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Polarized threefolds with non-zero effective anti-adjoint divisors

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Introduction.

In this paper we study the structure of a polarized threefold (Y, L) such that Y is a normal projective threefold with at worst $\boldsymbol{Q}$-factorial terminal singularities and that its anti-adjoint divisor $-(K_Y+L)$ is linearly equivalent to a non-zero effective Weil divisor. For a polarized variety, the property of its adjoint divisor has been studied by many algebraic geometers (see e.g., T. Fujita [**F**] Chapter II). In principle, the adjoint divisor has positivity properties (ampleness, spannedness, nefness and so on) except a few cases in which (Y, L) has some special structure (see e.g., loc. cit.). Our assumption means that the adjoint bundle is far from being positive, so we expect that the structure of such polarized threeholds can be investigated well.

The other motivation is to study non-normal Gorenstein Fano threefolds. M. Reid classified non-normal Gorenstein Del Pezzo surfaces in [**R**] via the minimal resolutions of their normalizations. Let X be a non-normal Gorenstein Del Pezzo surface, Y the normalization of X and L the ample Cartier divisor corresponding to the inverse image of the anti-dualizing sheaf ω_X^{-1}. By using the adjunction formula he showed that there exists a non-zero effective Weil divisor Δ such that $-K_Y \sim L+\Delta$. After the process above he took the minimal resolution of the normalization Y, and classified the all possibilities of it.

Now we consider the case of Gorenstein Fano threefolds. Let X be a Gorenstein Fano threefold, Y the normalization of X, and L the ample Cartier divisor on Y such that the invertible sheaf $\mathcal{O}_Y(L)$ is isomorphic to the inverse image of the anti-dualizing sheaf ω_X^{-1}. Then there exists a non-zero effective Weil divisor Δ such that $-K_Y \sim L+\Delta$ by using adjunction theory as before. Now we want to apply minimal model program. For this purpose we assume that Y has at worst $\boldsymbol{Q}$-factorial terminal singularities (if Y has worse singularities, minimal model program is difficult to apply). Such a pair (Y, L) is a sort of objects which we will study in this paper.

In section 1, we prepare several results on projective surfaces for the later use. The results and their proofs are very similar to the ones treated by M.

Reid in [**R**]. So we only give the outlines of the proofs here.

In section 2, we recall definitions and results in [**I**], and modify them slightly. For a polarized threefold (Y, L) in question, we construct a sequence of contraction morphisms of extremal rays $Y=Y_0 \to Y_1 \to Y_2 \to \cdots \to Y_r \to Z$ where Z is a variety of dimension less than three, and call Z a basis of (Y, L) (or Y) as in [**I**]. In the remainder of this paper, we consider the case of $\dim Z=2$ only. Under this assumption we prove that Z is a nonsingular surface and that there exists an ample vector bundle $\mathcal{E}$ of rank 2 on Z such that (Y, L) is isomorphic to $(\boldsymbol{P}_Z(\mathcal{E}), H)$ where H is the tautological divisor on $\boldsymbol{P}_Z(\mathcal{E})$.

In section 3, we collect examples of pairs Z and $\mathcal{E}$, where Z is a non-singular projective surface and $\mathcal{E}$ is an ample vector bundle of rank 2, such that $(\boldsymbol{P}_Z(\mathcal{E}), H)$ give us polarized threefolds with non-zero effective anti-adjoint divisors.

In the last section, we study the structure of (Y, L) in more detail. As proved in section 2, such threefold is a $\boldsymbol{P}^1$-bundle over a nonsingular projective surface Z associated to an ample vector bundle $\mathcal{E}$ of rank 2 on Z. The main theorem of this paper, Theorem (4.25), says that the examples in section 3 are the all possibilities for such Z and $\mathcal{E}$.

In this paper we use the standard notation as in [**H**]. All varieties are defined over the complex number field $\boldsymbol{C}$. The Hirzeburch surface of degree e is denoted by $\boldsymbol{F}_e$, its minimal section by C_0 and a fiber by f as in [**H**]. By $\boldsymbol{F}_{a,0}$, $(a>0)$, we denote the normal projective surface obtained by the contraction of the minimal section on the Hirzeburch surface $\boldsymbol{F}_a$ of degree a. An irreducible curve on a surface is called n-curve if it is a nonsingular rational curve with self-intersection number n. The restriction of an object to a closed subvariety is usually denoted by the subscript of the corresponding letter. For example, L_Γ stands for the restriction of a divisor L to a closed subvariety Γ. Only one exception is the case of canonical divisor. The symbol K_X does not denote the restriction of "K" to X but the canonical divisor of X. There are no danger of confusion in this paper.

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1. Preliminaries.

(1.1) In this section we prepare several results on some kind of projective surfaces for the later use. The proofs are almost the same as in [**R**]. So we only give the outlines of the proofs.

LEMMA (1.2). *Let S be a normal projective surface, which admits birational morphism $\varphi: S \to T$ to a nonsingular surface T. Assume that there exist an ample*

Cartier divisor L and an effective Weil divisor δ on S, such that $-K_S \sim L+\delta$. Then one of the following holds.

(1.2.1) *$\delta=0$ and S is a normal rational Gorenstein Del Pezzo surface.*

(1.2.2-i) *φ is an isomorphism and $S \cong T \cong \boldsymbol{P}^2$. $L \sim kH$, $\delta \sim (3-k)H$, $k=1$, or 2, where H is a line in $\boldsymbol{P}^2$, that is, $\mathcal{O}(H) \cong \mathcal{O}(1)$.*

(1.2.2-ii) *φ is an isomorphism and $S \cong T \cong \boldsymbol{F}_e$. $L \sim C_0+(e+k)\mathrm{f}$, $\delta \sim C_0+(2-k)\mathrm{f}$, $k=1$, or 2.*

(1.2.2-iii) *$S \cong \boldsymbol{F}_1$, $T \cong \boldsymbol{P}^2$ and $\varphi: S \to T$ is the morphism of blow-down. $L \sim C_0+(1+k)\mathrm{f}$ and $\delta \sim C_0+(2-k)\mathrm{f}$, $k=1$, or 2.*

OUTLINE OF THE PROOF. Let $f: \tilde{S} \to S$ be the minimal resolution of S. Then we can easily see the equality

$$-K_{\tilde{S}} \sim f^*L+\tilde{\delta}+E$$

where $\tilde{\delta}$ is the proper transform of δ by f and E is an effective integral Weil divisor whose support is contained in the exceptional locus of f. When the divisor $\tilde{\delta}+E$ is not equal to zero, we can prove that the surface $\tilde{S}$ is either $\boldsymbol{P}^2$ or a Hirzeburch surface $\boldsymbol{F}_e$, $(e \geqq 0)$ by the similar way as in [**R**]. Moreover we can compute f^*L explicitly. On the other hand, we have a birational morphism from $\tilde{S}$ to a nonsingular surface T. Therefore $\tilde{S} \cong S \cong T$ unless $\tilde{S} \cong S \cong \boldsymbol{F}_1$ and $T \cong \boldsymbol{P}^2$. These cases correspond to the case (1.2.2-i), (1.2.2-ii), and (1.2.2-iii). The equality $\tilde{\delta}+E=0$ implies that $\delta=0$ and $f^*K_S \sim K_{\tilde{S}}$. Therefore the normal surface S is a rational Gorenstein Del Pezzo surface by [**HW**]. This is the case of (1.2.1). □

COROLLARY (1.3). *Let S be a Gorenstein projective surface, which admits a birational morphism $\varphi: S \to T$ to a nonsingular surface T, and $\pi: V \to S$ the normalization of S (in the case that S is normal we put $V=S$). Assume that there exist an ample Cartier divisor L on S and an effective divisor δ on T such that $\omega_S^{-1} \cong \mathcal{O}_S(L) \otimes \varphi^*\mathcal{O}_T(\delta)$. Then one of the following holds.*

(1.3.1) *$\delta=0$ and S is a normal rational Gorenstein Del Pezzo surface.*

(1.3.2-i) *φ is an isomorphism and $S \cong T \cong \boldsymbol{P}^2$. $L \cong kH$, $\delta \sim (3-k)H$, $k=1$, or 2.*

(1.3.2-ii) *φ is an isomorphism and $S \cong T \cong \boldsymbol{F}_e$. $L \sim C_0+(e+k)\mathrm{f}$, $\delta \sim C_0+(2-k)\mathrm{f}$, $k=1$, or 2.*

(1.3.2-iii) *$S \cong \boldsymbol{F}_1$, $T \cong \boldsymbol{P}^2$ and $\varphi: S \to T$ is the morphism of blow-down. $L \sim C_0+2\mathrm{f}$, and $\delta \sim H$.*

(1.3.3) *S is a non-normal Gorenstein Del Pezzo surface, $V \cong \boldsymbol{F}_1$ $T \cong \boldsymbol{P}^2$, and $\varphi \cdot \pi: V \to T$ is the morphism of blow-down. Furthermore we have $\pi^*L \sim C_0+3\mathrm{f}$ and $\delta=0$.*

OUTLINE OF THE PROOF. By the adjunction formula there exists an effective divisor Δ on V such that $\mathcal{O}_V(K_V+\Delta)\cong\pi^*\omega_S$. Notice that $\Delta=0$ if and only if S is normal. Therefore we have $-K_V\sim\pi^*L+\pi^*\varphi^*\delta+\Delta$. Now we can apply the lemma above to the normal surface V. From the cases (1.2.1), (1.2.2-i) and (1.2.2-ii) we obtain the cases (1.3.1), (1.3.2-i) and (1.3.2-ii). For the case (1.2.2-iii), we have $V\cong\boldsymbol{F}_1$, $T\cong\boldsymbol{P}^2$, and $\pi^*\varphi^*\delta+\Delta\sim C_0+(2-k)\mathrm{f}$, $k=1$ or 2. If the divisor δ is not equal to zero, then we have $k=1$ and $\pi^*\varphi^*\delta+\Delta\sim C_0+\mathrm{f}$. Then $\Delta=0$ and S is normal, that is, $\pi: V\to S$ is isomorphism. This is the case of (1.3.2-iii). Because $\varphi\cdot\pi$ is isomorphic outside of C_0, non-normal locus of S is contained in the image of C_0 by π. Therefore we have the case (1.3.3) if δ is equal to zero. □

2. Contraction morphism associated to an ample decomposition.

(2.1) In this section we recall several results in [I] and apply them to our case.

DEFINITION (2.2) ([I], Definition 1.2). Let Y be a normal projective variety with at worst $\boldsymbol{Q}$-factorial terminal singularities, and D a Weil divisor on Y. A good decomposition of D is the following data:

(i) L is a nef Cartier divisor on Y
(ii) $\Delta>0$ is a non-zero effective divisor
(iii) D is linearly equivalent to $L+\Delta$, and
(iv) $L\cdot C>0$ for any irreducible curve C on Y with $\Delta\cdot C>0$.

The data $L+\Delta$ above is called a good decomposition of D.

DEFINITION (2.3). For a good decomposition $L+\Delta$ of a Weil divisor D we call it an ample decomposition of D if L is an ample Cartier divisor. Notice that (iv) is always the case for any ample Cartier divisor on Y.

REMARK (2.4). Let (Y, L) be a normal polarized 3-fold with non-zero effective anti-adjoint divisor, that is, $-(K_Y+L)$ is linearly equivalent to a non-zero effective Weil divisor Δ. Then $L+\Delta$ is an ample decomposition of $-K_Y$.

DEFINITION (2.5). Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities, and $-K_Y\sim L+\Delta$ an ample decomposition of $-K_Y$. An extremal ray R of Y with the property $\Delta\cdot R>0$ is called an extremal ray associated to the ample decomposition $-K_Y\sim L+\Delta$, and the contraction morphism of R is called a contraction morphism associated to the decomposition.

LEMMA (2.6) (cf. [I], Lemma 1.3). *Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities. Assume that $-K_Y$ admits an ample decomposition $L+\Delta$. Then there exists an extremal ray R associated to the* *ample decomposition above, and the contraction morphism $\varphi: Y \to Y'$ of R satisfies one of the following properties:*

(I) $\dim Y' \leqq 2$, *or*

(II) *φ is a birational contraction which contracts an irreducible divisor $D \cong \boldsymbol{F}_{a,0}$ $(a \geqq 1)$ to a nonsingular point on Y'. By setting $\Delta' = \varphi_* \Delta$ and $L' = -K_{Y'} - \Delta'$, Δ' is a non-zero effective Weil divisor on Y', L' is an ample Cartier divisor and $-K_{Y'} \sim L' + \Delta'$ is an ample decomposition of $-K_{Y'}$. Moreover $\varphi^* L' = L + aD$ and $\varphi^* \Delta' = \Delta + D$.*

PROOF. By Lemma 1.3 in [I], it is sufficient to show that L' is ample. Let C' be an irreducible curve on Y' and C the proper transform of C' in Y. By the equality $\varphi^* L' = L + aD$ and the inequality $D \cdot C \geqq 0$, we have

$$L' \cdot C' = \varphi^* L' \cdot C = (L + aD) \cdot C \geqq L \cdot C > 0 .$$

Hence L' is ample. □

(2.7) By the lemma above we get the following theorem.

THEOREM (2.8) (cf. [I], Theorem 1.3). *Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities. Assume that $-K_Y$ admits an ample decomposition $L + \Delta$. Then there exists a sequence of morphisms*

$$Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 \longrightarrow \cdots \longrightarrow Y_r \xrightarrow{\varphi_r} Z ,$$

satisfying the following properties.

(i) *Y_i has at worst $\boldsymbol{Q}$-factorial terminal singularities for every i $(0 \leqq i \leqq r)$.*

(ii) *$-K_{Y_i}$ admits an ample decomposition $L_i + \Delta_i$ for every i $(0 \leqq i \leqq r)$, where $L_0 = L$ and $\Delta_0 = \Delta$.*

(iii) *For every i, φ_i is a contraction morphism associated to the ample decomposition in* (ii), *which is a birational contraction described in the lemma above for $i < r$, and of fiber type for $i = r$ (therefore* $\dim Z \leqq 2$).

DEFINITION (2.9). The sequence $Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 \to \cdots \to Y_r \xrightarrow{\varphi_r} Z$ described in the theorem above is called a sequence of contractions associated to the ample decomposition $-K_Y \sim L + \Delta$. The non-negative integer r in the sequence above is called its length. We call the variety Z the basis of the sequence. In the case that we do not explicitly specify a sequence of contractions, we call Z a basis of Y with respect to the given ample decomposition.

(2.10) In the remainder of this paper we treat the cases that Y admits a basis Z with $\dim Z = 2$ with respect to the given ample decomposition of $-K_Y$.

(2.11) The following lemma is due to N. Nakayama. We can find the proof in [I].

LEMMA (2.12) (N. Nakayama, see Lemma 1.6 in [I]). *Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities, and $\varphi: Y \to S$ is the contraction morphism of an extremal ray of Y to a surface S. Assume that there exists a Cartier divisor L on Y such that $L \cdot l = 1$ for a general fiber l of φ. Then S is a nonsingular surface and $\varphi: Y \to S$ is a $\boldsymbol{P}^1$-bundle over S. More precisely, the direct image $\varphi_* \mathcal{O}_Y(L)$ is a vector bundle of rank 2 on S and Y is isomorphic to $\boldsymbol{P}_S(\varphi_* \mathcal{O}_Y(L))$ over S.*

(2.13) By virtue of the lemma above we can prove the following theorem.

THEOREM (2.14). *Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities, $-K_Y \sim L + \Delta$ an ample decomposition of $-K_Y$, and $Y = Y_0 \xrightarrow{\varphi_0} Y_1 \xrightarrow{\varphi_1} Y_2 \to \cdots \to Y_r \xrightarrow{\varphi_r} Z$ a sequence of contractions associated to the ample decomposition above. Suppose that* $\dim Z = 2$. *Then the length r must be zero, that is $Y = Y_r$. Moreover the surface Z is nonsingular, and $\varphi_r: Y (= Y_r) \to Z$ gives us a $\boldsymbol{P}^1$-bundle over Z. More precisely, there exists an ample vector bundle $\mathcal{E}$ of rank 2 on Z such that (Y, L) $(= (Y_r, L_r))$ is isomorphic to $(\boldsymbol{P}_Z(\mathcal{E}), H)$ over Z, where H is the tautological divisor on $\boldsymbol{P}_Z(\mathcal{E})$.*

PROOF. At first we prove that Z is a nonsingular surface and the final contraction $\varphi_r: Y_r \to Z$ gives us a $\boldsymbol{P}^1$-bundle over Z. Recall that φ_r is the contraction morphism of an extremal ray R_r associated to an ample decomposition $-K_{Y_r} \sim L_r + \Delta_r$. We have the property $\Delta_r \cdot R_r > 0$. Therefore a general fiber l of φ_r, which is $\boldsymbol{P}^1$, satisfies the inequality $\Delta_r \cdot l > 0$. Furthermore the intersection number $L_r \cdot l$ is a positive integer because L_r is ample Cartier divisor. Since $l \cong \boldsymbol{P}^1$, we conclude that $L_r \cdot l = 1$ and $\Delta_r \cdot l = 1$. Now we can apply Lemma (2.12), and then Z is a nonsingular surface and $\varphi_r: Y_r \to Z$ gives a $\boldsymbol{P}^1$-bundle.

If we assume that the length r is greater than 0. Then we have a part of the sequence $Y_{r-1} \xrightarrow{\varphi_{r-1}} Y_r \xrightarrow{\varphi_r} Z$. We have $\varphi_{r-1}^* L_r = L_{r-1} + a D_{r-1}$ for some integer $a \geqq 1$ and $\varphi_{r-1}(D_{r-1})$ is a point p on Y_r by Lemma (2.6), where D_{r-1} is the exceptional divisor of the morphism φ_{r-1}. Because $\varphi_r: Y_r \to Z$ is a $\boldsymbol{P}^1$-bundle we can take a fiber l passing through the point p. From the argument above, we have $L_r \cdot l = 1$. Let $\bar{l}$ be a proper transform of l on Y_{r-1}. Then $D_{r-1} \cdot \bar{l} > 0$ because l passes through the point p. Therefore we have

$$\begin{aligned} 1 = L_r \cdot l &= (\varphi_{r-1}^* L_r) \cdot \bar{l} \\ &= (L_{r-1} + a D_{r-1}) \cdot \bar{l} \\ &> L_{r-1} \cdot \bar{l}, \end{aligned}$$

and this contradicts to the fact that L_{r-1} is an ample Cartier divisor on Y_{r-1}. Thus we complete the proof. □

3. Examples.

(3.1) In this section we give examples of polarized 3-folds with non-zero effective anti-adjoint divisors. In the next section, these examples will turn out to be the all possible cases for such threefolds with a basis of dimension 2.

(3.2) At first we prepare a simple lemma for the later use.

LEMMA (3.3). *Let Z be a nonsingular surface and $\mathcal{E}$ be a vector bundle of rank 2 on Z. We set $Y=\boldsymbol{P}_Z(\mathcal{E})$ and denote the tautological divisor by L. Then there exists a non-zero effective divisor which is linearly equivalent to $-(K_Y+L)$ if and only if*

$$\mathrm{H}^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z)) \neq 0.$$

In particular, if $\mathcal{E}$ is ample in addition then (Y, L) is a nonsingular polarized 3-fold with non-zero effective anti-adjoint divisor.

PROOF. Notice that $-(K_Y+L)$ is not trivial because the restriction of it to a fiber is not trivial. Then it is easy from the formula

$$\mathcal{O}_Y(K_Y) \cong \mathcal{O}_Y(-2L)\otimes\varphi^*(\det \mathcal{E}\otimes\mathcal{O}_Z(K_Z)). \qquad \square$$

EXAMPLE (3.4). Let $Z=\boldsymbol{P}^2$, $\mathcal{E}=\mathcal{O}(k)\oplus\mathcal{O}(a)$, $k=1$, 2, or 3 and $a>0$, $Y=\boldsymbol{P}_Z(\mathcal{E})$ and L the tautological divisor. Notice that L is an ample divisor on Y. It is easy to check that $\mathrm{h}^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$.

EXAMPLE (3.5). Let $Z=\boldsymbol{P}^2$, $\mathcal{E}=\mathcal{T}_Z$, where $\mathcal{T}_Z$ is the tangent bundle of Z. Then it is well known that $\mathcal{T}_Z$ is an ample vector bundle of rank 2. Since $\det \mathcal{E}\cong\mathcal{O}_Z(-K_Z)$, we have $\mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(K_Z)\cong\mathcal{E}\cong\mathcal{T}_Z$. Therefore $\mathrm{H}^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$.

EXAMPLE (3.6). Let $Z=\boldsymbol{F}_e$, D an ample divisor on Z and $\mathcal{E}=\mathcal{O}_Z(D)\oplus\mathcal{O}_Z(C_0+(e+k)\mathrm{f})$, for $k=1$ or 2. By easy calculation we have

$$\mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z) \cong \mathcal{O}_Z(C_0+(2-k)\mathrm{f})\oplus\mathcal{O}_Z(-D+2C_0+(2+e)\mathrm{f}),$$

and then we can easily see that $\mathrm{h}^0(\mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$.

EXAMPLE (3.7). Let $Z=\boldsymbol{F}_e$, for $e>0$ and D a divisor on Z with the properties that $D\cdot\mathrm{f}\geqq 1$ and $D\cdot C_0=0$. It is easy to see that D is base point free and that we have $D\cdot C>0$ for every irreducible curve C different from C_0 (cf. [**H**]). We have the following exact sequence

$$0 \longrightarrow \mathcal{O}_Z(D-2C_0-(e+2)\mathrm{f}) \longrightarrow \mathcal{O}_Z(D-C_0-(e+2)\mathrm{f}) \longrightarrow \mathcal{O}_{C_0}(-2) \longrightarrow 0.$$

We have $\mathrm{h}^2(Z, \mathcal{O}_Z(D-2C_0-(e+2)\mathrm{f}))=\mathrm{h}^0(Z, \mathcal{O}_Z(-D))=0$ by Serre duality and the fact that $(-D)\cdot\mathrm{f}<0$. Then we obtain the surjection $\mathrm{H}^1(Z, \mathcal{O}_Z(D-C_0-(e+2)\mathrm{f}))\rightarrow$

$H^1(C_0, \mathcal{O}_{C_0}(-2))$. On the other hand, we have the isomorphisms $H^1(Z, \mathcal{O}_Z(D-C_0-(e+2)f))\cong \mathrm{Ext}^1(\mathcal{O}_Z(C_0+(e+2)f), \mathcal{O}_Z(D))$ and $H^1(C_0, \mathcal{O}_{C_0}(-2))\cong \mathrm{Ext}^1(\mathcal{O}_{C_0}(2), \mathcal{O}_{C_0})$. Therefore we can take a vector bundle $\mathcal{E}$ of rank 2 satisfying the exact sequence

$$(3.7.1) \qquad 0 \longrightarrow \mathcal{O}_Z(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(C_0+(e+2)f) \longrightarrow 0$$

such that the restriction of the exact sequence above to the curve C_0

$$0 \longrightarrow \mathcal{O}_{C_0} \longrightarrow \mathcal{E}_{C_0} \longrightarrow \mathcal{O}_{C_0}(2) \longrightarrow 0$$

does not split. Notice that $\mathcal{E}_{C_0}\cong\mathcal{O}_{C_0}(1)\oplus\mathcal{O}_{C_0}(1)$, and then $\mathcal{E}_{C_0}$ is ample on C_0. It is easy to see that $h^0(Z, \mathcal{E}\otimes(\det\mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$ by considering the exact sequence obtained by tensoring $(\det\mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z)=\mathcal{O}_Z(-D+C_0)$. Therefore we only have to prove that $\mathcal{E}$ is ample. Let $Y=\boldsymbol{P}_Z(\mathcal{E})$, L the tautological divisor on Y, and Γ the section corresponding to the surjection $\mathcal{E}\to\mathcal{O}_Z(C_0+(e+2)f)$. It suffices to prove that L is ample on Y. By the exact sequence (3.7.1) we have the following equalities

$$L \sim \Gamma+\varphi^*D,$$

$$L_\Gamma \sim C_0+(e+2)f.$$

It is easy to see that L has no base points outside of Γ from the first equality above and from the fact that the divisor D is base point free. Moreover it is well known that $L_\Gamma\cong C_0+(e+2)f$ is base point free. Therefore we know that L is base point free because $H^1(Y, \mathcal{O}_Y(L-\Gamma))\cong H^1(Z, \mathcal{O}_Z(D))=0$ (cf. [**H**]). Then we can define the morphism $\varphi_{|L|}$ associated to the linear system $|L|$. On the other hand, $\mathcal{E}_C$ is ample for every irreducible curve C on Z different from C_0, because of the inequality $D\cdot C>0$ and the exact sequence (3.7.1). We remarked that $\mathcal{E}_{C_0}$ is ample. Then it is easy to see that $L\cdot C>0$ for every irreducible curve C on Y. Therefore $\varphi_{|L|}$ is finite morphism and then L is ample.

EXAMPLE (3.8). Let Z be a nonsingular Del Pezzo surface, D an ample divisor on Z, and $\mathcal{E}=\mathcal{O}_Z(D)\oplus\mathcal{O}_Z(-K_Z)$. Then the vector bundle $\mathcal{E}$ is ample. We have

$$\mathcal{E}\otimes(\det\mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z) \cong \mathcal{O}_Z\oplus\mathcal{O}_Z(-D-K_Z),$$

and then $h^0(Z, \mathcal{E}\otimes(\det\mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$.

EXAMPLE (3.9) Let $S=\boldsymbol{F}_1$ and $Z=\boldsymbol{P}^2$. We denote the morphism of blow down by $\sigma: S\to Z$. Let us take a divisor D on S with the properties that $D\cdot f\geqq 2$ and $D\cdot C_0=-1$. We can easily see the inequality $D\cdot C>0$ for every irreducible curve C on S, different from C_0. We have an exact sequence

$$0 \longrightarrow \mathcal{O}_S(D-2C_0-2f) \longrightarrow \mathcal{O}_S(D-C_0-2f) \longrightarrow \mathcal{O}_{C_0}(-2) \longrightarrow 0,$$

from which we obtain the surjection $H^1(S, \mathcal{O}_S(D-C_0-2f)) \to H^1(C_0, \mathcal{O}_{C_0}(-2))$ because $h^2(S, \mathcal{O}_S(D-2C_0-2f))=h^0(S, \mathcal{O}_S(-D-f))=0$ by the inequalities $(-D-f)\cdot f<0$. Therefore we can take a vector bundle with the exact sequence

$$(3.9.1) \qquad 0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{O}_S(C_0+2f) \longrightarrow 0$$

such that the restriction of the exact sequence above to the curve C_0 does not split. Then it is easy to see that $\tilde{\mathcal{E}}_{C_0}$ is trivial. By Schwarzenberger's theorem [**S**] there exists a vector bundle $\mathcal{E}$ on Z such that $\tilde{\mathcal{E}} \cong \sigma^*\mathcal{E}$. By the fact that $D\cdot C>0$ for every irreducible curve C on S different from C_0, $\mathcal{E}$ turns out to be an ample vector bundle on Z. On the other hand, we have

$$\sigma^*\mathcal{E}\otimes(\det \sigma^*\mathcal{E})^{-1}\otimes\mathcal{O}_S(-\sigma^*K_Z) \cong \tilde{\mathcal{E}}\otimes\mathcal{O}_S(-D+2C_0+f),$$

and then

$$\begin{aligned} H^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z)) &\cong H^0(S, \sigma^*\mathcal{E}\otimes(\det \sigma^*\mathcal{E})^{-1}\otimes\mathcal{O}_S(-\sigma^*K_Z)) \\ &\cong H^0(S, \tilde{\mathcal{E}}\otimes\mathcal{O}_S(-D+2C_0+f)) \\ &\neq 0 \end{aligned}$$

from the exact sequence (3.9.1).

EXAMPLE (3.10). Let S, Z and $\sigma: S\to Z$ be as in Example (3.9). Taking a divisor D with the property that $D\cdot f\geqq 3$ and $D\cdot C_0=-2$. By the similar argument as in Example (3.9), we can find a vector bundle $\tilde{\mathcal{E}}$ of rank 2 on S satisfying the exact sequence

$$0 \longrightarrow \mathcal{O}_S(D) \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{O}_S(C_0+3f) \longrightarrow 0$$

such that the restriction $\tilde{\mathcal{E}}_{C_0}$ to the curve C_0 is trivial. Moreover we can easily see that $D\cdot C>0$ for every irreducible curve C different from C_0. Therefore we can find an ample vector bundle $\mathcal{E}$ of rank 2 on Z such that $\tilde{\mathcal{E}}\cong\sigma^*\mathcal{E}$. We can easily see that $H^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z))\neq 0$ as in Example (3.9).

EXAMPLE (3.11). Let Z be a nonsingular Del Pezzo surface, S a nonsingular weak Del Pezzo surface (that is, $-K_S$ is nef and big), and $\sigma: S\to Z$ a birational morphism from S to Z such that all (-2)-curves on S are exceptional divisors of σ. Then there exists an effective divisor E, whose support is contained in the exceptional locus of σ, such that $K_S\sim\sigma^*K_Z+E$. Notice that K_S and E have the same intersection numbers with the σ-exceptional divisors. Let us assume that we are given an ample divisor D on Z. We set $\tilde{D}=\sigma^*D+E$. We can easily see that for any σ-exceptional (-1)-curve C there exists a nonsingular rational curve l on S such that $l^2\geqq 0$ and $C\cdot l=0$. Then we have $(C-\tilde{D})\cdot l=-\tilde{D}\cdot l=-(\sigma^*D+E)\cdot l\leqq -1$ because D is ample, E is effective and E and l have no common component. Therefore we have $H^0(S, \mathcal{O}_S(C-\tilde{D}))=0$ for every

σ-exceptional (-1)-curve C. Since $\tilde{D}\cdot C=-1$ for any σ-exceptional curve C, we have an exact sequence

$$0 \longrightarrow \mathcal{O}_S(\tilde{D}+K_S-C) \longrightarrow \mathcal{O}_S(\tilde{D}+K_S) \longrightarrow \mathcal{O}_C(-2) \longrightarrow 0.$$

Then we have a surjection $\mathrm{H}^1(S, \mathcal{O}_S(\tilde{D}+K_S)) \to \mathrm{H}^1(C, \mathcal{O}_C(-2))$ because $\mathrm{h}^2(S, \mathcal{O}_S(\tilde{D}+K_S-C))=\mathrm{h}^0(S, \mathcal{O}_S(C-\tilde{D}))=0$ by Serre duality and the argument above. Therefore we can take a vector bundle $\tilde{\mathcal{E}}$ of rank 2 on S satisfying the exact sequence

$$0 \longrightarrow \mathcal{O}_S(\tilde{D}) \longrightarrow \tilde{\mathcal{E}} \longrightarrow \mathcal{O}_S(-K_S) \longrightarrow 0 \tag{3.11.1}$$

whose restriction to any σ-exceptional (-1)-curve C does not split. Hence $\tilde{\mathcal{E}}_C$ is trivial for every σ-exceptional (-1)-curve C. Also, we can easily see that $\tilde{\mathcal{E}}_C$ is trivial for every (-2)-curve C. Then there exists a vector bundle $\mathcal{E}$ of rank 2 on Z such that $\sigma^*\mathcal{E}\cong\tilde{\mathcal{E}}$ by applying Schwarzenberger's theorem **[S]** successively.

(3.12) Now we will see that $\mathcal{E}$ is ample and satisfies the condition in Lemma (3.3). At first let us remark that for any curve C on Z, the restriction $\mathcal{E}_C$ of $\mathcal{E}$ to the curve C is ample because $\tilde{D}\cdot\bar{C}>0$, where $\bar{C}$ is the proper transform of C by σ. Let $Y=\boldsymbol{P}_Z(\mathcal{E})$, $\tilde{Y}=\boldsymbol{P}_S(\tilde{\mathcal{E}})$, and L and $\tilde{L}$ the tautological divisors on Y and $\tilde{Y}$ respectively. Then we have a cartesian square

$$\begin{array}{ccc} \tilde{Y} & \xrightarrow{\tau} & Y \\ {\scriptstyle\psi}\downarrow & & \downarrow{\scriptstyle\varphi} \\ S & \xrightarrow[\sigma]{} & Z \end{array}$$

and denote the morphisms $Y\to Z$, $\tilde{Y}\to S$ and $\tilde{Y}\to Y$ by φ, ψ and τ respectively. We denote the section corresponding to the surjection $\tilde{\mathcal{E}}\to\mathcal{O}_S(-K_S)$ by $\tilde{\Gamma}$ and the image $\tau(\tilde{\Gamma})$ by Γ. Then there exists a divisor E', whose support is contained in the exceptional locus of σ, such that the equality $\tau^*\Gamma\sim\tilde{\Gamma}+\psi^*E'$. By using the base change theorem we obtain the equality

$$\begin{aligned} \sigma^*\varphi_*\mathcal{O}_Y(L-\Gamma) &\cong \psi_*\tau^*\mathcal{O}_Y(L-\Gamma) \\ &\cong \psi_*\mathcal{O}_{\tilde{Y}}(\tilde{L}-\tilde{\Gamma}-\psi^*E') \\ &\cong \mathcal{O}_S(\tilde{D}-E') \\ &\cong \mathcal{O}_S(\sigma^*D+E-E') \end{aligned}$$

and then E and E' is numerically equivalent relative to σ. Thus $E\sim E'$ because their supports are contained in the exceptional locus of σ and the intersection matrix of the exceptional divisors are negative definite. Hence we have the following equalities by easy calculations:

$$\det \mathcal{E} \cong \mathcal{O}_Z(D-K_Z)$$

$$\mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z) \cong \mathcal{E}\otimes\mathcal{O}_Z(-D)$$

$$L \sim \Gamma+\varphi^*D$$

$$-K_Y \sim L+\Gamma$$

$$-K_\Gamma \sim L_\Gamma$$

$$-K_{\tilde{Y}} \sim \tilde{L}+\tilde{\Gamma}$$

$$-K_{\tilde{\Gamma}} \sim \tilde{L}_{\tilde{\Gamma}}$$

$$\tau^*\Gamma \sim \tilde{\Gamma}+\psi^*E$$

$$\sigma^*(\mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z)) \cong \tilde{\mathcal{E}}\otimes\mathcal{O}_S(E-\tilde{D}).$$

Thus we have

$$\mathrm{H}^0(Z, \mathcal{E}\otimes(\det \mathcal{E})^{-1}\otimes\mathcal{O}_Z(-K_Z)) \cong \mathrm{H}^0(S, \tilde{\mathcal{E}}\otimes\mathcal{O}_S(E-\tilde{D})) \neq 0$$

from the exact sequence (3.11.1). Now it is sufficient to prove that L is an ample divisor. For every irreducible curve C on Y, we have $L\cdot C>0$, because the restriction of $\mathcal{E}$ on every irreducible curve in Z is ample as we remarked before. Therefore it suffices to show that mL is base point free for some positive integer m. Notice that L_Γ is ample because $(L_\Gamma)^2=(\tilde{L}_{\tilde{\Gamma}})^2=(-K_S)^2>0$.

(3.13) Since $-K_\Gamma \sim L_\Gamma$ is ample, the surface Γ is a rational Gorenstein Del Pezzo surface. Moreover we have a birational morphism $\tau_{\tilde{\Gamma}}: \tilde{\Gamma}\to\Gamma$ such that $\tau_{\tilde{\Gamma}}^*(-K_\Gamma)\sim -K_{\tilde{\Gamma}}$, Therefore Γ is normal from the classification of non-normal Gorenstein Del Pezzo surfaces by M. Reid [**R**]. Notice that Γ has at worst rational double points as its singular points. Also, it is easy to see that $\tau_{\tilde{\Gamma}}$ is the minimal resolution of Γ.

(3.14) Since D is an ample divisor on Z, and L_Γ is an ample divisor on Γ, there exists a positive integer m such that we have the followings:

$$\mathrm{H}^1(Z, \mathcal{O}_Z(mD)) = 0,$$

mD is base point free,

mL_Γ is base point free.

Now we prove the following lemma.

LEMMA (3.15). *In the situation above, we have*

$$\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(mL-k\Gamma)) = 0$$

for every integer k *with* $0\leqq k\leqq m+1$.

PROOF. Because Γ has only rational double points, we have an isomorphism

$$\mathrm{H}^1(\Gamma, \mathcal{O}_\Gamma(mL-k\Gamma)) \cong \mathrm{H}^1(\tilde{\Gamma}, \tau_{\tilde{\Gamma}}^*\mathcal{O}_\Gamma(mL-k\Gamma))$$

for every integer k. Using the equalities in (3.12) we obtain

$$\tau_{\tilde{\Gamma}}^*\mathcal{O}_\Gamma(mL-k\Gamma)-K_{\tilde{\Gamma}} \sim (m+1-k)(-K_{\tilde{\Gamma}})+k(\phi^*\sigma^*D)_{\tilde{\Gamma}}.$$

Because $-K_{\tilde{\Gamma}}$ is nef and D is ample, the sheaf $\tau_{\tilde{\Gamma}}^*\mathcal{O}_\Gamma(mL-k\Gamma)-K_{\tilde{\Gamma}}$ is nef and big for integer k with $0 \leqq k \leqq m+1$. Hence we conclude the results by Kawamata-Vieweg vanishing theorem. □

(3.16) We have

$$\mathrm{H}^1(Y, \mathcal{O}_Y(mL-m\Gamma)) \cong \mathrm{H}^1(Z, \mathcal{O}_Z(mD)) = 0$$

by the asumption. Now we can easily see that $\mathrm{H}^1(Y, \mathcal{O}_Y(mL-k\Gamma))=0$ for $0 \leqq k \leqq m$ by descending induction on k. Therefore we obtain the surjection

$$\mathrm{H}^0(Y, \mathcal{O}_Y(mL)) \longrightarrow \mathrm{H}^0(\Gamma, \mathcal{O}_\Gamma(mL)).$$

Thus we know that

$$\mathrm{Bs}|mL| \cap \Gamma = \emptyset.$$

On the other hand,

$$mL \sim m\Gamma + \varphi^*(mD)$$

and mD is base point free. Therefore it turns out that mL is base point free.

4. Case of two-dimensional basis.

(4.1) Let Y be a normal projective 3-fold with at worst $\boldsymbol{Q}$-factorial terminal singularities and $-K_Y \sim L+\Delta$ an ample decomposition of $-K_Y$. Throughout this section, we assume that there exists a basis Z with $\dim Z=2$ with respect to the ample decomposition above.

(4.2) Because of Theorem (2.14), there exists an ample vector bundle $\mathcal{E}$ of rank 2 such that Y is isomorphic to $\boldsymbol{P}_Z(\mathcal{E})$ and L is identified with the tautological divisor via this isomorphism.

(4.3) Since $\Delta \cdot l=1$, there exists a prime divisor Γ on Y with $\Gamma \cdot l=1$, and $\Delta=\Gamma+\varphi^*\delta$ for some effective divisor δ on Z. From the equality in (4.1) we have

$$-K_Y \sim L+\Gamma+\varphi^*\delta. \tag{4.3.1}$$

Then Γ is a Gorenstein surface and the restriction $\varphi_\Gamma : \Gamma \rightarrow Z$ of φ to Γ is a birational morphism. Furthermore we have

(4.3.2) $$-K_{\Gamma}\sim-(K_Y+\Gamma)_{\Gamma}\sim L_{\Gamma}+\varphi_{\Gamma}^{*}\delta$$

by the adjunction formula and substituting the equality (4.3.1). Therefore we can apply Corollary (1.3) to Γ, and we get the possibilities described in the Corollary (1.3).

(4.4) On the other hand we divide the problem into four parts according as Γ is a section or not, and as δ is equal to zero or not. Combining these possibilities, we have the following cases. Let us remark that we identify Γ and Z by φ_{Γ} when Γ is a section.

(I-1) Γ is a section and $\Gamma\cong Z\cong \boldsymbol{P}^2$. $L_{\Gamma}\sim kH$, $\delta\sim(3-k)H$, $k=1$, or 2, where H is a line in $\boldsymbol{P}^2\cong\Gamma\cong Z$, that is, $\mathcal{O}_Z(H)\cong\mathcal{O}_Z(1)$.

(I-2) Γ is a section and $\Gamma\cong Z\cong \boldsymbol{F}_e$, $L_{\Gamma}\cong C_0+(e+k)\mathrm{f}$, $\delta\sim C_0+(2-k)\mathrm{f}$, $k=$ 1, or 2.

(II) Γ is a section, $\delta=0$, and $Z\cong\Gamma$ is a nonsingular Del Pezzo surface.

(III) $\Gamma\cong\boldsymbol{F}_1$, $Z\cong\boldsymbol{P}^2$ and φ_{Γ} is the morphism of blow-down. $L_{\Gamma}\sim C_0+2\mathrm{f}$, $\delta\sim H$ on $Z\cong\boldsymbol{P}^2$.

(IV-1) Γ is a non-normal Del Pezzo surface, $Z\cong\boldsymbol{P}^2$, and $\delta=0$. Setting $\pi:\tilde{\Gamma}\to\Gamma$ the normalization of Γ, then $\tilde{\Gamma}\cong\boldsymbol{F}_1$ and the composition $\varphi_{\Gamma}\cdot\pi:\tilde{\Gamma}\to Z$ is the morphism of blow-down. Moreover $\pi^{*}L_{\Gamma}\sim C_0+3\mathrm{f}$.

(IV-2) Γ is not a section and normal (possibly nonsingular) rational Gorenstein Del Pezzo surface, and $\delta=0$.

(4.5) Case (I-1). In this case, $\mathcal{E}=\varphi_{*}\mathcal{O}_Y(L)$ is an ample vector bundle of rank 2 on $Z\cong\boldsymbol{P}^2$ and $Y\cong\boldsymbol{P}_Z(\mathcal{E})$. We have an exact sequence

$$0\longrightarrow\mathcal{O}_Y(L-\Gamma)\longrightarrow\mathcal{O}_Y(L)\longrightarrow\mathcal{O}_{\Gamma}(L)\longrightarrow 0$$

and then applying φ_{*}, we get an exact sequence

$$0\longrightarrow\varphi_{*}\mathcal{O}_Y(L-\Gamma)\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}_Z(k)\longrightarrow 0\quad k=1,\,2,$$

because $\Gamma\cong Z$ and $L_{\Gamma}\sim kH$. Furthermore $\varphi_{*}\mathcal{O}_Y(L-\Gamma)$ is a line bundle by the continuity theorem. Therefore $\mathcal{E}\cong\mathcal{O}_Z(k)\oplus\mathcal{O}_Z(a)$, $a>0$, because the exact sequence above always splits on $Z\cong\boldsymbol{P}^2$, and because $\mathcal{E}$ is ample. This is the case of Example (3.4).

(4.6) Case (I-2). By the similar argument as in (4.5), we get an exact sequence

(4.6.1) $$0\longrightarrow\mathcal{O}_Z(D)\longrightarrow\mathcal{E}\longrightarrow\mathcal{O}_Z(C_0+(e+k)\mathrm{f})\longrightarrow 0\quad k=1,\,2,$$

where $\mathcal{E}$ is an ample vector bundle of rank 2 on $Z\cong\boldsymbol{F}_e$ with the property $Y\cong\boldsymbol{P}_Z(\mathcal{E})$. We write down $D\sim aC_0+b\mathrm{f}$. By restricting the exact sequence above to $\mathrm{f}\cong\boldsymbol{P}^1$. We get an exact sequence

$$0\longrightarrow\mathcal{O}_{\mathrm{f}}(a)\longrightarrow\mathcal{E}_{\mathrm{f}}\longrightarrow\mathcal{O}_{\mathrm{f}}(1)\longrightarrow 0$$

and then

$$a+1=\det \mathcal{E}_{\mathrm{f}} \geqq \operatorname{rank} \mathcal{E}=2$$

since $\mathcal{E}$ is ample. Hence we have $a \geqq 1$. By restricting (4.6.1) on $C_0 \cong \boldsymbol{P}^1$, we get

$$0 \longrightarrow \mathcal{O}_{C_0}(b-ae) \longrightarrow \mathcal{E}_{C_0} \longrightarrow \mathcal{O}_{C_0}(k) \longrightarrow 0$$

and then, we have

$$(4.6.2) \qquad b-ae \geqq 2-k \quad k=1,\ 2.$$

Thus we have

$$(4.6.3) \qquad b-ae-k \geqq 2-2k \geqq -2.$$

On the other hand, we can see that

$$\begin{aligned}
&\dim \operatorname{Ext}^1(\mathcal{O}_Z(C_0+(e+k)\mathrm{f}),\ \mathcal{O}_Z(D)) \\
&= \mathrm{h}^1(Z,\ \mathcal{O}_Z(D-C_0-(e+k)\mathrm{f})) \\
&= \mathrm{h}^1(\boldsymbol{P}^1,\ \operatorname{Sym}^{(a-1)}(\mathcal{O} \oplus \mathcal{O}(-e)) \otimes \mathcal{O}(b-e-k)) \\
&= \begin{cases} 0 & \text{if } b-ae-k>-2 \\ 1 & \text{if } b-ae-k=-2, \end{cases}
\end{aligned}$$

by easy computation. Thus, if $b-ae-k>-2$, then $\mathcal{E} \cong \mathcal{O}_Z(D) \oplus \mathcal{O}_Z(C_0+(e+k))$, $k=1,\ 2$ by the equality above and the exact sequence (4.6.1). This is the case of Example (3.6). In the case that $b-ae-k=-2$, we have $k=2$ and $D \cdot C_0=0$ by the equalities (4.6.2) and (4.6.3). Therefore $\mathcal{E}$ satisfies the exact sequence

$$0 \longrightarrow \mathcal{O}_Z(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(C_0+(e+2)\mathrm{f}) \longrightarrow 0,$$

and the restriction $\mathcal{E}_{C_0}$ to the curve C_0 is isomorphic to $\mathcal{O}_{C_0}(1) \oplus \mathcal{O}_{C_0}(1)$. Thus this is the case of Example (3.7).

(4.7) Case (II). By the similar argument as in (4.5) we have an exact sequence

$$(4.7.1) \qquad 0 \longrightarrow \mathcal{O}_Z(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(-K_Z) \longrightarrow 0.$$

Let us remark that this exact sequence corresponds to an element of

$$\operatorname{Ext}^1(\mathcal{O}_Z(-K_Z),\ \mathcal{O}_Z(D)) \cong \mathrm{H}^1(Z,\ \mathcal{O}_Z(D+K_Z)) \cong (\mathrm{H}^1(Z,\ \mathcal{O}_Z(-D)))^{\vee}$$

by Serre duality, where $^{\vee}$ denotes the dual vector space.

At first, we study the case that $\rho(Z)>2$, where $\rho(Z)$ denotes the Picard number of Z.

Since Z is a nonsingular Del Pezzo surface, we have

$$\overline{NE}(Z) = \sum_j \boldsymbol{R}_{\geqq 0}[E_j], \tag{4.7.2}$$

where $\boldsymbol{R}_{\geqq 0}[E_j]$ is an extremal ray generated by an extremal curve E_j. Notice that the assumption $\rho(Z)>2$ implies that all E_j's are (-1)-curves.

Restricting the exact sequence (4.7.1) to a (-1)-curve E,

$$0 \longrightarrow \mathcal{O}_E(D) \longrightarrow \mathcal{E}_E \longrightarrow \mathcal{O}_E(1) \longrightarrow 0$$

because $K_Z \cdot E = -1$. Therefore we have $D \cdot E \geqq 1$ because $\mathcal{E}_E$ is ample. By (4.7.2) and Kleiman's criteria, D turns out to be an ample divisor, and then $\mathrm{H}^1(Z, \mathcal{O}_Z(-D))=0$ by Kodaira vanishing theorem. Hence we have $\mathcal{E} \cong \mathcal{O}_Z(D) \oplus \mathcal{O}_Z(-K_Z)$. This is the case of Example (3.8).

(4.8) Next we study the case that $\rho(Z) \leqq 2$. In this case we have only three possibilities that $Z \cong \boldsymbol{P}^2$, $Z \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, and $Z \cong \boldsymbol{F}_1$.

(4.9) If $Z \cong \boldsymbol{P}^2$, then $\mathcal{E} \cong \mathcal{O}_Z(D) \oplus \mathcal{O}_Z(-K_Z)$ from the exact sequence (4.7.1), and D must be an ample divisor. Therefore we have $\mathcal{E} \cong \mathcal{O}(a) \oplus \mathcal{O}(3)$, $a>0$, which is the case of Example (3.4).

(4.10) For the case that $Z \cong \boldsymbol{P}^1 \times \boldsymbol{P}^1$, we obtain an exact sequence

$$0 \longrightarrow \mathcal{O}_Z(D) \longrightarrow \mathcal{E} \longrightarrow \mathcal{O}_Z(2l_1+2l_2) \longrightarrow 0 \tag{4.10.1}$$

where l_j's are fibers of the projections $p_j\colon \boldsymbol{P}^1 \times \boldsymbol{P}^1 \rightarrow \boldsymbol{P}^1$ for $j=1, 2$. Restricting the exact sequence above to $l_j \cong \boldsymbol{P}^1$ as before, we obtain inequalities $D \cdot l_j \geqq 0$ for $j=1, 2$. If the inequalities $D \cdot l_j > 0$ hold for both j, then D is an ample divisor, and $\mathcal{E} \cong \mathcal{O}_Z(D) \oplus \mathcal{O}_Z(2l_1+2l_2)$ by the fact that $\mathrm{H}^1(Z, \mathcal{O}_Z(-D))=0$ because of the Kodaira vanishing theorem. This is the case of Example (3.8).

(4.11) If the equality $D \cdot l_2=0$ in the case of (4.10), $D \cong \alpha l_2$ for a nonnegative integer α. In this case we have $\mathcal{E}_l \cong \mathcal{O}_l(1) \oplus \mathcal{O}_l(1)$ for every fiber l of p_2 from the exact sequence (4.10.1) and from the fact that $\mathcal{E}$ is ample. Therefore we obtain an isomorphism $\mathcal{E} \cong \mathcal{O}_Z(l_1) \otimes p_2^* \varepsilon$ where ε is a vector bundle of rank 2 on $\boldsymbol{P}^1$. Since ε splits into a direct sum of two invertible sheaves, we have a decomposition $\mathcal{E} \cong \mathcal{O}_Z(l_1+al_2) \oplus \mathcal{O}_Z(l+bl_2)$, where a and b are positive integers because of the ampleness of $\mathcal{E}$. From the exact sequence (4.10.1) we have $\alpha+2=a+b$. Moreover the exact sequence (4.10.1) does not split, because D is not ample. Then we have $\mathrm{h}^1(Z, \mathcal{O}(-D))=\mathrm{h}^1(Z, \mathcal{O}_Z(-\alpha l_2)) \neq 0$. Therefore we have an inequality $\alpha \geqq 2$. We can take a nonsingular irreducible member C of the linear system $|2l_1+l_2|$, because $2l_1+l_2$ is very ample. Then we can easily see that C is a rational curve because of the adjunction formula. Restricting the exact sequence (4.10.1) to C we obtain the following exact sequence

$$0 \longrightarrow \mathcal{O}_C(2\alpha) \longrightarrow \mathcal{O}_C(2a+1) \oplus \mathcal{O}_C(2b+1) \longrightarrow \mathcal{O}_C(6) \longrightarrow 0,$$

which can not split. Therefore we must have $\mathrm{h}^1(C, \mathcal{O}_C(2\alpha-6)) \neq 0$, and then $\alpha \leqq 2$. Combining it with the inequality before, we obtain $\alpha=2$. By restricting the exact sequence (4.10.1) to the curve l_1 we can easily see that $a=b=2$. Thus this is the case of Example (3.6).

(4.12) For the case that $Z\cong \boldsymbol{F}_1$, we obtain inequalities $D\cdot C_0\geqq 1$ and $D\cdot \mathrm{f}\geqq 0$ by the similar argument as before. Moreover if $D\cdot \mathrm{f}>0$ holds, then D is ample, and $\mathcal{E}\cong \mathcal{O}_Z(D)\oplus \mathcal{O}_Z(-K_Z)$. This is the case of Example (3.8). In the case that $D\cdot \mathrm{f}=0$ we have $D\sim \alpha \mathrm{f}$ and the exact sequence (4.7.1) does not split. Therefore we have $\alpha\geqq 2$ as in (4.11). Moreover we can easily see the isomorphism $\mathcal{E}\cong \mathcal{O}_Z(C_0+a\mathrm{f})\oplus \mathcal{O}_Z(C_0+b\mathrm{f})$ by restricting the exact sequence (4.7.1) to the fiber f. By restricting (4.7.1) to the minimal section C_0 and to a nonsingular member of the linear system $|C_0+\mathrm{f}|$, we can see that $\alpha=2$ and $\mathcal{E}\cong \mathcal{O}_Z(C_0+2\mathrm{f})\oplus \mathcal{O}_Z(C_0+3\mathrm{f})$ by the similar computation as in (4.11). This is the case of Example (3.6).

(4.13) Case (III). We denote the surface $\boldsymbol{F}_1$ by S and the morphism of blow-down by $\sigma: S\to \boldsymbol{P}^2\cong Z$. Taking the fiber product, we get a $\boldsymbol{P}^1$-bundle $\tilde{Y}=Y\times_Z S\cong \boldsymbol{P}_S(\sigma^*\mathcal{E})$ over S. We denote the projections $\tilde{Y}\to Y$ and $\tilde{Y}\to S$ by τ and ψ respectively. Then we have a section $\tilde{\Gamma}$ of $\psi: \tilde{Y}\to S$ whose image by τ is equal to Γ, that is, $\tilde{\Gamma}$ is the proper transform of Γ by τ. Notice that $\tau_{\tilde{\Gamma}}: \tilde{\Gamma}\to \Gamma$ is an isomorphism. Therefore we have $(\tau^*L)_{\tilde{\Gamma}}=\tau^*_{\tilde{\Gamma}}(L_\Gamma)=C_0+2\mathrm{f}$ where C_0 is the minimal section of $S\cong \boldsymbol{F}_1$. Then we obtain an exact sequence

$$0\longrightarrow \mathcal{O}_{\tilde{Y}}(\tau^*L-\tilde{\Gamma})\longrightarrow \mathcal{O}_{\tilde{Y}}(\tau^*L)\longrightarrow \mathcal{O}_{\tilde{\Gamma}}(C_0+2\mathrm{f})\longrightarrow 0$$

and applying the functor ψ_* we get the following exact sequence

$$(4.13.1)\qquad 0\longrightarrow \mathcal{O}_S(D)\longrightarrow \sigma^*\mathcal{E}\longrightarrow \mathcal{O}_S(C_0+2\mathrm{f})\longrightarrow 0$$

where D is a divisor on S. Restricting the last exact sequence to a fiber $\mathrm{f}\cong \boldsymbol{P}^1$ and using the fact that $(\sigma^*\mathcal{E})_{\mathrm{f}}$ is ample, we obtain an inequality $D\cdot \mathrm{f}\geqq 1$. Restricting it to the minimal section C_0 and using the fact that $(\sigma^*\mathcal{E})_{C_0}$ is trivial, we get the equality $D\cdot C_0=-1$. If we have the inequality $D\cdot \mathrm{f}\geqq 2$, then this is the case of Example (3.9).

(4.14) As for the case that we have the equality $D\cdot \mathrm{f}=1$, we have $D\sim C_0$. From the exact sequence (4.13.1) we obtain that $\sigma^*\mathcal{E}_{\mathrm{f}}\cong \mathcal{O}_{\mathrm{f}}(1)\oplus \mathcal{O}_{\mathrm{f}}(1)$. Therefore we obtain an isomorphism $\sigma^*\mathcal{E}\cong \mathcal{O}_S(C_0+a\mathrm{f})\oplus \mathcal{O}_S(C_0+b\mathrm{f})$. By restricting this isomorphism to the curve C_0, we have $a=b=1$, that is, $\mathcal{E}\cong \mathcal{O}_Z(1)\oplus \mathcal{O}_Z(1)$, which is the case of Example (3.4).

(4.15) Case (IV-1). In this case we set up S, σ, $\tilde{Y}$, etc. as in (4.13). Then we can realize the surface $\tilde{\Gamma}$, which is the normalization of Γ, as a section of $\psi: \tilde{Y}\to S$ and the canonical morphism $\pi: \tilde{\Gamma}\to \Gamma$ as the restriction of σ to $\tilde{\Gamma}$. By the similar argument as in (4.13) we obtain an exact sequence

$$(4.15.1)\qquad 0\longrightarrow \mathcal{O}_S(D)\longrightarrow \sigma^*\mathcal{E}\longrightarrow \mathcal{O}_S(C_0+3\mathrm{f})\longrightarrow 0.$$

Restricting it to f and C_0, we obtain an inequality $D\cdot \mathrm{f}\geqq 1$ and an equality

$D\cdot C_0=-2$. If we have the inequality $D\cdot \mathrm{f}\geqq 3$, then this is the case of Example (3.10).

(4.16) If we assume that $D\cdot \mathrm{f}=2$, then $D=2C_0$. We can easily see that $\mathcal{E}$ is a uniform vector bundle on $Z\cong \boldsymbol{P}^2$, because of the fact that $\mathcal{E}$ is ample and $\det \mathcal{E}\cong \mathcal{O}_Z(3)$. Let us remark that $\mathrm{H}^i(Z, \mathcal{E}\otimes\mathcal{O}_Z(k))\cong \mathrm{H}^i(S, \sigma^*\mathcal{E}\otimes\mathcal{O}_S(k(C_0+\mathrm{f})))$, for every integer k and $i=0$, 1, and 2. From the exact sequence obtained by tensoring $\mathcal{O}_S(-2(C_0+\mathrm{f}))$ with (4.15.1), we can easily see that $\mathrm{h}^1(S, \sigma^*\mathcal{E}\otimes \mathcal{O}_S(-2(C_0+\mathrm{f})))\neq 0$. Therefore the vector bundle $\mathcal{E}$ does not split into the direct sum of two invertible sheaves. Hence $\mathcal{E}$ must be isomorphic to the tangent bundle of $Z\cong \boldsymbol{P}^2$ by Van de Ven's theorem [**OSS**]. On the other hand, we have $c_2(\mathcal{E})=c_2(\sigma^*\mathcal{E})=4$ from the exact sequence (4.15.1). This is a contradiction. Therefore we have $D\cdot \mathrm{f}\neq 2$.

(4.17) For the case that $D\cdot \mathrm{f}=1$, we have $D=C_0-\mathrm{f}$. In this case, we obtain $\mathcal{E}\cong \mathcal{O}_Z(1)\oplus\mathcal{O}_Z(1)$ by the similar argument as in (4.14). Thus this is the case of Example (3.4).

(4.18) Case (IV-2). Let $\pi: S\to \Gamma$ be the minimal resolution of Γ (if Γ itself is nonsingular we regard $\pi=id$.) and σ the composition of π and φ_Γ, that is, $\sigma=\varphi_\Gamma\cdot\pi$. Let us denote the exceptional divisors of σ by $\{E_i\}_{i\in I}$. Then we have an effective divisor $E=\sum_i a_iE_i$ such that $K_S\sim \sigma^*K_Z+E$. In this case we have $L_\Gamma\sim -K_\Gamma$, and $\pi^*K_\Gamma\sim K_S$ as studied in [**HW**]. For an irreducible curve C on Z we have $(-K_S)\cdot\bar{C}>0$, because of the equality $\pi^*K_\Gamma\sim K_S$, where $\bar{C}$ is the proper transform of C by σ. Therefore we have $(-K_Z)\cdot C>0$ for any irreducible curve C on Z. Moreover we can easily check the equality $(-K_Z)^2=(-K_S)^2-K_S\cdot E$, and then we have $(-K_Z)^2>0$ because $(-K_S)^2>0$, $-K_S$ is nef and E is effective. Therefore $-K_Z$ turns out to be ample, and then the surface Z is a nonsingular Del Pezzo surface. Let $\tilde{Y}=S\times_Z Y\cong \boldsymbol{P}_Z(\tau^*\mathcal{E})$ and $\tilde{L}$ the tautological divisor on $\tilde{Y}$. We denote the projections $\tilde{Y}\to Y$ and $\tilde{Y}\to S$ by τ and ψ respectively. Then we have $\tilde{L}\cong\tau^*L$. There exists a section $\tilde{\Gamma}$, which is the proper transform of Γ, obtained by the morphism $i\cdot\pi: S\to Y$ where $i:\Gamma\to Y$ is the inclusion. Since we have $(\tau^*\tilde{L})_{\tilde{\Gamma}}\cong\tau^*_{\tilde{\Gamma}}L_\Gamma\cong -\tau^*_{\tilde{\Gamma}}K_\Gamma\cong -K_S$ via the identification of $\tilde{\Gamma}$ with S, we obtain an exact sequence

$$0\longrightarrow \mathcal{O}_S(\tilde{D})\longrightarrow \sigma^*\mathcal{E}\longrightarrow \mathcal{O}_S(-K_S)\longrightarrow 0,$$

where $\tilde{D}$ is a divisor on S. Restricting the exact sequence above to an irreducible curve C on S, we get the following conditions according to the property of the curve C

(4.18.1) For a σ-exceptional (-1)-curve C on S, we have $\tilde{D}\cdot C=-1$.

(4.18.2) For a (-1)-curve C whose image by σ is still a curve on Z, we have $\tilde{D}\cdot C\geqq 1$.

(4.18.3) For a (-2)-curve C, we have $\tilde{D}\cdot C=0$.
(4.18.4) For a 0-curve C on S, we have $\tilde{D}\cdot C\geqq 0$.

On the other hand, E satisfies the same equalities in (4.18.1) and (4.18.3) because E is numerically equivalent to K_S relative to σ. Therefore we have $(\tilde{D}-E)\cdot C=0$ for every σ-exceptional curve C. Then there exists a divisor D on Z, such that $\tilde{D}-E=\sigma^*D$. From the inequality (4.18.2) we have $D\cdot C\geqq 1$ for every (-1)-curve C on Z. Let us remark that $\det\mathcal{E}\cong\mathcal{O}_Z(D-K_Z)$ by easy calculation.

(4.19) If the divisor D is ample, then this is the case of Example (3.11). If $\rho(Z)>2$, then D is ample because $\overline{NE}(Z)$ is generated by (-1)-curves.

(4.20) From now on we assume that D is not ample. Then the Picard number $\rho(Z)\leqq 2$ as remarked above.

(4.21) For the case that $Z=\boldsymbol{F}_1$, we have $D\cdot C_0\geqq 1$ and $D\cdot \mathrm{f}\geqq 0$ from the argument in (4.18). Therefore $D\cdot\mathrm{f}=0$ because D is not ample. Then $(\det\mathcal{E})\cdot\mathrm{f}=(D-K_Z)\cdot\mathrm{f}=2$ and $\mathcal{E}_{\mathrm{f}}\cong\mathcal{O}_{\mathrm{f}}(1)\oplus\mathcal{O}_{\mathrm{f}}(1)$. Thus we obtain an isomorphism

$$\mathcal{E}\cong\mathcal{O}_Z(C_0+a\mathrm{f})\oplus\mathcal{O}_Z(C_0+b\mathrm{f})$$

where a and b are integers with $2\leqq a\leqq b$. From the fact that

$$\mathrm{H}^0(Z,\ \mathcal{E}\otimes(\det\mathcal{E})^{-1}\otimes\mathcal{O}(-K_Z))\neq 0$$

a must be equal to 2 or 3. This is a case of Example (3.6).

(4.22) The case that $Z=\boldsymbol{P}^1\times\boldsymbol{P}^1$. Similarly we have $D\cdot l_i\geqq 0$ for $i=1,2$ where l_i is a fiber of the projection $p_i:\boldsymbol{P}^1\times\boldsymbol{P}^1\to\boldsymbol{P}^1$. Since D is not ample, we may assume that $D\cdot l_2=0$. Then $(\det\mathcal{E})\cdot l_2=(D-K_Z)\cdot l_2=2$. By the similar argument above we obtain

$$\mathcal{E}\cong\mathcal{O}_Z(l_1+l_2)\oplus\mathcal{O}_Z(l_1+al_2)$$

or

$$\mathcal{E}\cong\mathcal{O}_Z(l_1+2l_2)\oplus\mathcal{O}_Z(l_1+al_2)$$

where a is a positive integer in each case. So they are cases of Example (3.6).

(4.23) For the case that $Z=\boldsymbol{P}^2$, we can easily see that $\mathcal{O}_Z(D)\cong\mathcal{O}_Z$ or $\mathcal{O}_Z(-1)$, and $\det\mathcal{E}\cong\mathcal{O}_Z(3)$ or $\mathcal{O}_Z(2)$ respectively because $\det\mathcal{E}\cdot H=D\cdot H-K_Z\cdot H=D\cdot H+3\geqq 2$, where H denotes the line in $Z=\boldsymbol{P}^2$. Then it is easy to see that $\mathcal{E}$ is a uniform vector bundle on $\boldsymbol{P}^2$. Therefore we have three possibilities that

$$\mathcal{E}\cong\mathcal{O}_Z(1)\oplus\mathcal{O}_Z(1)$$

$$\mathcal{E}\cong\mathcal{O}_Z(1)\oplus\mathcal{O}_Z(2)$$

$$\mathcal{E}\cong\mathcal{T}_Z$$

by Van de Ven's theorem [**OSS**] and the condition on $\det \mathcal{E}$. Then they are the cases of Example (3.4) and (3.5).

(4.24) Combining all the possibilities above together, we obtain the following theorem.

THEOREM (4.25). *Let (Y, L) be a polarized threefold such that Y is a normal variety with at worst $\boldsymbol{Q}$-factorial terminal singularities, and that the anti-adjoint divisor $-(K_Y+L)$ is linearly equivalent to a non-zero effective divisor Δ. Assume that Y has a basis of dimension* 2 *with respect to the ample decomposition $-K_Y \sim L+\Delta$. Then there exists a pair of a nonsingular projective surface Z and an ample vector bundle $\mathcal{E}$ of rank* 2 *on Z which is described in one of the Examples* (3.4)-(3.11), *such that (Y, L) is isomorphic to $(\boldsymbol{P}_Z(\mathcal{E}), H)$ where H denotes the tautological divisor on $\boldsymbol{P}_Z(\mathcal{E})$.*

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