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On the capacity of singularity sets admitting no exceptionally ramified meromorphic functions

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1. Introduction.

For a totally disconnected compact set E in the extended z-plane $\hat{C}$, we denote by M_E the totality of meromorphic functions each of which is defined in the domain complementary to E and has E as the set of transcendental singularities. A meromorphic function $f(z)$ of M_E is said to be exceptionally ramified at a singularity $\zeta \in E$, if there exist values w_i, $1 \leqq i \leqq q$, and positive integers $\nu_i \geqq 2$, $1 \leqq i \leqq q$, with

$$\sum_{i=1}^{q}\left(1-\frac{1}{\nu_i}\right) > 2,$$

such that, in some neighborhood of ζ, the multiplicity of any w_i-point of $f(z)$ is not less than ν_i. Recently, we have shown that, for Cantor sets E with successive ratios $\{\xi_n\}$ satisfying $\xi_{n+1}=o(\xi_n^2)$, any function of M_E cannot be exceptionally ramified at any singularity $\zeta \in E$ (Theorem in [5]). The capacity (in this note, capacity means always logarithmic capacity) of these Cantor sets E is zero, because they satisfy the necessary and sufficient condition

$$\sum_{n=1}^{\infty}\frac{1}{2^n}\log\frac{1}{\xi_n} = \infty$$

to be of capacity zero.

The purpose of this note is to give Cantor sets E of positive capacity improving the above theorem. We shall prove

THEOREM. *Let E be a Cantor set with successive ratios $\{\xi_n\}$ satisfying the condition*

$$\xi_{n+1} = o(\xi_n^{r_0}), \qquad r_0=(1+\sqrt{33})/4,$$

then any function of M_E cannot be exceptionally ramified at any singularity $\zeta \in E$.

We set $\xi_{n+1}=\xi_n^r$ $(n=1, 2, 3, \cdots)$ with r, $r_0<r<2$. Then $\{\xi_n\}$ satisfies the condition of the theorem and

$$\sum_{n=1}^{\infty} \frac{1}{2^n} \log \frac{1}{\xi_n} < +\infty ,$$

so that the Cantor set E having this $\{\xi_n\}$ as the successive ratios is one wanted.

2. Preliminaries.

2.1. Let f be an exceptionally ramified meromorphic function in a domain G in the extended z-plane having three totally ramified values $\{w_i\}_{i=1,2,3}$ with $\{\nu_i\}_{i=1,2,3}$ such that $\sum_{i=1}^{3}(1-(1/\nu_i))>2$, and let R be a doubly connected subdomain of G with $\bar{R}\subset G$ which is bounded by analytic curves Γ_1 and Γ_2. Suppose that $f(\Gamma_1)$ and $f(\Gamma_2)$ are contained in discs D_1 and D_2. Since f is exceptionally ramified, we have the following lemma from Lemma 2 in [**2**].

LEMMA 1. *Under the above setting,*

$$D_1 \cap D_2 \neq \emptyset \quad \textit{and} \quad f(\bar{R}) \subset D_1 \cup D_2 .$$

Now let Δ be a triply connected subdomain of G with $\bar{\Delta}\subset G$ which is bounded by analytic curves $\{\Gamma_j\}_{j=1,2,3}$. We assume that they satisfy the following three conditions (1), (2) and (3):

(1) There exist mutually disjoint simply connected domains $\{D_j\}_{j=1,\dots,\alpha}$ $(1\leqq\alpha\leqq 3)$, the boundary curves ∂D_j being sectionally analytic, with

$$|D_j| < \frac{1}{2} \min_{k\neq m} \chi(w_k, w_m)$$

such that the images $\{f(\Gamma_i)\}_{i=1,2,3}$ are covered with $\{D_j\}_{j=1,\dots,\alpha}$ and each D_j contains $f(\Gamma_i)$ for at least one i, where $\chi(w_k, w_m)$ denotes the chordal distance between w_k and w_m and $|D_j|$ denotes the diameter of D_j.

(2) The number n of roots of the equation $f(z)=w$ in Δ is constant and $\geqq 1$ for $w\in\hat{C}-\bigcup_{j=1}^{\alpha}\bar{D}_j$.

(3) f has no ramified values on each boundary ∂D_j.

We remove from Δ all relatively noncompact components of $\{f^{-1}(\bar{D}_j)\}_{j=1,\dots,\alpha}$ with respect to Δ. Then there remains an open set, each component of which cannot be simply or doubly connected because of Lemma 2 in [**2**]. Hence the open set is a triply connected subdomain Δ' of Δ, whose boundary curves Γ'_j are homotopic to Γ_j $(j=1, 2, 3)$. The following 1), 2), 3) and 4) hold (see Lemma 3 in [**2**]).

1) The Riemannian image of Δ' under f belongs to one of the 25 classes listed in Table 1, where classes (8), (9), (19) and (22) are empty as we have shown recently in [**5**]. (This is the reason why we deleted these four classes from Table 1 by lining through them.)

2) f has no ramified values other than $\{w_i\}_{i=1,2,3}$ in Δ'.

3) Each component of $\Delta-\Delta'$ is doubly connected and its image is contained in one of $\{D_j\}_{j=1,\cdots,\alpha}$.

4) Each D_j contains one of the totally ramified values $\{w_i\}_{i=1,2,3}$.

Table 1.

	ν_1	ν_2	ν_3	m_1 $l_{1,j}$	m_2 $l_{2,j}$	m_3 $l_{3,j}$	n	σ_1	σ_2	σ_3
1	2	4	5	3 $l_{1,j}=2$	1 $l_{2,1}=4$	1 $l_{3,1}=5$	6	0	2 {1, 1}	1 {1}
2	2	4	5	4 $l_{1,j}=2$	2 $l_{2,j}=4$	1 $l_{3,1}=5$	8	0	0	3 {1, 1, 1}
3	2	3	7	4 $l_{1,j}=2$	2 $l_{2,j}=3$	1 $l_{3,1}=7$	8	0	2 {1, 1}	1 {1}
4	2	3	7	4 $l_{1,j}=2$	3 $l_{2,j}=3$	1 $l_{3,1}=7$	9	1 {1}	0	2 {1, 1}
5	2	3	7	5 $l_{1,j}=2$	3 $l_{2,j}=3$	1 $l_{3,1}=7$	10	0	1 {1}	2 {1, 2}
6	2	3	7	5 $l_{1,j}=2$	3 $l_{2,j}=3$	1 $l_{3,1}=8$	10	0	1 {1}	2 {1, 1}
7	2	3	7	5 $l_{1,j}=2$	3 $\{l_{2,1}, l_{2,2}, l_{2,3}\} = \{3, 3, 4\}$	1 $l_{3,1}=7$	10	0	0	3 {1, 1, 1}
8	2	3	7	6 $l_{1,j}=2$	4 $l_{2,j}=3$	1 $l_{3,1}=7$	12	0	0	3 {1, 1, 3}
9	2	3	7	6 $l_{1,j}=2$	4 $l_{2,j}=3$	1 $l_{3,1}=7$	12	0	0	3 {1, 2, 2}
10	2	3	7	6 $l_{1,j}=2$	4 $l_{2,j}=3$	1 $l_{3,1}=8$	12	0	0	3 {1, 1, 2}
11	2	3	7	6 $l_{1,j}=2$	4 $l_{2,j}=3$	1 $l_{3,1}=9$	12	0	0	3 {1, 1, 1}

12	2	3	7	8 $l_{1,j}=2$	5 $l_{2,j}=3$	2 $l_{3,j}=7$	16	0	1 $\{1\}$	2 $\{1, 1\}$
13	2	3	7	9 $l_{1,j}=2$	6 $l_{2,j}=3$	2 $l_{3,j}=7$	18	0	0	3 $\{1, 1, 2\}$
14	2	3	7	9 $l_{1,j}=2$	6 $l_{2,j}=3$	2 $\{l_{3,1}, l_{3,2}\}$ $=\{7, 8\}$	18	0	0	3 $\{1, 1, 1\}$
15	2	3	7	12 $l_{1,j}=2$	8 $l_{2,j}=3$	3 $l_{3,j}=7$	24	0	0	3 $\{1, 1, 1\}$
16	3	3	4	1 $l_{1,1}=3$	1 $l_{2,1}=3$	0	3	0	0	3 $\{1, 1, 1\}$
17	2	4	5	2 $l_{1,j}=2$	1 $l_{2,1}=4$	0	4	0	0	3 $\{1, 1, 2\}$
18	2	3	7	2 $l_{1,j}=2$	1 $l_{2,1}=3$	0	4	0	1 $\{1\}$	2 $\{1, 3\}$
~~19~~	~~2~~	~~3~~	~~7~~	~~2 $l_{1,j}=2$~~	~~1 $l_{2,1}=3$~~	~~0~~	~~4~~	~~0~~	~~1 $\{1\}$~~	~~2 $\{2, 2\}$~~
20	2	3	7	1 $l_{1,1}=2$	1 $l_{2,1}=3$	0	3	1 $\{1\}$	0	2 $\{1, 2\}$
21	2	3	7	3 $l_{1,j}=2$	2 $l_{2,j}=3$	0	6	0	0	3 $\{1, 1, 4\}$
~~22~~	~~2~~	~~3~~	~~7~~	~~3 $l_{1,j}=2$~~	~~2 $l_{2,j}=3$~~	~~0~~	~~6~~	~~0~~	~~0~~	~~3 $\{1, 2, 3\}$~~
23	2	3	7	3 $l_{1,j}=2$	2 $l_{2,j}=3$	0	6	0	0	3 $\{2, 2, 2\}$

24	2 2 2 2	3 7 4 5	7 3 5 4	1 $l_{1,1}=2$	0	0	2	0	1 {2}	2 {1, 1}
25	2 2 3	3 4 3	7 5 4	0	0	0	1	1 {1}	1 {1}	1 {1}

NOTATIONS. m_i: the number of w_i-points of $f(z)$ in $\varDelta'$ ($i=1, 2, 3$).

$\{l_{i,j}\}_{j=1,\cdots,m_i}$: the multiplicities of w_i-points.

σ_i: the number of Γ'_j in $\{\Gamma'_j\}_{j=1,2,3}$ with $f(\Gamma'_j)=\partial D_k$, $D_k \ni w_i$, where $\sigma_i=0$ means that none of $\{D_j\}_{j=1,\cdots,\alpha}$ contains w_i.

σ_3
2
{1, 2}

means that two of $\{\Gamma'_j\}_{j=1,2,3}$ are mapped onto ∂D_k, $D_k \ni w_3$, and one of them has an image curve winding once around w_3, while the other has an image curve winding twice.

2.2. We form a Cantor set in the usual manner. Let $\{\xi_n\}$ be a sequence of positive numbers satisfying $0<\xi_n<2/3$, $n=1, 2, 3, \cdots$. We remove first an open interval of length $(1-\xi_1)$ from the interval $I_{0,1}$: $[-1/2, 1/2]$, so that on both sides there remain closed intervals of length $\xi_1/2 \equiv \eta_1$, which are denoted by $I_{1,1}$ and $I_{1,2}$. Inductively we remove an open interval of length $(1-\xi_n)\prod_{p=1}^{n-1}\eta_p$, with $\eta_p=(1/2)\xi_p$ ($p=1, 2, 3, \cdots$), from each interval $I_{n-1,k}$ of length $\prod_{p=1}^{n-1}\eta_p$, $k=1, 2, 3, \cdots, 2^{n-1}$, so that on both sides there remain closed intervals of length $\prod_{p=1}^{n}\eta_p$, which are denoted by $I_{n,2k-1}$ and $I_{n,2k}$. By repeating this procedure endlessly, we obtain an infinite sequence of closed intervals $\{I_{n,k}\}_{n=1,2,\cdots,\,k=1,2,\cdots 2^n}$. The set given by

$$E=\bigcap_{n=1}^{\infty}\bigcup_{k=1}^{2^n} I_{n,k}$$

is called the Cantor set on the interval $I_{0,1}$ with successive ratios $\{\xi_n\}$.

Set

$$R_{n,k}=\left\{z\,;\ \prod_{p=1}^{n}\eta_p<|z-z_{n,k}|<\frac{1}{3}\prod_{p=1}^{n-1}\eta_p\right\}$$

and

$$\Gamma_{n,k}=\left\{z\,;\,|z-z_{n,k}|=\prod_{p=1}^{n-1}\eta_p\sqrt{\frac{\eta_n}{3}}\right\},$$

where $z_{n,k}$ is the midpoint of $I_{n,k}$. Denoting by $\mu_n=\mu(R_{n,k})$ the harmonic modulus of $R_{n,k}$, we have

$$\mu_n=\log\frac{1}{3\eta_n}=\log\frac{2}{3\xi_n}.$$

Assuming that $\lim_{n\to\infty}\xi_n=0$, we have

LEMMA 2 (Lemma 4 in [2]). *Let f be an exceptionally ramified meromorphic function in the domain $G=\hat{C}-E$. Then, for sufficiently large n, we have*

$$|f(\Gamma_{n,k})|<M\exp(-\mu_n/2),$$

where M is a positive constant depending only on E and f.

Let f be exceptionally ramified in the domain $G=\hat{C}-E$. By our previous result ([3]), f has just three totally ramified values $\{w_i\}_{i=1,2,3}$. Since $|f(\Gamma_{n,k})|<M\exp(-\mu_n/2)=M\sqrt{3\xi_n/2}=\delta_n$ by Lemma 2, we can take a spherical disc $D_{n,k}$ of radius δ_n containing $f(\Gamma_{n,k})$. We denote by $\Delta_{n,k}$ the triply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$. Taking n so large that $\delta_n<(1/12)\cdot\min_{i\neq j}\chi(w_i, w_j)$, we consider the union $D=\bar{D}_{n,k}\cup\bar{D}_{n+1,2k-1}\cup\bar{D}_{n+1,2k}$, which consists of at most three, say α, components.

If $\alpha=1$, that is, D is connected, it is possible that D is doubly connected, and we take a disc $\tilde{D}_1$ of radius at most $\delta_n+2\delta_{n+1}$ containing D. If $\alpha=2$ or 3, we denote the components of D by $\{\tilde{D}_j\}_{j=1,\cdots,\alpha}$, which are simply connected.

When $\alpha=1$ and f takes in $\Delta_{n,k}$ no values outside $\tilde{D}_1$, $\bar{f}(\Delta_{n,k})\subset\tilde{D}_1$, we say that $\Delta_{n,k}$ is degenerate(f). When $\alpha=1$ and f takes in $\Delta_{n,k}$ values outside $\tilde{D}_1$ or when $\alpha=2$ or 3, we say that $\Delta_{n,k}$ is non-degenerate(f). Then f, $\Delta_{n,k}$ and $\{\tilde{D}_j\}_{j=1,\cdots,\alpha}$ satisfy three conditions (1), (2) and (3) stated in 2.1, so that by 4) stated there, each $\tilde{D}_j$ contains one w_j^* of the totally ramified values $\{w_i\}_{i=1,2,3}$ and the union $\bigcup_{j=1}^{\alpha}\tilde{D}_j\supset D$ is contained in $\bigcup_{i=1}^{3}D(w_i, 2(\delta_n+2\delta_{n+1}))$, where we denote by $D(w,\delta)$ the spherical disc of radius δ and with center at w. We assume $2\delta_{n+1}<\delta_n$ and set $\tilde{D}_j'=D(w_j^*, 4\delta_n)$, $j=1,\cdots,\alpha$. Then f, $\Delta_{n,k}$ and $\{\tilde{D}_j'\}_{j=1,\cdots,\alpha}$ again satisfy three conditions (1), (2) and (3), so that there exists a triply connected subdomain $\Delta_{n,k}'$ of $\Delta_{n,k}$ such that 1), 2), 3) and 4) stated there hold. The Riemannian image $S_{n,k}$ of $\Delta_{n,k}'$ under f belongs to one of the classes of Table 1. The boundary curves of $\Delta_{n,k}'$ are denoted by $\check{\gamma}_{n,k}$, $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$ being homotopic to $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$ and $\Gamma_{n+1,2k}$, respectively. Each γ of them has an image curve winding around some w^* of w_1, w_2 and w_3, and we denote its winding number by $s(\gamma)$. The value w^* corresponds to one $\tilde{w}$ of three totally ramified values for the class in Table 1 to which $S_{n,k}$ belongs, and we can read the ν-value, the minimum of the multiplicities of $\tilde{w}$-points, in Table 1, which we denote by $\nu(\gamma)$.

Suppose now that $S_{n,k}$ belongs to a class other than (23). Reading Table 1, we see that the image curves of at least two of $\check{\gamma}_{n,k}$, $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$ have the winding number 1. Hence $s(\hat{\gamma}_{n+1,2k-1})=1$ or $s(\hat{\gamma}_{n+1,2k})=1$, say $s(\hat{\gamma}_{n+1,2k})=1$, where we assume $\nu(\hat{\gamma}_{n+1,2k-1})\leqq\nu(\hat{\gamma}_{n+1,2k})$ if $s(\hat{\gamma}_{n+1,2k-1})=s(\hat{\gamma}_{n+1,2k})=1$. The adjacent $\Delta_{n+1,2k}$ is degenerate(f) or non-degenerate(f). Suppose that $\Delta_{n+1,2k}$ is non-degenerate(f). Then $\hat{\gamma}_{n+1,2k}$ and $\check{\gamma}_{n+1,2k}$ wind around the same totally ramified value w^* and bound a doubly connected domain where f takes the value w^*. Since $f(\hat{\gamma}_{n+1,2k})\subset D(w^*, 4\delta_n)$ and $f(\check{\gamma}_{n+1,2k})\subset D(w^*, 4\delta_{n+1})$, we see from Lemma 1 that f takes no values outside $D(w^*, 4\delta_n)$ in the doubly connected domain bounded by $\hat{\gamma}_{n+1,2k}$ and $\check{\gamma}_{n+1,2k}$. By the argument principle, we have

$$s(\hat{\gamma}_{n+1,2k})+s(\check{\gamma}_{n+1,2k}) \geqq \max\{\nu(\hat{\gamma}_{n+1,2k}),\ \nu(\check{\gamma}_{n+1,2k})\},$$

that is,

$$s(\check{\gamma}_{n+1,2k}) \geqq \max\{\nu(\hat{\gamma}_{n+1,2k}),\ \nu(\check{\gamma}_{n+1,2k})\} -1,$$

because $s(\hat{\gamma}_{n+1,2k})=1$. From Table 1, we see that only the pairs $\{\Delta_{n,k}, \Delta_{n+1,2k}\}$ listed below satisfy this inequality.

Table 2.

$\Delta_{n,k}$			$\Delta_{n+1,2k}$		
class	$\nu(\hat{\gamma}_{n+1,2k})$	$s(\hat{\gamma}_{n+1,2k})$	class	$\nu(\check{\gamma}_{n+1,2k})$	$s(\check{\gamma}_{n+1,2k})$
			(4)	2	1
(20)	2	1	(20)	2	1
			(25)	2	1
(3)	3	1			
(5)	3	1			
(18)	3	1	(24)	3	2
(24)	3	1			
(25)	3	1			

REMARK. The pair of $\Delta_{n,k}((20), 2, 1)$ and $\Delta_{n+1,2k}((24), 3, 2)$ satisfies the inequality, but, under the assumption that f is exceptionally ramified, we can omit it, because $S_{n,k}$ and $S_{n+1,2k}$ have branch points of multiplicity 2 over distinct totally ramified values.

From Table 1, we see that, if $\Delta_{n+1,2k}$ of the right side of Table 2 is of class (4), (20) or (24), one of $\hat{\gamma}_{n+2,4k-1}$ and $\hat{\gamma}_{n+2,4k}$, say $\hat{\gamma}_{n+2,4k}$, satisfies $s(\hat{\gamma}_{n+2,4k})=1$ and $\nu(\hat{\gamma}_{n+2,4k})=7$, and if it is of class (25), $s(\hat{\gamma}_{n+2,4k})=1$ and $\nu(\hat{\gamma}_{n+2,4k})\geqq 5$. Therefore $\Delta_{n+2,4k}$ must be degenerate(f). Thus we have

LEMMA 3 (Lemma 2 in [**5**]). *If $\Delta_{n,k}$ is non-degenerate(f) and belongs to a class other than the class* (23), *then for at least one of $\hat{\gamma}_{n+1,2k-1}$ and $\hat{\gamma}_{n+1,2k}$, say $\hat{\gamma}_{n+1,2k}$, $s(\hat{\gamma}_{n+1,2k})=1$. If the adjacent $\Delta_{n+1,2k}$ is non-degenerate(f), then for at least one of $\hat{\gamma}_{n+2,4k-1}$ and $\hat{\gamma}_{n+2,4k}$, say $\hat{\gamma}_{n+2,4k}$, $s(\hat{\gamma}_{n+2,4k})=1$ and the adjacent $\Delta_{n+2,4k}$ is degenerate(f).*

We shall state a theorem due to Teichmüller for the moduli of ring domains as a lemma, which we shall often use later.

LEMMA 4. *If a ring domain R in $\boldsymbol{C}$ separates two points* 0 *and $r_1 e^{i\theta_1}$ from two points $r_2 e^{i\theta_2}$ and ∞ ($r_1>0$, $r_2>0$), then*

$$\text{har. mod. } R \leqq \log\left(16\frac{r_2}{r_1}+8\right)$$

(*cf. Lehto and Virtanen* [**4**], *pp.* 54-62).

3. Proof of Theorem.

3.1. Now we shall prove our theorem. Contrary suppose that a function f of M_E is exceptionally ramified at a singularity $\zeta_0\in E$. As mentioned after Lemma 2, f has just three totally ramified values $\{w_i\}_{i=1,2,3}$ near ζ_0 with $\{\nu_i\}_{i=1,2,3}$, satisfying

$$\sum_{i=1}^{3}\left(1-\frac{1}{\nu_i}\right)>2,$$

where we may assume without any loss of generality that $w_1=\infty$, $w_2=1$ and $w_3=0$. From our assumption $\xi_{n+1}=o(\xi_n^{r_0})$, $r_0=(1+\sqrt{33})/4$, we can take n_0 so large that $\delta_n=M\sqrt{3\xi_n/2}<\sqrt{2}/24$ and $\delta_{n+1}<(1/2)\delta_n$ for $n\geqq n_0$. Here we may assume that Γ_{n_0,k_0} surrounds ζ_0 and f is exceptionally ramified in the part G_0 of $G=\hat{\boldsymbol{C}}-E$ surrounded with Γ_{n_0,k_0}. Then if $\Delta_{n,k}$ in G_0 is degenerate(f), $f(\bar{\Delta}_{n,k})$ is contained in a disc $\tilde{D}_{n,k}$ of radius at most $\delta_n+2\delta_{n+1}<2\delta_n$.

Now suppose that all $\Delta_{n,k}$ in G_0 are degenerate(f). The image $f(\bar{\Delta}_{n_0,k_0})$ is contained in $\tilde{D}_{n_0,k_0}$. Since $\tilde{D}_{n_0,k_0}\cap\tilde{D}_{n_0+1,2k_0-1}\neq\emptyset$ and $\tilde{D}_{n_0,k_0}\cap\tilde{D}_{n_0+1,2k_0}\neq\emptyset$, $f(\bar{\Delta}_{n_0,k_0}\cup\bar{\Delta}_{n_0+1,2k_0-1}\cup\bar{\Delta}_{n_0+1,2k_0})$ is contained in a disc D_2 of radius at most $2\delta_{n_0}+4\delta_{n_0+1}<2\delta_{n_0}(1+2^0)$ and with the same center w_0 as $\tilde{D}_{n_0,k_0}$. If

$f(\bar{\Delta}_{n_0,k_0}\cup(\bigcup_{p=1}^{m}(\bigcup_k' \bar{\Delta}_{n_0+p,k})))$ is contained in a disc D_m of radius at most $2\delta_{n_0}(1+\sum_{p=1}^{m}(1/2^{p-1}))$ and with center at w_0, then $f(\bar{\Delta}_{n_0,k_0}\cup(\bigcup_{p=1}^{m+1}(\bigcup_k' \bar{\Delta}_{n_0+p,k})))$ is contained in a disc D_{m+1} of radius at most $2\delta_{n_0}(1+\sum_{p=1}^{m}(1/2^{p-1}))+4\delta_{n_0+m+1}<2\delta_{n_0}(1+\sum_{p=1}^{m+1}(1/2^{p-1}))$ and with center at w_0, because $D_m\cap\tilde{D}_{n_0+m+1,k}\neq\emptyset$ for each $\Delta_{n_0+m+1,k}$ in G_0, where $\bigcup_k'\Delta_{n_0+p,k}$ means the union taken over all the $\Delta_{n_0+p,k}$'s in G_0. By induction, we conclude that $f(G_0)$ is contained in a disc of radius at most $2\delta_{n_0}(1+\sum_{p=1}^{\infty}(1/2^{p-1}))=6\delta_{n_0}<\sqrt{2}/4$. This means that f is bounded in G_0. Since E is of linear measure zero, each point of E in the domain surrounded with Γ_{n_0,k_0} must be a removable singularity for f (cf. Besicovitch [1]). This contradicts our assumption that $f\in M_E$. Thus we see that there are infinitely many $\Delta_{n,k}$ in G_0 being non-degenerate(f).

We take such a domain $\Delta_{n,k}$. If $\Delta_{n,k}$ belongs to a class other than (23), we may assume from Lemma 3 that $s(\hat{\gamma}_{n+1,2k})=1$ and the adjacent $\Delta_{n+1,2k}$ is degenerate(f). We shall show that $f(\Gamma_{n+1,2k})\subset D(w_i, 8\delta_{n+1})$ and $f(\Gamma_{n+2,4k-1})\cup f(\Gamma_{n+2,4k})\subset D(w_i, 8\delta_{n+2})$ for some $w_i\in\{w_i\}_{i=1,2,3}$.

For $\Delta_{m,l}$ being non-degenerate(f), the union $D=\bar{D}_{m,l}\cup\bar{D}_{m+1,2l-1}\cup\bar{D}_{m+1,2l}$ is contained in $\bigcup_{i=1}^{3}D(w_i, 2(\delta_m+2\delta_{m+1}))\subset\bigcup_{i=1}^{3}D(w_i, 4\delta_m)$ as mentioned after we stated Lemma 2. Therefore, if $f(\Gamma_{m,l})\not\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_m)$, then $\Delta_{m,l}$ is degenerate(f) and $f(\bar{\Delta}_{m,l})$ is contained in a disc $\tilde{D}_{m,l}$ of radius at most $2\delta_m$. We have $\tilde{D}_{m,l}\cap\bigcup_{i=1}^{3}D(w_i, 4\delta_m)=\emptyset$. Since $2\delta_{m+1}<\delta_m$, we see that $f(\Gamma_{m+1,2l-1})\not\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_{m+1})$ and $f(\Gamma_{m+1,2l})\not\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_{m+1})$ so that $\Delta_{m+1,2l-1}$ and $\Delta_{m+1,2l}$ both are degenerate(f). Then, by induction, we see all $\Delta_{p,q}$ in the part of G surrounded with $\Gamma_{m,l}$ are degenerate(f). However, this is impossible as we have seen above. Hence $f(\Gamma_{m,l})\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_m)$. We see now that $f(\Gamma_{m,l})\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_m)$, whether $\Delta_{m,l}$ is non-degenerate(f) or degenerate(f). From this fact, $f(\Gamma_{n+1,2k})\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_{n+1})$ and $f(\Gamma_{n+2,4k-1})\cup f(\Gamma_{n+2,4k})\subset\bigcup_{i=1}^{3}D(w_i, 8\delta_{n+2})$. However, $\Delta_{n+1,2k}$ is degenerate(f) and so we see that $f(\Gamma_{n+1,2k})\subset D(w_i, 8\delta_{n+1})$ and $f(\Gamma_{n+2,4k-1})\cup f(\Gamma_{n+2,4k})\subset D(w_i, 8\delta_{n+2})$ for some $w_i\in\{w_i\}_{i=1,2,3}$. We may assume $w_i=w_3=0$.

Set

$$\hat{\Gamma}_{n,k}^{(s)}=\{z\,;\,|z-z_{n,k}|=(1/3)\xi_{n-1}^{s}Y_{n-1}\}\quad\text{and}\quad\hat{\Gamma}_{n,k}^{(0)}=\hat{\Gamma}_{n,k}\,,$$

where $Y_n=\prod_{p=1}^{n}\eta_p=(\prod_{p=1}^{n}\xi_p)/2^n$ and $0\leqq 2s\leqq r_0-1$. By the Cauchy integral formula,

$$f'(z)=\frac{1}{2\pi i}\int_{\partial\Delta_{n+1,2k}}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta\,,\qquad z\in\hat{\Gamma}_{n+2,4k-1}^{(s)}\cup\hat{\Gamma}_{n+2,4k}^{(s)}\,,$$

so that

$$|f'(z)|\leqq\frac{1}{2\pi}\Big(\int_{\Gamma_{n+1,2k}}+\int_{\Gamma_{n+2,4k-1}}+\int_{\Gamma_{n+2,4k}}\Big)\frac{|f(\zeta)|}{|\zeta-z|^2}|d\zeta|.$$

Since $f(\Gamma_{n+1,2k})\subset D(0, 8\delta_{n+1})$ and $f(\Gamma_{n+2,4k-1})\cup f(\Gamma_{n+2,4k})\subset D(0, 8\delta_{n+2})$,

$$|f'(z)| \leqq \frac{1}{2\pi}\cdot\frac{8\delta_{n+1}}{\sqrt{1-(8\delta_{n+1})^2}}\cdot\frac{1}{\{\sqrt{\xi_{n+1}/6}\,Y_n(1-\sqrt{3\xi_{n+1}}/2)\}^2}\cdot 2\pi\sqrt{\xi_{n+1}/6}\,Y_n$$

$$+2\cdot\frac{1}{2\pi}\cdot\frac{8\delta_{n+2}}{\sqrt{1-(8\delta_{n+2})^2}}\cdot\frac{1}{\{(1/3)\xi_{n+1}^s Y_{n+1}(1-3\sqrt{\xi_{n+2}/6\xi_{n+1}^{-s}})\}^2}\cdot 2\pi\sqrt{\xi_{n+2}/6}\,Y_{n+1}$$

$$\leqq \frac{192\ M}{Y_n}+\frac{288\ M}{Y_n}\cdot\frac{\xi_{n+2}\xi_{n+1}^{-(1+2s)}}{(1-3\sqrt{\xi_{n+2}/6\xi_{n+1}^{-s}})^2}$$

$$<\frac{384\ M}{Y_n},$$

for sufficiently large n, because $\delta_n\to 0$, $\xi_n\to 0$, $\xi_{n+2}^{1/2}\xi_{n+1}^{-s}=o(\xi_{n+1}^{(r_0-2s)/2})=o(\xi_{n+1}^{1/2})$ and $\xi_{n+2}\xi_{n+1}^{-(1+2s)}=o(\xi_{n+1}^{r_0-(1+2s)})=o(1)$. Hence, for $z, z'\in\hat{\Gamma}_{n+2,4k-j}^{(s)}$ $(j=0, 1)$,

$$|f(z)-f(z')| \leqq \int_{\hat{\Gamma}_{n+2,4k-j}^{(s)}} |f'(z)|\,|dz|$$

$$\leqq \frac{384\ M}{Y_n}\cdot 2\pi\frac{1}{3}\xi_{n+1}^s Y_{n+1} = 128\pi M\xi_{n+1}^{1+s} \equiv \hat{\delta}_{n+2}^{(s)}.$$

This inequality implies that the images $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})$ and $f(\hat{\Gamma}_{n+2,4k}^{(s)})$ are contained in discs $\hat{D}_{n+2,4k-1}^{(s)}$ and $\hat{D}_{n+2,4k}^{(s)}$ of radius at most $\hat{\delta}_{n+2}^{(s)}$, respectively. We shall show that $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})\cup f(\hat{\Gamma}_{n+2,4k}^{(s)})\subset D(0, \xi_{n+1}^{1+(s/2)})$ for sufficiently large n. Consider the triply connected domain $\hat{\Delta}_{n+2,4k-1}$ bounded by $\hat{\Gamma}_{n+2,4k-1}^{(s)}$, $\Gamma_{n+3,8k-3}$ and $\Gamma_{n+3,8k-2}$, where $f(\Gamma_{n+3,8k-3})$ and $f(\Gamma_{n+3,8k-2})$ are contained in discs $D_{n+3,8k-3}$ and $D_{n+3,8k-2}$ of radius at most $\delta_{n+3}=M\sqrt{(3/2)\xi_{n+3}}=o(\xi_{n+1}^{1+(s/2)})$. If $\hat{\Delta}_{n+2,4k-1}$ is non-degenerate(f), then the union $D=\bar{\hat{D}}_{n+2,4k-1}^{(s)}\cup\bar{D}_{n+3,8k-3}\cup\bar{D}_{n+3,8k-2}$ is contained in $\bigcup_{i=1}^3 D(w_i, 2(\hat{\delta}_{n+2}^{(s)}+2\delta_{n+3}))$, so that $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})\subset D(0, \xi_{n+1}^{1+(s/2)})$ for sufficiently large n, because $\hat{\delta}_{n+2}^{(s)}=O(\xi_{n+1}^{1+s})=o(\xi_{n+1}^{1+(s/2)})$ and $\delta_{n+3}=o(\xi_{n+1}^{1+(s/2)})$. Therefore if $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})\not\subset D(0, \xi_{n+1}^{1+(s/2)})$, then $\hat{\Delta}_{n+2,4k-1}$ is degenerate(f) and $f(\hat{\Delta}_{n+2,4k-1})\subset D$, where D is connected and $|D|\leqq 2(\hat{\delta}_{n+2}^{(s)}+2\delta_{n+3})=o(\xi_{n+1}^{1+(s/2)})$. Thus $f(\Gamma_{n+3,8k-j})\not\subset D(0, 8\delta_{n+3})$ $(j=3, 2)$. It is obvious that $f(\Gamma_{n+3,8k-j})\cap\{D(\infty, 8\delta_{n+3})\cup D(1, 8\delta_{n+3})\}=\emptyset$ $(j=3, 2)$, but, as we have seen above, any $\Gamma_{m,l}$ satisfies $f(\Gamma_{m,l})\subset D(w_i, 8\delta_m)$ for some $w_i\in\{w_i\}_{i=1,2,3}$. Contradiction. Thus we have $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})\subset D(0, \xi_{n+1}^{1+(s/2)})$. Quite similarly, we have $f(\hat{\Gamma}_{n+2,4k}^{(s)})\subset D(0, \xi_{n+1}^{1+(s/2)})$.

We consider now the part of Riemannian image of the quadruply connected domain bounded by $\Gamma_{n,k}$, $\Gamma_{n+1,2k-1}$, $\hat{\Gamma}_{n+2,4k-1}^{(s)}$ and $\hat{\Gamma}_{n+2,4k}^{(s)}$ under f over the annulus $R=\{w\,;\,\xi_{n+1}^{1+(s/2)}<\chi(0, w)<1/2\}$, $s>0$. Since $s(\gamma_{n+1,2k})=1$, f has no ramified values other than $\{w_i\}_{i=1,2,3}$ in $\Delta'_{n,k}$ and $f(\hat{\Gamma}_{n+2,4k-1}^{(s)})\cup f(\hat{\Gamma}_{n+2,4k}^{(s)})\subset D(0, \xi_{n+1}^{1+(s/2)})$, its component $\tilde{R}$ containing $f(\gamma_{n+1,2k})$ covers R univalently, so that $\tilde{R}$ is also an annulus and its harmonic modulus is equal to that of R. The inverse image $f^{-1}(\tilde{R})$ is a ring domain separating $\Gamma_{n,k}\cup\Gamma_{n+1,2k-1}$ from $\hat{\Gamma}_{n+2,4k-1}^{(s)}\cup\hat{\Gamma}_{n+2,4k}^{(s)}$. By Lemma 4, we have

$$\log\left(16\frac{Y_n(1-\xi_{n+1})}{Y_{n+1}}+8\right)\geqq \text{har. mod. } \tilde{R}$$

$$=\log\frac{1/2\sqrt{1-(1/2)^2}}{\xi_{n+1}^{1+(s/2)}/\sqrt{1-(\xi_{n+1}^{1+(s/2)})^2}}$$

and hence

$$32/\xi_{n+1}\geqq 1/2\xi_{n+1}^{1+(s/2)}, \quad \text{so that} \quad \xi_{n+1}^{s/2}>1/64\,.$$

Thus there are only finitely many $\Delta_{n,k}$ in G_0 being non-degenerate(f) which belong to classes other than the class (23), for, otherwise, the inequality holds for infinitely many n contradicting our assumption $\xi_{n+1}=o(\xi_n^{r_0})$. Now we may assume that all $\Delta_{n,k}$ in G_0 being non-degenerate(f) are of class (23).

3.2. Let $\Delta_{n,k}$ be non-degenerate(f) and belong to the class (23). Then the image $f(\partial\Delta_{n,k})$ of the boundary of $\Delta_{n,k}$ is contained in one of $\{D(w_i, 4\delta_n)\}_{i=1,2,3}$, say $D(w_3, 4\delta_n)$, $w_3=0$. Both of adjacent $\Delta_{n+1,2k-1}$ and $\Delta_{n+1,2k}$ are degenerate(f). In fact, if $\Delta_{n+1,2k-1}$ is non-degenerate(f), $s(\hat{\gamma}_{n+1,2k-1})=s(\check{\gamma}_{n+1,2k-1})=2$ because $\Delta_{n+1,2k-1}$ is also of class (23), and f takes the totally ramified value $w_3=0$ with $\nu_3=7$. Because $f(\hat{\gamma}_{n+1,2k-1})\cup f(\check{\gamma}_{n+1,2k-1})\subset D(0, 4\delta_n)$, the image of the doubly connected domain bounded by $\hat{\gamma}_{n+1,2k-1}$ and $\check{\gamma}_{n+1,2k-1}$ is also contained in $D(0,4\delta_n)$ by Lemma 1, consequently f has no poles there, and hence we have $s(\hat{\gamma}_{n+1,2k-1})+s(\check{\gamma}_{n+1,2k-1})\geqq 7$ by the argument principle. It is absurd, and hence $\Delta_{n+1,2k-1}$ is degenerate(f). Similarly we see that $\Delta_{n+1,2k}$ is also degenerate(f). Now at least one of $\Delta_{n+2,4k-1}$ and $\Delta_{n+2,4k}$, say $\Delta_{n+2,4k}$, is degenerate(f). Contrary suppose that both of them are non-degenerate(f). Then they are of class (23) and f has the totally ramified value $w_3=0$ in the domain bounded by $\hat{\gamma}_{n+1,2k}$, $\check{\gamma}_{n+2,4k-1}$ and $\check{\gamma}_{n+2,4k}$, but has no poles there. In fact $\Delta_{n+1,2k}$ is degenerate(f) and hence f might have poles only in the doubly connected domain bounded by $\Gamma_{n+2,4k-1}$ and $\check{\gamma}_{n+2,4k-1}$ or $\Gamma_{n+2,4k}$ and $\check{\gamma}_{n+2,4k}$, while this is impossible because of Lemma 1. Therefore

$$s(\hat{\gamma}_{n+1,2k})+s(\check{\gamma}_{n+2,4k-1})+s(\check{\gamma}_{n+2,4k})\geqq 7$$

by the argument principle again, while $s(\hat{\gamma}_{n+1,2k})=s(\check{\gamma}_{n+2,4k-1})=s(\check{\gamma}_{n+2,4k})=2$. Contradiction. The other $\Delta_{n+2,4k-1}$ is degenerate(f) or non-degenerate(f) and of class (23).

Set

$$\check{\Gamma}_{n,k}=\{z\,;\,|z-z_{n,k}|=Y_n\}\,.$$

We shall show that the diameter of $f(\check{\Gamma}_{n+2,4k})$ is $O(\xi_{n+1}\xi_{n+2})$. By the Cauchy integral formula,

$$f'(z)=\frac{1}{2\pi i}\int_{\partial(\Delta_{n+1,2k}\cup\Gamma_{n+2,4k}\cup\Delta_{n+2,4k})}\frac{f(\zeta)}{(\zeta-z)^2}d\zeta, \qquad z\in\check{\Gamma}_{n+2,4k},$$

so that

$$\begin{aligned}|f'(z)| &\leqq \frac{1}{2\pi}\Big(\int_{\Gamma_{n+1,2k}}+\int_{\Gamma_{n+2,4k-1}}+\int_{\Gamma_{n+3,8k-1}}+\int_{\Gamma_{n+3,8k}}\Big)\frac{|f(\zeta)|}{|\zeta-z|^2}|d\zeta| \\ &\leqq \frac{1}{2\pi}\cdot\frac{8\delta_{n+1}}{\sqrt{1-(8\delta_{n+1})^2}}\cdot\frac{1}{\{\sqrt{\xi_{n+1}/6}\,Y_n(1-\sqrt{3\xi_{n+1}/2})\}^2}\cdot 2\pi\sqrt{\xi_{n+1}/6}\,Y_n \\ &\quad+\frac{1}{2\pi}\cdot\frac{8\delta_{n+2}}{\sqrt{1-(8\delta_{n+2})^2}}\cdot\frac{1}{\{Y_{n+1}(1-4\sqrt{\xi_{n+2}/6})\}^2}\cdot 2\pi\sqrt{\xi_{n+2}/6}\,Y_{n+1} \\ &\quad+2\cdot\frac{1}{2\pi}\cdot\frac{8\delta_{n+3}}{\sqrt{1-(8\delta_{n+3})^2}}\cdot\frac{1}{\{(Y_{n+2}/2)(1-2\sqrt{\xi_{n+3}/6})\}^2}\cdot 2\pi\sqrt{\xi_{n+3}/6}\,Y_{n+2} \\ &\leqq \frac{192M}{Y_n}+\frac{32M\xi_{n+2}}{Y_{n+1}}+\frac{256M\xi_{n+3}}{Y_{n+2}},\end{aligned}$$

because $\Delta_{n+1,2k}$ and $\Delta_{n+2,4k}$ are both degenerate(f) and so $f(\Gamma_{n+1,2k})\subset D(0, 8\delta_{n+1})$, $f(\Gamma_{n+2,4k-1})\subset D(0, 8\delta_{n+2})$ and $f(\Gamma_{n+3,8k-1})\cup f(\Gamma_{n+3,8k})\subset D(0, 8\delta_{n+3})$. Hence

$$\begin{aligned}|f(z)-f(z')| &\leqq \int_{\check{\Gamma}_{n+2,4k}}|f'(z)||dz| \\ &\leqq \Big(\frac{192M}{Y_n}+\frac{32M\xi_{n+2}}{Y_{n+1}}+\frac{256M\xi_{n+3}}{Y_{n+2}}\Big)2\pi Y_{n+2} \\ &= 32\pi M(3\xi_{n+1}\xi_{n+2}+\xi_{n+2}^2+16\xi_{n+3}) \\ &< 2^6\cdot 3\pi M\xi_{n+1}\xi_{n+2}\equiv M'\xi_{n+1}\xi_{n+2}\equiv\check{\delta}_{n+2},\end{aligned}$$

because $\xi_{n+2}^2=o(\xi_{n+1}\xi_{n+2})$ and $\xi_{n+3}=o(\xi_{n+2}^{r_0})=o(\xi_{n+1}^{r_0(r_0-1)}\xi_{n+2})=o(\xi_{n+1}\xi_{n+2})$. This implies that the diameter of the image $f(\check{\Gamma}_{n+2,4k})$ is contained in a disc $\check{D}_{n+2,4k}$ of radius at most $\check{\delta}_{n+2}$. We note here that if $\Delta_{n+2,4k-1}$ is degenerate(f), the curve $\check{\Gamma}_{n+2,4k-1}$ has the same property.

3.3. To show that $f(\check{\Gamma}_{n+2,4k})\subset D(0, 8\check{\delta}_{n+2})$, we shall prove first

LEMMA 5. *If $\Delta_{m,l}$ belongs to the class* (23), *f has no zeros in the doubly connected domain bounded by $\hat{\Gamma}_{m,l}$ and $\check{\gamma}_{m,l}$ and $f(\hat{\Gamma}_{m,l})\subset D(0, \xi_{m-1})$, then the image of the curve $\tilde{\Gamma}_{m,l}=\{z\,;\,|z-z_{m,l}|=(1/\sqrt{6})\xi_m^{1/2}Y_{m-1}\xi_{m-1}^{-1/4}\}$ is contained in $D(0, 24\pi^2\xi_{m-1}^{1/2}\xi_m)$.*

PROOF. For small $d>0$, we denote by S_d the covering surface of class (23) over $\hat{C}-\bar{D}(0, d)$. When $d=4\delta_m$, S_d is the Riemannian image $S_{m,l}$ of the subdomain $\Delta'_{m,l}$ of $\Delta_{m,l}$. As the limit surface as $d\to 0$, we have a six-sheeted covering surface of $\hat{C}-\{0\}$ having three pinholes over 0. We stop up these holes and obtain a six-sheeted covering surface Φ of $\hat{C}$, which is planar and has three branch points of multiplicity 2 over $w_1=\infty$, two branch points of multiplicity 3 over $w_2=1$ and three branch points of multiplicity 2 over $w_3=0$. Let $w=\varphi(\omega)$ be a conformal mapping of the extended ω-plane onto Φ with $\varphi(0)=\varphi(1)=\varphi(\infty)=0$. Consider S_d, $d=4\delta_m$, as a subdomain of Φ. Its inverse image $\varphi^{-1}(S_d)$ is a triply connected domain $\hat{C}-\cup_{i=1}^3 B_i$, where $\partial B_1=\varphi^{-1}\circ f(\hat{\gamma}_{m+1,2l-1})$, $\partial B_2=\varphi^{-1}\circ f(\hat{\gamma}_{m+1,2l})$ and $\partial B_3=\varphi^{-1}\circ f(\check{\gamma}_{m,l})$. We may assume that $B_1\ni\omega_1=0$, $B_2\ni\omega_2=1$ and $B_3\ni\omega_3=\infty$. If m is sufficiently large, that is, d is sufficiently small, for each i, ∂B_i is nearly a circle of chordal radius $\alpha_i\sqrt{d}$ and with center at ω_i, where $\{\alpha_i\}_{i=1,2,3}$ are positive constants not depending on d and hence on m. The annulus $R=\{\omega\,;\,2\alpha_3\sqrt{d}<\chi(\omega,\infty)<1/\sqrt{5}\}$ separates $B_1\cup B_2$ from B_3, so that its image $f^{-1}\circ\varphi(R)$ is a ring domain in $\Delta'_{m,l}\subset\Delta_{m,l}$ separating $\Gamma_{m+1,2l-1}\cup\Gamma_{m+1,2l}$ from $\Gamma_{m,l}$ and has the same harmonic modulus as R. We set

$$r=\min\{|z-z_{m,l}|\,;\,z\in\check{\gamma}_{m,l}\}.$$

By Lemma 4, we have

$$\log\left(16\frac{r}{Y_m/2}+8\right)\geqq \text{har. mod. } R$$

$$=\log\frac{\sqrt{1-(2\alpha_3\sqrt{d}\,)^2}/2\alpha_3\sqrt{d}}{2}.$$

Hence

$$32r/Y_m\geqq(1/8\alpha_3\sqrt{d}\,)-8\geqq 1/16\alpha_3\sqrt{d}\,,$$

so that we have

$$r\geqq Y_m/2^9\alpha_3\sqrt{d}\;=KY_{m-1}\xi_m^{3/4}$$

with a constant K not depending on m. Similarly we have $r_i\leqq K_iY_m\xi_m^{1/4}$ with constants K_i not depending on m, where $r_i=\max\{|z-z_{m+1,2l-i}|\,;\,z\in\hat{\gamma}_{m+1,2l-i}\}$, $i=0, 1$. Therefore the ring domain $\{z\,;\,Y_m<|z-z_{m,l}|<KY_{m-1}\xi_m^{3/4}\}\subset\Delta'_{m,l}$ for sufficiently large m and its image under $\varphi^{-1}\circ f$ separates $B_1\cup B_2$ from B_3. Thus we have again by Lemma 4

$$16\min\{|\omega|\,;\,\omega\in\varphi^{-1}\circ f(\gamma_{m,l})\}\geqq K/\xi_m^{1/4}=K'/\sqrt{d}\;,$$

where $\gamma_{m,l}$ denotes the circle $|z-z_{m,l}|=KY_{m-1}\xi_m^{3/4}$. This means that $|f(z)|\leqq\alpha d=4\alpha\delta_m$ on $\gamma_{m,l}$, where α does not depend on m.

Since f has no zeros and no poles in the domain bounded by $\hat{\Gamma}_{m,l}$ and $\gamma_{m,l}$ and $s(\check{\gamma}_{m,l})=2$, the image curve of any closed curve in this domain being homotopic to $\Gamma_{m,l}$ winds twice around 0. Therefore $f^{1/2}$ is single-valued there. By the Cauchy integral formula,

$$\frac{df^{1/2}}{dz}(z)=\frac{1}{2\pi i}\Big(\int_{\hat{\Gamma}_{m,l}}-\int_{\gamma_{m,l}}\Big)\frac{f^{1/2}(\zeta)}{(\zeta-z)^2}d\zeta, \qquad z\in\tilde{\Gamma}_{m,l}.$$

We have

$$\left|\frac{df^{1/2}}{dz}(z)\right|\leqq\frac{1}{2\pi}\Big(\frac{(2\xi_{m-1})^{1/2}}{(Y_{m-1}/3-(1/\sqrt{6})\xi_m^{1/2}Y_{m-1}\xi_{m-1}^{-1/4})^2}\cdot 2\pi\frac{Y_{m-1}}{3}$$
$$+\frac{(4\alpha\delta_m)^{1/2}}{((1/\sqrt{6})\xi_m^{1/2}Y_{m-1}\xi_{m-1}^{-1/4}-KY_{m-1}\xi_m^{3/4})^2}\cdot 2\pi KY_{m-1}\xi_m^{3/4}\Big)$$
$$\leqq\frac{6}{Y_{m-1}}\xi_{m-1}^{1/2},$$

for sufficiently large m. Thus the length of the curve $f^{1/2}(\tilde{\Gamma}_{m,l})$ is dominated by

$$\int_{\tilde{\Gamma}_{m,l}}\left|\frac{df^{1/2}}{dz}(z)\right||dz|\leqq\frac{6}{Y_{m-1}}\xi_{m-1}^{1/2}\cdot 2\pi\frac{1}{\sqrt{6}}\xi_m^{1/2}Y_{m-1}\xi_{m-1}^{-1/4}$$
$$=2\sqrt{6}\,\pi\xi_{m-1}^{1/4}\xi_m^{1/2}.$$

Since the curve $f^{1/2}(\tilde{\Gamma}_{m,l})$ winds once around 0, we see that $|f^{1/2}(z)|\leqq 2\sqrt{6}\,\pi\xi_{m-1}^{1/4}\xi_m^{1/2}$ and hence $|f(z)|\leqq 24\pi^2\xi_{m-1}^{1/2}\xi_m$ on $\tilde{\Gamma}_{m,l}$. Thus $f(\tilde{\Gamma}_{m,l})\subset D(0, 24\pi^2\xi_{m-1}^{1/2}\xi_m)$. Our proof is complete.

Now we can show that $f(\check{\Gamma}_{n+2,4k})\subset D(0, 8\check{\delta}_{n+2})$, $\check{\delta}_{n+2}=M'\xi_{n+1}\xi_{n+2}$. Contrary suppose that $f(\check{\Gamma}_{n+2,4k})\not\subset D(0, 8\check{\delta}_{n+2})$. Then $\check{D}_{n+2,4k}\cap D(0, 4\check{\delta}_{n+2})=\emptyset$, where $\check{D}_{n+2,4k}\supset f(\check{\Gamma}_{n+2,4k})$ is a disc of radius at most $\check{\delta}_{n+2}$. Obviously $s(\check{\Gamma}_{n+2,4k})=0$ and we see similarly as before that one of $\Delta_{n+3,8k-1}$ and $\Delta_{n+3,8k}$, say $\Delta_{n+3,8k}$, is degenerate(f) and $f(\check{\Gamma}_{n+3,8k})$ is contained in a disc $\check{D}_{n+3,8k}$ of radius at most $\check{\delta}_{n+3}=M'\xi_{n+2}\xi_{n+3}$. Assume that $\Delta_{n+3,8k-1}$ is non-degenerate(f) and of class (23). Then f has no poles in the domain Δ bounded by $\check{\Gamma}_{n+2,4k}$, $\check{\gamma}_{n+3,8k-1}$ ($f(\check{\gamma}_{n+3,8k-1})=\partial D(0, 4\delta_{n+3})$, $\delta_{n+3}=\sqrt{3/2}M\xi_{n+3}^{1/2}$) and $\check{\Gamma}_{n+3,8k}$, because $\Delta_{n+2,4k}$ and $\Delta_{n+3,8k}$ are degenerate(f) and f has no poles in the domain bounded by $\Gamma_{n+3,8k-1}$ and $\check{\gamma}_{n+3,8k-1}$ by Lemma 1. If $\check{D}_{n+3,8k}\not\ni 0$, then $s(\check{\Gamma}_{n+3,8k})=0$, so that f has two zeros of order 1 or a zero of order 2 in Δ, while $w_3=0$ is a totally ramified value of f with $\nu_3=7$. Hence $0\in\check{D}_{n+3,8k}\subset D(0, 4\check{\delta}_{n+2})\cap D(0, 4\delta_{n+3})$. We take the component Δ' of $f^{-1}(\hat{C}-\check{D}_{n+3,8k})\cap\Delta$ having $\check{\Gamma}_{n+2,4k}$ as a boundary component. The boundary $\partial\Delta'$ has a boundary component $\check{\Gamma}$ with $f(\check{\Gamma})=\partial D_{n+3,8k}$, which separates $\check{\Gamma}_{n+2,4k}$ from $\check{\Gamma}_{n+3,8k}$ in Δ. We orientate $\check{\Gamma}$ positively with respect to the domain Δ'. Then $f(\check{\Gamma})$ winds around 0 in the negative direction, so that, if $\check{\Gamma}$ separates $\check{\Gamma}_{n+2,4k}$ from $\check{\gamma}_{n+3,8k-1}$ too and Δ' is bounded by $\check{\Gamma}_{n+2,4k}$ and $\check{\Gamma}$, then f has at least one pole in Δ', because the winding number of $\check{\Gamma}_{n+2,4k}$ is 0. Hence it is only possible that $\partial\Delta'$ consists of $\check{\Gamma}_{n+2,4k}$, $\check{\gamma}_{n+3,8k-1}$ and $\check{\Gamma}$ with winding numbers 0, 2 and -2 around 0, respectively, and f has no zeros in

Δ'. Since $\Delta_{n+2,4k}$ is degenerate(f), $f(\hat{\Gamma}_{n+3,8k-1})\subset D(0, \xi_{n+2})$ and we see from Lemma 5 that $f(\check{\Gamma}_{n+3,8k-1})\subset D(0, 24\pi^2\xi_{n+2}^{1/2}\xi_{n+3})\subset D(0, 4\check{\delta}_{n+2})$. Thus $f(\check{\Gamma}_{n+2,4k})\subset \check{D}_{n+2,4k}$, $\check{D}_{n+2,4k}\cap D(0, 4\check{\delta}_{n+2})=\emptyset$ and $f(\check{\Gamma}\cup\check{\Gamma}_{n+3,8k-1})\subset D(0, 4\check{\delta}_{n+2})$. Hence f is not bounded in $\Delta'\subset\Delta$, while f has no poles in Δ. Thus $\Delta_{n+3,8k-1}$ must be degenerate(f), so that $f(\check{\Gamma}_{n+3,8k-1})$ is contained in a disc $\check{D}_{n+3,8k-1}$ of radius at most $\check{\delta}_{n+3}$ and $\check{D}_{n+2,4k}\cup\check{D}_{n+3,8k-1}\cup\check{D}_{n+3,8k}$ is connected. Hence $f(\check{\Gamma}_{n+3,8k-1})\not\subset D(0, 8\check{\delta}_{n+3})$ and $f(\check{\Gamma}_{n+3,8k})\not\subset D(0, 8\check{\delta}_{n+3})$. By induction, we see that f is bounded in the part of $G=\hat{\boldsymbol{C}}-E$ surrounded with $\check{\Gamma}'_{n+2,4k}$. This contradicts our assumption $f\in M_E$. We have now that $f(\check{\Gamma}_{n+2,4k})\subset D(0, 8\check{\delta}_{n+2})$.

3.4. Recall that $\Delta_{n,k}$ is non-degenerate(f) and of class (23), $\Delta_{n+1,2k}$ and $\Delta_{n+2,4k}$ are degenerate(f) so that $f(\check{\Gamma}_{n+2,4k})\subset D(0, 8\check{\delta}_{n+2})$, $\check{\delta}_{n+2}=M'\xi_{n+1}\xi_{n+2}$, and $\Delta_{n+2,4k-1}$ is degenerate(f) so that $f(\check{\Gamma}_{n+2,4k-1})\subset D(0, 8\check{\delta}_{n+2})$, or non-degenerate($f$) and of class (23). We denote by $\hat{\gamma}$ the curve in $\Delta_{n,k}$ such that $f(\hat{\gamma})=\{w\,;\,|w|=1/2\}$ and it is homotopic to $\hat{\gamma}_{n+1,2k}$, and by Δ the domain bounded by $\hat{\gamma}$, $\Gamma_1=\check{\Gamma}_{n+2,4k-1}$ and $\Gamma_2=\check{\Gamma}_{n+2,4k}$ if $\Delta_{n+2,4k-1}$ is degenerate(f), or the domain bounded by $\hat{\gamma}$, $\check{\gamma}_{n+2,4k-1}$ and $\Gamma_2=\check{\Gamma}_{n+2,4k}$ if $\Delta_{n+2,4k-1}$ is of class (23). Assuming that $\Delta_{n+2,4k-1}$ is of class (23), we consider the component Δ' of $f^{-1}(\hat{\boldsymbol{C}}-D(0, 8\check{\delta}_{n+2}))\cap\Delta$ having $\hat{\gamma}$ as a boundary component. The boundary $\partial\Delta'$ has a boundary component Γ' with $f(\Gamma')=\partial D(0, 8\check{\delta}_{n+2})$ which separates $\hat{\gamma}$ and $\check{\gamma}_{n+2,4k-1}$ from Γ_2 or $\hat{\gamma}$ from $\check{\gamma}_{n+2,4k-1}$ and Γ_2. In the latter case, Δ' is the ring domain bounded by $\hat{\gamma}$ and Γ' and its Riemannian image under f covers divalently the ring domain $R=\{w\,;\,8\check{\delta}_{n+2}<\chi(0, w)<1/\sqrt{5}\}$, so that its harmonic modulus is equal to one half of that of R, that is, $(1/2)\log(\sqrt{1-(8\check{\delta}_{n+2})^2}/16\check{\delta}_{n+2})$. Since Δ' separates $\{z_{n+2,4k-1}, z_{n+2,4k}\}$ from $\{z_{n+1,2k-1}, \infty\}$, we have by Lemma 4

$$\log\left(16\frac{Y_n(1-\eta_{n+1})}{Y_{n+1}(1-\eta_{n+2})}+8\right)\geqq\frac{1}{2}\log\frac{\sqrt{1-(8\check{\delta}_{n+2})^2}}{16\check{\delta}_{n+2}},$$

so that $2^{12}/\xi_{n+1}^2\geqq 1/2^5M'\xi_{n+1}\xi_{n+2}$, that is, $o(\xi_{n+1}^{r_0-1})\geqq 1/2^{17}M'$. It is impossible for sufficiently large n. Therefore only the former case is possible and Δ' is bounded by $\hat{\gamma}$, $\check{\gamma}_{n+2,4k-1}$ and Γ' with winding numbers 2, 2 and -4 around 0, respectively, and f has no zeros there. Since $\Delta_{n+1,2k}$ is degenerate(f), $f(\hat{\Gamma}_{n+2,4k-1})\subset D(0, \xi_{n+1})$ and we see from Lemma 5 that $f(\tilde{\Gamma}_{n+2,4k-1})\subset D(0, 24\pi^2\xi_{n+1}^{1/2}\xi_{n+2})$. We set $\Gamma_1=\tilde{\Gamma}_{n+2,4k-1}$ in the case that $\Delta_{n+2,4k-1}$ is of class (23). Noting that $f(\Gamma_1)\cup f(\Gamma_2)\subset D(0, 24\pi^2\xi_{n+1}^{1/2}\xi_{n+2})$, we consider the component Δ'' of $f^{-1}(\hat{\boldsymbol{C}}-D(0, 24\pi^2\xi_{n+1}^{1/2}\xi_{n+2}))\cap\Delta$ having $\hat{\gamma}$ as a boundary component. The boundary $\partial\Delta''$ has two boundary components Γ_1'' and Γ_2'' with $f(\Gamma_1'')=f(\Gamma_2'')=\partial D(0, 24\pi^2\xi_{n+1}^{1/2}\xi_{n+2})$, being homotopic to Γ_1 and Γ_2, respectively, or a boundary component Γ'' with $f(\Gamma'')=\partial D(0, 24\pi^2\xi_{n+1}^{1/2}\xi_{n+2})$ separating $\hat{\gamma}$ from Γ_1 and Γ_2. Quite similarly as before we see that only the former case is possible. Then Δ'' is bounded by $\hat{\gamma}$, Γ_1'' and Γ_2'' and its Riemannian image $\tilde{R}$ under f covers the ring domain $R'=\{w\,;\,24\pi^2\xi_{n+1}^{1/2}\xi_{n+2}<\chi(0,w)<1/\sqrt{5}\}$ divalently. By the Hurwitz formula, $\tilde{R}$ has just one branch point of order 2, whose projection we denote by w^*. Since the part of $\tilde{R}$ over $\{w\,;\,|w^*|<|w|<1/2\}$ is doubly connected, we have by Lemma 4

$$\log\left(16\frac{Y_n(1-\eta_{n+1})}{Y_{n+1}(1-\eta_{n+2})}+8\right)\geqq\frac{1}{2}\log(1/2|w^*|),$$

that is,

$$|w^*|>\xi_{n+1}^2/2^{13}.$$

The inverse image of the circle $\{w\,;\,|w|=|w^*|\}$ in Δ'' is an eightshaped closed curve crossing at the point z^* with $f(z^*)=w^*$, that is, it consists of two simple closed curves C_1 and C_2 with $C_1\cap C_2=\{z^*\}$, being homotopic to Γ_1 and Γ_2, respectively. Since $s(C_2)=s(\Gamma_2)=1$, one of $\Delta_{n+3,8k-1}$ and $\Delta_{n+3,8k}$, say $\Delta_{n+3,8k}$, is degenerate(f) so that $f(\check{\Gamma}_{n+3,8k})\subset D(0,8\check{\delta}_{n+3})$, $\check{\delta}_{n+3}=M'\xi_{n+2}\xi_{n+3}$, and $\Delta_{n+3,8k-1}$ is degenerate(f) so that $f(\check{\Gamma}_{n+3,8k-1})\subset D(0,8\check{\delta}_{n+3})$, or non-degenerate$(f)$ and of class (23). We denote by D the domain bounded by C_2, $C=\check{\Gamma}_{n+3,8k-1}$ and $C'=\check{\Gamma}_{n+3,8k}$ if $\Delta_{n+3,8k-1}$ is degenerate(f), or the domain bounded by C_2, $\check{\gamma}_{n+3,8k-1}$ and $C'=\check{\Gamma}_{n+3,8k}$ if $\Delta_{n+3,8k-1}$ is of class (23). Assuming that $\Delta_{n+3,8k-1}$ is of class (23), we consider the component D' of $f^{-1}(\hat{C}-D(0,8\check{\delta}_{n+3}))\cap D$ having C_2 as a boundary component. The boundary $\partial D'$ has a boundary component $\tilde{C}$ with $f(\tilde{C})=\partial D(0,8\check{\delta}_{n+3})$ which separates C_2 and $\check{\gamma}_{n+3,8k-1}$ from C' or C_2 from $\check{\gamma}_{n+3,8k-1}$ and C'. In the latter case, D' is the ring domain bounded by C_2 and $\tilde{C}$ and its Riemannian image under f covers univalently the ring domain $\{w\,;\,8\check{\delta}_{n+3}<\chi(0,w)<|w^*|/\sqrt{1+|w^*|^2}\}$. Since D' separates $\{z_{n+3,8k-1},z_{n+3,8k}\}$ from $\{z_{n+2,4k-1},\infty\}$, we have by Lemma 4

$$\log\left(16\frac{Y_{n+1}(1-\eta_{n+2})}{Y_{n+2}(1-\eta_{n+3})}+8\right)\geqq\log\frac{|w^*|\sqrt{1-(8\check{\delta}_{n+3})^2}}{8\check{\delta}_{n+3}},$$

so that

$$2^6/\xi_{n+2}\geqq|w^*|/2^4M'\xi_{n+2}\xi_{n+3}\geqq\xi_{n+1}^2/2^{17}M'\xi_{n+2}\xi_{n+3}.$$

Hence we have $o(\xi_{n+1}^{r_0^2-2})>1/2^{23}M'$, where $r_0^2-2>0$. It is absurd.

Thus $\tilde{C}$ separates C_2 and $\check{\gamma}_{n+3,8k-1}$ from C', D' is bounded by C_2, $\check{\gamma}_{n+3,8k-1}$ and $\tilde{C}$ with winding numbers 1, 2 and -3 around 0, respectively, and f has no zeros there. Since $\Delta_{n+2,4k}$ is degenerate(f), $f(\hat{\Gamma}_{n+3,8k-1})\subset D(0,\xi_{n+2})$ and we see from Lemma 5 that $f(\tilde{\Gamma}_{n+3,8k-1})\subset D(0,24\pi^2\xi_{n+2}^{1/2}\xi_{n+3})$. We set $C=\tilde{\Gamma}_{n+3,8k-1}$ in the case that $\Delta_{n+3,8k-1}$ is of class (23).

Noting that $f(C)\cup f(C')\subset D(0,24\pi^2\xi_{n+2}^{1/2}\xi_{n+3})$, we consider D'' of $f^{-1}(\hat{C}-D(0,24\pi^2\xi_{n+2}^{1/2}\xi_{n+3}))\cap D$ having C_2 as a boundary component. The Riemannian image of D'' under f covers univalently the ring domain $\{w\,;\,24\pi^2\xi_{n+2}^{1/2}\xi_{n+3}<\chi(0,w)<|w^*|/\sqrt{1+|w^*|^2}\}$, so that D'' is a ring domain with harmonic modulus

$\log(|w^*|\sqrt{1-(24\pi^2\xi_{n+2}^{1/2}\xi_{n+3})^2}/24\pi^2\xi_{n+2}^{1/2}\xi_{n+3})$. Since D'' separates $\{z_{n+3,8k-1}, z_{n+3,8k}\}$ from $\{z_{n+2,4k-1}, \infty\}$, we have by Lemma 4

$$2^6/\xi_{n+2} \geqq |w^*|/48\pi^2\xi_{n+2}^{1/2}\xi_{n+3} \geqq \xi_{n+1}^2/2^{17}\cdot 3\xi_{n+2}^{1/2}\xi_{n+3},$$

so that $o(\xi_{n+1}^{r_0\{r_0-(1/2)\}-2}) \geqq 1/2^{23}\cdot 3$, where $r_0\{r_0-(1/2)\}-2=0$. It is absurd and now our proof of the theorem is complete.

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