times(H^1\cap L^q)$ provided that the following (AI) or (AII) is satisfied.

(AI) The case of power nonlinearity (1.3):

$$N\geqq 1,\quad 1<\rho\leqq 1+\frac{2(1-\delta)}{N} \quad \text{and} \quad b_1(1+|x|+t)^{-\delta}\leqq b(x, t)\leqq b_2$$

for some $0\leqq\delta<1$ and $b_1, b_2>0$. If $\lambda=0$, we additionally require

$$b(x, t) \text{ is nonincreasing in } t\geqq 0. \tag{1.8}$$

(AII) The case of cubic convolution (1.4):

$$N\geqq 1 \quad \text{and} \quad 0<\gamma<N,\quad \gamma\leqq 1.$$

On the other hand, we shall show that the energy does not deacy for a class of small initial data in $H^2\times(H^1\cap L^q)$ provided that the following (AIII) or (AIV) is satisfied.

(AIII) The case of power nonlinearity (1.3): If $\lambda=0$,

$$N\geqq 2,\quad \rho>1+\frac{2(1-\delta)}{N-1} \quad \text{and} \quad 0\leqq b(x, t)\leqq b_3(1+|x|)^{-\delta},$$

and if $\lambda>0$,

$$N \geqq 1, \quad \rho > 1 + \frac{2(1-\delta)}{N} \quad \text{and} \quad 0 \leqq b(x, t) \leqq b_3(1+|x|)^{-\delta}$$

for some $0 \leqq \delta \leqq 1$ and $b_3 > 0$.

(AIV) The case of cubic convolution (1.4): If $\lambda=0$,

$$N \geqq 3 \quad \text{and} \quad \frac{N}{N-1} < \gamma < N,$$

and if $\lambda > 0$,

$$N \geqq 2 \quad \text{and} \quad 1 < \gamma < N.$$

Comparing (AI) and (AIII), or (AII) and (AIV), we see that in case $\lambda > 0$, the number

$$\rho = 1 + \frac{2(1-\delta)}{N} \quad \text{or} \quad \gamma = 1 \tag{1.9}$$

is a critical value which divides the energy decay and nondecay. However, in case $\lambda=0$, the problems remain unsolved so far for ρ or γ in the interval

$$1 + \frac{2(1-\delta)}{N} < \rho \leqq 1 + \frac{2(1-\delta)}{N-1} \quad \text{or} \quad 1 < \gamma \leqq \frac{N}{N-1}. \tag{1.10}$$

There are several works on the decay and nondecay problems of energy for the Cauchy problem (1.1), (1.2). In case $\lambda=0$, a nondecay result has been stated in Mochizuki [**7**] under (AIII) (the detail of proof is not given there). In case $\lambda > 0$, the energy decay is proved by Nakao [**13**] under (AI) with $b(x, t) \equiv 1$, and a nondecay result is proved by Motai [**12**] under (AIII) with $b(x, t) \equiv 1$. Thus, in these cases our results are not new. However, our proof is simple and unified. Especially, the key requirement in [**13**] that the initial data $w(0)$ have compact support is removed in this paper. This paper gives the first results on the energy decay problem in case $\lambda=0$.

In case of linear dissipation, the decay and nondecay problems have been solved even for $\lambda=0$ by the works of Matsumura [**6**] and Mochizuki [**8**]. It is proved in [**6**] that the energy decays if $b_1(1+|x|+t)^{-1} \leqq b(x, t) \leqq b_2$ and $b_t(x, t) \leqq 0$. On the other hand, it is proved in [**8**] that the energy does not in general decay if $0 \leqq b(x, t) \leqq b_3(1+|x|)^{-\delta}$ with $\delta > 1$. Thus, we have the critical value $\delta=1$ in this case. Note that Matsumura's result is also restricted to the compactly supported initial data. Its noncompact version is given in [**9**]. In Rauch-Taylor [**14**] another approach is developed for a linear dissipation with $b(x, t)$ of compact support.

Once we know the cases where the energy does not decay, it becomes an interesting problem to know the asymptotic behavior of solutions as $t \to \infty$. A partial answer to this problem has been given in [**12**] (for a linear dissipation, see [**8**], [**9**]). We are able to obtain similar conclusions also for the case of

$\lambda=0$ or of cubic convolution. The results will be summarized in [**10**].

Our argument on the decay property is based on the weighted energy inequality. On the other hand, to obtain the nondecay results we combine the usual energy estimate for $w(t)$ and uniform decay estimates for the free wave equation

$$w_{0tt}-\Delta w_0+\lambda w_0=0, \qquad (x, t)\in \boldsymbol{R}^N\times(0, \infty). \tag{1.11}$$

Thus, we shall follow the same line of proof of our previous works.

The rest of the paper is organized as follows: In the next §2 we give a semi-abstract sufficient condition on the dissipative term with which the energy decay occurs. The result is applied in §§3 and 4 to the problem (1.1), (1.2) with dissipative term (1.3) and to the problem (1.1), (1.2) with dissipative term (1.4), respectively. Finally, in §4 we summarize nondecay results of the energy.

2. A weighted energy estimate and the energy decay.

Let $\varphi(s)$, $s\geqq 0$, be a smooth function satisfying

$$\varphi(s)\geqq 1, \quad \varphi'(s)>0, \quad \varphi''(s)\leqq 0 \quad \text{and} \quad \lim_{s\to\infty}\varphi(s)=\infty\,; \tag{2.1}$$

$$\lambda\varphi(s)^2-\varphi(s)\varphi''(s)\geqq k^2\varphi'(s)^2 \qquad \text{for some } k>1\,. \tag{2.2}$$

With this $\varphi(s)$ we define a weighted energy of solutions at time t as follows:

$$\|w(t)\|_{E_\varphi}^2=\frac{1}{2}\int_{R^N}[\varphi(r+t)\{w_t^2+|\nabla w|^2+\lambda w^2\}-\varphi''(r+t)w^2]dx\,, \tag{2.3}$$

where $r=|x|$. In order to show the energy decay property, the initial data are required other than (1.6) to satisfy

$$\{w_1, w_2\}\in E_\varphi\,, \qquad \text{i.e., } \|w(0)\|_{E_\varphi}^2<\infty\,. \tag{2.4}$$

Multiply by $\{\varphi(r+t)w\}_t$ on both sides of (1.1). It then follows that

$$X_t+\nabla\cdot Y+Z=0\,, \tag{2.5}$$

where

$$X=\frac{1}{2}\varphi\{w_t^2+|\nabla w|^2+\lambda w^2\}-\frac{1}{2}\varphi''w^2+\varphi'w_tw\,,$$

$$Y=-\varphi\nabla ww_t-\varphi'\nabla ww+\frac{x}{2r}\varphi''w^2\,,$$

$$Z=\varphi\beta(x, t, w_t)w_t^2+\varphi'\beta(x, t, w_t)w_tw$$
$$+\frac{1}{2}\varphi'\{|\nabla w|^2-3w_t^2+2w_rw_t\}+\frac{1}{2}\left\{\lambda\varphi'-\frac{N-1}{r}\varphi''\right\}w^2\,.$$

LEMMA 1. *We choose the initial data to satisfy* (1.6) *and* (2.4). *Then the* *corresponding solution* $w(t)$ *of* (1.1) *admits the inequality*

$$\frac{k-1}{k}\|w(t)\|_{E_\varphi}^2+\int_0^t\int_{R^N}\varphi\beta(x,\tau,w_t)w_t^2dxd\tau$$
$$\leqq\int_0^t\int_{R^N}\varphi'\{2w_t^2+\beta(x,\tau,w_t)|w_tw|\}dxd\tau+\frac{k+1}{k}\|w(0)\|_{E_\varphi}^2 \quad (2.6)$$

for each $t>0$.

PROOF. Let $B(R)=\{x;|x|<R\}$ and $S(R)=\partial B(R)$. We integrate (2.5) over $B(R)\times(0,t)$. Then since

$$Z(x,t)\geqq\varphi\beta w_t^2-\varphi'\{2w_t^2+\beta|w_tw|\}$$

by (2.1), integration by parts gives

$$\int_{B(R)}X(x,t)dx+\int_0^t\int_{B(R)}\varphi\beta w_t^2dxd\tau+\int_0^t\int_{S(R)}\frac{x}{r}\cdot Y(x,\tau)dSd\tau$$
$$\leqq\int_0^t\int_{B(R)}\varphi'\{2w_t^2+\beta|w_tw|\}dxd\tau+\int_{B(R)}X(x,0)dx. \quad (2.7)$$

By (2.2),

$$|\varphi'w_tw|\leqq\frac{1}{2k}\varphi\{w_t^2+\lambda w^2\}-\frac{1}{2k}\varphi''w^2.$$

Thus, we have

$$X(x,t)\geqq\frac{k-1}{2k}[\varphi\{w_t^2+|\nabla w|^2+\lambda w^2\}-\varphi''w^2], \quad (2.8)$$

$$X(x,0)\leqq\frac{k+1}{2k}[\varphi\{w_2^2+|\nabla w_1|^2+\lambda w_1^2\}-\varphi''w_1^2]. \quad (2.9)$$

Similarly, we have

$$\left|\frac{x}{r}\cdot Y(x,\tau)\right|\leqq\frac{k+1}{2k}[\varphi\{w_t^2+w_r^2+\lambda w^2\}-\varphi''w^2]. \quad (2.10)$$

Since $\varphi'(r+t)$ is bounded, it follows from properties (i)$\sim$(iii) of the solution $w(t)$ that

$$\int_0^t\int_{R^N}\varphi'\{2w_t^2+\beta(x,\tau,w_t)|w_tw|\}dxd\tau<\infty$$

for each $t>0$. It follows from (2.4) and (2.9) that

$$\int_{R^N}X(x,0)dx\leqq\frac{k+1}{k}\|w(0)\|_{E_\varphi}^2<\infty.$$

Moreover, since $\varphi(r)=O(r)$ as $r\to\infty$, (2.10) and properties (i)$\sim$(iii) imply that

$$\liminf_{R\to\infty}\int_0^t\int_{S(R)}\left|\frac{x}{r}\cdot Y(x,\tau)\right|dSd\tau=0.$$

Thus, applying (2.8) and letting $R\to\infty$ in (2.7), we conclude the assertion of the lemma. □

THEOREM 1. *Let $w(t)$ be as in the above lemma. Suppose the following: There exist constants $p>1$, $J_1(\varphi)>0$ and $K_1(w(0), \varphi)>0$ such that the a priori inequalities*

$$\int_0^t\int_{R^N} 2\varphi' w_t^2 dx d\tau \leqq J_1(\varphi)^{(p-1)/(p+1)}\left(\int_0^t\int_{R^N}\varphi\beta w_t^2 dx d\tau\right)^{2/(p+1)}, \tag{2.11}$$

$$\int_0^t\int_{R^N}|\varphi'\beta w_t w|dx d\tau \leqq K_1(w(0), \varphi)^{1/(p+1)}\left(\int_0^t\int_{R^N}\varphi\beta w_t^2 dx d\tau\right)^{p/(p+1)}, \tag{2.12}$$

hold for any $t>0$. Then we have

$$\begin{aligned}\|w(t)\|_{E_\varphi}^2 &\leqq K(w(0), \varphi)\\ &\equiv C\{\|w(0)\|_{E_\varphi}^2+J_1(\varphi)+K_1(w(0), \varphi)\} < \infty\end{aligned} \tag{2.13}$$

for some $C>0$. Thus, the energy of $w(t)$ decays like

$$\|w(t)\|_E^2 \leqq K(w(0), \varphi)\varphi(t)^{-1} \quad \text{as } t\to\infty. \tag{2.14}$$

PROOF. We apply (2.11) and (2.12) to (2.6) and use the Young inequality. Then for any $\varepsilon>0$ there exists a $C(\varepsilon)>0$ such that

$$\begin{aligned}&\frac{k-1}{k}\|w(t)\|_{E_\varphi}^2+(1-2\varepsilon)\int_0^t\int_{R^N}\varphi\beta w_t^2 dx d\tau\\ &\leqq \frac{k+1}{k}\|w(0)\|_{E_\varphi}^2+C(\varepsilon)J_1(\varphi)+C(\varepsilon)K_1(w(0), \varphi).\end{aligned}$$

Choosing $\varepsilon<1/2$ in this inequality, we conclude (2.13).

(2.14) is obvious since we have $\varphi(t)\leqq\varphi(t+\tau)$. □

3. A dissipative term of power nonlinearity.

In this and next §§ we consider the concrete examples (1.3) and (1.4), under (AI) and (AII), respectively. We shall show that Theorem 1 can be applied to these two examples. As the weight function $\varphi(s)$, we choose one of the following two functions.

$$\varphi_1(s) = \{\log(a+s)\}^{\mu}, \quad \mu>0 \quad \text{and} \quad \log a > \max\{1, 2\mu-1\}, \tag{3.1}$$

$$\varphi_2(s) = (\tilde{a}+s)^{\nu}, \quad 0<\nu<1 \quad \text{and} \quad \tilde{a}^2 > \max\left\{\frac{2\nu^2-\nu}{\lambda}, 1\right\}. \tag{3.2}$$

As is easily seen, these functions satisfy conditions (2.1) and (2.2).

As the Sobolev embedding shows,

$$H^k \hookrightarrow L^q \quad \text{if } 2 \leqq q \leqq \frac{2N}{N-2k} \quad (N>2k), \; <\infty \; (N \leqq 2k). \tag{3.3}$$

As is easily seen, this condition with $k=1$ is satisfied by q in (1.6) if we require (AI) or (AII). Thus, in this and next §§, the initial data is chosen to satisfy

$$\{w_1, w_2\} \in \{H^2 \times H^1\} \cap E_{\varphi_i}. \tag{3.4}$$

In this § we shall show the following theorem.

THEOREM 2. (i) *Assume* (AI). *If we choose*

$$0 < \mu < \frac{2}{\rho-1} \tag{3.5}$$

in (3.1), *then there exists a constant* $K(w(0), \varphi_1)>0$ *such that*

$$\|w(t)\|_E^2 \leqq K(w(0), \varphi_1)\{\log(a+t)\}^{-\mu}. \tag{3.6}$$

(ii) *Assume* $\lambda>0$ *and* $\rho<1+2(1-\delta)/N$ *in* (AI). *If we choose*

$$0 < \nu < \frac{2(1-\delta)}{\rho-1} - N \quad \text{and} \quad \nu \leqq 1 \tag{3.7}$$

in (3.2), *then we have a better decay estimate*

$$\|w(t)\|_{E_\lambda}^2 \leqq K(w(0), \varphi_2)(\tilde{a}+t)^{-\nu}. \tag{3.8}$$

We shall verify the a priori inequalities (2.11) and (2.12).
(2.11) will be based on the following inequality.

$$\int_0^t \int_{R^N} \varphi' w_t^2 dx d\tau \leqq \left(\int_0^t \int_{R^N} \varphi b |w_t|^{\rho+1} dx d\tau\right)^{2/(\rho+1)}$$
$$\times \left(\int_0^t \int_{R^N} (\varphi b)^{-2/(\rho-1)} (\varphi')^{(\rho+1)/(\rho-1)} dx d\tau\right)^{(\rho-1)/(\rho+1)}. \tag{3.9}$$

LEMMA 2. (i) (2.11) *with* $p=\rho$ *and* $\varphi=\varphi_1$ *holds if we require*

$$1 < \rho \leqq 1+\frac{2(1-\delta)}{N} \quad \text{and} \quad 0 < \mu < \frac{2}{\rho-1}. \tag{3.10}$$

(ii) (2.11) *with* $p=\rho$ *and* $\varphi=\varphi_2$ *holds if we require*

$$1 < \rho < 1+\frac{2(1-\delta)}{N} \quad \text{and} \quad 0 < \nu < \frac{2(1-\delta)}{\rho-1} - N. \tag{3.11}$$

PROOF. We have only to show that

$$J_1(\varphi_i) \equiv \int_0^\infty \int_{R^N} (\varphi_i b)^{-2/(\rho-1)} (\varphi_i')^{(\rho+1)/(\rho-1)} dx d\tau < \infty. \tag{3.12}$$

(i) By the condition on $b(x, t)$ and (3.1),

$$(\varphi_1 b)^{-2/(\rho-1)}(\varphi_1')^{(\rho+1)/(\rho-1)} \leqq C\{\log(a+r+t)\}^{\mu-(\rho+1)/(\rho-1)}(a+r+t)^{-1-2(1-\delta)/(\rho-1)}.$$

Since $-2(1-\delta)/(\rho-1) \leqq -N$ and $\mu-(\rho+1)/(\rho-1)<-1$ by (3.10), we have

$$J_1(\varphi_1) \leqq C\int_0^\infty \{\log(a+\tau)\}^{\mu-(\rho+1)/(\rho-1)}d\tau \int_0^\infty (a+r+\tau)^{-2}dr$$

$$\leqq \int_0^\infty \{\log(a+\tau)\}^{\mu-(\rho+1)/(\rho-1)}(a+\tau)^{-1}d\tau < \infty .$$

Thus, (3.12) with $i=1$ follows.

(ii) We have similarly

$$(\varphi_2 b)^{-2/(\rho-1)}(\varphi_2')^{(\rho+1)/(\rho-1)} \leqq C(\tilde{a}+r+t)^{\nu-1-2(1-\delta)/(\rho-1)} .$$

The right side becomes integrable in $\boldsymbol{R}^N\times(0, \infty)$ by (3.11). Thus, (3.12) with $i=2$ also follows. □

Next we shall show (2.12) based on the following inequality.

$$\int_0^t\int_{\boldsymbol{R}^N} |\varphi'\beta w_t w|\,dxd\tau \leqq \left(\int_0^t\int_{\boldsymbol{R}^N}\varphi b|w_t|^{\rho+1}dxd\tau\right)^{\rho/(\rho+1)}$$

$$\times\left(\int_0^t\int_{\boldsymbol{R}^N}\varphi^{-\rho}b(\varphi'|w|)^{\rho+1}dxd\tau\right)^{1/(\rho+1)}. \qquad (3.13)$$

LEMMA 3. (i) *In case* (1.8) *being satisfied, if we require*

$$1<\rho \leqq \frac{N+2}{N-2} \quad (N \geqq 3),\ <\infty \quad (N=1,\, 2) \quad \text{and} \quad 0<\mu<\rho\,, \qquad (3.14)$$

then (2.12) *with* $p=\rho$ *and* $\varphi=\varphi_1$ *holds, where*

$$K_1(w(0), \varphi_1) = J_2(\varphi_1)\{\|w(0)\|_E^2+\|w_1\|_{H^1}^{\rho+1}\}.$$

(ii) *In case* $\lambda>0$, *if we require*

$$1<\rho \leqq \frac{N+2}{N-2} \quad (N \geqq 3),\ <\infty \quad (N=1,\, 2) \quad \text{and} \quad \mu>0 \ [\text{or } 0<\nu<\rho], \quad (3.15)$$

then (2.12) *with* $p=\rho$ *and* $\varphi=\varphi_i$ $(i=1,\, 2)$ *holds, where*

$$K_1(w(0), \varphi_i) = J_3(\varphi_i)\|w(0)\|_E^{\rho+1} .$$

PROOF. Note that

$$\int_0^t\int_{\boldsymbol{R}^N}\varphi_i^{-\rho}b(\varphi_i'|w|)^{\rho+1}dxd\tau \leqq \int_0^t \varphi_i(\tau)^{-\rho}\varphi_i'(\tau)^{\rho+1}d\tau\int_{\boldsymbol{R}^N}b|w(\tau)|^{\rho+1}dx. \quad (3.16)$$

(i) If we use (1.8), then as is easily seen,

$$b|w(\tau)|^{\rho+1} \leqq C\left\{\tau^\rho\int_0^\tau b|w_t(t)|^{\rho+1}dt+b|w(0)|^{\rho+1}\right\}.$$

Integrate the both sides over $\boldsymbol{R}^N$. Then the energy equation (1.7), the boundedness of $b(x, t)$ and the Sobolev embedding (3.3) with $k=1$, $q=\rho+1$ imply

$$\int_{R^N} b|w(\tau)|^{\rho+1}dx \leqq C(a+\tau)^\rho\{\|w(0)\|_E^2+\|w_0\|_{L^{\rho+1}}^{\rho+1}\}$$
$$\leqq C(a+\tau)^\rho\{\|w(0)\|_E^2+\|w_1\|_{H^1}^{\rho+1}\}.$$

Substitute this in (3.16) with $i=1$. Then since

$$J_2(\varphi_1) \equiv C\int_0^\infty \varphi_1(\tau)^{-\rho}\varphi_1'(\tau)^{\rho+1}(a+\tau)^\rho d\tau$$
$$\leqq C\int_0^\infty \{\log(a+\tau)\}^{\mu-\rho-1}(a+\tau)^{-1}d\tau < \infty$$

by (3.1) and (3.13), we conclude assertion (i).

(ii) In this case, since $\lambda>0$, the boundedness of $b(x, t)$ and the Sobolev embedding show that under (3.14),

$$\int_{R^N} b|w(\tau)|^{\rho+1}dx \leqq C\|w(\tau)\|_{H^1}^{\rho+1} \leqq C\|w(0)\|_E^{\rho+1}.$$

On the other hand, it follows from (3.1) [or (3.2)] and (3.15) that

$$J_3(\varphi_i) \equiv C\int_0^\infty \varphi_i(\tau)^{-\rho}\varphi_i'(\tau)^{\rho+1}d\tau < \infty.$$

Substituting these inequalities in (3.16), we conclude assertion (ii). □

PROOF OF THEOREM 2. Assertion (i) is a result of Lemmas 2 (i), 3 and Theorem 1, where $K(w(0), \varphi_1)$ is given by (2.13) with

$$K_1(w(0), \varphi_1) = \begin{cases} J_2(\varphi_1)\{\|w(0)\|_E^2+\|w_1\|_{H^1}^{\rho+1}\} & \text{if } \lambda=0 \\ J_3(\varphi_1)\|w(0)\|_E^{\rho+1} & \text{if } \lambda>0. \end{cases}$$

Assertion (ii) is a result of Lemmas 2 (ii), 3 (ii) and Theorem 1, where $K_1(w(0), \varphi_2)$ in (2.13) is given by $J_3(\varphi_2)\|w(0)\|_E^{\rho+1}$.

4. A dissipative term of cubic convolution.

In this § we shall prove the following theorem.

THEOREM 3. (i) *Assume* (AII). *If we choose*

$$0<\mu<1 \tag{4.1}$$

in (3.1), *then there exists a constant* $K(w(0), \varphi_1)>0$ *such that*

$$\|w(t)\|_E^2 \leqq K(w(0), \varphi_1)\{\log(a+t)\}^{-\mu}. \tag{4.2}$$

(ii) *Assume* $\lambda>0$ *and* $\gamma<1$ *in* (AII). *If we choose*

$$0<\nu<1-\gamma \tag{4.3}$$

in (3.2), *then there exists a constant* $K(w(0), \varphi_2)>0$ *such that*

$$\|w(t)\|_E^2 \leqq K(w(0), \varphi_2)(\tilde{a}+t)^{-\nu}. \tag{4.4}$$

We shall also verify the a priori inequalities (2.11) and (2.12). For this aim we introduce the homogeneous Sobolev space $\dot{H}^{\sigma,p}$ with $p\geqq 1$ and $\alpha>-N$, which consists of all functions such that

$$\|f\|_{\dot{H}^{\sigma,p}} \equiv \|\mathcal{F}^{-1}\{|\xi|^\alpha \hat{f}(\xi)\}\|_{L^p} < \infty,$$

where $\hat{f}$ is the Fourier transform of $f(x)$ and $\mathcal{F}^{-1}$ means its inverse transformation. In case $p=2$, we simply write $\dot{H}^\sigma=\dot{H}^{\sigma,2}$.

LEMMA 4. *The following equality and inequalities hold.*

$$\int_{\boldsymbol{R}^N}(V_\gamma * w_t^2)w_t^2 dx = \|V_{(N+\gamma)/2} * w_t^2\|_{L^2}^2, \tag{4.5}$$

$$\int_0^t\int_{\boldsymbol{R}^N}\varphi' w_t^2 dx d\tau \leqq \left(\int_0^t \varphi(\tau)^{-1}\eta(\tau)^2\|\zeta(\tau)\|_{\dot{H}^{(N-\gamma)/2}}^2 d\tau\right)^{1/2} \times\left(\int_0^t\int_{\boldsymbol{R}^N}\varphi(V_\gamma * w_t^2)w_t^2 dx d\tau\right)^{1/2}, \tag{4.6}$$

$$\left|\int_0^t\int_{\boldsymbol{R}^N}\varphi'(V_\gamma * w_t^2)w_t w\, dx d\tau\right| \leqq \left(\int_0^t\int_{\boldsymbol{R}^N}\varphi(V_\gamma * w_t^2)w_t^2 dx d\tau\right)^{3/4} \times\left(\int_0^t \varphi(\tau)^{-3}\varphi'(\eta)^4\|V_{(N+\gamma)/2} * w(\tau)^2\|_{L^2}^2 d\tau\right)^{1/4}. \tag{4.7}$$

Here in (4.6), $\eta(\tau)=\{\log(a+\tau)\}^{\mu-1}$, $\zeta(r, t)=(1+r^2+\tau^2)^{-1/2}$ *if* $\varphi=\varphi_1$, *and* $\eta(\tau)=1$, $\zeta(r, t)=(1+r^2+t^2)^{-(1-\nu)/2}$ *if* $\varphi=\varphi_2$.

PROOF. The Parseval equality shows

$$\int_{\boldsymbol{R}^N}(V_\gamma * w_t^2)w_t^2 dx = \int_{\boldsymbol{R}^N}|V_{(-N+\gamma)/2}(\xi)\widehat{w_t^2}(\xi)|^2 d\xi = \|V_{(N+\gamma)/2} * w_t^2\|_{L^2}^2$$

(cf. e.g., [**1**], [**2**]). This is (4.5). We similarly have

$$\int_0^t\int_{\boldsymbol{R}^N}\varphi' w_t^2 dx d\tau \leqq \int_0^t \eta(\tau)\int_{\boldsymbol{R}^N}\zeta w_t^2 dx d\tau$$

$$\leqq \int_0^t \eta(\tau)\|\zeta(\tau)\|_{\dot{H}^{(N-\gamma)/2}}\|V_{(N+\gamma)/2} * w_t^2\|_{L^2} d\tau$$

Thus, the Schwarz inequality and (4.5) show (4.6) since we have $\varphi(\tau)\leqq\varphi(r+\tau)$.
Finally, we have

$$\left|\int_0^t\int_{R^N}\varphi'(V_\gamma*w_t^2)\,w_t w\,dx\,d\tau\right|$$
$$\leqq \int_0^t\varphi'(\tau)\left(\int_{R^N}[V_{(N+\gamma)/2}*w_t^2]^2dx\right)^{1/2}\left(\int_{R^N}[V_{(N+\gamma)/2}*(w_t w)]^2dx\right)^{1/2}d\tau$$
$$\leqq \int_0^t\varphi'(\tau)\left(\int_{R^N}[V_{(N+\gamma)/2}*w_t^2]^2dx\right)^{3/4}\left(\int_{R^N}[V_{(N+\gamma)/2}*w^2]^2dx\right)^{1/4}d\tau\,,$$

where in the first inequality we have used the fact $0<\varphi'(r+\tau)\leqq\varphi'(\tau)$. Thus, (4.5) and the Hölder inequality show (4.7) since we have $\varphi(\tau)\leqq\varphi(r+\tau)$. □

LEMMA 5. *Let* $0<\gamma<N$, $\beta>\gamma/2$ $(N\geqq 2)$ *or* $\beta>1/2$ $(N=1)$. *Then*

$$\|(1+r^2+t^2)^{-\beta/2}\|^2_{\dot{H}^{(N-\gamma)/2}}\leqq C(1+t)^{-2\beta+\gamma}\,.$$

PROOF. First consider the case $N\geqq 2$. Let k be the integer satisfying $(N-\gamma)/2\leqq k<(N-\gamma)/2+1$. If we choose p to satisfy

$$\frac{1}{p}-\frac{k}{N}=\frac{1}{2}-\frac{N-\gamma}{2N}\,, \tag{4.8}$$

then the Sobolev embedding shows that

$$\|(1+r^2+t^2)^{-\beta/2}\|_{\dot{H}^{(N-\gamma)/2}}\leqq C\|(1+r^2+t^2)^{-\beta/2}\|_{\dot{H}^{k,p}}\,.$$

As is shown in [**1**; Theorem 6.3.1], $\|f\|_{\dot{H}^{k,p}}\cong\sum_{|\alpha|=k}\|\nabla^\alpha f\|_{L^p}$. Thus, we have

$$\|(1+r^2+t^2)^{-\beta/2}\|_{\dot{H}^{k,p}}\leqq C\sum_{|\alpha|=k}\|\nabla^\alpha(1+r^2+t^2)^{-\beta/2}\|_{L^p}$$
$$\leqq C(1+t)^{-\beta-k+N/p}\,.$$

This and (4.8) show the desired inequality.

Next let $N=1$. Then since $(1+r^2+t^2)^{-\beta/2}\in L^2$, by use of the Parseval equality and the Hölder inequality, we obtain

$$\|(1+r^2+t^2)^{-\beta/2}\|_{\dot{H}^{(1-\gamma)/2}}\leqq\|(1+r^2+t^2)^{-\beta/2}\|_{L^2}^{1-(1-\gamma)/2}\|(a+r^2+t^2)^{-\beta/2}\|_{\dot{H}^1}^{(1-\gamma)/2}$$
$$\leqq C(1+t)^{(-\beta+1/2)(1-(1-\gamma)/2)}(1+t)^{(-\beta-1/2)(1-\gamma)/2}\,.$$

Thus, the desired inequality follows. □

Now we shall show the a priori inequality (2.11).

LEMMA 6. (i) (2.11) *with* $p=3$ *and* $\varphi=\varphi_1$ *holds if we require*

$$0<\gamma\leqq 1 \quad and \quad 0<\mu<1\,. \tag{4.9}$$

(ii) (2.11) *holds with* $p=3$ *and* $\varphi=\varphi_2$ *if we require*

$$0<\gamma<1 \quad and \quad 0<\nu<1-\gamma\ (N\geqq 2),\ <\min\left\{1-\gamma,\ \frac{1}{2}\right\}\ (N=1)\,. \tag{4.10}$$

PROOF. By (4.6) we have only to show that

$$J_1(\varphi_i)=\int_0^\infty \varphi_i(\tau)^{-1}\eta(\tau)^2\|\zeta(\tau)\|^2_{\dot{H}^{(N-\gamma)/2}}d\tau<\infty\,. \tag{4.11}$$

(i) Noting the definition of $\eta(t)$ and $\zeta(r, t)$, we have from (3.1), (4.9) and Lemma 5 with $\beta=1$,

$$J_1(\varphi_1)\leqq C\int_0^\infty \{\log(a+\tau)\}^{\mu-2}(1+\tau)^{-2+\gamma}d\tau<\infty\,.$$

Thus, (4.11) with $i=1$ follows.

(ii) In this case we have from (3.2), (4.10) and Lemma 5 with $\beta=1-\nu$,

$$J_1(\varphi_2)\leqq C\int_0^\infty (1+\tau)^{-2+\nu+\gamma}d\tau<\infty\,.$$

Thus, (4.11) with $i=2$ follows. □

Next we shall show inequality (2.12).

LEMMA 7. (i) *If we require*

$$0<\gamma<N,\quad \gamma\leqq 4 \quad \textit{and}\quad 0<\mu<3\,, \tag{4.12}$$

then (2.12) *with* $p=3$ *and* $\varphi=\varphi_1$ *holds, where*

$$K_1(w(0),\varphi_1)=J_2(\varphi_1)\{\|w(0)\|_E^2+\|w_1\|_{H^1}^4\}\,.$$

(ii) *In the case* $\lambda>0$, *if we require*

$$0<\gamma<N,\quad \gamma\leqq 4\quad \textit{and}\quad \mu>0\quad [\textit{or}\ 0<\nu<3]\,, \tag{4.13}$$

then (2.12) *holds with* $p=3$ *and* $\varphi=\varphi_i$ $(i=1, 2)$, *where*

$$K_1(w(0),\varphi_i)=J_3(\varphi_i)\|w(0)\|_E^4\,.$$

PROOF. (i) The Hardy-Littlewood inequality (cf., [**1**] or [**2**]) and the Sobolev embedding (3.3) with $k=1$, $q=4N/(2N-\gamma)$ assert that

$$\|V_{(N+\gamma)/2}*w^2\|_{L^2}\leqq C\|w^2\|_{L^{2N/(2N-\gamma)}}\leqq C\|w\|_{H^1}^2\,. \tag{4.14}$$

Here we have used the condition $0\leqq\gamma\leqq 4$ to verify

$$\frac{1}{2}\geqq\frac{2N-\gamma}{4N}\geqq\frac{N-2}{2N}\,.$$

As is easily seen,

$$[V_{(N+\gamma)/2}*w(\tau)^2]^2\leqq C\Big\{\tau^3\int_0^\tau [V_{(N+\gamma)/2}*w_t(t)^2]^2dt+[V_{(N+\gamma)/2}*w_1^2]^2\Big\}\,.$$

Integrate the both sides over $\boldsymbol{R}^N$. Then (4.5), (1.7) and (4.13) imply

$$\|V_{(N+\gamma)/2}*w(\tau)^2\|_{L^2}^2 \leqq C\left\{\tau^3\int_0^\tau\int_{R^N}(V_\gamma*w_t^2)w_t^2dxdt+C^2\|w_1^2\|_{H^1}^4\right\}$$

$$\leqq C(a+\tau)^3\{\|w(0)\|_E^2+\|w_1\|_{H^1}^4\}.$$

Substitute this in (4.7). Then since

$$J_2(\varphi_1) \equiv C\int_0^\infty \varphi_1(\tau)^{-3}\varphi_1'(\tau)^4(a+\tau)^3d\tau$$

$$\leqq C\int_0^\infty \{\log(a+\tau)\}^{\mu-4}(a+\tau)^{-1}d\tau < \infty$$

by (3.1) and (4.12), we conclude assertion (i).

(ii) In this case, since $\lambda>0$, we have from (4.14),

$$\|V_{(N+\gamma)/2}*w(t)^2\|_{L^2} \leqq C\|w(t)\|_{L^q}^2 \leqq C\|w(t)\|_{H^1}^2 \leqq C\|w(0)\|_E^2 .$$

On the other hand, by (4.12) we easily see

$$J_3(\varphi_i) \equiv C\int_0^\infty \varphi(\tau)^{-1}\varphi'(\tau)^2d\tau < \infty .$$

Substituting these inequalities in (4.7), we conclude assertion (ii). □

PROOF OF THEOREM 3. As in the case of Theorem 2, Lemmas 6, 7 and Theorem 1 show the both assertions of the theorem. □

5. The cases where the energy decay does not occur.

In this § we shall show that under (AIII) or (AIV) the energy decay does not in general occur. Our results will be based on the uniform decay estimates for the free wave equation (1.11).

Let $U_0(t)$, $t\in \boldsymbol{R}$, be the unitary operator in the energy space E which represents the solution of the free wave equation (1.11):

$$\{w_0(t),\ w_{0t}(t)\} = U_0(t)f \tag{5.1}$$

with initial data $f=\{f_1,\ f_2\}\in E$. Then as is well known we have

LEMMA 8. *Let* $f\in H^k\times H^{k-1}$ $(k=1, 2, \cdots)$. *Then*

$$U_0(t)f \in \bigcap_{j=0}^{k-1} C^j(\boldsymbol{R}\,;\ H^{k-j}\times H^{k-j-1}). \tag{5.2}$$

The uniform decay estimates for $U_0(t)f$ which will be used in this § are summarized in the following lemma.

LEMMA 9. (i) *In case* $\lambda=0$, *let* f *be in the class* $\|f\|_{\Gamma,m+2,2}<\infty$, *where*

$$\|f\|_{\Gamma, m+2, 2} = \sum_{|\alpha|, |\beta| \leq m+2} \|x^\alpha \nabla^\beta f_1\|_{L^2} + \sum_{|\alpha|, |\beta| \leq m+1} \|x^\alpha \nabla^\beta f_2\|_{L^2} \tag{5.3}$$

with $m=[N/2]$ *the least integer not less than* $N/2$. *Then we have*

$$|[U_0(t)f]_2(x)| \leqq C(1+|r|+t)^{-(N-1)/2}(1+|r-t|)^{-1/2}\|f\|_{\Gamma, m+2, 2}. \tag{5.4}$$

(ii) *In case* $\lambda>0$, *let* $f \in H^{m+1,1} \times H^{m,1}$, *where* m *is as above and* $H^{k,1}$ *is the Sobolev space with norm* $\|u\|_{H^{k,1}} = \sum_{|\alpha| \leq k} \|\nabla^\alpha u\|_{L^1}$. *Then we have*

$$|[U_0(t)f]_2(x)| \leqq C(1+|t|)^{-N/2}\|f\|_{H^{m+3,1} \times H^{m+2,1}}. \tag{5.5}$$

PROOF. Assertion (i) easily follows from the results of Klainerman [**3**]. For (ii) see e. g., Marshall-Strauss-Wainger [**5**; Lemma 1]. □

Now multiply by $w_{0t}=[U_0(t)f]_2$ on both sides of (1.1), and integrate by parts over $\boldsymbol{R}^N \times (\sigma, t)$, where $0 \leqq \sigma < t$. Then we have

$$2(w(t), w_0(t))_E - 2(w(\sigma), w_0(\sigma))_E = -\int_\sigma^t \int_{\boldsymbol{R}^N} \beta(x, \tau, w_t) w_t w_{0t} dx d\tau. \tag{5.6}$$

Here $(\cdot, \cdot)_E$ is the inner-product in the energy space E:

$$(w, w_0)_E = \frac{1}{2}\int_{\boldsymbol{R}^N} \{w_t w_{0t} + \nabla w \cdot \nabla w_0 + \lambda w w_0\} dx .$$

LEMMA 10. *Corresponding to the nonlinearities* (1.3) *and* (1.4), *we choose* $p=\rho$ *and* $=3$. *Then*

$$\int_\sigma^t \int_{\boldsymbol{R}^N} |\beta(x, \tau, w_t) w_t w_{0t}| dx d\tau \leqq \left(\int_\sigma^t \int_{\boldsymbol{R}^N} \beta(x, \tau, w_t) w_t^2 dx d\tau\right)^{p/(p+1)}$$
$$\times \left(\int_\sigma^t \int_{\boldsymbol{R}^N} \beta(x, \tau, w_{0t}) w_{0t}^2 dx d\tau\right)^{1/(p+1)}. \tag{5.7}$$

PROOF. In case of the power nonlinearity (1.3), the assertion easily follows from the Hölder inequality. In case of the cubic convolution (1.4), we can follow the line of proof of inequality (4.7). □

Let V be the space of all functions $u=u(x, t)$ such that

$$\int_0^\infty \int_{\boldsymbol{R}^N} \beta(x, t, u) u^2 dx dt < \infty . \tag{5.8}$$

The energy equation (1.7) shows that $w_t(t) \in V$ under (1.6). Moreover, by use of Lemma 9, we can prove the following

LEMMA 11. *Assume* (AIII) *or* (AIV), *and let* f *be as in Lemma* 9. *Then we have* $w_{0t}(t)=[U_0(t)f]_2 \in V$.

PROOF. First consider (1.3). If $\lambda=0$, by means of (5.4),

$$\int_0^\infty\int_{R^N}\beta(x,\ t,\ w_{0t})w_{0t}^2dxdt$$

$$\leqq\int_0^\infty\int_0^\infty b_3(1+r)^{-\delta+N-1}|w_{0t}|^{\rho+1}drdt$$

$$\leqq C\int_0^\infty(1+t)^{-(N-1)(\rho-1)/2-\delta}dt\int_0^\infty(1+|r-t|)^{-(\rho+1)/2}dr\,.$$

Since $\rho>1+2(1-\delta)/(N-1)$, the right side becomes finite, and the assertion holds. If $\lambda>0$, by means of (5.5),

$$\int_0^\infty\int_{R^N}\beta(x,\ t,\ w_{0t})w_{0t}^2dxdt$$

$$\leqq\int_0^\infty\sup_x|w_{0t}(x,\ t)|^{\rho+1-\eta}dt\int_{R^N}b(x,\ t)|w_{0t}(x,\ t)|^\eta dx$$

$$\leqq C\int_0^\infty(1+t)^{-N(\rho+1-\eta)/2}dt$$

$$\times\left(\int_{R^N}[b_3(1+r)]^{-2\delta/(2-\eta)}dx\right)^{(2-\eta)/2}\left(\int_{R^N}w_{0t}(x,\ t)^2dx\right)^{\eta/2}.$$

Since $\rho>1+2(1-\delta)/N$ in this case, we can choose η to satisfy

$$\frac{N}{2}(\rho+1-\eta)>1\quad\text{and}\quad 2>\eta>2-\frac{2\delta}{N}.$$

Thus, the assertion also holds.

Next consider (1.4). It follows from (4.5) and (4.14) with $w(t)=w_0(t)$ that

$$\int_0^\infty\int_{R^N}\beta(x,\ t,\ w_{0t})w_{0t}^2dxdt\leqq\int_0^\infty\left(\int_{R^N}|w_{0t}(x,\ t)|^{4N/(2N-\gamma)}dx\right)^{(2N-\gamma)/N}dt\,.$$

Apply the results of Lemma 9 in this inequality and use the conditions (AIV). Then we similarly conclude the finiteness of the right side, and the assertion of the lemma is verified to hold. □

REMARK 1. Inequality (5.5) holds for $\lambda=0$ if we replace N of the right side by $N-1$. We can use this inequality to obtain similar results if our concern is restricted to (1.3) with $b(x,\ t)\equiv1$ or (1.4).

We are now ready to prove the following theorem.

THEOREM 4. *Assume* (AIII) *or* (AIV) *and let* $f\in H^2\times(H^1\cap L^q)$, $f\neq0$, *satisfy the conditions of Lemma* 9. *We choose* $\sigma\geqq0$ *so that*

$$\int_\sigma^\infty\int_{R^N}\beta(x,\ t,\ [U_0(t)f]_2)[U_0(t)f]_2^2dxdt<2^{\rho+1}\|f\|_E^2\,, \tag{5.8}$$

and suppose

$$U_0(\sigma)f \in H^2\times(H^1\cap L^q), \tag{5.9}$$

where p *is as given in Lemma* 10 *and* q *is as in* (1.6). *If* $w^\sigma(t)$ *is the solution of the initial value problem*

$$\begin{cases} w^\sigma_{tt}-\Delta w^\sigma+\lambda w^\sigma+\beta(x, t, w^\sigma_t)w^\sigma_t = 0, & (x, t)\in \boldsymbol{R}^N\times(\sigma, \infty) \\ \{w^\sigma, w^\sigma_t\}(\sigma)=U_0(\sigma)f, \end{cases} \tag{5.10}$$

then its energy $\|w^\sigma(t)\|_E$ *never decays as* $t\to\infty$.

REMARK 2. Let $f\in H^k\times H^{k-1}$, where

$$k \geqq \frac{N+2}{2}-\frac{N}{q}. \tag{5.11}$$

Then by Lemma 8 and the Sobolev embedding, we see that $w_{0t}(t)=[U_0(t)f]_2 \in L^q$ for any $t\geqq 0$.

PROOF. Lemma 8 and (5.9) guarantee that the initial data $U_0(\sigma)f$ satisfies (1.6). Thus, (5.10) has a unique solution satisfying (i)$\sim$(iii) given in Introduction.

Contrary to the conclusion, assume that $\|w^\sigma(t)\|_E\to 0$ as $t\to\infty$. Then it follows from (5.6), (5.7) and the unitarity of $U_0(t)$ in E that

$$2(w^\sigma(\sigma), w_0(\sigma))_E = 2\|f\|_E^2 = \int_\sigma^\infty\int_{\boldsymbol{R}^N}\beta(x, t, w^\sigma_t)w^\sigma_t w_{0t}dxdt$$

$$\leqq \left(\int_\sigma^\infty\int_{\boldsymbol{R}^N}\beta(x, t, w^\sigma_t)w^{\sigma 2}_t dxdt\right)^{p/(p+1)}\left(\int_\sigma^\infty\int_{\boldsymbol{R}^N}\beta(x, t, w_{0t})w^2_{0t}dxdt\right)^{1/(p+1)}.$$

Using the unitarity of $U_0(t)$ in E and (1.7), we have from this the inequality

$$2^{p+1}\|f\|_E^2 \leqq \int_\sigma^\infty\int_{\boldsymbol{R}^N}\beta(x, t, [U_0(t)f]_2)[U_0(t)f]_2^2 dxdt,$$

which contradicts to (5.8). The theorem is thus proved. □

As a corollary of this theorem we have

THEOREM 5. *Assume* (AIII) *or* (AIV) *and let* $f\in H^2\times(H^1\cap L^q)$, $f\neq 0$, *satisfy the conditions of Lemma* 9. *We choose* $\varepsilon>0$ *as small that*

$$\varepsilon^{p-1}\int_0^\infty\int_{\boldsymbol{R}^N}\beta(x, t, [U_0(t)f]_2)[U_0(t)f]_2^2 dxdt < 2^{p+1}\|f\|_E^2. \tag{5.12}$$

If $w(t)$ *is the solution of* (1.1) *with initial data* $\{w_1, w_2\}=\varepsilon f$, *then its energy* $\|w(t)\|_E^2$ *never decays as* $t\to\infty$.

PROOF. If we assume the energy decay, then as in the above proof, we have

$$2^{p+1}\|\varepsilon f\|_E^2 \leqq \int_0^\infty\int_{\boldsymbol{R}^N}\beta(x,\ t,\ w_{0t})w_{0t}^2 dxdt,$$

where $w_0(t)$ is the solution of the free wave equation (1.11) with initial data εf. Thus, $w_{0t}=\varepsilon[U_0(t)f]_2$. It then follows from this inequality that

$$2^{p+1}\|f\|_E^2 \leqq \varepsilon^{p-1}\int_0^\infty\int_{\boldsymbol{R}^N}\beta(x,\ t,\ [U_0(t)f]_2)[U_0(t)f]_2^2 dxdt.$$

This contradicts to (5.12), and the theorem is proved. □

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