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On the number of apparent singularities of the Riemann-Hilbert problem on Riemann surface

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1. Introduction.

Let X be a compact Riemann surface of genus g. For a given finite subset S of X, let $\mathcal{D}(S)$ be the set of linear differential equations on X with meromorphic coefficients having singularities S. In this article we investigate the monodromy map from $\mathcal{D}(S)$ to the set of representations (up to conjugacy)

$$\pi_1(X-S) \longrightarrow \mathrm{GL}(n, \boldsymbol{C}).$$

The Riemann-Hilbert problem is, roughly speaking, the question whether this map is surjective or not. During this century, many mathematicians have given affirmative answers under various situations. To solve this problem, we have to think of a differential equation with some singularities, besides given S, such that around each of these singularities the monodromy is trivial. Such singularities are called *apparent* singularities. An estimate for the number of these apparent singularities was made by M. Ohtsuki [**9**] a decade ago. He used the formulation given by P. Deligne [**1**]. Deligne's formulation is explained as follows. For a holomorphic vector bundle E over X constructed by using a given representation, this problem is reduced to find a holomorphic line subbundle of E. In fact any line bundle with sufficiently small degree can be realized as a subbundle of E. To improve the estimate, however, we need a line subbundle of E with larger degree. Ohtsuki used the Riemann-Roch theorem to estimate the largest possible degree.

Recently a similar situation to Deligne's formulation is considered in a different context by P. Kronheimer and T. Mrowka [**7**]. The main tool of their argument is the Riemann-Roch-Grothendieck theorem.

In § 2 we shall use the Riemann-Roch-Grothendieck theorem for our context to improve Ohtsuki's estimate.

THEOREM A. *Let X be a compact Riemann surface of genus g and $p_1, \cdots, p_m$ distinct points in X. Assume that*

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{GL}(n, \boldsymbol{C})$$

is an irreducible representation and that, for each i, the image of ρ for a sufficiently small circle around p_i is semi-simple. Then there are holomorphic line bundles V, W over X and a nonzero differential operator

$$Q : \Gamma(X, \mathcal{M}(V)) \longrightarrow \Gamma(X, \mathcal{M}(W)),$$

(for a vector bundle E over X, $\mathcal{M}(E)$ is the sheaf of meromorphic sections of E), satisfying the following properties.

- *The monodromy of differential equation $Qt=0$ is isomorphic to ρ.*
- *The number of apparent singularities is at most*

$$(n-1)(g-1)+\frac{n(n-1)}{2}(m+2g-2).$$

In §3 we consider a similar problem for meromorphic projective connections and their monodromies on X.

THEOREM B. *Let X be a compact Riemann surface of genus g and $p_1, \cdots, p_m$ distinct points in X. Assume that*

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{PSL}(2, \boldsymbol{C})$$

is an irreducible representation and that, for each i, the image of ρ for a sufficiently small circle around p_i is semi-simple. Then there exists a meromorphic projective connection satisfying the following two properties.

- *Its monodromy is isomorphic to ρ.*
- *The number of apparent singularities is at most $m+3g-3$.*

In [**4**], K. Iwasaki gave a geometric interpretation of generalized Painlevé equation on Riemann surface. By using Theorem B, we solve Iwasaki's conjecture (Problem 5.11 of [**4**, p. 495]) of the surjectivity of the monodromy map of meromorphic projective connections and show that this map is a 2^g-fold covering.

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2. The Riemann-Hilbert problem.

2.1. Deligne's solution.

Let $p_1, \cdots, p_m$ be distinct points in X, and ρ a representation

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{GL}(n, \boldsymbol{C}).$$

In §2 we consider the following problems.

(a) Find a linear differential equation on X with meromorphic coefficients with the monodromy isomorphic to ρ.

(b) Estimate the number of apparent singularities.

Here a linear differential equation with meromorphic coefficients means a linear differential operator

$$Q : \Gamma(X, \mathcal{M}(V)) \longrightarrow \Gamma(X, \mathcal{M}(W)),$$

for some holomorphic vector bundles V, W over X.

In § 2.1 we shall deal with (a) according to P. Deligne [1, 9].

Let γ_i be the homotopy class of a sufficiently small counter-clockwise-oriented circle C_i around p_i. From now on, we assume that ρ is irreducible and that $\rho(\gamma_i)$ is semi-simple for all i.

2.1.1. Construction of a vector bundle.

Let E be a vector bundle with flat connection ∇_0 over $X-\{p_1, \cdots, p_m\}$ induced by ρ. Since $\rho(\gamma_i)$ is semi-simple, we can take a $n\times n$ matrix B_i which satisfies

$$\rho(\gamma_i) = \exp(-2\pi\sqrt{-1}\,B_i).$$

In this step we construct a vector bundle $\tilde{E}$ over X whose restriction on $X-\{p_1, \cdots, p_m\}$ is E. To do so, choose a sufficiently small local coordinate system (U, z) around p_1 such that p_i is not in U for $i=2, \cdots, m$. Consider the meromorphic connection ∇_1 for the trivial vector bundle $U\times \boldsymbol{C}^n$ over U of the form

$$\nabla_1 = d+\frac{B_1}{z}dz.$$

Since the holonomy of ∇_1 along C_1 is $\rho(\gamma_1)$, we obtain a new vector bundle over $X-\{p_2, \cdots, p_m\}$ patching $E|_{U-\{p_1\}}$ and $U\times \boldsymbol{C}^n$ together so that the flat connection ∇_0 of E coincides with ∇_1 over $U-\{p_1\}$. Thus we obtain a vector bundle over $X-\{p_2, \cdots, p_m\}$ with a meromorphic connection which has a pole at p_1. Applying similar procedures for small local coordinate systems around $p_2, \cdots, p_m$, we obtain a vector bundle $\tilde{E}$ over X and its meromorphic connection ∇ with poles at $p_1, \cdots, p_m$.

The value of $e=\deg(\tilde{E})$ depends on the choice of $B_1, \cdots, B_m$. It is easily seen, for example by using the Chern-Weil theorem, that

$$c_1(\tilde{E}) = -\sum_{i=1}^{m} \mathrm{tr}(B_i).$$

2.1.2. The Riemann-Roch theorem.

In this step we find a pair of (L, φ) such that L is a holomorphic line bundle over X and that φ is a nonzero holomorphic cross section of $L\otimes\tilde{E}^*$.

The existence is easy to see by the Riemann-Roch theorem. For any line bundle L of degree l, we have

$$\begin{aligned}&\dim \mathrm{H}^0(X;\, L\otimes\tilde{E}^*)-\dim \mathrm{H}^1(X;\, L\otimes\tilde{E}^*)\\&=\deg(L\otimes\tilde{E}^*)+\mathrm{rank}(L\otimes\tilde{E}^*)(1-g)\\&=nl-e+n(1-g)\,.\end{aligned}$$

So if $nl-e$ is sufficiently large, we have

$$\mathrm{H}^0(X;\, L\otimes\tilde{E}^*)\neq 0\,,$$

that is to say, there is a nonzero holomorphic cross section φ of $L\otimes\tilde{E}^*$. Now fix such a pair (L, φ).

2.1.3. Construction of a differential operator.

Here we shall construct a linear differential equation whose monodromy is isomorphic to ρ.

As a first step, we shall construct a differential operator on a small open set U of $X-\{p_1, \cdots, p_m\}$. Let z be a local coordinate on U, $s_1, \cdots, s_n$ linearly independent, holomorphic and parallel sections of E on U, and $\phi\colon L|_U\to U\times \boldsymbol{C}$ a trivialization of L over U. Set

$$Qt=\det\begin{pmatrix}\phi\circ t & \phi\circ\langle\varphi, s_1\rangle & \cdots & \phi\circ\langle\varphi, s_n\rangle\\ D\phi\circ t & D\phi\circ\langle\varphi, s_1\rangle & \cdots & D\phi\circ\langle\varphi, s_n\rangle\\ & \cdots & \cdots & \\ D^n\phi\circ t & D^n\phi\circ\langle\varphi, s_1\rangle & \cdots & D^n\phi\circ\langle\varphi, s_n\rangle\end{pmatrix},$$

where D is d/dz, t is a local section of $\mathcal{M}(L)$ and $\langle\,,\rangle\colon(L\otimes\tilde{E}^*)\otimes\tilde{E}\to L$ is the natural pairing. Then Q is a differential operator over U, but direct calculations show that Q is well-defined globally:

$$Q:\Gamma(X, \mathcal{M}(L))\longrightarrow\Gamma(X, \mathcal{M}(L^{\otimes(n+1)}\otimes\det(\tilde{E}^*)\otimes\kappa^{\otimes n(n+1)/2}))$$

where κ is the canonical line bundle of X. It is clear that Q is holomorphic on $X-\{p_1, \cdots, p_m\}$. We shall see that Q has a pole at p_ι. By definition, we can write locally

$$Qt=A_0D^n\phi\circ t+A_1D^{n-1}\phi\circ t+\cdots+A_n\phi\circ t\,,$$

where

$$A_0=(-1)^n\det\begin{pmatrix}\phi\circ\langle\varphi, s_1\rangle & \cdots & \phi\circ\langle\varphi, s_n\rangle\\ \vdots & & \vdots\\ D^{n-1}\phi\circ\langle\varphi, s_1\rangle & \cdots & D^{n-1}\phi\circ\langle\varphi, s_n\rangle\end{pmatrix}.$$

Around p_ι, noticing that s_j's are parallel, we have

$$\begin{aligned} D\langle\varphi, s_j\rangle &= \langle\nabla_D\varphi, s_j\rangle \\ &= \frac{1}{z}\langle B_i\varphi, s_j\rangle \\ &\qquad + (\text{regular term}). \end{aligned}$$

Here we regard $L|_U$ as $U\times \boldsymbol{C}$ and ∇ means the natural connection of $L\otimes\tilde{E}^*|_U$ determined by D and ∇.

Then we obtain

$$A_0 = (-1)^n \det \begin{pmatrix} \langle\varphi, s_1\rangle & \cdots & \langle\varphi, s_n\rangle \\ \frac{1}{z}\langle B_i\varphi, s_1\rangle & \cdots & \frac{1}{z}\langle B_i\varphi, s_n\rangle \\ & \cdots & \\ \frac{1}{z^{n-1}}\langle B_i^{n-1}\varphi, s_1\rangle & \cdots & \frac{1}{z^{n-1}}\langle B_i^{n-1}\varphi, s_n\rangle \end{pmatrix}$$

$$+(\text{lower singularities}),$$

and it is clear that the right-half-side of above equation is independent of the choice of $s_1, \cdots, s_n$ (up to non-zero constant). Thus A_0 has a pole at each p_i of order at most $n(n-1)/2$.

It is shown in [**9**, Proposition] that the irreducibility of ρ implies that the order of Q is n, i.e., A_0 is nonzero.

2.2. Apparent singularities.

In this subsection we shall deal with (b) of § 2.1, that is, the number of apparent singularities of the equation $Qs=0$.

Summing up § 2.1, we notice the following points.

(A) Q exists if $nl-e+n(1-g)>0$.

(B) A_0 is holomorphic on $X-\{p_1, \cdots, p_m\}$ and p_i is a pole of A_0 of order at most $n(n-1)/2$.

(C) The zeros of A_0 are apparent singularities.

Now we estimate the number of the zeros of A_0 (counted with multiplicity). Since A_0 is a meromorphic section of $L^{\otimes n}\otimes\det(\tilde{E}^*)\otimes\kappa^{\otimes n(n-1)/2}$,

$$\begin{aligned} &\#(\text{Zeros of } A_0) - \#(\text{Poles of } A_0) \\ &= \deg(L^{\otimes n}\otimes\det(\tilde{E}^*)\otimes\kappa^{\otimes n(n-1)/2}) \\ &= nl-e+n(n-1)(g-1). \end{aligned} \tag{D}$$

From (A), (B), (C), and (D), M. Ohtsuki obtained

THEOREM 1 ([**9**]). *Let X be a compact Riemann surface of genus g and $p_1, \cdots, p_m$ distinct points in X. Assume that*

$$\rho : \pi_1(X - \{p_1, \cdots, p_m\}) \longrightarrow \mathrm{GL}(n, \boldsymbol{C})$$

is an irreducible representation and that, for each i, the image of ρ for a sufficiently small circle around p_i is semi-simple. Then there is a holomorphic line bundles L, a rank 2 vector bundle $\tilde{E}$ over X, and a nonzero differential operator

$$Q : \Gamma(X, \mathcal{M}(L)) \longrightarrow \Gamma(X, \mathcal{M}(L^{\otimes(n+1)} \otimes \det(\tilde{E}^*) \otimes \kappa^{\otimes n(n+1)/2}))$$

satisfying the following properties.

- *The monodromy of differential equation $Qt=0$ is isomorphic to ρ.*
- *The number of apparent singularities is at most*

$$n(g-1)+1+\frac{n(n-1)}{2}(m+2g-2). \tag{1}$$

2.3. Better estimate.

In § 2.1 we considered only the degree of holomorphic line bundle over X. That is to say, for any line bundle of sufficiently large degree l such that $nl-e+n(1-g)>0$, we can construct a differential equation whose monodromy is isomorphic to ρ. There may exist a holomorphic line bundle L such that $\mathrm{H}^0(X; L\otimes\tilde{E}^*)\neq 0$ even if the degree l of L satisfies $nl-e+n(1-g)\leqq 0$. If such L exists, it is possible to get a better estimate for the number of apparent singularities. The following argument is an extension of that of Kronheimer-Mrowka [7, § 9]. They considered rank 2 vector bundles, while we consider vector bundles of general rank.

Fix an integer l. Recall that all of the holomorphic line bundles of degree l over a Riemann surface of genus g is parameterized by its Jacobi manifold $\mathcal{J}_l = T^g$. It is well known that there is a universal bundle of this family $\mathcal{P}_l$ over $\mathcal{J}_l \times X$ called Poincaré line bundle which is uniquely determined if we assume its restriction on $\mathcal{J}_l \times \{p\}$ is trivial for a fixed $p\in X$. Let α_i, β_i ($i=1, \cdots, g$) be a set of generators of $\mathrm{H}^1(X)$ satisfying

$$\alpha_i\beta_j = \delta_{ij}, \quad \alpha_i\alpha_j = 0, \quad \text{and} \quad \beta_i\beta_j = 0 \quad (i, j = 1, \cdots, g).$$

We denote by ω the standard generator of $\mathrm{H}^2(X)$. The following formula is well-known

$$c_1(\mathcal{P}_l) = l(1\otimes\omega)+\varepsilon. \tag{2}$$

Here ε is expressed as

$$\varepsilon = \sum_{i=1}^{g} (a_i\otimes\tilde{\alpha}_i + b_i\otimes\tilde{\beta}_i),$$

for some a_i, b_i in $\mathrm{H}^1(\mathcal{J}_l)$. Then a_i, b_i are dual of α_i, β_i respectively.

Let P be the family of Dolbeault operators $\bar{\partial}_a$ on line bundles $\mathcal{P}_l|_{\{a\}\times X}\otimes\tilde{E}^*$. By contradiction we shall show that if $-nl+e+n(g-1)<g$ there is an element

a in $\mathcal{J}_l$ satisfying

$$\dim \mathrm{H}^0(X; \mathcal{P}_l|_{\{a\}\times X}\otimes\tilde{E}^*) \neq 0.$$

Take l and e satisfying $0\leqq -nl+e+n(g-1)<g$. Suppose $\mathrm{H}^0(X; \mathcal{P}_l|_{\{a\}\times X}\otimes\tilde{E}^*)=0$ for any $a\in\mathcal{J}_l$. Then $\dim \mathrm{H}^1(X; \mathcal{P}_l|_{\{a\}\times X}\otimes\tilde{E}^*)$ is constant, and hence

$$-\operatorname{ind} P = \coprod_{a\in\mathcal{J}_l} \mathrm{H}^1(X; \mathcal{P}_l|_{\{a\}\times X}\otimes\tilde{E}^*)$$

is regarded as a vector bundle, not only as a virtual vector bundle, over $\mathcal{J}_l$ with rank $-nl+e+n(g-1)$. In particular, for $k=-nl+e+n(g-1)+1$, we have

$$c_k(-\operatorname{ind} P) = 0.$$

On the other hand we can calculate $c_k(-\operatorname{ind} P)$ by using the Riemann-Roch-Grothendieck theorem.

From (2), we have

$$ch(\mathcal{P}_l) = 1+\{l(1\otimes\omega)+\varepsilon\}-\Theta\otimes\omega,$$

where

$$\Theta = -\sum_{i=1}^{g} a_i\smile b_i.$$

Let $\pi: \mathcal{J}_l\times X\to\mathcal{J}_l$ be the natural projection. The Riemann-Roch-Grothendieck theorem implies

$$\begin{aligned} ch(-\operatorname{ind} P) &= \pi_!\, ch(\mathcal{P}_l\otimes\tilde{E}^*)\boldsymbol{Td}(\mathcal{J}_l\times X) \\ &= [(1+\{l(1\otimes\omega)+\varepsilon\}-\Theta\otimes\omega)(n-l(1\otimes\omega))(1+(1-g)(1\otimes\omega))]/[X] \\ &= -nl+e+n(g-1)+n\Theta. \end{aligned}$$

Here we used the formulae

$$\begin{aligned} ch(\tilde{E}^*) &= n-l(1\otimes\omega), \\ \boldsymbol{Td}(\mathcal{J}_l\times X) &= 1+(1-g)(1\otimes\omega). \end{aligned}$$

Using the Newton formula, we obtain

$$c_j(-\operatorname{ind} P) = \frac{n^j}{j!}\Theta^j.$$

Since

$$\begin{aligned} \Theta^j &= 0 \quad \text{if } j>g \\ &\neq 0 \quad \text{if } 0\leqq j\leqq g, \end{aligned}$$

$c_k(-\operatorname{ind} P)=0$ if and only if

$$k = -nl+e+n(g-1)+1 > g.$$

This contradicts the condition $0 \leqq -nl+e+n(g-1)<g$. Summing up, we have proved

(A′) (L, Q) exists if $nl-e+n(1-g)>-g$.

From (A′), (B), (C), and (D) we obtain

THEOREM 2. *Under the same circumstances of Theorem* 1, *there exists a pair* (L, Q) *that has at most*

$$(n-1)(g-1)+\frac{n(n-1)}{2}(m+2g-2) \tag{3}$$

apparent singularities.

Our estimate (3) is less g than that of (1). This follows from the fact that we think about the family of line bundles and the difference g comes from the dimension of the parameter space $\mathcal{J}_l$.

3. Meromorphic projective connections and apparent singularities.

3.1. Meromorphic projective connections.

In this subsection we review the definition of meromorphic projective connections on X and its relation with SL-operators.

DEFINITION 3. A meromorphic projective connection on X is a collection of the following data.

- A holomorphic $\boldsymbol{CP}^1$-bundle P over X with a meromorphic flat connection whose poles are $p_1, \cdots, p_m$.
- A non-parallel holomorphic cross section s of P.

We refer as the projective monodromy representation of a meromorphic projective connection to the representation

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{PSL}(2, \boldsymbol{C})$$

induced by the meromorphic flat connection.

The apparent singularities of a meromorphic projective connection (P, s) is defined by the points at which s is tangent to parallel leaves of P.

Assume that L is a holomorphic line bundle over X.

DEFINITION 4. Let $\mathcal{U}=\{(U_\alpha, z_\alpha)\}$ be a coordinate covering of X, and $\xi=(\xi_\alpha)$ the transition function of L according to $\mathcal{U}$. A Fuchsian SL-operator (simply call SL-operator) on L for $\mathcal{U}$ is a collection $Q=(Q_\alpha)$ of second order linear meromorphic Fuchsian differential operators which obey the following two conditions.

- For each α,

$$Q_\alpha = -D_\alpha^2 + q_\alpha ,$$

where $D_\alpha = d/dz_\alpha$ and q_α is a meromorphic function on U_α.
- If $h_\alpha = \xi_{\alpha\beta} h_\beta$ where h_α and h_β are meromorphic functions on U_α and U_β, respectively, then $Q_\alpha h_\alpha = 0$ is equivalent to $Q_\beta h_\beta = 0$ as differential equations.

Let Q (respectively Q') is an SL-operator on L for a coordinate covering $\mathcal{U}$ (respectively $\mathcal{U}'$). Q and Q' are said to be equivalent if $Q \cup Q'$ is an SL-operator on L for the coordinate covering $\mathcal{U} \cup \mathcal{U}'$. An SL-operators on L is an equivalent class of an SL-operator on L for some coordinate covering of X.

The following proposition indicates the relation between these two concepts.

PROPOSITION 5 ([4]). *For a holomorphic line bundle L over X with* $\deg(L) = 1-g$, *there exists a one-to-one correspondence between the set of SL-operators on L and the set of meromorphic projective connections on X.*

Hereafter we identify SL-operators with meromorphic projective connections via above correspondence. We omit details about SL-operators, which can be found in [**4**, **5**].

3.2. Geometric meaning.

Here we investigate a geometric meaning of meromorphic projective connections whose singularities are all apparent. Most of the following arguments are due to M. Furuta and K. Iwasaki.

Let (P, s) be a meromorphic projective connection. For simplicity, we assume in this subsection, that the flat connection of P has no poles. Let $\rho : \pi_1(X) \to \mathrm{PSL}(2, \boldsymbol{C})$ be the monodromy representation of ρ. We denote the apparent singularities by $q_1, \cdots, q_m$.

Let $\tilde{X}$ be the universal covering of X, $\tilde{P}$ the pull-back of P through the projection $\pi : \tilde{X} \to X$, and $\tilde{s} : \tilde{X} \to \tilde{P}$ the pull-back of s. Since $\pi_1(\tilde{X})$ is trivial, $\tilde{P}$ is a trivial flat $\boldsymbol{CP}^1$-bundle $\tilde{X} \times \boldsymbol{CP}^1$. Then the developing map $\tau : \tilde{X} \to \boldsymbol{CP}^1$ is defined by

$$\tau = pr \circ \tilde{s} ,$$

where $pr : \tilde{P} = \tilde{X} \times \boldsymbol{CP}^1 \to \boldsymbol{CP}^1$ is the canonical projection.

It is clear, by definition, that τ commutes with the action of $\pi_1(X)$ on $\tilde{X}$ through ρ, and that the differential of τ is nondegenerate except at points of $\pi^{-1}\{q_1, \cdots, q_m\}$.

The ramification index $d : X \to \boldsymbol{Z}$ is defined as follows. For any $p \in X$, if $p = q_j$ then $d(p) - 1$ is the order of zero of the differential of τ at a point in

$\pi^{-1}(q_j)$, and otherwise $d(p)=1$. As mentioned above, τ is $\pi_1(X)$-equivariant, so $d(p)$ is well-defined.

Let $\mathcal{U}=\{U_p\}_{p\in X}$ be an open covering of X satisfying the following condition.

- $p\in U_p$.
- If $p\neq q_j$, then $q_j\notin U_p$.
- $\pi^{-1}(U_p)$ is the disjoint union of open sets each of which is biholomorphic to U_p.

For each $p\in X$, fix a local biholomorphic section $\sigma_p: U_p\to\tilde{X}$, a local inverse of π. If necessary, replacing U_p by a smaller one, we can take a local coordinate z_p around p on U_p such that if we choose some coordinate w around $\tau\circ\sigma_p(p)\in \boldsymbol{CP}^1$ then the local representation of $\tau\circ\sigma_p$ is $w=z_p^{d(p)}$. Therefore, since τ is $\pi_1(X)$-equivariant, we get a 0-cochain $(z_p)\in C^0(\mathcal{U}, \mathcal{O})$ ($\mathcal{O}$ is the sheaf of holomorphic functions on X) which obeys (up to PSL(2, $\boldsymbol{C}$) conjugacy)

$$z_p^{d(p)} = g_{pq} z_q^{d(q)},$$

where $g_{pq}\in Z^1(\mathcal{U}, \mathrm{PSL}(2, \boldsymbol{C}))$ is the representation cocycle of P. This implies that the apparent singularities of meromorphic projective connection correspond to *branched points* of Čech representation.

If the projective connection is holomorphic, i.e., there is no apparent singularities, then $d(p)=1$ for any $p\in X$, so (U_p, z_p) determines a projective structure (see, e.g. [3]). Hence our geometric description of meromorphic projective connection with only apparent singularities is a natural extension of holomorphic one.

3.3. Estimate for apparent singularities.

In this subsection we consider the following Riemann-Hilbert type question. For a given representation

$$\rho: \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{PSL}(2, \boldsymbol{C}),$$

find a meromorphic projective connection whose projective monodromy is isomorphic to ρ, and estimate the number of apparent singularities. That is to say, find a (P, s) satisfying the followings.

- The monodromy representation of P is isomorphic to ρ.
- The number of points at which s is tangent to a parallel leaf of P (counted with multiplicity) is as small as possible.

From now on, we assume that ρ is irreducible and semi-simple around p_i for each i.

By using a parallel argument of § 2.1.1, replacing vector bundles by $\boldsymbol{CP}^1$-bundle, we can construct a holomorphic $\boldsymbol{CP}^1$-bundle P with a meromorphic connection determined by ρ. The fundamental group of $X-\{p_1, \cdots, p_m\}$ is generated by $\alpha_1, \cdots, \alpha_g, \beta_1, \cdots, \beta_g, \gamma_1, \cdots, \gamma_m$ with the only relation

$$\gamma_m \cdots \gamma_1[\alpha_g, \beta_g] \cdots [\alpha_1, \beta_1] = 1 \in \pi_1(X-\{p_1, \cdots, p_m\}).$$

Here we set $[\alpha, \beta]=\alpha\beta\alpha^{-1}\beta^{-1}$ and denote by γ_j the homotopy class of a sufficiently small clockwise-oriented circle around p_j. Set $R_j=\rho(\alpha_j)$, $S_j=\rho(\beta_j)$, $T_j=\rho(\gamma_j)$, and fix a lift $\tilde{R}_j, \tilde{S}_j, \tilde{T}_j \in \mathrm{SL}(2, \boldsymbol{C})$ of $R_j, S_j, T_j \in \mathrm{PSL}(2, \boldsymbol{C})$, respectively.

To begin with, we consider the case $m \geqq 1$. In this case, choosing a good lift of T_1, we can assume that

$$\tilde{T}_m \cdots \tilde{T}_1[\tilde{R}_g, \tilde{S}_g] \cdots [\tilde{R}_1, \tilde{S}_1] = \mathrm{Id} \in \mathrm{SL}(2, \boldsymbol{C}).$$

In this situation, P is the projective bundle of some holomorphic vector bundle E with a meromorphic flat connection whose poles are $p_1, \cdots, p_m$, and the meromorphic flat connection of P is induced by that of E.

From the argument of § 2.3, we can find a holomorphic line bundle $\tilde{L}$ which obeys

$$\mathrm{H}^0(X, \tilde{L}^{-1} \otimes E) \neq 0,$$
$$-2\deg(\tilde{L})+\deg(E) \leqq g-1.$$

Taking the tensor product of $\tilde{L}$ and some line bundle on X, we get a line subbundle L of E with $2\deg(L) \geqq \deg(E)+1-g$.

Fix such L and denote the inclusion $L \hookrightarrow E$ by φ, then we get a meromorphic projective connection (P, s) where $s: X \to P$ is defined by φ. Since ρ is irreducible, s is not parallel.

Choose $p \in X-\{p_1, \cdots, p_m\}$ and let (U, z) be a local coordinate system around p such that $U \subset X-\{p_1, \cdots, p_m\}$. Fix trivializations of L and E over U, where the trivialization of E is defined by an $\mathrm{SL}(2, \boldsymbol{C})$-frame on U. According to these trivializations, φ is represented by

$$\varphi: (z, v) \longmapsto \begin{pmatrix} \varphi_1(z, v) \\ \varphi_2(z, v) \end{pmatrix}.$$

Of course, $\varphi_i(z, v)$ is linear for the second variable v. It is clear that $q \in U$ is an apparent singularity if and only if either φ_1/φ_2 or φ_2/φ_1 is critical at $z=z(q)$, that is,

$$\left(\frac{d}{dz}\varphi_1\right)\varphi_2-\varphi_1\left(\frac{d}{dz}\varphi_2\right) = 0$$

at $(z(q), v)$ for all v. Hence apparent singularities are the zeros of a meromorphic cross section ψ of $\det(L^{-1} \otimes E) \otimes \kappa$ which is locally expressed by

$$\left(\frac{d}{dz}\varphi_1\right)\varphi_2-\varphi_1\left(\frac{d}{dz}\varphi_2\right).$$

We can use a similar argument to § 2.1.3 to show that p_j is a pole of ψ of order at most 1. Hence the number of poles of ψ (counted with multiplicities) is at most m. Therefore the number of apparent singularities is at most

$$\begin{aligned}&\deg(\det(L^{-1}\otimes E)\otimes\kappa)+m\\ &= -2\deg(L)+\deg(E)+4(g-1)+m\\ &\leqq m+3g-3.\end{aligned}$$

Secondly we think about the case $m=0$. In the spin case, i.e.,

$$[\tilde{R}_g, \tilde{S}_g]\cdots[\tilde{R}_1, \tilde{S}_1] = \mathrm{Id} \in \mathrm{SL}(2, \boldsymbol{C}),$$

the above argument can be taken.

In the non-spin case, i.e.,

$$[\tilde{R}_g, \tilde{S}_g]\cdots[\tilde{R}_1, \tilde{S}_1] = -\mathrm{Id} \in \mathrm{SL}(2, \boldsymbol{C}),$$

we can regard P as a projective bundle associated not to a holomorphic vector bundle, but to some *orbibundle*†. Such a holomorphic orbibundle V over X is constructed as below.

Fix $p\in X$ and a sufficiently small local coordinate (U, z) around p. Let $\mathcal{V}^c(U)=$"$(\boldsymbol{Z}_2, \tilde{U})\to U$" be a ramified covering $\tilde{U}\ni w\mapsto z=w^2\in U$ such that the action of $\boldsymbol{Z}_2$ is $w\mapsto -w$. For $X-\{p\}$, $\mathcal{V}^c(X-\{p\})$ is "$(1, X-\{p\})\to X-\{p\}$" where 1 is the trivial group. Thus we get a orbifold structure $\mathcal{V}^c$ over X. Next let $\mathcal{B}(U)=(\boldsymbol{Z}_2, \tilde{\pi}_U : \tilde{U}\times\boldsymbol{C}^2\to\tilde{U})$ be a trivial vector bundle with $\boldsymbol{Z}_2$-action $(w, v)\mapsto(-w, -v)$. For $X-\{p\}$, $\mathcal{B}(X-\{p\})$ is $(1, \tilde{\pi}_{X-\{p\}} : \tilde{V}\to X-\{p\})$ where $\tilde{V}$ is the flat bundle associated to ρ. It is easy to see that P is just the projectification of the orbibundle V determined by $\mathcal{B}$.

If we find a holomorphic orbibundle W with rank 1 whose $\boldsymbol{Z}_2$-action on the fibre at p is non-trivial, and a nonzero holomorphic cross section φ of $\mathrm{Hom}(W, V)$, then we get a meromorphic projective connection induced by (W, φ). However $\mathrm{Hom}(W, V)$ is a holomorphic vector bundle ($\boldsymbol{Z}_2$-actions are canceled each other), so the situation is just the same as the spin case. Hence we have proved the following

THEOREM 6. *Let X be a compact Riemann surface of genus g and $p_1, \cdots, p_m$ distinct points in X. Assume that*

$$\rho : \pi_1(X-\{p_1, \cdots, p_m\}) \longrightarrow \mathrm{PSL}(2, \boldsymbol{C})$$

is an irreducible representation and semi-simple around each p_i. Then there exists a meromorphic projective connection satisfying the following properties.

† For the general definitions of orbifolds and orbibundles, see [**6**]. Here we use the terms *orbifold* and *orbibundle* instead of *V-manifold* and *V-bundle* of [**6**].

- *Its projective monodromy is isomorphic to* ρ.
- *The number of apparent singularities is at most* $m+3g-3$.

3.4. The monodromy preserving deformation.

In this subsection, we shall briefly review the monodromy preserving deformation, and examine the projective monodromy map by using Theorem 6.

Iwasaki considered the following situation (for details, see [4]). Choose $m\in \boldsymbol{N}$ such that $n=m+3g-3>0$. Let us introduce a total order $\succ$ into $\boldsymbol{C}$ by

$$\lambda \succ \mu \overset{\text{def}}{\Longleftrightarrow} \begin{cases} \text{either } \operatorname{Re}(\lambda) > \operatorname{Re}(\mu) \\ \text{or } \operatorname{Re}(\lambda) = \operatorname{Re}(\mu),\ \operatorname{Im}(\lambda) \geqq \operatorname{Im}(\mu). \end{cases}$$

We put

$$\boldsymbol{C}_+ = \{\theta \in \boldsymbol{C} \mid \theta \succ 0\}.$$

For a fixed $\theta=(\theta_1, \cdots, \theta_m)\in(\boldsymbol{C}_+-\boldsymbol{Z}_+)^m$, let $E(m\,;\theta)_{irr}$ be the set of all *irreducible* SL-operators with ordered $m+n$ regular singularities such that the characteristic exponents of the first m singularities are determined by θ and the last n singularities are apparent with multiplicity 1. Then $E(m\,;\theta)_{irr}$ becomes an analytic space. We denote by $R(m\,;\theta)_{irr}$ the set of irreducible representations (up to conjugacy) from $\pi_1(X-S)$ to $\mathrm{PSL}(2, \boldsymbol{C})$ whose *eigenvalues* around each punctured point are determined by θ, where S runs the set of ordered m distinct points of X. Then $R(m\,;\theta)_{irr}$ is a complex manifold. Iwasaki considered a certain nonsingular open subset $\mathcal{E}(m\,;\theta)_{irr}$ of $E(m\,;\theta)_{irr}$, and showed that the projective monodromy map

$$PM: \mathcal{E}(m\,;\theta)_{irr} \longrightarrow R(m\,;\theta)_{irr}$$

is locally biholomorphic. A canonical foliation is induced on $\mathcal{E}(m\,;\theta)_{irr}$ via this monodromy map PM and this foliation describes *the monodromy preserving deformation* on $\mathcal{E}(m\,;\theta)_{irr}$.

Together with Theorem 5.9 of [4, p. 494] which says that the image of *excited states* by PM is nowhere dense in $R(m\,;\theta)_{irr}$, our Theorem 6 gives an answer to Problem 5.11 of [4, p. 495].

COROLLARY 7. *The above monodromy map PM is a covering map if the base space is restricted to some Zariski dense open subset* $\tilde{R}(m\,;\theta)$ *of* $R(m\,;\theta)_{irr}$.

We know about this covering map further by using a similar argument to Kronheimer-Mrowka [7, § 9 (iii)].

PROPOSITION 8. *At generic points of* $\tilde{R}(m\,;\theta)$ *PM is a* 2^g*-fold covering.*

The next lemma about transversality indicates this proposition by the same way of [7, Lemma 9.15], and we shall only prove the lemma.

LEMMA 9. *For a generic point* ρ *in* $\tilde{R}(m;\theta)$, *any meromorphic projective connection* (P, s) *in* $PM^{-1}(\rho)$ *is induced by some pair of vector bundles* $L\subset E$ (*respectively pair of orbibundles* $W\subset V$) *just as in* § 3.3, *which satisfies*

(a) $\deg(L^{-1}\otimes E)=g-1$ (*respectively* $\deg(\mathrm{Hom}(V, W))=g-1$).

(b) $\dim \mathrm{H}^0(L^{-1}\otimes E)=1$ (*respectively* $\dim \mathrm{H}^0(\mathrm{Hom}(V, W))=1$).

(c) *The natural map*

$$\mathrm{H}^1(\mathcal{O}) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(L^{-1}\otimes E), \mathrm{H}^1(L^{-1}\otimes E))$$

$$(\textit{respectively}\quad \mathrm{H}^1(\mathcal{O}) \longrightarrow \mathrm{Hom}(\mathrm{H}^0(\mathrm{Hom}(V, W)), \mathrm{H}^1(\mathrm{Hom}(V, W))))$$

is onto.

This lemma corresponds to Lemma 9.16 of [**7**]. It dealt with stable bundles, while we consider flat SL(2, $\boldsymbol{C}$)-bundles.

PROOF. For simplicity we shall discuss vector bundle case only. In orbibundle case a similar argument can be taken.

(a) Firstly we assume that $m=0$. Here we denote $\tilde{R}(0;*)$ simply by $\tilde{R}$. Let $\{\rho_n\}$ be a following sequence in $\tilde{R}$.

- The sequence $\{\rho_n\}$ converges to ρ in Zariski topology of $\tilde{R}$.
- There is a meromorphic projective connection in $PM^{-1}(\rho_n)$ induced by $L_n\subset E_n$ with $\deg(L_n^{-1}\otimes E_n)=-2\deg(L_n)<g-1$ for each n.

Recall that, for an algebraic family of vector bundles, the set of points corresponding to stable bundles is Zariski open (see [**10**, Lemma 6]). Hence $\tilde{R}_s:=\{\tau\in\tilde{R}\,|\,\text{The vector bundle induced by } \tau \text{ is stable.}\}$ is Zariski open in $\tilde{R}$, and we may assume that ρ is in $\tilde{R}_s$ and that E_n's are stable. In particular we get

$$0<\deg(L_n^{-1}\otimes E_n)=-2\deg(L_n)<g-1.$$

Since the fibre of the map from flat SL(2, $\boldsymbol{C}$)-bundles to flat PSL(2, $\boldsymbol{C}$)-bundles is finite, we may assume that the sequence $\{E_n\}$ converges to some flat SL(2, $\boldsymbol{C}$)-bundle E in the fibre of ρ.

Then we get a sequence $\{(L_n, E_n)\}$ in

$$\mathcal{J}_{\ell<l<0}\times\{\text{a neighborhood of } E \text{ in the moduli of flat SL}(2, \boldsymbol{C})\text{-bundles}\},$$

where $\ell=(1/2)(1-g)$ and $\mathcal{J}_{\ell<l<0}$ is the disjoint union of Jacobi manifolds $\mathcal{J}_l$ for $\ell<l<0$. Since $\mathcal{J}_{\ell<l<0}$ is compact, we may assume that the sequence $\{(L_n, E_n)\}$ converges to (L, E). In particular the degree of L coincides with that of L_n for sufficiently large n, that is,

$$\deg(L^{-1}\otimes E)=-2\deg(L)<g-1.$$

Hence we have proved (a) in the case of $m=0$.

In the case of $m>0$, we can prove similarly using the concept of parabolic stable bundles (for parabolic bundles, see [8]). Here the parameter θ of $\tilde{R}(m;\theta)$ corresponds to the weights of parabolic bundles.

(b) is proved by using a similar argument to (a) and the proof of (c) is just the same as that of [7, Lemma 9.16(c)].

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