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Generalized #-unknotting operations

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Introduction.

We shall work in the P.L. and locally flat category. We discuss oriented knots and links in S^3. Two knots are equivalent if there is an ambient isotopy of S^3 carrying one knot to the other.

H. Murakami [6] showed that any knot can be changed into a trivial knot by repeatedly altering a diagram of the knot as in Figure 0.

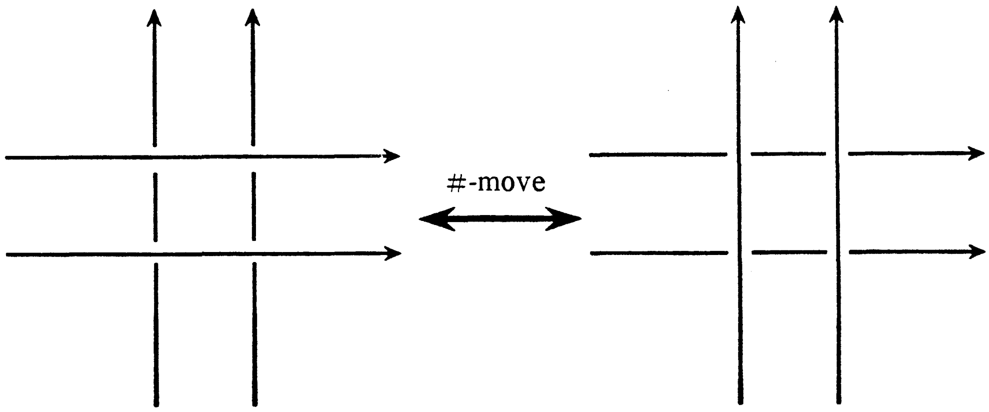


Figure 0.

This move on a diagram is called the #-move or the #-unknotting operation. In this note, generalizing this, we define for any prime p, a *$\#^p$-move* on a knot diagram as shown in Figure 1. Note that even if p is fixed, x and y in Figure 1 may vary. (It is easy to define $\#^p$-moves for any integers p. However, if p' is a factor of p, then a $\#^p$-move is also a $\#^{p'}$-move. We thus consider $\#^p$-moves only for prime numbers p.) The #-unknotting operation and the pass-move [4] are examples of $\#^2$-moves.

We shall show that for any prime p any knot can be transformed into a trivial knot by a finite sequence of $\#^p$-moves (Theorem 1.1). (In fact, if p is odd, a combination of a certain $\#^p$-move and Reidemeister moves achieves a crossing change.) Then we can define the $\#^p$-unknotting number $u^p(K)$ much like the ordinary unknotting number. Since a family of $\#^p$-moves is a wide variety of diagramatic changes, one might initially think that every knot can be untied

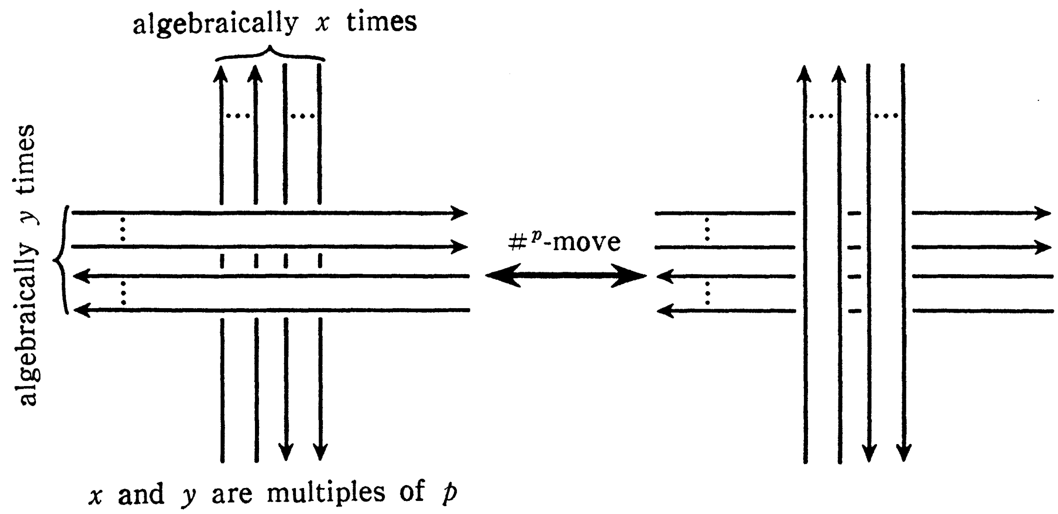


Figure 1.

by a single $\#^p$-move for some p and/or there might be an upper bound for the values of $\#^p$-unknotting numbers. However, we shall show that:

PROPOSITION 1.6. *Given n and p, there is a knot K such that $u^p(K) \geqq n$.*

PROPOSITION 2.7. *There is a knot K such that $u^p(K) > 1$ for any p.*

Let M be $S^2 \times S^2$ with a puncture. In § 2, $\#^p$-moves are related to certain disks properly embedded in M, and studied using results of 4-dimensional topology. As an application, in § 3, we consider whether every link in $\partial M \cong S^3$ bounds disjoint disks in M. It is already known that every knot bounds a disk in M (Norman [8], Suzuki [10]). We shall show that this does not hold for a 2-component link (Proposition 3.6). We only find an obstruction of links being slice in M for certain, not all, links.

PROBLEM. Find an obstruction for links to bound disjoint disks in $S^2 \times S^2$ with a puncture.

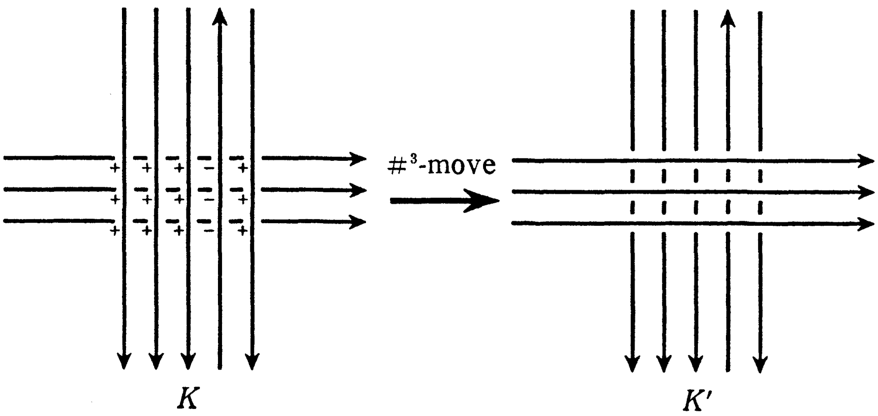
We summarize the notation used in this note. All manifolds will be assumed to be oriented. For a manifold M, $-M$ denotes M with the opposite orientation. If M^4 is a closed 4-manifold, puncM^4 denotes M^4 with an open 4-ball deleted; the orientation of $\partial(\text{punc}M^4)$ is the one induced from puncM^4. For a knot K in S^3, we write $\bar{K}$ for the knot $-K$ in $-S^3$. We write O for a trivial knot in S^3.

1. $\#^p$-Moves.

If a diagram of a knot K' is a result of one $\#^p$-move on a diagram of a knot K, then we write $K \overset{\#^p}{\to} K'$.

THEOREM 1.1. *For any prime p, a diagram of any knot can be transformed into a diagram of a trivial knot by a finite sequence of $\#^p$-moves.*

Before proving the theorem we define a function $\varphi_p(K, K')$ for $K \overset{\#^p}{\to} K'$. If a diagram of K' is obtained from that of K by a single $\#^p$-move, then we define $\varphi_p(K, K')$ to be the sum of signs of the changed crossings. See Figure 2. Note that $\varphi_p(K, K')$ does not depend on the orientation of K. However, $\varphi_p(K, K')$ seems to depend on a diagram of K and the $\#^p$-move to apply. Theorem 1.2 below says that $\varphi_p(K, K')$ depends only on p, K and K'. The proof will be given in § 2.



$\varphi_3(K, K')=9$

Figure 2.

THEOREM 1.2. *Suppose $K \overset{\#^p}{\to} K'$. Then for any $\#^p$-move transforming a diagram of K into that of K', $\varphi_p(K, K')$ takes the same value.*

COROLLARY 1.3. (1) *If $K \overset{\#^p}{\to} K'$, then $K' \overset{\#^p}{\to} K$ and $\varphi_p(K', K)=-\varphi_p(K, K')$.*
(2) *If K and K' are amphicheiral knots such that $K \overset{\#^p}{\to} K'$, then $\varphi_p(K, K')=0$.*

PROOF OF COROLLARY 1.3. We only prove (2). Let $\tilde{K}$ be a diagram of K which a single $\#^p$-move transforms into K'. Change all crossings of $\tilde{K}$ and the orientation; then the sign of each crossing changes and $\tilde{K}$ becomes a diagram of $\bar{K}$. It follows $\bar{K} \overset{\#^p}{\to} \bar{K}'$ with $\varphi_p(\bar{K}, \bar{K}')=-\varphi_p(K, K')$. Since K and K' are amphicheiral, the equality implies $\varphi_p(K, K')=0$. □

We now give the proof of Theorem 1.1.

PROOF OF THEOREM 1.1. If $p=2$, then $\#^2$-moves contain the #-unknotting operation in [**6**]. Thus a $\#^2$-move is an unknotting operation.

If p is odd, then Figure 3 demonstrates how a combination of a certain $\#^p$-move and isotopies achieves a crossing change. □

Given two knots K, K', define the *$\#^p$-Gordian distance* $d_G^p(K, K')$ to be the minimum number of $\#^p$-moves which can transform a diagram of K to that of K'. Given a knot K, define the *$\#^p$-unknotting number* $u^p(K)$ to be $d_G^p(K, O)$. The proof of Theorem 1.1 then implies the following.

COROLLARY 1.4. *If p is an odd prime, then $d_G(K, K') \geqq d_G^p(K, K')$ where d_G is the Gordian distance defined in* [**6**]. *In particular, $u(K) \geqq u^p(K)$ where $u(K)$ is the ordinary unknotting number of K.*

EXAMPLE 1. By Corollary 1.4 the $\#^p$-unknotting number of the figure eight knot 4_1 is 1 if $p>2$. On the other hand, if $p=2$, Figure 4 describes a sequence $4_1 \xrightarrow{\#^2} \overline{3_1} \xrightarrow{\#^2} O$ where $\overline{3_1}$ is the right handed trefoil. Hence $u^2(4_1) \leqq 2$. We also see that $\varphi_2(4_1, \overline{3_1})=0$ and $\varphi_2(\overline{3_1}, O)=4$. In § 2 we shall see that $u^2(4_1)=2$.

EXAMPLE 2. Let $T(p, q)$ be the (p, q) torus knot. Since a $2n$-full twist of p parallel strings can be realized by a single $\#^p$-move (Figure 5), $T(p, 2np \pm 1) \xrightarrow{\#^p} T(p, \pm 1) \cong O$. Thus $u^p(T(p, 2np \pm 1))=1$ for any n, where $\varphi_p(T(p, 2np \pm 1), O) =2np$.

It is a standard technique to find lower bounds of unknotting numbers in terms of the minimum number of generators of the first homology group of a covering space [**13**], [**7**]. In this direction Nakanishi pointed out the following estimates.

PROPOSITION 1.5. *Let X_p be the p-fold cyclic branched covering of S^3 along a knot K. Let $e_p(K)$ be the minimum number of generators of $H_1(X_p)$.*

$$\text{Then} \quad d_G^p(K, K') \geqq \frac{|e_p(K)-e_p(K')|}{3p},$$

$$u^p(K) \geqq \frac{e_p(K)}{3p}.$$

PROOF. First note that a $\#^p$-move is realized by three surgeries as shown in Figure 6. The linking number of each surgery circle and the knot is a multiple of p. Hence the preimage of each surgery circle in X_p has p components. In general, a single Dehn surgery changes the minimum number of generators of the first homology group of an ambient manifold by at most one

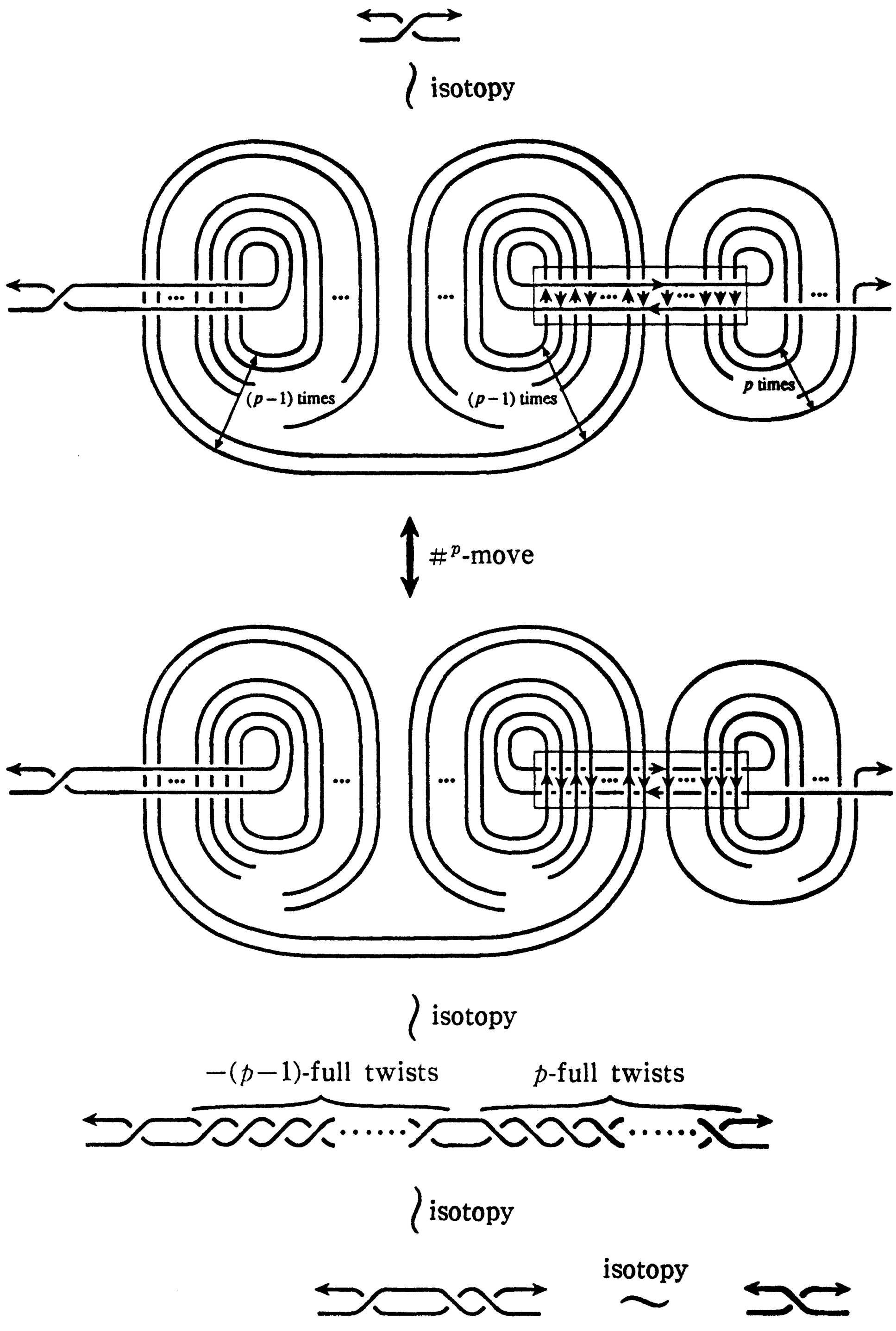

isotopy
(p−1) times
(p−1) times
p times
#^p-move
isotopy
−(p−1)-full twists
p-full twists
isotopy
isotopy

Figure 3.

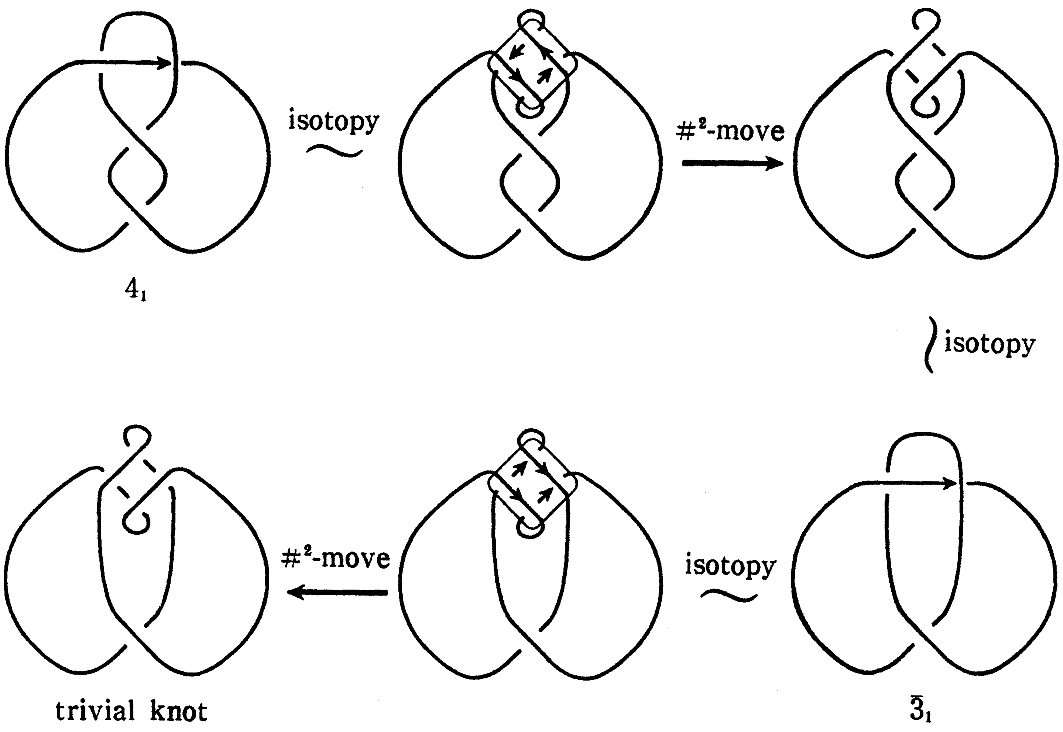


Figure 4.

[3, Lemma 3]. Thus, if $K \overset{\#^p}{\to} K'$, then $|e_p(K)-e_p(K')| \leqq 3p$. The proposition easily follows. □

The estimates in Proposition 1.5 will be far from best possible, but are enough to prove:

PROPOSITION 1.6. *For any n and prime p, there is a knot whose $\#^p$-unknotting number is greater than or equal to n.*

PROOF. By Proposition 1.5 it suffices to prove that for given p and n there is a knot K such that $e_p(K) \geqq 3pn$. The figure eight knot 4_1 has a Seifert matrix $S=\begin{pmatrix} -1 & 0 \\ 1 & 1 \end{pmatrix}$; for example see [1, p. 320]. Since $\det S=-1$, $M_m=I-(S^T S^{-1})^m$ is a presentation matrix for the first homology group of the m-fold cyclic branched covering along 4_1. Hence, if $\det M_m \neq \pm 1$, then $e_m(4_1) \geqq 1$. A calculation shows that $\det M_m=2-(\alpha^m+\beta^m)$, where $\alpha=(3+\sqrt{5})/2$, $\beta=(3-\sqrt{5})/2$ are the eigenvalues of $S^T S^{-1}=\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Since $\alpha>2$, $2-(\alpha^m+\beta^m) \neq \pm 1$ for $m \geqq 2$. Thus, $e_m(4_1) \geqq 1$ for $m \geqq 2$, and so $e_p(\#^{3pn} 4_1) \geqq 3pn$ for any prime p and n. □

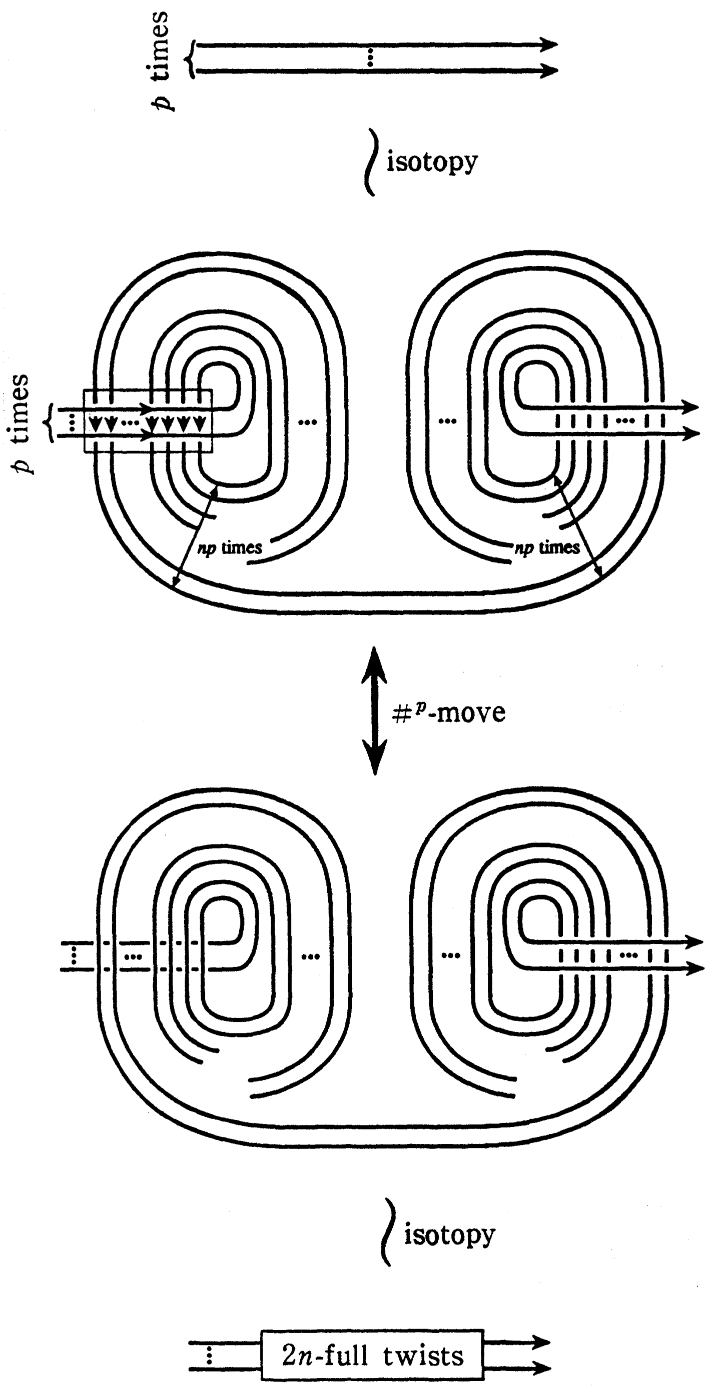

p times
isotopy
p times
np times
np times
#^p-move
isotopy
2n-full twists

Figure 5.

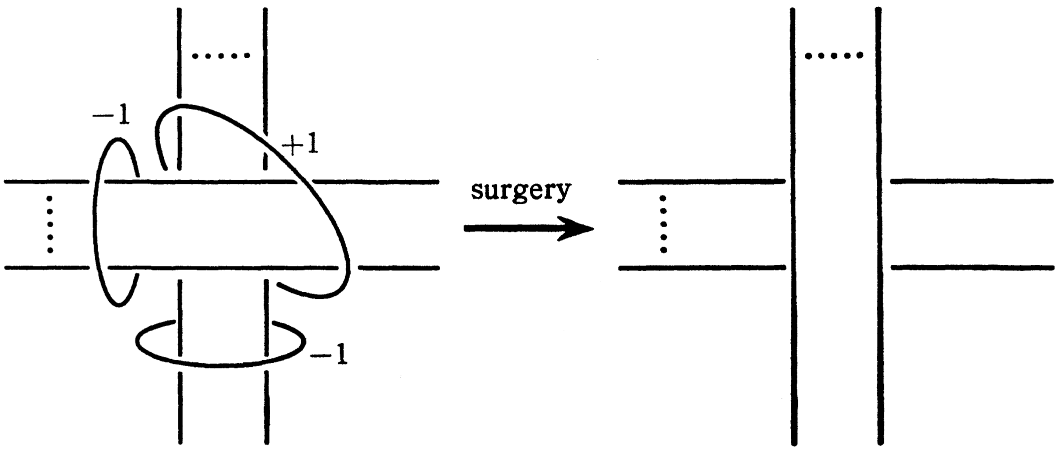


Figure 6.

2. $\#^p$-moves from a 4-dimensional point of view.

In this section, we show that $\varphi_p(K, K')$ in § 1 is well-defined and study its properties via 4-dimensional topology. As shown below, $\varphi_p(K, K')$ approximates $\sigma_p(K')-\sigma_p(K)$, where σ_p is Tristram's p-signature [11].

PROPOSITION 2.1. *If* $K\xrightarrow{\#^p}K'$, *then the following hold.*

(1) $\left|\frac{4}{p^2}\left[\frac{p}{2}\right]\left(p-\left[\frac{p}{2}\right]\right)\varphi_p(K, K')+\sigma_p(K)-\sigma_p(K')\right|\leqq 2,$

where $[x]$ *is the greatest integer not exceeding* x.

(2) $\frac{1}{4}\varphi_2(K, K')\equiv \mathrm{Arf}(K)+\mathrm{Arf}(K') \bmod 2.$

Note that the coefficient of φ_p in the inequality of (1) above equals 1 if $p=2$, $(p^2-1)/p^2$ if $p>2$.

REMARK 1. If a knot K' is obtained from a knot K by a $\#$-unknotting operation [6], then we have $K\xrightarrow{\#^p}K'$ with $\varphi_2(K, K')=\pm 4$. It follows from Proposition 2.1(1) that $\sigma_2(K')-\sigma_2(K)=-2, -4, -6$ if $\varphi_2(K, K')=-4$, and $\sigma_2(K')-\sigma_2(K)=2, 4, 6$ if $\varphi_2(K, K')=4$. This recovers [**6**, Theorem 3.2], which is proved by using a Goeritz matrix.

REMARK 2. Recall that Tristram's p-signature $\sigma_p(K)$ is the signature of the Hermitian matrix $V(\xi)=(1-\xi)M+(1-\bar{\xi})M^T$ where M is a Seifert matrix of a knot K and $\xi=\exp([p/2]2\pi i/p)$. Note that $2\pi/3\leqq[p/2]2\pi/p\leqq\pi$. The matrix $V(\xi)$ is singular if and only if ξ is a root of the Alexander polynomial $\Delta(t)$ of K. The signature of $V(z)$ for $z\in S^1$ is continuous at $z=z_0$ if $V(z_0)$ is a non-singular matrix. Thus, if the arguments of the roots of $\Delta(t)$ do not lie in

$[2\pi/3, \pi]$, then Tristram's p-signatures of K do not depend on p.

As an application of Proposition 2.1 we show:

PROPOSITION 2.2. $u^2(4_1)=2$.

PROOF. We know that $u^2(4_1)\leqq 2$ (Example 1 in §1). Assume for a contradiction that $4_1 \overset{\#^2}{\to} O$. Since 4_1 is amphicheiral, Corollary 1.3(2) implies $\varphi_2(4_1, O)=0$. Then, applying Proposition 2.1(2) gives $\mathrm{Arf}(4_1)=0$, a contradiction. □

LEMMA 2.3. *If $K \overset{\#^p}{\to} K'$, then there exists a properly embedded 2-disk Δ in $M=\mathrm{punc}(S^2\times S^2)$ such that*

(1) *$\partial\Delta\subset\partial M$ is $\bar{K}\# K'$,*

(2) *$[\Delta]\in H_2(M, \partial M)$ is divisible by p, and*

(3) *the intersection number $[\Delta]\cdot[\Delta]$ equals $2\varphi_p(K, K')$.*

PROOF. Suppose that $\bar{K}\# K$ is in the boundary of a 4-ball D^4. Note that $\bar{K}\# K \overset{\#^p}{\to} \bar{K}\# K'$ and that $\bar{K}\# K$ bounds a 2-disk Δ in D^4. Figure 7 shows that doing 0-surgeries along l_1 and l_2 have the same effect on $\bar{K}\# K$ as the $\#^p$-move. Attach 2-handles h_1^2 and h_2^2 to D^4 with framings 0 along l_1 and l_2 respectively. Then $M=D^4\cup h_1^2\cup h_2^2$ is homeomorphic to $\mathrm{punc}(S^2\times S^2)$ and $(\partial M, \partial\Delta)\cong(S^3, \bar{K}\# K')$. Orient l_1, l_2 so that $\mathrm{lk}(l_1, l_2)=1$, and set $x=\mathrm{lk}(l_1, \bar{K}\# K)$ and $y=\mathrm{lk}(l_2, \bar{K}\# K)$. Then Δ represents $x\alpha+y\beta\in H_2(M, \partial M)$ where α, $\beta\in H_2(M, \partial M)$ are represented by the cocores of h_1^2, h_2^2, respectively. It follows that $[\Delta]\cdot[\Delta]=2xy=2\varphi_p(K, K')$. By the definition of $\#^p$-moves x and y are multiples of p, thus $[\Delta]$ is divisible by p. □

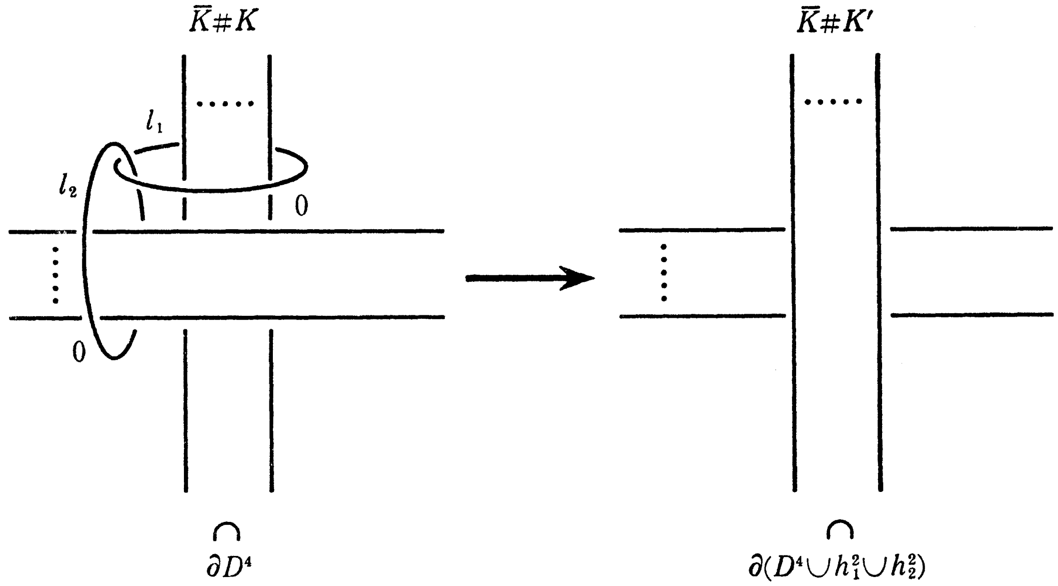


Figure 7.

Lemma 2.3 relates a $\#^p$-move to an embedded disk in punc$(S^2\times S^2)$. Then Theorem 1.2 and Proposition 2.1 follow from the theorems in 4-dimensional topology, Theorems 2.4 and 2.5 below. Theorem 2.4 is originally due to Viro [**12**]. It is also obtained by letting $d=p$ and $a=[p/2]$ in the inequality of Gilmer [**2**, Remarks (a) on p. 371]. Theorem 2.5 is really Robertello's definition of the Arf invariant [**9**].

THEOREM 2.4. *Let M be a compact, oriented 4-manifold with $\partial M\cong\emptyset$ or $\cong S^3$, and F a properly embedded, oriented surface in M with $\partial F\cong\emptyset$ or $\cong S^1$. If $[F]\in H_2(M, \partial M; Z)$ is divisible by a prime integer p, then we have*

$$\left|\frac{2}{p^2}\left[\frac{p}{2}\right]\left(p-\left[\frac{p}{2}\right]\right)[F]\cdot[F]-\sigma_p(\partial F)-\sigma(M)\right| \leqq \dim H_2(M; Z_p)+2\,\mathrm{genus}(F)\,.$$

THEOREM 2.5. *Let M and F be as in Theorem 2.4. If* genus$(F)=0$ *and F represents a characteristic element of $H_2(M, \partial M)$, then the following holds.*

$$\frac{[F]\cdot[F]-\sigma(M)}{8}\equiv \mathrm{Arf}(\partial F) \bmod 2\,,$$

where Arf$(\partial F)=0$ *for* $\partial F=\emptyset$.

PROOF OF THEOREM 1.2. Suppose that $\varphi_p(K, K')$ takes two values n_1 and n_2. By Lemma 2.3, for each n_i $(i=1, 2)$, there is a properly embedded 2-disk Δ_i in $M_i=$punc$(S^2\times S^2)$ such that: (1) $\partial\Delta_i\subset\partial M_i$ is $\bar{K}\# K'$, (2) $[\Delta_i]$ is divisible by p, and (3) $[\Delta_i]\cdot[\Delta_i]=2n_i$.

Set $M=M_1\cup_f(-M_2)$, $\Sigma=\Delta_1\cup_f(-\Delta_2)$, where f is an orientation reversing diffeomorphism from $(\partial M_1, \partial\Delta_1)$ to $(-\partial M_2, -\partial\Delta_2)$. Then $M\cong\#^2(S^2\times S^2)$, $\Sigma\cong S^2$, $[\Sigma]\in H_2(M, \partial M)$ is divisible by p and $[\Sigma]\cdot[\Sigma]=2(n_1-n_2)$.

If $p=2$, then Theorems 2.4, 2.5 give

$$|n_1-n_2|=\left|\frac{[\Sigma]\cdot[\Sigma]}{2}\right|\leqq 4,$$

and

$$\frac{(n_1-n_2)}{4}=\frac{[\Sigma]\cdot[\Sigma]}{8}\equiv 0 \bmod 2\,.$$

This implies $n_1=n_2$.

Suppose p is an odd prime. By Theorem 2.4,

$$\left|\frac{(n_1-n_2)(p^2-1)}{p^2}\right|=\left|\frac{[\Sigma]\cdot[\Sigma](p^2-1)}{2p^2}\right|\leqq 4\,.$$

By the definition of a $\#^p$-move, both n_1 and n_2 are multiples of p^2. If $n_1\neq n_2$, then

$$p^2-1 \leqq \left|\frac{(n_1-n_2)(p^2-1)}{p^2}\right| \leqq 4.$$

This contradicts $p>2$. It follows $n_1=n_2$. □

PROOF OF PROPOSITION 2.1. By Lemma 2.3 and Theorem 2.4, we have

$$\left|\frac{4}{p^2}\left[\frac{p}{2}\right]\left(p-\left[\frac{p}{2}\right]\right)\varphi_p(K, K')-\sigma_p(\overline{K}\# K')\right| \leqq 2.$$

Since $\sigma_p(\overline{K}\# K')=-\sigma_p(K)+\sigma_p(K')$ for any prime integer p, we have (1) of the proposition. Proposition 2.1(2) follows from Lemma 2.3 and Theorem 2.5. We omit the detail. □

COROLLARY 2.6. *If there is a 'triangle' sequence of $\#^p$-moves $K_0 \overset{\#^p}{\to} K_1 \overset{\#^p}{\to} K_2 \overset{\#^p}{\to} K_0$, then $\varphi_p(K_0, K_1)+\varphi_p(K_1, K_2)+\varphi_p(K_2, K_0)=0$.*

PROOF. For simplicity, set $\alpha=(4/p^2)[p/2](p-[p/2])$, $K_3=K_0$, and $x=\sum_{i=1}^{3}\varphi_p(K_{i-1}, K_i)$. Apply Proposition 2.1(1) to sequences $K_{i-1} \overset{\#^p}{\to} K_i$ and add those three inequalities; then

$$\sum_{i=1}^{3} |\alpha\varphi_p(K_{i-1}, K_i)+\sigma_p(K_{i-1})-\sigma_p(K_i)| \leqq 6.$$

Hence $|\alpha x+\sigma_p(K_0)-\sigma_p(K_3)|\leqq 6$, so that $|x|\leqq 6/\alpha$. If $p>2$, $6/\alpha=6p^2/(p^2-1) \leqq 27/4$; otherwise, $6/\alpha=6$. Since φ_p and thus x are multiples of p^2, it follows $x=0$ for $p>2$ as desired.

If $p=2$, then $x=0, \pm 4$. On the other hand, adding the three equalities obtained by applying Proposition 2.1(2) to $K_{i-1} \overset{\#^2}{\to} K_i$ $(1\leqq i\leqq 3)$, we obtain $x/4\equiv \mathrm{Arf}(K_0)+\mathrm{Arf}(K_3)\equiv 0 \bmod 2$. Hence, $x=0$. □

REMARK. Corollary 2.6 does not necessarily hold for an 'n-gon' sequence of $\#^p$-moves if $n>3$. By Example 1 in §1 there is a sequence $4_1 \overset{\#^2}{\to} \overline{3_1} \overset{\#^2}{\to} O$ such that $\varphi_2(4_1, \overline{3_1})=0$ and $\varphi_2(\overline{3_1}, O)=4$. By the amphicheirality of 4_1, changing all the crossings in Figure 4 yields a sequence $4_1 \overset{\#^2}{\to} 3_1 \overset{\#^2}{\to} O$ such that $\varphi_2(4_1, 3_1)=0$, $\varphi_2(3_1, O)=-4$, where 3_1 is the left handed trefoil. We thus obtain a '4-gon' sequence $4_1 \overset{\#^2}{\to} \overline{3_1} \overset{\#^2}{\to} O \overset{\#^2}{\to} 3_1 \overset{\#^2}{\to} 4_1$ such that $\varphi_2(4_1, \overline{3_1})+\varphi_2(\overline{3_1}, O)+\varphi_2(O, 3_1)+\varphi_2(3_1, 4_1)=8\neq 0$. □

PROPOSITION 2.7. *There is a knot K such that* $\min_p u_p(K)=2$.

PROOF. We show that $5_2\# 5_2$ is the desired knot. Set $K=5_2$. Figure 8 shows that $O \overset{\#^p}{\to} K$ with $\varphi_p(O, K)=0$ for any p. This extends to a sequence

$O \xrightarrow{\#^p} K \xrightarrow{\#^p} K \# K$ with $\varphi_p(O, K)=\varphi_p(K, K\# K)=0$. Suppose $K\# K \xrightarrow{\#^p} O$ for some p. Then, by those sequences and Corollary 2.6 we have $\varphi_p(K\# K, O)=-\varphi_p(O, K)-\varphi_p(K, K\# K)=0$. Proposition 2.1(1) then implies $|\sigma_p(K\# K)|\leqq 2$. Since $\sigma_p(K)$ is even, $\sigma_p(K)=0$ for some p. This is absurd because $\sigma_p(K)=2$ for any prime p as proved below. It is known that $\sigma_2(K)=2$ [**1**, p. 312], so it suffices to see $\sigma_p(K)=\sigma_2(K)$. Now the roots of the Alexander polynomial $2t^2-3t+2$ of 5_2 are $e^{i\theta}$ where $\cos\theta=3/4$, so $\theta\notin[2\pi/3, \pi]$. Hence, by Remark 2 after Proposition 2.1, $\sigma_p(5_2)=\sigma_2(5_2)=2$. □

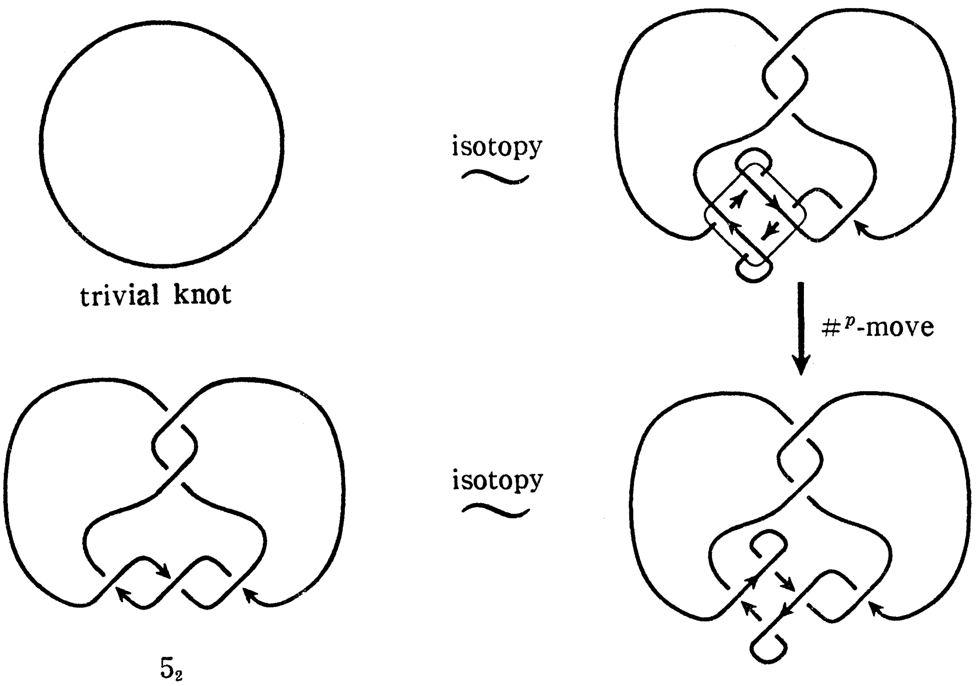


Figure 8.

3. Non-slice links in punc($S^2\times S^2$).

To construct a non-slice link in punc($S^2\times S^2$), we first define a $\#^2$-move for knot concordance classes. The definition is based on the 4-dimensional properties of $\#^2$-moves stated in Lemma 2.3.

Definition 3.1. Let C, C' be knot concordance classes. We write $C \xrightarrow{\#^2} C'$ if there are a properly embedded disk $\Delta\subset$punc($S^2\times S^2$) and knots $K\in C$, $K'\in C'$ satisfying the following:

(1) $\partial\Delta\subset\partial M$ is a knot $\bar{K}\# K'$

(2) $[\Delta]\in H_2(M, \partial M)$ is divisible by 2, i.e., characteristic.

Definition 3.2. Let C, C' be knot concordance classes. If $C \xrightarrow{\#^2} C'$, then define $\varphi(C, C')$ to be a half of the intersection number $[\Delta]\cdot[\Delta]$ where Δ is the disk in Definition 3.1.

REMARKS. (1) If $K \overset{\#^2}{\to} K'$ for knots K, K', then Lemma 2.3 implies that $[K] \overset{\#^2}{\to} [K']$ and $\varphi([K], [K'])=\varphi_2(K, K')$ where $[*]$ denotes knot concordance class.

(2) Suppose $C \overset{\#^2}{\to} C'$ for some knot concordance classes C, C'; then for any knots $K \in C$, $K' \in C'$ there is a disk Δ in punc$(S^2 \times S^2)$ satisfying (1) and (2) in Definition 3.1.

The disk Δ in Definition 3.1 satisfies conditions (1), (2) of Lemma 2.3. Therefore the proofs of Theorem 1.2 and Proposition 2.1 readily imply the following results on a $\#^2$-move of concordance classes.

PROPOSITION 3.3. *Let C, C' be knot concordance classes. If $C \overset{\#^2}{\to} C'$, then $\varphi(C, C')$ does not depend on the choice of a disk Δ and representatives of C, C'.*

PROPOSITION 3.4. *Let C, C' be knot concordance classes, and knots K, K' their representatives, respectively. If $C \overset{\#^2}{\to} C'$, then*

(1) $|\varphi(C, C')+\sigma_2(K)-\sigma_2(K')| \leqq 2$,

(2) $\dfrac{1}{4}\varphi(C, C') \equiv \mathrm{Arf}(K)+\mathrm{Arf}(K') \bmod 2$.

In § 2 it is shown that the figure eight knot 4_1 cannot be untied by a single $\#^2$-move (Proposition 2.2). Here we show that $[4_1] \overset{\#^2}{\to} [O]$ is impossible for knot concordance classes. In other words, the following holds.

PROPOSITION 3.5. *The figure eight knot does not bound a disk in* punc$(S^2 \times S^2)$ *representing a characteristic element.*

PROOF. If the figure eight knot 4_1 bounded a disk in punc$(S^2 \times S^2)$ representing a characteristic element, then $[4_1] \overset{\#^2}{\to} [O]$. Reversing the orientation of punc$(S^2 \times S^2)$, we obtain $[\overline{4_1}] \overset{\#^2}{\to} [\bar{O}]$ with $\varphi([4_1], [O])=-\varphi([\overline{4_1}], [\bar{O}])$. Since 4_1 and O are amphicheiral, $\varphi([4_1], [O])=0$. It then follows from Proposition 3.4(2) that $\mathrm{Arf}(4_1)=0$, which is absurd. □

PROPOSITION 3.6. *There is a 2-component link in* ∂(punc$(S^2 \times S^2)$) *which does not bound disjoint disks in* punc$(S^2 \times S^2)$.

The rest of this section is devoted to proving this proposition. We define a band sum of a link as follows. Let L be a link in S^3, and $f: I \times I \to S^3$ an embedding such that $f(I \times I) \cap L = f(\partial I \times I)$. We assume that if L is oriented, $f(I \times I)$ and L induce the opposite orientations to $L \cap f(I \times I)$. Then the link $L \cup f(I \times I) - f(I \times \mathrm{int}\, I)$ is said to be the *band sum* of L along the band $f(I \times I)$.

LEMMA 3.7. *Let $L=K_1\cup K_2$ be a 2-component link with* $\mathrm{lk}(K_1, K_2)$ *even. Let K_3 be the band sum of L via arbitrary band connecting K_1 and K_2. If none of K_i bounds a disk in* $\mathrm{punc}(S^2\times S^2)$ *representing a characteristic element, then L cannot bound two disjoint disks in* $\mathrm{punc}(S^2\times S^2)$.

PROOF. Suppose for a contradiction that L bounds disjoint disks D_1, D_2 in $M=\mathrm{punc}(S^2\times S^2)$. Let α and β be generators of $H_2(M, \partial M)$, and set $[D_i]=x_i\alpha+y_i\beta$, $i=1, 2$. Then K_3 bounds a 2-disk D_3 in M representing $[D_3]=[D_1]+[D_2]=(x_1+x_2)\alpha+(y_1+y_2)\beta$. Since $D_1\cap D_2=\emptyset$, $\mathrm{lk}(K_1, K_2)=[D_1]\cdot[D_2]=x_1y_2+x_2y_1$ is even. Then $x_1y_2\equiv x_2y_1\equiv 1 \bmod 2$ or $x_1y_2\equiv x_2y_1\equiv 0$. The former implies x_i, y_i are all odd, and hence $[D_3]$ is characteristic, a contradiction. Suppose the latter holds. Without loss of generality $x_1\equiv 0 \bmod 2$. Since $[D_1]$, $[D_2]$ are not characteristic, it follows that $y_1\equiv 1$, $x_2\equiv 0$, and $y_2\equiv 1$. However, this implies $[D_3]$ is characteristic, a contradiction. □

To construct such a link as in Lemma 3.7, we use a result from the theory of spatial theta curves. A *labelled theta curve* is a graph θ with two vertices labelled v_1, v_2, and three edges labelled 1, 2, 3. A *spatial theta curve* is the image of an embedding of a labelled theta curve into S^3. The *i-th constituent knot* of a spatial theta curve is the union of the two edges labelled j and k where $\{i, j, k\}=\{1, 2, 3\}$. As for the representability of constituent knots, Kinoshita [**5**] proved:

THEOREM 3.8. *Given knots K_1, K_2, K_3, there is a spatial theta curve whose three constituent knots are equivalent to K_1, K_2, K_3.*

See the Appendix for a concise proof using a canonical diagram of knots.

PROOF OF PROPOSITION 3.6. Using Theorem 3.8, take a spatial theta curve, G, such that each constituent knot is equivalent to the figure eight knot. Let K be one of the constituent knots of G, and e the edge not contained in K. Take a band B in S^3 which connects K to itself and its centerline is the edge e. Then the band sum of K along B is a 2-component link, say $K_1\cup K_2$ with $K_i\cong 4_1$. By twisting the band B, if necessary, we may assume that the linking number of K_1 and K_2 is even. Since $B\cap K_i$ is an arc for $i=1, 2$, we can regard the disk B as a band connecting K_1 and K_2. Then, the band sum of the link $K_1\cup K_2$ along B becomes K. Since the figure eight knot does not bound a disk in $\mathrm{punc}(S^2\times S^2)$ representing a characteristic element, the link $K_1\cup K_2$ satisfies the hypothesis in Lemma 3.7. Therefore, by Lemma 3.7 this is a non-slice link in $\mathrm{punc}(S^2\times S^2)$. □

Appendix. Proof of Theorem 3.8.

We first show the claim below by using a canonical diagram of knots due to Suzuki, Terasaka, and Yamamoto.

CLAIM. *Given a knot K, there is a spatial theta curve such that one of its constituent knots is equivalent to K and the other two are trivial knots.*

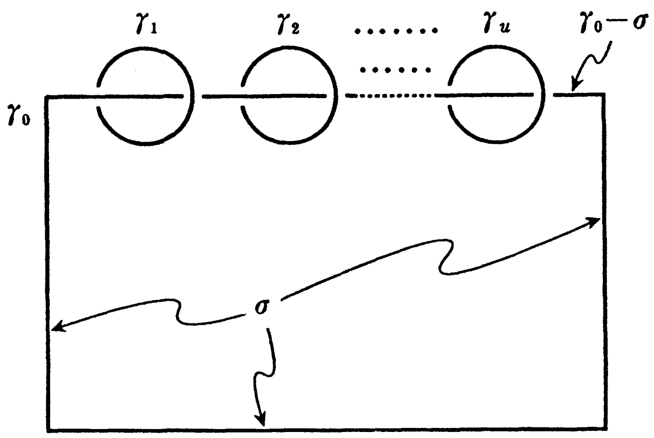
PROOF OF CLAIM. Let $L=\gamma_0\cup\gamma_1\cup\cdots\cup\gamma_u$ be the link in the diagram of Figure 9, where $u=u(K)$, and let σ be the union of the left, right and lower sides of the rectangle γ_0. Let $\Delta_1, \cdots, \Delta_u$ be mutually disjoint disks in S^3 such that $\partial\Delta_i=\gamma_i$ and $\Delta_i\cap\gamma_0$ is a single point off σ for all i. Suzuki [**10**] showed that the knot K can be expressed as a band sum of L along mutually disjoint u bands $B_1, \cdots, B_u$ with the following properties (1) (2):

(1) B_i connects γ_i and σ for $i=1, \cdots, u$,

(2) $B_i\cap\operatorname{int}\Delta_j=\emptyset$ for all i, j.

Moreover, Yamamoto [**15**] improved these in such a way that

(3) when γ_0 is counterclockwise oriented, the u subarcs $B_1\cap\sigma, B_2\cap\sigma, \cdots, B_u\cap\sigma$ are located on σ in this order.



L

Figure 9.

This diagram is said to be a *canonical diagram* K. An example is Figure 10(a).

Now attach an edge, e, to this diagram, say $\tilde{K}$, of K so that $e\cap\tilde{K}=\partial e=\partial\sigma$, $e\cap(\Delta_i\cup B_i)=\emptyset$ for all i, and the spatial theta curve $\gamma_0\cup e$ lies on some plane after an ambient isotopy. (Cf. Figure 10(b).) Then, the constituent knots of the spatial theta curve $\tilde{K}\cup e$ are $e\cup(\gamma_0-\sigma)$, $e\cup(\tilde{K}-(\gamma_0-\sigma))$, $\tilde{K}$; the knot types are O, O, K respectively. Hence, $\tilde{K}\cup e$ is the desired theta curve in Claim. □

Let K_k $(1\leqq k\leqq 3)$ be arbitrary knots. By Claim, for $1\leqq i<j\leqq 3$ there is a spatial theta curve $f_{ij}: \theta\to S^3$ such that its k-th constituent knot is equivalent

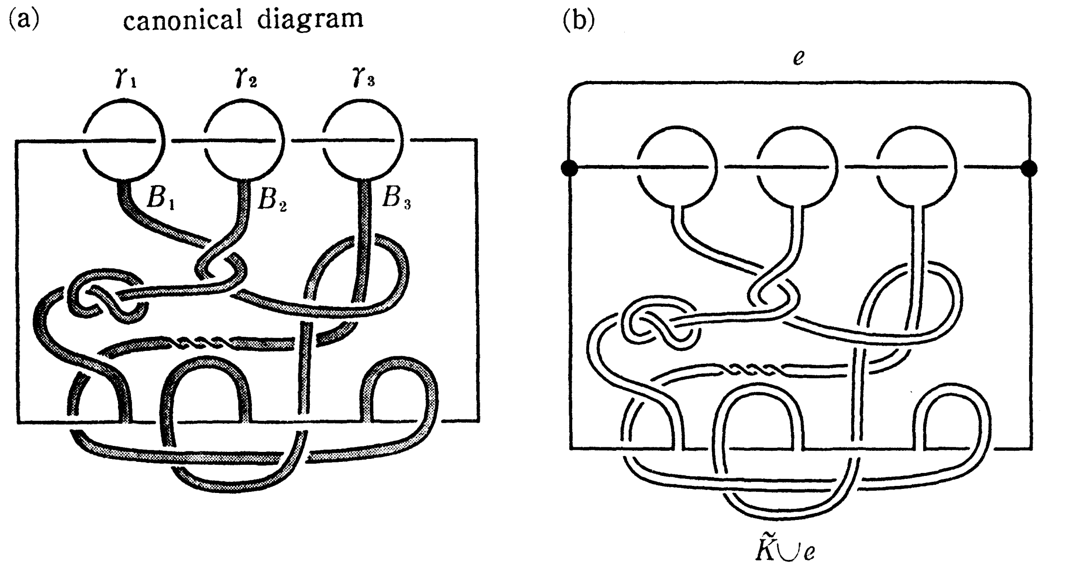


Figure 10.

to O if $k\in\{i, j\}$, and K_k otherwise. Take the vertex connected sum of the three spatial theta curves $f_{12}(\theta)$, $f_{23}(\theta)$, $f_{13}(\theta)$ (Figure 11). (For the definition of a vertex connected sum refer to [**14**].)

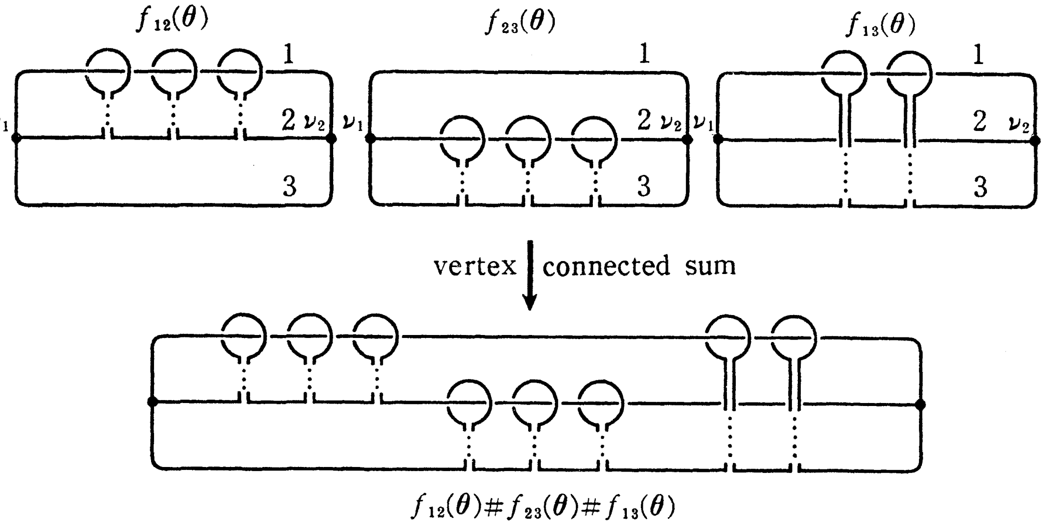


Figure 11.

Then the first constituent knot of the resulting theta curve is $O\# K_1\# O\cong K_1$; the second one $O\# O\# K_2\cong K_2$; the third one $K_3\# O\# O\cong K_3$. Therefore, $f_{12}(\theta)\# f_{23}(\theta)\# f_{13}(\theta)$ is the desired spatial theta curve. □

References

[1] G. Burde and H. Zieschang, Knots, de Gruyter Studies in Math., vol. 5, Walter de Gruyter, Berlin, 1985.

[2] P.M. Gilmer, Configurations of surfaces in 4-manifolds, Trans. Amer. Math. Soc., **264** (1981), 353-380.

[3] J. Hoste, Y. Nakanishi and K. Taniyama, Unknotting operations involving trivial tangles, Osaka J. Math., **27** (1990), 555-566.

[4] L. Kauffman, On knots, Ann. of Math. Stud., vol. **115**, Princeton Univ. Press, Princeton, New Jersey, 1987.

[5] S. Kinoshita, On θ_n-curves in R^3 and their constituent knots, Topology and Computer Science, (ed. S. Suzuki), Kinokuniya, Tokyo, Japan, 1987, pp. 211-216.

[6] H. Murakami, Some metrics on classical knots, Math. Ann., **270** (1985), 35-45.

[7] Y. Nakanishi, A note on unknotting number, Math. Sem. Notes, Kobe Univ., **9** (1981), 99-108.

[8] R.A. Norman, Dehn's Lemma for certain 4-manifolds, Invent. Math., **7** (1969), 143-147.

[9] R. Robertello, An Arf invariant of knot cobordism, Comm. Pure Appl. Math., **18** (1965), 543-555.

[10] S. Suzuki, Local knots of 2-spheres in 4-manifolds, Proc. Japan Acad., **45** (1969), 34-38.

[11] A.G. Tristram, Some cobordism invariants for links, Proc. Cambridge Philos. Soc., **66** (1969), 251-264.

[12] O. Ya Viro, Link types in codimension-2 with boundary, Uspekhi Mat. Nauk, **30** (1970), 231-232, (Russian).

[13] H. Wendt, Die gordische Auflösung von Knoten, Math. Z., **42** (1937), 680-696.

[14] K. Wolcott, The knotting of theta curves and other graphs in S^3, Geometry and Topology, (eds. C. McCrory and T. Shifrin), Marcel Dekker, New York, 1987, pp. 325-346.

[15] M. Yamamoto, Knots in spatial embeddings of the complete graph on four vertices, Topology Appl., **36** (1990), 291-298.

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