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# Global real analytic angle parameters for Teichmüller spaces

Dedicated to the memory of Professor Kiyoshi Niino

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## 1. Introduction.

A Fuchsian group $G$ acting on the unit disk $\boldsymbol{D}$ is of *type* $(g, 0, m)$, if the quotient space $\boldsymbol{D}/G$ is a Riemann surface of genus $g$ with $m$ holes. This Riemann surface is also called of *type* $(g, 0, m)$. From now on, let $(g, 0, m)$ satisfy $2g+m \geqq 3$. This condition means that $G$ is non-elementary. Teichmüller space $T(g, 0, m)$ is the set of equivalence classes of marked Fuchsian groups of type $(g, 0, m)$ and a global real analytic manifold of dimension $6g+3m-6$. There are various methods parametrizing $T(g, 0, m)$. For example, $T(g, 0, m)$ is parametrized global real analytically by some lengths of closed geodesics and intersection angles between geodesics on a Riemann surface represented by a marked Fuchsian group (see Keen [**5**]). Such lengths and angles are called *length parameters* and *angle parameters*, respectively.

It is known that length parameters parametrize $T(g, 0, m)$ global real analytically (see for example, [**3**], [**6**], [**8**], [**9**] and [**18**]). We denote by $N_1(g, 0, m)$ the minimal number of length parameters which describe $T(g, 0, m)$ global real analytically. Recently, Schmutz [**14**] showed that

$$N_1(g, 0, 0) = \dim(T(g, 0, 0))+1 .$$

Thus in the case of $T(2, 0, 0)$, the minimal number of length parameters is seven. In the same time, the author [**10**] also obtained this result independently and described this parameter space. The length parameter spaces are represented by some complicated polynomials.

In the hyperbolic geometry, a triangle is determined by three lengths of sides or three interior angles. Hence we can deduce that $T(g, 0, m)$ is parametrized global real analytically by some angle parameters. We denote by $N_2(g, 0, m)$ the minimal number of angle parameters which describe $T(g, 0, m)$ global real analytically. For describing the deformations of figures, angles are more suitable than lengths. Thus it seems that the information from angles is richer than the one from lengths. Hence we conjecture that

$$N_2(g, 0, m) \leqq N_1(g, 0, m).$$

Since a polynomial with respect to cosines and sines can be changed to a linear form, the angle parameter spaces become possibly easier than the length parameter spaces. Forthcoming paper, we shall show that

$$N_2(g, 0, m) = \dim(T(g, 0, m)) \text{ in the case of } (g, 0, m) \neq (2, 0, 0).$$

In Section 2, we define the one-half power of a Möbius transformation. This is useful for considering the geometry of Möbius transformations. In Section 3, we show the geometry of hyperbolic transformations and give some criterions of canonical systems of generators of types (0, 0, 3) and (1, 0, 1). In Section 4, we investigate the relations of the sides and the interior angles of some triangle which is determined by the axes of hyperbolic transformations. Then we obtain global real analytic angle parameters for $T(1, 0, 1)$ and describe this angle parameter space. Also we show that

$$N_2(1, 0, 1) = \dim(T(1, 0, 1)).$$

We also consider the positions and lengths of some geodesics on a Riemann surface of type (1, 0, 1). In Section 5, we decompose a marked Fuchsian group of type (2, 0, 0) into two basic marked Fuchsian groups of type (1, 0, 1) and consider how to combine these two groups by one angle parameter. Then we show that

$$N_2(2, 0, 0) \leqq \dim(T(2, 0, 0))+1$$

and this angle parameter space is simpler than the length parameter space stated in [**10**].

## 2. Preliminaries.

The group of Möbius transformations preserving $\boldsymbol{D}$, $M(\boldsymbol{D})$, is the group of isometries of $\boldsymbol{D}$ with respect to the Poincaré metric $d$. For distinct two points $p_1$ and $p_2$ in $\bar{\boldsymbol{D}}$, let $L(p_1, p_2)$ be the full geodesic through $p_1$ and $p_2$ with the direction from $p_1$ to $p_2$, where this direction is sometimes ignored. $L(p_1, p_2)$ divides $\bar{\boldsymbol{D}}$ into two parts. The right-hand part and the left-hand part are denoted by $r-L(p_1, p_2)$ and $l-L(p_1, p_2)$, respectively.

An elliptic element $A \in M(\boldsymbol{D})$ has the sole fixed point in $\boldsymbol{D}$. We denote it by $fp(A)$. A hyperbolic element $A \in M(\boldsymbol{D})$ has the *attracting fixed point*, $q(A)$, and the *repelling fixed point*, $p(A)$, which are characterized by $q(A)=\lim_{n\to\infty} A^n(z)$ and $p(A)=\lim_{n\to\infty} A^{-n}(z)$ for any $z \in \boldsymbol{D}$. The *axis* of $A$, $ax(A)=L(p(A), q(A))$, and the *translation length* of $A$, $tl(A)=\inf\{d(z, A(z))\,|\,z\in \boldsymbol{D}\}$, are characterized by

$$ax(A)=\{z\in \boldsymbol{D}\mid d(z, A(z))=tl(A)\},$$

$$\cosh\frac{tl(A)}{2}=\frac{|tr\,A|}{2}.$$

We remark that $q(A)=p(A^{-1})$ and $l-ax(A)=r-ax(A^{-1})$.

Let $A$ be a hyperbolic element of a Fuchsian group $G$ acting on $\boldsymbol{D}$. Then $ax(A)$ projects on a closed geodesic on $\boldsymbol{D}/G$ whose length is $tl(A)$ and corresponds to $|tr\,A|$ real analytically.

To define a marked Fuchsian group, we give the following:

PROPOSITION 2.1 (Keen [**5**]). *Let $G$ be a Fuchsian group of type $(g, 0, m)$. Then $G$ has a system of generators*

$$\Sigma=(A_1, B_1, \cdots, A_g, B_g, E_1, \cdots, E_m);$$

$$E_m E_{m-1}\cdots E_1 C_g C_{g-1}\cdots C_1=identity,$$

*where $A_j$, $B_j$, $C_j=[B_j, A_j]=B_j^{-1}A_j^{-1}B_jA_j$ $(j=1, \cdots, g)$ and $E_k$ $(k=1, \cdots, m)$ are hyperbolic with axes illustrated as in Figure 2.1, and if $g=0$ (resp. $m=0$), then $A_j$, $B_j$ and $C_j$ (resp. $E_k$) are omitted.*

A system $\Sigma$ mentioned in Proposition 2.1 is called a *canonical system of generators* of $G$. A pair $(G, \Sigma)$ of $G$ and this system $\Sigma$ is called a *marked Fuchsian group*. Two marked Fuchsian groups $(G_1, \Sigma_1)$ and $(G_2, \Sigma_2)$ are *equivalent*, if $G_2=hG_1h^{-1}$ and $\Sigma_2=h\Sigma_1h^{-1}$ for some $h\in M(\boldsymbol{D})$. Teichmüller space $T(g, 0, m)$ is the set of equivalence classes of $(G, \Sigma)$ of type $(g, 0, m)$.

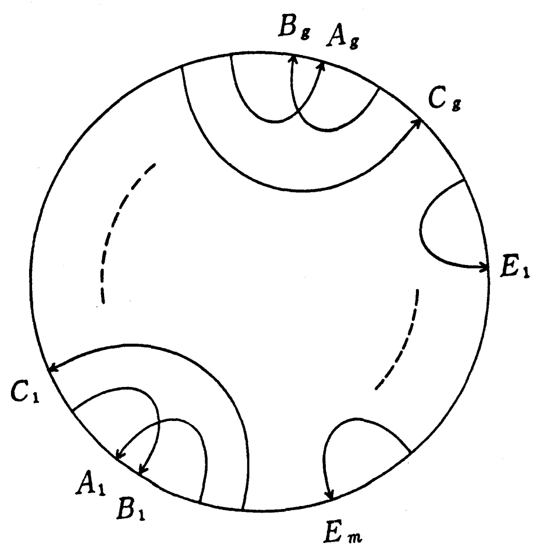


FIGURE 2.1.

One of the matrix representations of a Möbius transformation $A$ is denoted by $\tilde{A}$. Since the matrix representations of $A$ are $\pm\tilde{A}$, $\tilde{A}$ is determined up to the sign. For two Möbius transformations $A$ and $B$, the matrix $[\tilde{B}, \tilde{A}]$ is invariant under the choice of $\tilde{A}$ and $\tilde{B}$. Then we remark the following:

REMARK 2.2. The matrix $[\tilde{B}, \tilde{A}]$ is uniquely determined by $A$ and $B$.

The following equations of commutator traces of $X$, $Y$, $Z=(YX)^{-1}\in SL(2, \boldsymbol{C})$ are useful: for $\varepsilon$, $\eta\in\{\pm 1\}$,

$$\begin{aligned} tr[X, Y] &= tr[X^{\varepsilon}, Y^{\eta}] = tr[Y^{\varepsilon}, X^{\eta}] \\ &= tr[Y^{\varepsilon}, Z^{\eta}] = tr[Z^{\varepsilon}, Y^{\eta}] \\ &= tr[Z^{\varepsilon}, X^{\eta}] = tr[X^{\varepsilon}, Z^{\eta}]. \end{aligned}$$

Finally, we define the one-half power of $A\in M(\boldsymbol{D})$. For this purpose, we notice the following:

LEMMA 2.3. *Let $A$ be an element of $M(\boldsymbol{D})$. If $A$ is hyperbolic or parabolic, then $X\in M(\boldsymbol{D})$ satisfying $X^2=A$ is uniquely determined. Otherwise, such $X$ is not uniquely determined.*

In fact, if $A$ is the elliptic element with the angle of rotation $\theta$ and the fixed point $z_0\in \boldsymbol{D}$, then $X$ is the elliptic element with the angle of rotation $\theta/2$ or $\pi+\theta/2$ and the fixed point $z_0\in\boldsymbol{D}$. If $A$ is the identity, then $X$ is any elliptic element of order 2 or the identity.

DEFINITION. Let $A\in M(\boldsymbol{D})$ be hyperbolic or parabolic. Then $X\in M(\boldsymbol{D})$ satisfying $X^2=A$ is called the *one-half power* of $A$ and denoted by $A^{1/2}$.

$A^{1/2}$ is determined by $A$ as follows:

PROPOSITION 2.4. *Let $A\in M(\boldsymbol{D})$ be hyperbolic or parabolic. If $\tilde{A}$ is the matrix representation of $A$ with negative trace (resp. positive trace), then the matrix representations of $A^{1/2}$ are*

$$\frac{\pm 1}{\sqrt{|tr\,A|+2}}(\tilde{A}-I) \quad (resp.\ \frac{\pm 1}{\sqrt{|tr\,A|+2}}(\tilde{A}+I)).$$

*Thus*

$$|tr\,A^{1/2}| = \sqrt{|tr\,A|+2}.$$

This is shown by a simple calculation.

Since $(A^{1/2})^{-1}=(A^{-1})^{1/2}$, they are denoted by $A^{-1/2}$.

## 3. The geometry of hyperbolic transformations.

In this section, we state some results of the geometry of Möbius transformations related to our parametrizations, without proofs. We refer to [13] in detail.

First we state the positions of the axes of two hyperbolic transformations.

LEMMA 3.1. *Let $A, B \in M(\boldsymbol{D})$ be hyperbolic. Then $ax(A)$ and $ax(B)$ intersect, if and only if $tr[\tilde{B}, \tilde{A}]<2$.*

LEMMA 3.2. *Let $A, B \in M(\boldsymbol{D})$ be hyperbolic elements with intersecting axes. Then eight elements $A^{\varepsilon}B^{\eta}$, $B^{\varepsilon}A^{\eta}$; $\varepsilon, \eta \in \{\pm 1\}$ are hyperbolic. Let $p$ be the intersection point of $ax(A)$ and $ax(B)$. Then*

$$ax(BA) = L(A^{-1/2}(p), B^{1/2}(p)),$$

$$ax(B^{-1}A) = L(A^{-1/2}(p), B^{-1/2}(p)),$$

$$d(A^{-1/2}(p), B^{1/2}(p)) = \frac{tl(BA)}{2},$$

$$d(A^{-1/2}(p), B^{-1/2}(p)) = \frac{tl(B^{-1}A)}{2}.$$

*Especially, $ax(A)$, $ax(B)$ and $ax(BA)$ determine the triangle with vertices $p$, $A^{-1/2}(p)$ and $B^{1/2}(p)$ (see Figure 3.1).*

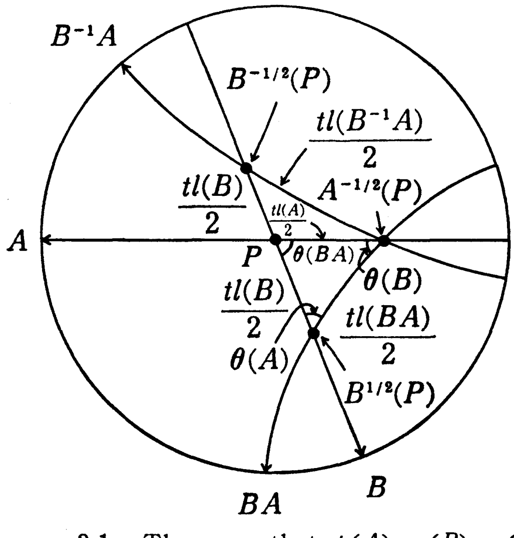


FIGURE 3.1. The case that $p(A)$, $q(B)$, $q(A)$ and $p(B)$ are arranged clockwise in order on the circle at infinity.

Let $X, Y \in M(\boldsymbol{D})$ be hyperbolic. If $Z=(YX)^{-1}$ is not hyperbolic, then $ax(X)$ and $ax(Y)$ do not intersect, by Lemma 3.2. If $Z$ is hyperbolic, then $ax(X)$,

$ax(Y)$ and $ax(Z)$ are characterized as follows:

LEMMA 3.3. *Let $X$, $Y$ and $Z$ be hyperbolic elements of $M(\boldsymbol{D})$ satisfying $ZYX$=identity. Then the axes of $X$, $Y$ and $Z$ are positioned as one of the following*:

(a) *three axes are disjoint,*

(b) *three axes are parallel (namely, they have one common endpoint on the circle at infinity) or coincident,*

(c) *three axes do not intersect at one point and any two axes intersect each other.*

*Thus, if some two axes are disjoint, parallel, coincident or intersecting, then three axes are also in the same situation. Furthermore, the orientations of the axes are determined as in Figure 3.2, where the pair $(U, V, W)$ is any permutation of $X$, $Y$ and $Z$. These cases are characterized by $tr\,\tilde{X}$, $tr\,\tilde{Y}$ and $tr\,\tilde{Y}\tilde{X}$ as follows*:

($a_1$) $\Leftrightarrow tr\,\tilde{X}\ tr\,\tilde{Y}\ tr\,\tilde{Y}\tilde{X}<0$,

($a_2$) $\Leftrightarrow tr\,\tilde{X}\ tr\,\tilde{Y}\ tr\,\tilde{Y}\tilde{X}>0$, $tr[\tilde{Y}, \tilde{X}]>2$,

(b) $\Leftrightarrow tr\,\tilde{X}\ tr\,\tilde{Y}\ tr\,\tilde{Y}\tilde{X}>0$, $tr[\tilde{Y}, \tilde{X}]=2$,

(c) $\Leftrightarrow tr\,\tilde{X}\ tr\,\tilde{Y}\ tr\,\tilde{Y}\tilde{X}>0$, $tr[\tilde{Y}, \tilde{X}]<2$.

REMARK 3.4. $tr[\tilde{Y}, \tilde{X}]=tr^2\tilde{X}+tr^2\tilde{Y}+tr^2\tilde{Y}\tilde{X}-tr\,\tilde{X}\,tr\,\tilde{Y}\,tr\,\tilde{Y}\tilde{X}-2$ and $tr\,\tilde{X}\ tr\,\tilde{Y}\ tr\,\tilde{Y}\tilde{X}$ are invariant under the choice of matrix representations. In the case of ($a_1$), we have $tr[\tilde{Y}, \tilde{X}]>18$.

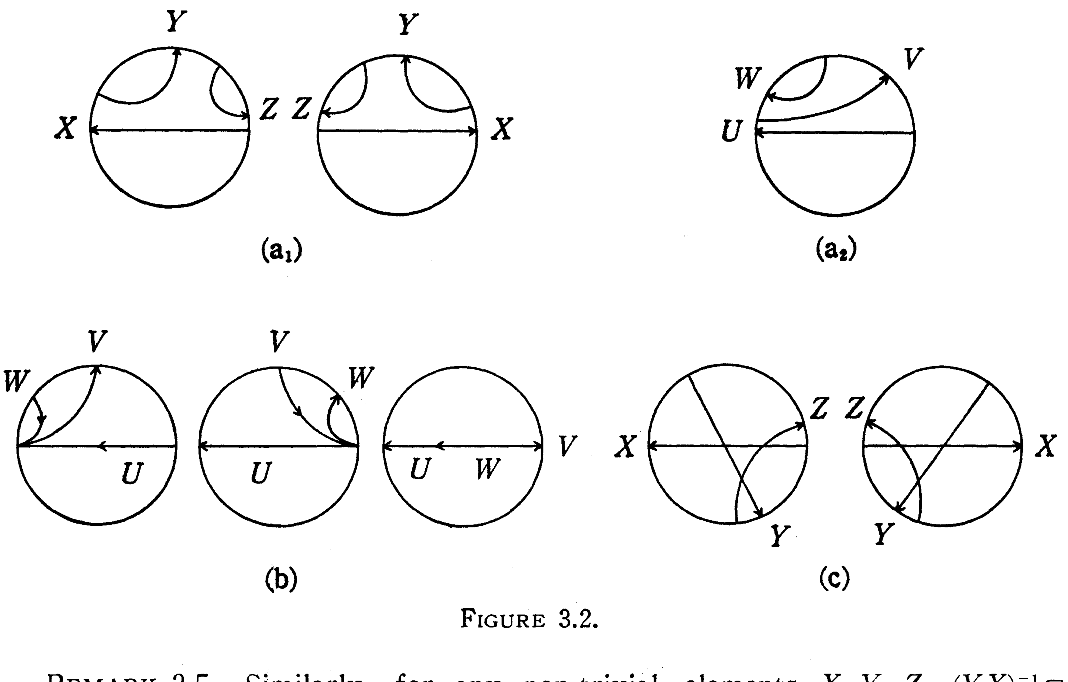


FIGURE 3.2.

REMARK 3.5. Similarly, for any non-trivial elements $X$, $Y$, $Z=(YX)^{-1}\in M(\boldsymbol{D})$, the positions of their fixed points and the *directions* of their actions are characterized by such three traces (see [13]).

Next, we give some criterions of canonical systems of generators of types (0, 0, 3) and (1, 0, 1).

LEMMA 3.6. *Let* $X, Y, Z=(YX)^{-1}\in M(\boldsymbol{D})$ *be hyperbolic. The pair* $(X, Y, Z)$ *is a canonical system of generators of type* (0, 0, 3) *if and only if* $p(X)$, $q(X)$, $p(Y)$, $q(Y)$, $p(Z)$ *and* $q(Z)$ *are arranged clockwise in order on the circle at infinity.*

From Lemmas 3.3 ($a_1$) and 3.6, we obtain the following:

LEMMA 3.7. *Let* $X, Y, Z=(YX)^{-1}\in M(\boldsymbol{D})$ *be hyperbolic. Either* $(X, Y, Z)$ *or* $(Z^{-1}, Y^{-1}, X^{-1})$ *is a canonical system of generators of type* (0, 0, 3) *if and only if*

$$tr\,\tilde{X}\,tr\,\tilde{Y}\,tr\,\tilde{Y}\tilde{X}<0.$$

LEMMA 3.8. *Let* $A, B\in M(\boldsymbol{D})$ *be hyperbolic. The pair* $(A, B, [B, A]^{-1})$ *is a canonical system of generators of type* (1, 0, 1) *if and only if the following two conditions are satisfied*:

(i) $A$ *and* $B$ *have intersecting axes such that* $p(A)$, $q(B)$, $q(A)$ *and* $p(B)$ *are arranged clockwise in order on the circle at infinity,*

(ii) $[B, A]$ *is hyperbolic.*

This lemma is also shown by Keen's results of [4], [5] and [6]. From Lemmas 3.3 (c) and 3.8, we obtain the following:

LEMMA 3.9. *Let* $A, B\in M(\boldsymbol{D})$ *be hyperbolic. Either* $(A, B, [B, A]^{-1})$ *or* $(A^{-1}, B, [B, A^{-1}]^{-1})$ *is a cononical system of generators of type* (1, 0, 1) *if and only if*

$$tr\,\tilde{A}\,tr\,\tilde{B}\,tr\,\tilde{B}\tilde{A}>0 \quad and \quad tr[\tilde{B}, \tilde{A}]<-2.$$

Finally, we state the following result.

PROPOSITION 3.10. *Let* $A, B\in M(\boldsymbol{D})$ *be hyperbolic elements with intersecting axes. Let* $p$ *be the intersection point of these axes. Let* $R\in M(\boldsymbol{D})$ *be elliptic of order* 2 *with the fixed point* $p$.

(i) *The axes of* $A^{\varepsilon}B^{\eta}$, $B^{\varepsilon}A^{\eta}$; $\varepsilon, \eta\in\{\pm 1\}$ *determine the quadrilateral with sides* $tl(BA)/2$, $tl(B^{-1}A)/2$, $tl(BA)/2$ *and* $tl(B^{-1}A)/2$, *and vertices* $A^{-1/2}(p)$, $B^{1/2}(p)$, $A^{1/2}(p)$ *and* $B^{-1/2}(p)$.

*Further, suppose that* $C=[B, A]$ *is hyperbolic and* $p(A)$, $q(B)$, $q(A)$ *and* $p(B)$ *are arranged clockwise in order on the circle at infinity. Then the following holds*:

(ii) $(A, B^{-1}A^{-1}B, C^{-1})$, $(BA, B^{-1}A^{-1}, C^{-1})$ *and* $(A^{-1}BA, B^{-1}, C^{-1})$ *are canonical systems of generators of type* (0, 0, 3).

(iii) *$A$, $B$, $C$ and $R$ satisfy the following*:

$$A = RA^{-1}R = [R,\ A^{1/2}],$$
$$B = RB^{-1}R = [R,\ B^{1/2}],$$
$$C^{1/2} = RBA,$$
$$\tilde{R} = \frac{\pm 1}{\sqrt{det(\tilde{B}\tilde{A}-\tilde{A}\tilde{B})}}(\tilde{B}\tilde{A}-\tilde{A}\tilde{B}).$$

(iv) *$C^{-1/2}A$, $C^{-1/2}B^{-1}$ and $C^{-1/2}BA$ are elliptic of order 2 satisfying*

$$fp(C^{-1/2}A) = (ABA)^{-1/2}(p) = (BA)^{-1/2}A^{-1/2}(p),$$
$$fp(C^{-1/2}B^{-1}) = A^{-1/2}(p),$$
$$ax(ABA) = L(fp(C^{-1/2}A),\ p).$$

(v) *Let $A_{1/2}$ (resp. $A_{-1/2}$) be elliptic of order 2 with the fixed point $A^{1/2}(p)$ (resp. $A^{-1/2}(p)$), namely, $A_{1/2}=A^{1/2}RA^{-1/2}$ and $A_{-1/2}=A^{-1/2}RA^{1/2}$. Similarly, $B_{1/2}$ and $B_{-1/2}$ are defined. Then we have*

$$A = RA_{-1/2} = A_{1/2}R,$$
$$B = RB_{-1/2} = B_{1/2}R,$$
$$BA = B_{1/2}A_{-1/2},$$
$$AB = A_{1/2}B_{-1/2},$$
$$C = B_{-1/2}A_{1/2}B_{1/2}A_{-1/2}.$$

*Especially, $C$ is determined by four elliptic transformations of order* **2** *whose fixed points are four vertices of this quadrilateral* (*see Figure* 3.3).

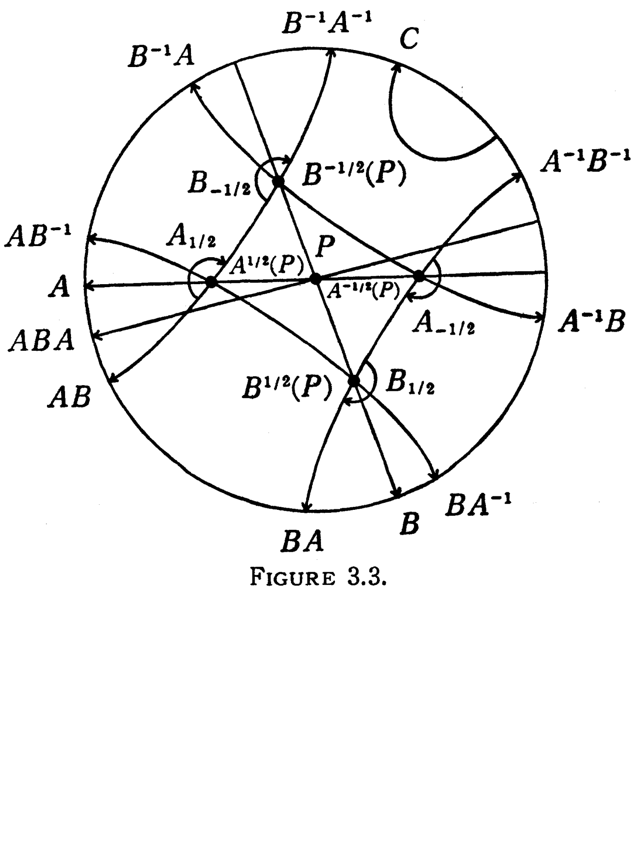


FIGURE 3.3.

## 4. A parametrization of $T(1, 0, 1)$.

In this section, we show a parametrization of $T(1, 0, 1)$ by three angle parameters.

First, we recall the cosine formula of a hyperbolic triangle.

LEMMA 4.1 (Beardon [2, p. 148]). *Let $\alpha$, $\beta$ and $\gamma$ be interior angles of a hyperbolic triangle. Let $c$ be the side opposite to $\gamma$. Then*

$$\cosh c=\frac{\cos\gamma+\cos\alpha\cos\beta}{\sin\alpha\sin\beta}.$$

Let $\Sigma_{(1,0,1)}=(A, B, C^{-1})$; $C=[B, A]$ be a canonical system of generators of type (1, 0, 1). Let $p$ be the intersection point of $ax(A)$ and $ax(B)$. Then by Lemma 3.2, $ax(A)$, $ax(B)$ and $ax(BA)$ determine the triangle $T$ with vertices $p$, $A^{-1/2}(p)$ and $B^{1/2}(p)$. Let $\theta(A)$, $\theta(B)$ and $\theta(BA)$ be interior angles of $T$ as in Figure 3.1.

LEMMA 4.2. *Traces of $A$, $B$, $BA$ and $[B, A]$ are determined by these three angles as follows:*

$$tr\tilde{A}=\frac{2\varepsilon(\cos\theta(A)+\cos\theta(B)\cos\theta(BA))}{\sin\theta(B)\sin\theta(BA)},$$

$$tr\tilde{B}=\frac{2\eta(\cos\theta(B)+\cos\theta(BA)\cos\theta(A))}{\sin\theta(BA)\sin\theta(A)},$$

$$tr\tilde{B}\tilde{A}=\frac{2\varepsilon\eta(\cos\theta(BA)+\cos\theta(A)\cos\theta(B))}{\sin\theta(A)\sin\theta(B)},$$

$$tr[\tilde{B}, \tilde{A}]=2-4F(\theta(A), \theta(B), \theta(BA))^2,$$

*where $\varepsilon$, $\eta\in\{\pm 1\}$ and*

$$F(x, y, z):=\frac{\cos^2x+\cos^2y+\cos^2z+2\cos x\cos y\cos z-1}{\sin x\sin y\sin z}.$$

PROOF. Since $ax(A)$ and $ax(B)$ intersect, Lemma 3.3 implies that $tr\tilde{A}\, tr\tilde{B}\, tr\tilde{B}\tilde{A}>0$. Then we set $tr\tilde{A}=\varepsilon|tr A|$, $tr\tilde{B}=\eta|tr B|$ and $tr\tilde{B}\tilde{A}=\varepsilon\eta|tr BA|$. By Lemma 3.2, three sides of $T$ are $tl(A)/2$, $tl(B)/2$ and $tl(BA)/2$. Thus the cosine formula implies that

$$tr\tilde{A}=\varepsilon|tr A|=2\varepsilon\cosh\frac{tl(A)}{2}=\frac{2\varepsilon(\cos\theta(A)+\cos\theta(B)\cos\theta(BA))}{\sin\theta(B)\sin\theta(BA)}.$$

Similarly, $tr\tilde{B}$ and $tr\tilde{B}\tilde{A}$ are obtained. And $tr[\tilde{B}, \tilde{A}]$ is determined by these three traces. Q.E.D.

Since $tr[\tilde{B}, \tilde{A}] \leqq 2$ and $[B, A]$ is hyperbolic, we have

$$tr[\tilde{B}, \tilde{A}] < -2.$$

This is equivalent to $F(\theta(A), \theta(B), \theta(BA))^2 > 1$. In general, $F(x, y, z)$ satisfies the following:

LEMMA 4.3. *Let $x, y, z \in (0, \pi)$ satisfy $x+y+z<\pi$. Then*

$$G(x, y, z) := \cos^2 x + \cos^2 y + \cos^2 z + 2\cos x \cos y \cos z - 1 > 0,$$

$$F(x, y, z) = \frac{G(x, y, z)}{\sin x \sin y \sin z} > 0,$$

$$F(x, y, z) < 1 \quad \textit{for some } x, y \textit{ and } z,$$

$$\frac{\cos x + \cos y \cos z}{\sin y \sin z} = \sqrt{\frac{G(x, y, z)}{\sin^2 y \sin^2 z} + 1} > 1.$$

PROOF. For example, $F(11\pi/12, \pi/36, \pi/36) < 1$. Since

$$\left(\frac{\cos x + \cos y \cos z}{\sin y \sin z}\right)^2 - 1 = \frac{G(x, y, z)}{\sin^2 y \sin^2 z},$$

we only prove that $G(x, y, z) > 0$. $G(x, y, z)$ is symmetric with respect to $x$, $y$ and $z$. Then we can assume $x \geqq y \geqq z$ without loss of generality.

$$\begin{aligned} G(x, y, z) &= (\cos z + \cos x \cos y)^2 - \sin^2 x \sin^2 y \\ &= (\cos z + \cos(x+y))(\cos z + \cos(x-y)). \end{aligned}$$

Since $0 \leqq x-y < x+y < \pi - z < \pi$, we obtain $\cos(x-y) > \cos(x+y) > \cos(\pi - z) = -\cos z$. Hence we have $G(x, y, z) > 0$. Q.E.D.

LEMMA 4.4. *The following statements are equivalent*:
(i) $C = [B, A]$ *is hyperbolic,*
(ii) $tr[\tilde{B}, \tilde{A}] < -2$,
(iii) $F(\theta(A), \theta(B), \theta(BA)) > 1$.

The following lemma is useful for determining matrix representations.

LEMMA 4.5. *Let $U \in M(\boldsymbol{D})$ be hyperbolic with $q(U) = e^{i\theta}$ and $p(U) = -e^{i\theta}$. Then $\tilde{U}$ is determined by $|tr U|$ as follows*:

$$\tilde{U} = \frac{\pm 1}{2}\begin{pmatrix} |tr U| & e^{i\theta}\sqrt{|tr U|^2 - 4} \\ e^{-i\theta}\sqrt{|tr U|^2 - 4} & |tr U| \end{pmatrix}.$$

This is shown by a simple calculation.

Now we parametrize $T(1, 0, 1)$ by $\theta(A)$, $\theta(B)$ and $\theta(BA)$.

LEMMA 4.6. *Let normalize* $\Sigma_{(1,0,1)}$ *such that* $q(A)=-1$, $p(A)=1$ *and* $p=0$. *Then the matrix representations of* $A$ *and* $B$ *are*

$$\frac{\pm 1}{\sin\theta(B)\sin\theta(BA)}\begin{pmatrix}\cos\theta(A)+\cos\theta(B)\cos\theta(BA) & -\sqrt{G(\theta(A),\ \theta(B),\ \theta(BA))}\\ -\sqrt{G(\theta(A),\ \theta(B),\ \theta(BA))} & \cos\theta(A)+\cos\theta(B)\cos\theta(BA)\end{pmatrix}$$

*and*

$$\frac{\pm 1}{\sin\theta(BA)\sin\theta(A)}\begin{pmatrix}\cos\theta(B)+\cos\theta(BA)\cos\theta(A) & e^{-i\theta(BA)}\sqrt{G(\theta(A),\ \theta(B),\ \theta(BA))}\\ e^{i\theta(BA)}\sqrt{G(\theta(A),\ \theta(B),\ \theta(BA))} & \cos\theta(B)+\cos\theta(BA)\cos\theta(A)\end{pmatrix},$$

*respectively. Thus* $\theta(A)$, $\theta(B)$ *and* $\theta(BA)$ *determine* $\Sigma_{(1,0,1)}$ *real analytically up to conjugation by any Möbius transformation. Hence they are global real analytic angle parameters for* $T(1,0,1)$.

PROOF. By our normalization, $q(B)=e^{-i\theta(BA)}$ and $p(B)=-e^{-i\theta(BA)}$. Thus Lemmas 4.2 and 4.5 imply that $\tilde{A}$ and $\tilde{B}$ are obtained such that each entry is a real analytic function of $\theta(A)$, $\theta(B)$ and $\theta(BA)$. Since these angles are intersection angles between geodesics, they correspond to intersection angles between geodesics on the Riemann surface represented by $\Sigma_{(1,0,1)}$. Thus they are angle parameters. Q. E. D.

Lemma 4.6 implies that any triangle in $\boldsymbol{D}$ determines $A$, $B\in M(\boldsymbol{D})$ satisfying the condition (i) of Lemma 3.8 by three interior angles. Thus from Lemmas 3.8, 4.4 and 4.6, we obtain the following theorem.

THEOREM 4.7. $T(1,0,1)$ *is parametrized global real analytically by three angle parameters* $\theta(A)$, $\theta(B)$ *and* $\theta(BA)$ *which correspond to the interior angles of the triangle determined by* $ax(A)$, $ax(B)$ *and* $ax(BA)$ *as in Figure* 3.1. *Hence* $N_2(1,0,1)=\dim(T(1,0,1))$. *This parameter space is defined by*

$$\theta(A),\ \theta(B),\ \theta(BA)\in(0,\pi), \tag{4.1}$$

$$\theta(A)+\theta(B)+\theta(BA)<\pi, \tag{4.2}$$

$$F(\theta(A),\ \theta(B),\ \theta(BA))>1. \tag{4.3}$$

We consider some geodesics and angles on the Riemann surface $R$ represented by $\Sigma_{(1,0,1)}$. Let $(a, b, (aba^{-1}b^{-1})^{-1})$ be a canonical homotopy basis of the fundamental group of $R$ corresponding to $\Sigma_{(1,0,1)}$. We put same labels on a closed curve on $R$ and the closed geodesic freely homotopic to it (see Figure 4.1). Then $a$, $b$ and $ab$ are the projections of $ax(A)$, $ax(B)$ and $ax(BA)$, respectively. These three closed geodesics determine two triangles on $R$. These triangles are congruent, since they have same interior angles. $\theta(A)$, $\theta(B)$ and $\theta(BA)$ are three interior angles of these triangles. Let $q$ be the intersection point of $a$

and $b$. Let $r(a)$ be the unique point on $a$ satisfying $d(q, r(a))=tl(A)/2$, that is, two segments $a-\{q, r(a)\}$ have same length $tl(A)/2$. Similarly, $r(b)$ is defined. From Lemma 3.2, the following result is obtained.

PROPOSITION 4.8. *$ab$ intersects $a$ and $b$ at $r(a)$ and $r(b)$, respectively. Thus the geodesic through $r(a)$ and $r(b)$ is the closed geodesic $ab$. Two segments $ab-\{r(a), r(b)\}$ have same length $tl(BA)/2$. The anti-holomorphic involution of $R$ interchanges the above two triangles and fixes their vertices $q$, $r(a)$ and $r(b)$.*

Since $R$ is determined by $\theta(A)$, $\theta(B)$ and $\theta(BA)$, the length of a closed geodesic is parametrized by these angles. From Lemmas 4.2 and 4.3, we have

$$\cosh\frac{tl(A)}{2}=\frac{\cos\theta(A)+\cos\theta(B)\cos\theta(BA)}{\sin\theta(B)\sin\theta(BA)}=\sqrt{\frac{G(\theta(A), \theta(B), \theta(BA))}{\sin^2\theta(B)\sin^2\theta(BA)}+1}$$

and

$$\cosh\frac{tl(C)}{4}=\sqrt{\frac{\cosh\frac{tl(C)}{2}+1}{2}}=F(\theta(A), \theta(B), \theta(BA)).$$

Then we obtain the following:

PROPOSITION 4.9. *The lengths of $a$ and $aba^{-1}b^{-1}$ are*

$$2\operatorname{arc}\cosh\left(\frac{\cos\theta(A)+\cos\theta(B)\cos\theta(BA)}{\sin\theta(B)\sin\theta(BA)}\right)=2\operatorname{arc}\sinh\left(\frac{\sqrt{G(\theta(A), \theta(B), \theta(BA))}}{\sin\theta(B)\sin\theta(BA)}\right)$$

*and*

$$4\operatorname{arc}\cosh(F(\theta(A), \theta(B), \theta(BA))),$$

*respectively. Similarly, the lengths of $b$ and $ab$ are obtained.*

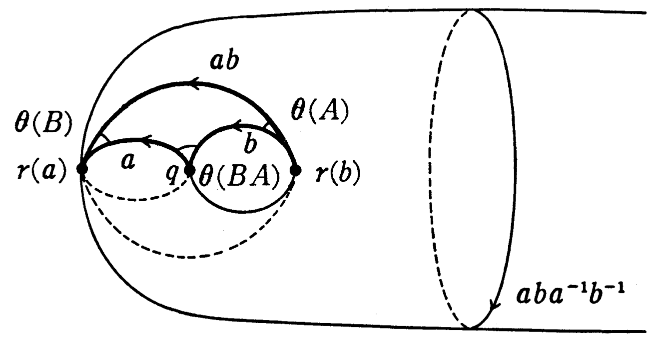


FIGURE 4.1.

## 5. A parametrization of $T(2, 0, 0)$.

In this section, we consider global real analytic angle parameters for $T(2, 0, 0)$. Using the combination theorem for two Fuchsian groups of type $(1, 0, 1)$, Lemma 4.6 and Theorem 4.7, we shall prove Theorem 5.3.

Let $\Sigma_{(2,0,0)}=(A_1, B_1, A_2, B_2)$ be a canonical system of generators of type $(2, 0, 0)$. Then $(A_1, B_1, C_1^{-1})$ and $(A_2, B_2, C_2^{-1})$ are canonical systems of generators of type $(1, 0, 1)$ and $C_1=C_2^{-1}$. Conversely, let $\Sigma_1=(A_1, B_1, C_1^{-1})$ and $\Sigma_2=(A_2, B_2, C_2^{-1})$ be canonical systems of generators of type $(1, 0, 1)$ satisfying $C_1=C_2^{-1}$. Then the combination theorem implies that the amalgamated product of two Fuchsian groups generated by $\Sigma_1$ and $\Sigma_2$ with the amalgamated subgroup generated by $C_1=C_2^{-1}$ is a Fuchsian group generated by a canonical system of generators $(A_1, B_1, A_2, B_2)$ of type $(2, 0, 0)$. Thus $\Sigma_{(2,0,0)}$ is uniquely determined, if and only if $\Sigma_1$ and $\Sigma_2$ satisfying $C_1=C_2^{-1}$ are uniquely determined.

We shall construct $\Sigma_1$ and $\Sigma_2$ satisfying $C_1=C_2^{-1}$ global real analytically by some angles between geodesics. Let $p_j$ be the intersection point of $ax(A_j)$ and $ax(B_j)$ for $j=1, 2$. Lemma 3.2 implies that the axes of $A_j$, $B_j$ and $B_jA_j$ determine the triangle $T_j$ with vertices $p_j$, $A_j^{-1/2}(p_j)$ and $B_j^{1/2}(p_j)$. Let $\theta(A_j)$, $\theta(B_j)$ and $\theta(B_jA_j)$ be three interior angles of $T_j$ as in Figure 5.1.

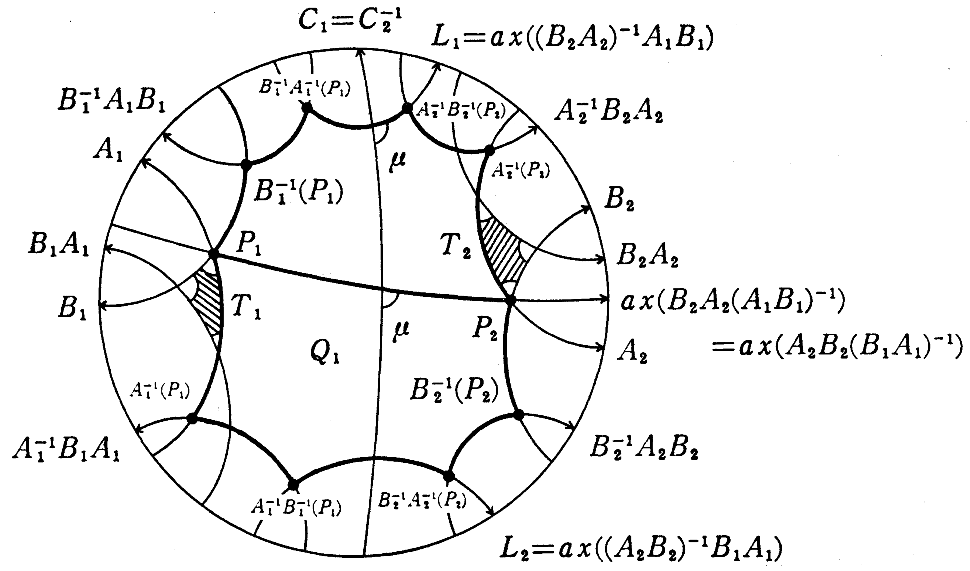


FIGURE 5.1.

Let normalize $A_1$ and $B_1$ such that

$$q(A_1)=-1, \quad p(A_1)=1 \quad \text{and} \quad p_1=0.$$

Then Lemma 4.6 implies that $\theta(A_1)$, $\theta(B_1)$ and $\theta(B_1A_1)$ determine $A_1$ and $B_1$.

Let $\hat{\Sigma}_2=(\hat{A}_2, \hat{B}_2, \hat{C}_2^{-1})$ be a canonical system of generators of type $(1, 0, 1)$

such that $q(\hat{A}_2)=-1$, $p(\hat{A}_2)=1$ and the intersection point of $ax(\hat{A}_2)$ and $ax(\hat{B}_2)$ is 0. Similarly, we define $\theta(\hat{A}_2)$, $\theta(\hat{B}_2)$ and $\theta(\hat{B}_2\hat{A}_2)$. Further, let $\hat{\Sigma}_2$ satisfy

$$\theta(\hat{A}_2)=\theta(A_2), \quad \theta(\hat{B}_2)=\theta(B_2) \quad \text{and} \quad \theta(\hat{B}_2\hat{A}_2)=\theta(B_2A_2).$$

Then Lemma 4.6 implies that $\hat{\Sigma}_2$ is determined by these three angles and conjugate to $\Sigma_2$. Thus we have

$$T\hat{\Sigma}_2T^{-1}=\Sigma_2 \quad \text{for some } T\in M(\boldsymbol{D}).$$

Since $q(C_2)=p(C_1)$ and $p(C_2)=q(C_1)$, $T$ satisfies

$$T(q(\hat{C}_2))=p(C_1) \quad \text{and} \quad T(p(\hat{C}_2))=q(C_1).$$

Then one more condition, for example,

$$T(\hat{A}_2\hat{B}_2(0))=A_2B_2(p_2)$$

determines $T$. $\hat{A}_2\hat{B}_2(0)$ and the fixed points of $C_1$ and $\hat{C}_2$ are obtained from above six angles. Thus, if $A_2B_2(p_2)$ is determined real analytically by some angles, then $T$ and $\Sigma_2$ are also determined real analytically by angles.

In order to show that some angles determine $A_2B_2(p_2)$, we notice the following:

LEMMA 5.1. *$A_j^{-1}B_j^{-1}(p_j)$, $B_j^{-1}A_j^{-1}(p_j)\in l-ax(C_j)$. Especially, $B_1^{-1}A_1^{-1}(p_1)\in l-ax(C_1)$ and $A_2^{-1}B_2^{-1}(p_2)\in r-ax(C_1)$.*

PROOF. Put $A=A_1$, $\tilde{A}=\tilde{A}_1$, $B=B_1^{-1}A_1^{-1}B_1$ and $\tilde{B}=\tilde{B}_1^{-1}\tilde{A}_1^{-1}\tilde{B}_1$. Then we have $BA=C_1$, $\tilde{B}\tilde{A}=[\tilde{B}_1, \tilde{A}_1]$ and

$$tr\,\tilde{A}\; tr\,\tilde{B}\; tr\,\tilde{B}\tilde{A}=tr^2\tilde{A}_1 tr[\tilde{B}_1, \tilde{A}_1]<0,$$

by Lemma 4.4. Thus by Lemma 3.3 $(a_1)$ and the positions of $ax(A_1)$ and $ax(C_1)$, we have

$$ax(B_1^{-1}A_1^{-1}B_1)\subset l-ax(C_1).$$

Since $B_1^{-1}A_1^{-1}(p_1)\in B_1^{-1}(ax(A_1^{-1}))=ax(B_1^{-1}A_1^{-1}B_1)$, we obtain $B_1^{-1}A_1^{-1}(p_1)\in l-ax(C_1)$ and $A_1^{-1}B_1^{-1}(p_1)=C_1^{-1}(B_1^{-1}A_1^{-1})(p_1)\in l-ax(C_1)$. Similarly, we obtain $B_2^{-1}A_2^{-1}(p_2)$, $A_2^{-1}B_2^{-1}(p_2)\in l-ax(C_2)$. Q.E.D.

This lemma implies that $ax(C_1)$ and the segment $[B_1^{-1}A_1^{-1}(p_1), A_2^{-1}B_2^{-1}(p_2)]$ intersect. Let $\mu$ be the intersection angle between them as in Figure 5.1. Let $L_1$ be the full geodesic containing this segment. Then $L_1$ is determined by $ax(C_1)$, $B_1^{-1}A_1^{-1}(p_1)$ and $\mu$. Set $L_2=C_1^{-1}(L_1)$. Then we have

$$B_2^{-1}A_2^{-1}(p_2)=C_1^{-1}(A_2^{-1}B_2^{-1}(p_2))\in L_2.$$

From the property of a hyperbolic transformation, we obtain the following:

LEMMA 5.2. *Let $z\in L_1\cap r-ax(C_1)$. If $z$ moves from the intersection point of $L_1$ and $ax(C_1)$ to infinity along this semi-infinite geodesic, then $d(z, C_1^{-1}(z))$ increases monotonically from $tl(C_1)$ to infinity. Thus there exists the unique point $z_0\in L_1\cap r-ax(C_1)$ satisfying*

$$d(z_0, C_1^{-1}(z_0)) = d(A_2^{-1}B_2^{-1}(p_2), B_2^{-1}A_2^{-1}(p_2)) = d(\hat{A}_2^{-1}\hat{B}_2^{-1}(0), \hat{B}_2^{-1}\hat{A}_2^{-1}(0)).$$

*Hence $A_2^{-1}B_2^{-1}(p_2)=z_0$ is determined real analytically by $A_1$, $B_1$, $\hat{A}_2$, $\hat{B}_2$ and $\mu$, namely, $\theta(A_j)$, $\theta(B_j)$ and $\theta(B_jA_j)$ $(j=1, 2)$ and $\mu$.*

By these arguments, $\Sigma_1$ and $\Sigma_2$ satisfying $C_1=C_2^{-1}$ are obtained real analytically from above seven angles. By our construction, any $\mu\in(0, \pi)$, $\Sigma_1$ and $\hat{\Sigma}_2$ satisfying

$$|tr\,\hat{C}_2| = |tr\,C_1| > 2$$

determine $\Sigma_2$ satisfying $C_1=C_2^{-1}$. This trace equation is equivalent to

$$tr[\tilde{B}_1, \tilde{A}_1] = tr[\tilde{B}_2, \tilde{A}_2] < -2.$$

Hence from Theorem 4.7, we obtain the following theorem.

THEOREM 5.3. *Seven angle parameters $\theta(A_j)$, $\theta(B_j)$, $\theta(B_jA_j)$ $(j=1, 2)$ and $\mu$ parametrize $T(2, 0, 0)$ global real analytically. Hence $N_2(2, 0, 0)\leq\dim(T(2, 0, 0))+1$. This parameter space is defined by*

$$\theta(A_j),\ \theta(B_j),\ \theta(B_jA_j),\ \mu\in(0, \pi)\quad (j=1, 2), \tag{5.1}$$

$$\theta(A_j)+\theta(B_j)+\theta(B_jA_j) < \pi\quad (j = 1, 2), \tag{5.2}$$

$$F(\theta(A_1), \theta(B_1), \theta(B_1A_1)) = F(\theta(A_2), \theta(B_2), \theta(B_2A_2)) > 1. \tag{5.3}$$

Finally, we consider the angles on a Riemann surface corresponding to these seven angle parameters.

Let $G$ be a Fuchsian group generated by $\Sigma_{(2,0,0)}$ and $R$ the Riemann surface represented by $G$. Let $R_j\in M(\boldsymbol{D})$ be elliptic of order 2 with the fixed point $p_j$ $(j=1, 2)$. Proposition 3.10 (iii) implies that $C_j^{1/2}=R_jB_jA_j=B_j^{-1}A_j^{-1}R_j$. Since $C_1^{1/2}=C_2^{-1/2}$, we have

$$L(p_1, p_2) = ax(R_2R_1) = ax(B_2A_2(A_1B_1)^{-1}) = ax(A_2B_2(B_1A_1)^{-1}),$$

$$\begin{aligned} L_1 &= L(B_1^{-1}A_1^{-1}(p_1), A_2^{-1}B_2^{-1}(p_2)) = L(C_1^{1/2}(p_1), C_1^{1/2}(p_2)) = C_1^{1/2}(L(p_1, p_2)) \\ &= ax(C_1^{1/2}R_2R_1C_1^{-1/2}) = ax((B_2A_2)^{-1}A_1B_1), \end{aligned}$$

$$L_2 = C_1^{-1}(L_1)=C_1^{-1/2}(L(p_1, p_2)) = ax((A_2B_2)^{-1}B_1A_1).$$

Especially, the intersection angle $\mu$ between $L_1$ and $ax(C_1)$ is equal to the one between $L(p_1, p_2)$ and $ax(C_1)$.

Let $(a_1, b_1, a_2, b_2)$ be a canonical homotopy basis of the fundamental group of $R$ corresponding to $\Sigma_{(2,0,0)}$. Then $\theta(A_j)$, $\theta(B_j)$ and $\theta(B_jA_j)$ are three interior angles of two triangles on $R$ determined by $a_j$, $b_j$ and $a_jb_j$. Let $q_j$, $r(a_j)$ and $r(b_j)$ be defined as in Section 4 $(j=1, 2)$. These six points are the fixed points of the hyperelliptic involution $J$ (namely, the Weierstrass points) of $R$.

The decagon $Q$ with the vertices $A_j^{-1}B_j^{-1}(p_j)$, $A_j^{-1}(p_j)$, $p_j$, $B_j^{-1}(p_j)$, $B_j^{-1}A_j^{-1}(p_j)$ $(j=1, 2)$ is the fundamental polygon for $G$ (see Keen [5]). From the above observation, $Q$ is also determined by ten axes of hyperbolic elements of $G$. (Let $Q_1$ be the hexagon as in Figure 5.1. Then we have $Q=Q_1\cup C_1^{1/2}(Q_1)$.) Two segments $[p_1, p_2]$ and $[B_1^{-1}A_1^{-1}(p_1), A_2^{-1}B_2^{-1}(p_2)]$ are contained in $Q$ and have same length $d(p_1, p_2)=tl(R_2R_1)/2=tl((B_2A_2)^{-1}A_1B_1)/2$. Thus the projection $pr(L_1)$ of $L_1$ is the simple closed geodesic through $q_1$ and $q_2$. We also have $pr(L_1)=pr(L_2)=pr(L(p_1, p_2))$. Since $pr(L_1)$ and $pr(ax(C_1))$ are invariant under $J$ and intersect twice, two intersection angles are same $\mu$. This $\mu$ corresponds to a *Fenchel-Nielsen twist parameter* (see Figure 5.2).

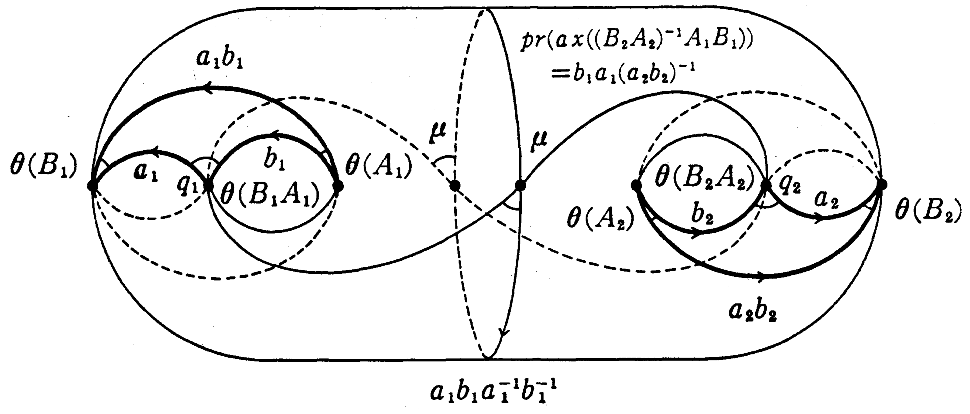


FIGURE 5.2.

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