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Asymptotic behavior of spectral functions of elliptic operators with Hölder continuous coefficients

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0. Introduction.

In the previous papers [9], [10], [11] we improved the remainder estimate in the asymptotic formula for the counting function $N(t)$ of a strongly elliptic operator A of order $2m$ defined in a bounded domain Ω of $\boldsymbol{R}^n$, whose coefficients of top order are Hölder continuous of exponent τ. The notation will be given in the next section.

Let $2m>n$. We obtained

$$N(t)=\mu_A(\Omega)t^{n/2m}+O(t^{(n-\theta)/2m}) \quad \text{as } t\to\infty \tag{0.1}$$

with $\theta=\tau/(\tau+1)$ when $0<\tau<\infty$. For the second-order operator we obtained (0.1) with $\theta=3\tau/(2\tau+3)$ when $0<\tau<3$. In a simple case we obtained (0.1) with $\theta=\tau$ when $0<\tau\leqq 1$.

Considering $U(t)=\int_{-\infty}^{\infty}e^{-st}d_sN(s)$ instead of $N(t)$, we obtained

$$U(t)=c_{n,m}\mu_A(\Omega)t^{-n/2m}+O(t^{(\tau-n)/2m}) \quad \text{as } t\to+0 \tag{0.2}$$

when $0<\tau<1$, and (0.2) with the remainder term replaced with $O(t^{(1-n)/2m}\log t^{-1})$ when $\tau=1$.

In this paper we will try to obtain similar remainder estimates for the asymptotic behavior of the spectral function $e(t, x, y)$ of A.

It is known that

$$e(t, x, x)=\mu_A(x)t^{n/2m}+O(\delta(x)^{-\theta}t^{(n-\theta)/2m}) \quad \text{as } t\to\infty \tag{0.3}$$

holds with any $\theta\in(0, \tau/(\tau+2))$ when $0<\tau<\infty$ (Tsujimoto [15]; it is necessary to assume the smoothness of coefficients of order $2m-1$ when $\tau>2$), and that (0.3) holds with $\theta=1$ when $\tau=\infty$ (Tsujimoto [16] and Brüning [4]) under some additional assumptions.

Improving the above result, we will obtain the asymptotic formula

$$e(t, x, x)=\mu_A(x)t^{n/2m}+O(\{\delta(x)^{-\theta}+\log_+(\delta(x)t^{1/2m})\}t^{(n-\theta)/2m}) \quad \text{as } t\to\infty \tag{0.4}$$

with $\theta=\tau/(\tau+1)$ when $0<\tau<\infty$ in the general case (Theorem B), and

$$(0.5)\qquad e(t, x, x) = \mu_A(x)t^{n/2m}+O(\delta(x)^{-\tau}t^{(n-\tau)/2m} +\log_+(\delta(x)t^{1/2m})\{t^{(n-\tau)/2m}+\delta(x)^{-1}t^{(n-1)/2m}\}) \quad \text{as } t\to\infty$$

when $0<\tau\leqq 1$ in a simple case (Theorem C).

Considering the asymptotic behavior of the heat kernel $U(t, x, y)=\int_{-\infty}^{\infty}e^{-ts}d_s e(s, x, y)$ instead of the spectral function, we will obtain the asymptotic formula

$$U(t, x, x) = c_{n,m}\mu_A(x)t^{-n/2m}+O(t^{(\tau-n)/2m}+\delta(x)^{-1}t^{(1-n)/2m}) \quad \text{as } t\to +0$$

when $0<\tau\leqq 1$ (Theorem A), which suggests that the remainder estimates in (0.4) and (0.5) should be improved.

For the proof of our main results we employ the resolvent kernel method (see [**2**], [**6**]). In the case of the heat kernel we make use of Lemma 4.5, in which the integral kernel is estimated by four kinds of operator norms.

In the case of the spectral function we need to estimate the resolvent kernel more elaborately. The key lies in Lemma 7.7, in which the integral kernel of an operator involved with the resolvent kernels is estimated by the Sobolev norms of the resolvent kernels. It is also important to use the information on the spectral function of an operator which is defined in a larger domain than Ω and which approximates A.

1. Main theorems.

Let Ω be a (not necessarily bounded) domain in the n-dimensional Euclidean space $\boldsymbol{R}^n$ with generic point $x=(x_1, \cdots, x_n)$. We denote by $\alpha=(\alpha_1, \cdots, \alpha_n)$ a multi-index of length $|\alpha|=\alpha_1+\cdots+\alpha_n$ and use the notation

$$D^\alpha = D_1^{\alpha_1}\cdots D_n^{\alpha_n}, \quad D_k = -i\partial/\partial x_k, \quad i=\sqrt{-1}.$$

For an integer $m\geqq 0$ we denote by $H^m(\Omega)$ the space of functions whose distributional derivatives of order up to m belong to $L^2(\Omega)$ and we introduce in it the usual norm

$$\|u\|_{m,\Omega} = \Big(\int_\Omega \sum_{|\alpha|\leq m}|D^\alpha u|^2 dx\Big)^{1/2}.$$

$H_0^m(\Omega)$ denotes the closure of $C_0^\infty(\Omega)$ in $H^m(\Omega)$.

For $\tau>0$ we define $\mathscr{B}^\tau(\Omega)$ as follows. Let $\tau=k+\theta$, where k is an integer and $0<\theta\leqq 1$. $\mathscr{B}^\tau(\Omega)$ is the space of functions u in Ω such that $D^\alpha u$ are bounded and continuous for $|\alpha|\leqq k$ and $|D^\alpha u(x)-D^\alpha u(y)|/|x-y|^\theta$ $(x, y\in\Omega,$

$x\neq y$) are bounded for $|\alpha|=k$. We set

$$|u|_{0,\Omega}=\sup_{x\in\Omega}|u(x)|,$$

$$|u|_{\tau,\Omega}=\sum_{|\alpha|\leq k}|D^{\alpha}u|_{0,\Omega}+\sum_{|\alpha|=k}\sup_{\substack{x,y\in\Omega\\ x\neq y}}\frac{|D^{\alpha}u(x)-D^{\alpha}u(y)|}{|x-y|^{\theta}}.$$

We consider a symmetric integro-differential sesquilinear form of order m:

$$B[u, v]=\int_{\Omega}\sum_{|\alpha|,|\beta|\leq m}a_{\alpha\beta}(x)D^{\alpha}u\overline{D^{\beta}v}\,dx$$

and a closed subspace V of $H^{m}(\Omega)$, and assume the following conditions.

(H0) $2m>n$; and the boundary $\partial\Omega$ of Ω is minimally smooth (see [13]).

(H1) $H_0^m(\Omega)\subset V\subset H^m(\Omega)$.

(H2) There are constants $C_0\geqq 0$ and $\delta>0$ such that

$$B[u, u]\geqq\delta\|u\|_{m,\Omega}^2-C_0\|u\|_{0,\Omega}^2\quad\text{for any } u\in V.$$

(H3) The coefficients $a_{\alpha\beta}(x)$ ($|\alpha|\leqq m$, $|\beta|\leqq m$) are bounded on Ω, and for some $\tau>0$ the coefficients of top order satisfy

$$a_{\alpha\beta}\in\mathscr{B}^{\tau}(\Omega)\quad(|\alpha|=|\beta|=m).$$

Let A be the self-adjoint operator associated with the variational triple $\{B, V, L^2(\Omega)\}$, that is, $u\in V$ belongs to $D(A)$, the domain of A, and $Au=f$ if and only if $B[u, v]=(f, v)$ is valid for any $v\in V$. Here $(\,,\,)$ denotes the inner product in $L^2(\Omega)$.

We use the following notation.

$$c_{n,m}=\int_0^{\infty}t^{n/2m}e^{-t}dt,\quad a(x,\xi)=\sum_{|\alpha|=|\beta|=m}a_{\alpha\beta}(x)\xi^{\alpha+\beta},$$

$$\mu_A(x)=(2\pi)^{-n}\int_{a(x,\xi)<1}d\xi,\quad \mu_A(\Omega)=\int_{\Omega}\mu_A(x)dx,$$

$$M_{\tau}=\sum_{|\alpha|,|\beta|\leq m}|a_{\alpha\beta}|_{0,\Omega}+\sum_{|\alpha|=|\beta|=m}|a_{\alpha\beta}|_{\tau,\Omega},$$

$$\delta(x)=\min\{\operatorname{dist}(x,\partial\Omega), 1\}\quad\text{for } x\in\Omega,$$

$$\log_+t=\max\{\log t, 0\}\quad\text{for } t>0.$$

THEOREM A. *Let* $0<\tau\leqq 1$. *We assume the conditions* (H0)-(H3). *Then there is a constant C depending only on n, m, δ, τ, C_0, M_τ and Ω such that*

$$|U(t, x, x)-c_{n,m}\mu_A(x)t^{-n/2m}|\leqq C(t^{(\tau-n)/2m}+\delta(x)^{-1}t^{(1-n)/2m})$$

holds for $x\in\Omega$ and $t>0$.

THEOREM B. *Let* $0<\tau<\infty$. *We assume the conditions* (H0)-(H3). *Then there is a constant* C *independent of* x *and* t *such that*

$$|e(t, x, x)-\mu_A(x)t^{n/2m}| \leqq C\{\delta(x)^{-\theta}+\log_+(\delta(x)t^{1/2m})\}t^{(n-\theta)/2m}$$

holds with $\theta=\tau/(\tau+1)$ *for* $x\in\Omega$ *and* $t>0$.

If we add the condition (H4) below, we can obtain a sharper result than Theorem B.

(H4) The coefficients $a_{\alpha\beta}(x)$ $(|\alpha|=|\beta|=m)$ can be written in the form $a_{\alpha\beta}(x)=b_{\alpha\beta}p(x)^{\alpha+\beta}$ with some real constants $b_{\alpha\beta}$ and $p(x)=(p_1(x), \cdots, p_n(x))$, where $p_j(x)$ is a function only of x_j for each integer $j\in[1, n]$.

THEOREM C. *Let* $0<\tau\leqq 1$. *We assume the conditions* (H0)-(H4). *Then there is a constant* C *depending only on* n, m, δ, τ, C_0, $\tilde{M}_\tau$ *and* Ω *such that*

$$|e(t, x, x)-\mu_A(x)t^{n/2m}| \leqq \begin{cases} C\{\delta(x)^{-\tau}+\log_+(\delta(x)t^{1/2m})\}t^{(n-\tau)/2m} & (0<\tau<1) \\ C\delta(x)^{-1}\{1+\log_+(\delta(x)t^{1/2m})\}t^{(n-1)/2m} & (\tau=1) \end{cases}$$

holds for $x\in\Omega$ *and* $t>0$, *where*

$$\tilde{M}_\tau=\sum_{|\alpha|=|\beta|=m}|b_{\alpha\beta}|+\sum_{j=1}^{n}|p_j|_{\tau,\Omega}+\sum_{|\alpha|+|\beta|<2m}|a_{\alpha\beta}|_{0,\Omega}.$$

2. Preliminaries.

In this section we introduce the notation and some remarks required in the proof of the main theorems.

REMARK 1. We use one and the same symbol C in order to denote constants which may differ from each other. When we specify that such a constant C is depending only on parameters, say, n and m, we denote it by $C_{n,m}$.

REMARK 2. Following Maruo and Tanabe [6], we extend A to a bounded linear operator on V to V^*, where V^* is the antidual of V, that is, the space of continuous conjugate linear functionals on V. This extended operator, which is again denoted by A, is defined by

$$B[u, v]=\langle Au, v\rangle \quad \text{for any } v\in V,$$

where $\langle\, ,\rangle$ stands for the duality between V^* and V. Identifying $L^2(\Omega)$ with its antidual, we may consider $V\subset L^2(\Omega)\subset V^*$ algebraically and topologically, and as is easily seen, V is a dense subspace of V^* under this convention. The resolvent of A thus extended is a bounded linear operator on V^* to V.

REMARK 3. According to Stein [13], there is an extension operator E: $\mathcal{B}^{\tau}(\Omega)\rightarrow\mathcal{B}^{\tau}(\boldsymbol{R}^n)$ such that $|Eu|_{\tau,\boldsymbol{R}^n}\leqq C_{n,\tau}|u|_{\tau,\Omega}$ for any $u\in\mathcal{B}^{\tau}(\Omega)$. Hence by Gårding's inequality there exist $\tilde{a}_{\alpha\beta}\in\mathcal{B}^{\tau}(\boldsymbol{R}^n)$ $(|\alpha|=|\beta|=m)$ and $C_1=C_{n,m,\delta,\tau,M_\tau}$ such that

$$\tilde{a}_{\alpha\beta}|_{\Omega}=a_{\alpha\beta},\quad |\tilde{a}_{\alpha\beta}|_{\tau,\boldsymbol{R}^n}\leqq C_{n,\delta,\tau,M_\tau},\tag{2.1}$$

$$\int_{\boldsymbol{R}^n}\sum_{|\alpha|=|\beta|=m}\tilde{a}_{\alpha\beta}(x)D^{\alpha}u\overline{D^{\beta}u}\,dx\geqq\frac{\delta}{2}\|u\|_{m,\boldsymbol{R}^n}^2-C_1\|u\|_{0,\boldsymbol{R}^n}^2 \quad \text{for } u\in H^m(\boldsymbol{R}^n).\tag{2.2}$$

REMARK 4. In the following we assume that $C_0=0$ in (H2) and therefore that

$$B[u,u]\geqq\delta\|u\|_{m,\Omega}^2 \quad \text{for any } u\in V.$$

Under this assumption we do not lose any generality in the proof of the main theorems.

We consider another set of a domain Ω_1, a sesquilinear form

$$B_1[u,v]=\int_{\Omega_1}\sum_{|\alpha|,|\beta|\leq m}a_{\alpha\beta}^1(x)D^{\alpha}u\overline{D^{\beta}v}\,dx$$

and a closed subspace V_1 of $H^m(\Omega_1)$ satisfying the following.

(H0)$_1$ $\Omega\subset\Omega_1$; and the boundary of Ω_1 is minimally smooth.

(H1)$_1$ $H_0^m(\Omega_1)\subset V_1\subset H^m(\Omega_1)$.

(H2)$_1$ $B_1[u,u]\geqq\delta\|u\|_{m,\Omega_1}^2$ holds for any $u\in V_1$, where δ is the same constant as in (H2).

(H3)$_1$ The coefficients $a_{\alpha\beta}^1(x)$ $(|\alpha|\leqq m,\ |\beta|\leqq m)$ are bounded.

Let A_1 be the operator associated with $\{B_1, V_1, L^2(\Omega_1)\}$. Let $\lambda\in\boldsymbol{C}\backslash[0,\infty)$, and set $d(\lambda)=\mathrm{dist}(\lambda,[0,\infty))$. Let $G_\lambda=(A-\lambda)^{-1}$ be the resolvent of A with kernel $G_\lambda(x,y)$, and let $G_\lambda^1=(A_1-\lambda)^{-1}$ be the resolvent of A_1 with kernel $G_\lambda^1(x,y)$.

To evaluate $G_\lambda(x,y)$ in comparison with $G_\lambda^1(x,y)$ we will often use the constants $N_1>0$, $N_2>0$ and $r\in(0,1]$ satisfying for $0\leqq j\leqq m$

$$\sum_{|\alpha|+|\beta|=j}|a_{\alpha\beta}-a_{\alpha\beta}^1|_{0,\Omega}\leqq N_1r^{j-2m},\tag{2.3}$$

$$\sum_{|\alpha|+|\beta|=j}(|a_{\alpha\beta}|_{0,\Omega}+|a_{\alpha\beta}^1|_{0,\Omega_1})\leqq N_2r^{j-2m}.\tag{2.4}$$

We take a function $\tilde{\varphi}\in C_0^\infty(\boldsymbol{R}^n)$ satisfying

$$\mathrm{supp}\,\tilde{\varphi}\subset\{x:|x|<1\},\quad \tilde{\varphi}(0)=1,\quad 0\leqq\tilde{\varphi}(x)\leqq 1.$$

We fix $x_0 \in \Omega$ arbitrarily and set $d=\delta(x_0)$ and $\varphi(x)=\tilde{\varphi}(\delta(x_0)^{-1}(x-x_0))$. Then we find that $\operatorname{supp}\varphi \subset \Omega$ and that

$$|\varphi^{(\alpha)}|_{0,\Omega} \leqq C_{n,m} d^{-|\alpha|} \quad \text{for } |\alpha| \leqq m. \tag{2.5}$$

We simply write $\| \ \|_{j,\Omega}$ by $\| \ \|_j$. We denote by $\| \ \|_X$ the norm in a Hilbert space X. We denote by $\|T\|_{X\to Y}$ the norm of T as a bounded linear operator on X to Y, and by $\mathcal{K}[T](x, y)$ the kernel of T if T is an integral operator.

3. Resolvent equation.

We shall derive the resolvent equation. For any $f \in V_1^*$ we set $w=G_\lambda^1 f \in V_1$. We note that $\varphi w \in \varphi V_1 \subset V$. Applying Leibniz's formula for $D^\alpha(\varphi w)$ and $D^\beta(\varphi v)$, we have for any $v \in V$

$$\begin{aligned}
\langle A(\varphi w), v\rangle &= \int_\Omega \sum_{\alpha,\beta} a_{\alpha\beta} D^\alpha(\varphi w)\overline{D^\beta v}\, dx \\
&= \int_\Omega \sum_{\alpha,\beta} a_{\alpha\beta}\Big(\varphi D^\alpha w + \sum_{\gamma<\alpha}\binom{\alpha}{\gamma} D^{\alpha-\gamma}\varphi D^\gamma w\Big)\overline{D^\beta v}\, dx \\
&= \int_\Omega \sum_{\alpha,\beta} a_{\alpha\beta}^1 D^\alpha w\, \overline{\varphi D^\beta v}\, dx + \int_\Omega \sum_{\alpha,\beta}(a_{\alpha\beta}-a_{\alpha\beta}^1)\varphi D^\alpha w\, \overline{D^\beta v}\, dx \\
&\quad + \int_\Omega \sum_{\alpha,\beta}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma} a_{\alpha\beta} D^{\alpha-\gamma}\varphi D^\gamma w\, \overline{D^\beta v}\, dx \\
&= \int_\Omega \sum_{\alpha,\beta} a_{\alpha\beta}^1 D^\alpha w\, \overline{D^\beta(\varphi v)} + \int_\Omega \sum_{\alpha,\beta}(a_{\alpha\beta}-a_{\alpha\beta}^1)\varphi D^\alpha w\, \overline{D^\beta v}\, dx \\
&\quad + \int_\Omega \sum_{\alpha,\beta}\sum_{\gamma<\alpha}\binom{\alpha}{\gamma} a_{\alpha\beta} D^{\alpha-\gamma}\varphi D^\gamma w\, \overline{D^\beta v}\, dx \\
&\quad - \int_\Omega \sum_{\alpha,\beta}\sum_{\gamma<\beta}\binom{\beta}{\gamma} a_{\alpha\beta}^1 D^{\beta-\gamma}\varphi D^\alpha w\, \overline{D^\gamma v}\, dx\,.
\end{aligned} \tag{3.1}$$

For i and j with $0 \leqq i \leqq m$ and $0 \leqq j \leqq m$ we define the operator P_{ij} on $H^i(\Omega_1)$ to $H^j(\Omega)^*$ by

$$\langle P_{ij} w, v\rangle = \int_\Omega \sum_{|\alpha|=i,\, |\beta|=j} (a_{\alpha\beta}-a_{\alpha\beta}^1)\varphi D^\alpha w\, \overline{D^\beta v}\, dx\,. \tag{3.2}$$

In view of (2.3) we have

$$|\langle P_{ij} w, v\rangle| \leqq C_{n,m} N_1 r^{i+j-2m} \|w\|_{i,\Omega_1} \|v\|_j\,, \tag{3.3}$$

that is,

$$\|P_{ij}\|_{H^i(\Omega_1)\to H^j(\Omega)^*} \leqq C_{n,m} N_1 r^{i+j-2m}. \tag{3.4}$$

For i, j and l with $0\leqq i\leqq m$, $0\leqq j\leqq m$, $1\leqq l\leqq m$ and $i+j+l\leqq 2m$ we define the operator Q_{ijl} on $H^{i}(\Omega_{1})$ to $H^{j}(\Omega)^{*}$ by

$$\langle Q_{ijl}w, v\rangle=\int_{\Omega}\sum_{\substack{\alpha,\beta,\gamma\\ |\gamma|=i,|\beta|=j\\ \alpha>\gamma,|\alpha-\gamma|=l}}\binom{\alpha}{\gamma}a_{\alpha\beta}D^{\alpha-\gamma}\varphi D^{\gamma}w\overline{D^{\beta}v}\,dx \tag{3.5}$$
$$-\int_{\Omega}\sum_{\substack{\alpha,\beta,\gamma\\ |\alpha|=i,|\gamma|=j\\ \beta>\gamma,|\beta-\gamma|=l}}\binom{\beta}{\gamma}a_{\alpha\beta}^{1}D^{\beta-\gamma}\varphi D^{\alpha}w\overline{D^{\gamma}v}\,dx\,.$$

In view of (2.4) and (2.5) we have

$$|\langle Q_{ijl}w, v\rangle|\leqq C_{n,m}N_{2}r^{i+j+l-2m}d^{-l}\|w\|_{i,\Omega_{1}}\|v\|_{j}, \tag{3.6}$$

that is,

$$\|Q_{ijl}\|_{H^{i}(\Omega_{1})\rightarrow H^{j}(\Omega)^{*}}\leqq C_{n,m}N_{2}r^{i+j+l-2m}d^{-l}. \tag{3.7}$$

We set

$$S_{\lambda}=G_{\lambda}\varphi-\varphi G_{\lambda}^{1},$$

which is a bounded linear operator on V_{1}^{*} to V.

LEMMA 3.1.

$$S_{\lambda}=-\sum_{\substack{0\leqq i\leqq m\\ 0\leqq j\leqq m}}G_{\lambda}P_{ij}G_{\lambda}^{1}-\sum_{\substack{0\leqq i\leqq m\\ 0\leqq j\leqq m\\ 1\leqq l\leqq\min\{m,2m-i-j\}}}G_{\lambda}Q_{ijl}G_{\lambda}^{1},$$

PROOF. Since $\operatorname{supp}\varphi\subset\Omega$, we have

$$\int_{\Omega}\sum_{\alpha,\beta}a_{\alpha\beta}^{1}D^{\alpha}w\overline{D^{\beta}(\varphi v)}\,dx-\lambda\langle\varphi w, v\rangle \tag{3.8}$$
$$=\langle(A_{1}-\lambda)w, \varphi v\rangle_{V_{1}^{*}\times V_{1}}=\langle f, \varphi v\rangle_{V_{1}^{*}\times V_{1}}=\langle\varphi f, v\rangle,$$

where $\langle\,,\rangle_{V_{1}^{*}\times V_{1}}$ stands for the duality between V_{1}^{*} and V_{1}. Combining (3.1), (3.2), (3.5) and (3.8), we get

$$(A-\lambda)(\varphi w)=\varphi f+\sum_{i,j}P_{ij}w+\sum_{i,j,l}Q_{ijl}w,$$

from which the lemma immediately follows. □

4. Estimates for the resolvent kernels—1.

In this section following Maruo and Tanabe [6], we shall estimate the kernel of S_{λ} by four kinds of operator norms $\|S_{\lambda}\|_{L^{2}(\Omega_{1})\rightarrow L^{2}(\Omega)}$, $\|S_{\lambda}\|_{L^{2}(\Omega)\rightarrow V}$ and so on.

LEMMA 4.1. *Let* $0\leqq j\leqq m$. *There is a constant* $C=C_{n,m,\Omega}$ *such that*

$$\|u\|_j\leqq C|\lambda|^{(j-m)/2m}(\|u\|_m+|\lambda|^{1/2}\|u\|_0)$$

for $u\in H^m(\Omega)$ *and* $|\lambda|\geqq 1$.

PROOF. See [6]. □

LEMMA 4.2. *There is a constant* $C=C_\delta$ *such that*

$$\|G_\lambda\|_{L^2(\Omega)\to L^2(\Omega)}\leqq\frac{1}{d(\lambda)},\quad \|G_\lambda\|_{L^2(\Omega)\to V}\leqq C\frac{|\lambda|^{1/2}}{d(\lambda)},$$

$$\|G_\lambda\|_{V^*\to L^2(\Omega)}\leqq C\frac{|\lambda|^{1/2}}{d(\lambda)},\quad \|G_\lambda\|_{V^*\to V}\leqq C\frac{|\lambda|}{d(\lambda)}.$$

PROOF. For any $f\in L^2(\Omega)$ we set $u=G_\lambda f$. From the proof of [6, Lemma 3.1] we have

$$\|u\|_0\leqq d(\lambda)^{-1}\|f\|_0, \tag{4.1}$$

$$\|u\|_m\leqq\sqrt{2\delta^{-1}}|\lambda|^{1/2}d(\lambda)^{-1}\|f\|_0 \tag{4.2}$$

$$\|u\|_m\leqq(1+\sqrt{2})\delta^{-1}|\lambda|d(\lambda)^{-1}\|f\|_{V^*}, \tag{4.3}$$

$$\|u\|_0^2\leqq\sqrt{2}\,d(\lambda)^{-1}\|f\|_{V^*}\|u\|_m. \tag{4.4}$$

Combining (4.3) and (4.4), we get

$$\|u\|_0^2\leqq(2+\sqrt{2})\delta^{-1}|\lambda|d(\lambda)^{-2}\|f\|_{V^*}^2. \tag{4.5}$$

Then the lemma follows from (4.1), (4.2), (4.3) and (4.5). □

LEMMA 4.3. *Let* $0\leqq j\leqq m$. *There is a constant* $C=C_{n,m,\delta,\Omega_1}$ *such that*

$$\|G_\lambda^1 f\|_{j,\Omega_1}\leqq C\frac{|\lambda|^{1/2+j/2m}}{d(\lambda)}\|f\|_{V_1^*},\quad \|G_\lambda^1 f\|_{j,\Omega_1}\leqq C\frac{|\lambda|^{j/2m}}{d(\lambda)}\|f\|_{0,\Omega_1}$$

for $f\in L^2(\Omega_1)$ *and* $|\lambda|\geqq 1$.

PROOF. The lemma easily follows from Lemma 4.1 and 4.2. □

LEMMA 4.4. *There is a constant* $C=C_{n,m,\delta,\Omega,\Omega_1}$ *such that*

$$\|S_\lambda\|_{L^2(\Omega_1)\to L^2(\Omega)}\leqq C\frac{|\lambda|}{d(\lambda)^2}K_{\lambda d},\quad \|S_\lambda\|_{L^2(\Omega_1)\to V}\leqq C\frac{|\lambda|^{3/2}}{d(\lambda)^2}K_{\lambda d},$$

$$\|S_\lambda\|_{V_1^*\to L^2(\Omega)}\leqq C\frac{|\lambda|^{3/2}}{d(\lambda)^2}K_{\lambda d},\qquad \|S_\lambda\|_{V_1^*\to V}\leqq C\frac{|\lambda|^2}{d(\lambda)^2}K_{\lambda d}$$

for $|\lambda|\geqq\max\{r^{-2m}, d^{-2m}\}$, *where*

$$K_{\lambda d}=N_1+N_2d^{-1}|\lambda|^{-1/2m}.$$

PROOF. For any $f\in L^2(\Omega_1)$ we set $u=S_\lambda f\in V$ and $I_\lambda(u)=\|u\|_m+|\lambda|^{1/2}\|u\|_0$. Since $|a-\lambda|\geqq(2|\lambda|)^{-1}d(\lambda)(a+|\lambda|)$ for $a>0$ [**10**, Lemma 2.6], we have

$$\text{(4.6)}\qquad |B[u,u]-\lambda(u,u)|\geqq\frac{d(\lambda)}{2|\lambda|}(B[u,u]+|\lambda|\,\|u\|_0^2)$$

$$\geqq\frac{d(\lambda)}{2|\lambda|}(\delta\|u\|_m^2+|\lambda|\,\|u\|_0^2)\geqq\frac{d(\lambda)}{4|\lambda|}\min\{\delta,1\}I_\lambda(u)^2.$$

On the other hand, we have

$$\text{(4.7)}\qquad |B[u,u]-\lambda(u,u)|=|\langle(A-\lambda)S_\lambda f,u\rangle|$$

$$=|\sum_{i,j}\langle P_{ij}G_\lambda^1 f,u\rangle+\sum_{i,j,l}\langle Q_{ijl}G_\lambda^1 f,u\rangle|\quad\text{(by Lemma 3.1)}$$

$$\leqq C\sum_{i,j}N_1r^{i+j-2m}\|G_\lambda^1 f\|_{i,\Omega_1}\|u\|_j$$

$$+C\sum_{i,j,l}N_2r^{i+j+l-2m}d^{-l}\|G_\lambda^1 f\|_{i,\Omega_1}\|u\|_j\quad\text{(by (3.3), (3.6))}$$

$$\leqq C\sum_{i,j}N_1|\lambda|^{(2m-i-j)/2m}\frac{|\lambda|^{i/2m}}{d(\lambda)}\|f\|_{0,\Omega_1}|\lambda|^{(j-m)/2m}I_\lambda(u)$$

$$+C\sum_{i,j,l}N_2|\lambda|^{(2m-i-j-l)/2m}|\lambda|^{(l-1)/2m}d^{-1}$$

$$\times\frac{|\lambda|^{i/2m}}{d(\lambda)}\|f\|_{0,\Omega_1}|\lambda|^{(j-m)/2m}I_\lambda(u)\quad\text{(by Lemma 4.1, 4.3)}$$

$$\leqq C\frac{|\lambda|^{1/2}}{d(\lambda)}(N_1+N_2d^{-1}|\lambda|^{-1/2m})\|f\|_{0,\Omega_1}I_\lambda(u),$$

where we used $l\geqq1$. Combining (4.6) and (4.7), we get

$$\|u\|_m+|\lambda|^{1/2}\|u\|_0\leqq C\frac{|\lambda|^{3/2}}{d(\lambda)^2}(N_1+N_2d^{-1}|\lambda|^{-1/2m})\|f\|_{0,\Omega_1},$$

from which the first two inequalities of the lemma follow. The last two inequalities are proved in the same way. □

LEMMA 4.5. *Let T be a bounded linear operator on V_1^* to V. Then T has a continuous integral kernel $K(x,y)$ in $\Omega\times\Omega_1$ and there is a constant $C=C_{n,m,\Omega,\Omega_1}$ such that*

$$|K(x,y)|\leqq C\|T\|_{V_1^*\to V}^{(n/2m)^2}\|T\|_{V_1^*\to L^2(\Omega)}^{(n/2m)(1-n/2m)}\|T\|_{L^2(\Omega_1)\to V}^{(n/2m)(1-n/2m)}\|T\|_{L^2(\Omega_1)\to L^2(\Omega)}^{(1-n/2m)^2}$$

for $x\in\Omega$ and $y\in\Omega_1$.

Furthermore $\|K(x,\cdot)\|_{m,\Omega_1}$ is a continuous function of x in Ω.

PROOF. Although in our case it does not always hold that $\Omega=\Omega_1$ and $V=V_1$, the existence and the continuity of $K(x, y)$ are proved in the same way as [1, Lemma 2.1], and the estimate for $K(x, y)$ is derived in the same way as [6, Lemma 3.2].

It remains to prove the continuity of $\|K(x, \cdot)\|_{m,\Omega_1}$. By virtue of (H0) there are constants $C=C_{n,m,\Omega}$ and $\theta=C_{n,m}>0$ such that for $u\in H^m(\Omega)$ and $x, x'\in\Omega$

$$|u(x)-u(x')| \leqq C|x-x'|^{\theta}\|u\|_m.$$

Then we have for any $f\in L^2(\Omega_1)$

$$\left|\int_{\Omega}(K(x, y)-K(x', y))f(y)dy\right| = |Tf(x)-Tf(x')|$$

$$\leqq C|x-x'|^{\theta}\|Tf\|_m \leqq C|x-x'|^{\theta}\|T\|_{V_1^*\to V}\|f\|_{V_1^*},$$

which yields

$$\|K(x, \cdot)-K(x', \cdot)\|_{V_1} \leqq C|x-x'|^{\theta}\|T\|_{V_1^*\to V}.$$

This completes the proof. □

LEMMA 4.6. *There is a constant* $C=C_{n,m,\delta,\Omega,\Omega_1}$ *such that*

$$|G_\lambda(x, x)-G_\lambda^1(x, x)| \leqq C\frac{|\lambda|^{n/2m+1}}{d(\lambda)^2}(N_1+N_2\delta(x)^{-1}|\lambda|^{-1/2m})$$

for $x\in\Omega$ *and* $|\lambda|\geqq\max\{r^{-2m}, \delta(x)^{-2m}\}$.

PROOF. Recalling the definition of φ and d, using Lemma 4.4 and 4.5, and setting $x=y=x_0$, we obtain the lemma for $x=x_0$. □

5. Estimates for the heat kernels.

Let $U_1(t, x, y)$ be the heat kernel of A_1.

LEMMA 5.1. *There is a constant* $C=C_{n,m,\delta,\Omega,\Omega_1}$ *such that*

$$|U(t, x, x)-U_1(t, x, x)| \leqq C(N_1+N_2\delta(x)^{-1}t^{1/2m})t^{-n/2m}$$

for $x\in\Omega$ *and* $0<t\leqq\min\{r^{2m}, \delta(x)^{2m}\}$.

PROOF. For $a>0$ we denote by $\Lambda(a)$ the oriented curve from $\infty e^{7\pi i/4}$ to $\infty e^{\pi i/4}$.

$$\Lambda(a) = \left\{\lambda : |\lambda|\geqq a, \arg\lambda=\frac{\pi}{4} \text{ or } \frac{7\pi}{4}\right\} \cup\left\{\lambda : |\lambda|=a, \frac{\pi}{4}\leqq\arg\lambda\leqq\frac{7\pi}{4}\right\}. \tag{5.1}$$

Noting that $\lambda \in \Lambda(t^{-1})$ implies $|\lambda| \geqq \max\{r^{-2m}, \delta(x)^{-2m}\}$, and using the formula

$$U(t, x, x) = \frac{1}{2\pi i}\int_{\Lambda(t^{-1})} e^{-t\lambda} G_\lambda(x, x) d\lambda$$

and Lemma 4.6, we have

$$\begin{aligned}
&|U(t, x, x) - U_1(t, x, x)| \\
&= \left|\frac{1}{2\pi i}\int_{\Lambda(t^{-1})} e^{-t\lambda}\{G_\lambda(x, x) - G_\lambda^1(x, x)\}\, d\lambda\right| \\
&\leqq \int_{\Lambda(t^{-1})} |e^{-t\lambda}|\, C|\lambda|^{n/2m+1} d(\lambda)^{-2}(N_1 + N_2\delta(x)^{-1}|\lambda|^{-1/2m})|d\lambda| \\
&\leqq C\int_{t^{-1}}^{\infty} e^{-ts/2} s^{n/2m-1}(N_1 + N_2\delta(x)^{-1}s^{-1/2m}) ds \\
&\quad + C\int_{\pi/4}^{7\pi/4} (t^{-1})^{n/2m-1}(N_1 + N_2\delta(x)^{-1}t^{1/2m})t^{-1} d\theta \\
&\leqq C(N_1 + N_2\delta(x)^{-1}t^{1/2m})t^{-n/2m},
\end{aligned}$$

which is the desired result. □

LEMMA 5.2. *There is a constant* $C = C_{n,m,\delta,\Omega}$ *such that*

$$|U(t, x, x)| \leqq Ct^{-n/2m}$$

for $x \in \Omega$ *and* $t > 0$.

PROOF. From Lemma 4.2 and 4.5 we have

$$|G_\lambda(x, y)| \leqq C_{n,m,\delta,\Omega}\frac{|\lambda|^{n/2m}}{d(\lambda)}. \tag{5.2}$$

Then the lemma is derived from (5.2) in the same way as Lemma 5.1. □

6. Proof of Theorem A.

We take a function $\phi \in C_0^\infty(\boldsymbol{R}^n)$ satisfying

$$\operatorname{supp}\phi \subset \{x : |x| < 1\}, \quad \int_{\boldsymbol{R}^n} \phi(x) dx = 1, \quad \int_{\boldsymbol{R}^n} x^\alpha \phi(x) dx \quad (1 \leqq |\alpha| < \tau)$$

(see [9] for the existence of ϕ) and set $\phi_\varepsilon(x) = \varepsilon^{-n}\phi(\varepsilon^{-1}x)$ for $\varepsilon > 0$. In this section we do not need the last condition for ϕ since $0 < \tau \leqq 1$. It will be essentially used in Section 9. We define a sesquilinear form B_ε by

$$B_\varepsilon[u, v] = \int_{\boldsymbol{R}^n} \sum_{|\alpha|=|\beta|=m} a_{\alpha\beta}^\varepsilon(x) D^\alpha u\, \overline{D^\beta v}\, dx, \quad a_{\alpha\beta}^\varepsilon = \phi_\varepsilon * \tilde{a}_{\alpha\beta}.$$

It is easily seen that for $|\alpha|=|\beta|=m$

$$|a_{\alpha\beta}^{\varepsilon}-\tilde{a}_{\alpha\beta}|_{0,R^n} \leqq C_{n,\delta,\tau,M_\tau}\varepsilon^{\tau}, \tag{6.1}$$

$$|D^{\gamma}a_{\alpha\beta}^{\varepsilon}|_{0,R^n} \leqq C_{n,\gamma,\delta,\tau,M_\tau}\varepsilon^{\min\{0,\tau-|\gamma|\}} \tag{6.2}$$

(see [9], [10]). Then in view of (2.2) there is $\varepsilon_0=C_{n,m,\delta,\tau,M_\tau}\in(0, 1]$ such that if $0<\varepsilon<\varepsilon_0$

$$B_\varepsilon[u, u] \geqq \frac{\delta}{3}\|u\|_{m,R^n}^2-C_1\|u\|_{0,R^n}^2 \quad \text{for } u\in H^m(\boldsymbol{R}^n). \tag{6.3}$$

For $\varepsilon\in(0, \varepsilon_0)$ let A_ε be the operator associated with $\{B_\varepsilon+C_1, H^m(\boldsymbol{R}^n), L^2(\boldsymbol{R}^n)\}$, let $G_\lambda^\varepsilon=(A_\varepsilon-\lambda)^{-1}$ with kernel $G_\lambda^\varepsilon(x, y)$, and let $U_\varepsilon(t, x, y)$ be the heat kernel of A_ε. Then A_ε is written in the form of

$$A_\varepsilon = \sum_{|\alpha|\leqq 2m} a_\alpha^\varepsilon(x)D^\alpha,$$

where the coefficients a_α^ε satisfy

$$|D^\gamma a_\alpha^\varepsilon|_{0,R^n} \leqq C_{n,m,\gamma,\delta,\tau,M_\tau}\varepsilon^{\min\{0,\tau-2m+|\alpha|-|\gamma|\}}. \tag{6.4}$$

LEMMA 6.1. *Let $0<\varepsilon<\varepsilon_0$. There are constants $C=C_{n,m,\delta}$ and $\delta'=C_{n,m,\delta,\tau,M_\tau}$ such that*

$$|G_\lambda^\varepsilon(x, y)| \leqq C|\lambda|^{n/2m-1}\exp(-\delta'|\lambda|^{1/2m}|x-y|)$$

for $x, y\in\boldsymbol{R}^n$ and $1\leqq|\lambda|\leqq 2d(\lambda)$.

PROOF. See [10]. □

For sake of simplicity of notation we set

$$E_m(t, x, a) = \exp\left\{-a\left(\frac{|x|^{2m}}{t}\right)^{1/(2m-1)}\right\}$$

for $x\in\boldsymbol{R}^n$, $t>0$ and $a>0$.

LEMMA 6.2. *Let $0<\varepsilon<\varepsilon_0$. There are constants $C=C_{n,m,\delta,\tau,M_\tau}$ and $\delta'=C_{n,m,\delta,\tau,M_\tau}$ such that*

$$|U_\varepsilon(t, x, y)| \leqq Ct^{-n/2m}e^{2t}E_m(t, x-y, \delta')$$

for $x, y\in\boldsymbol{R}^n$ and $t>0$.

PROOF. We set

$$\rho = \frac{|x-y|^{2m/(2m-1)}}{t^{1/(2m-1)}}, \quad a = \frac{\eta\rho}{t},$$

where $\eta>0$ will be specified later. Let $\Lambda(a)$ be the curve defined in (5.1). It is easily checked that if $\lambda\in\Lambda(a)$ then $1\leqq|\lambda-\sqrt{2}|\leqq 2d(\lambda-\sqrt{2})$ and $|\lambda-\sqrt{2}|\geqq|\lambda|/4$. Hence Lemma 6.1 gives

$$|G^{\varepsilon}_{\lambda-\sqrt{2}}(x, y)|\leqq C|\lambda|^{n/2m-1}\exp(-\delta'|\lambda|^{1/2m}|x-y|)$$

with appropriate constants $C=C_{n,m,\delta}$ and $\delta'=C_{n,m,\delta,\tau,M_\tau}$. Then noting that $|x-y|a^{1/2m}=\eta^{1/2m}\rho$, we have

$$\begin{aligned}
&|e^{-t\sqrt{2}}U_{\varepsilon}(t, x, y)| = |\mathcal{K}[e^{-t(A_{\varepsilon}+\sqrt{2})}](x, y)| \\
&= \left|\frac{1}{2\pi i}\int_{\Lambda(a)}e^{-t\lambda}G^{\varepsilon}_{\lambda-\sqrt{2}}(x, y)d\lambda\right| \\
&\leqq \frac{2}{2\pi}\int_a^{\infty}e^{-ts/2}Cs^{n/2m-1}\exp(-\delta'|x-y|s^{1/2m})ds \\
&\quad+\frac{1}{2\pi}\int_{\pi/4}^{7\pi/4}e^{at}Ca^{n/2m-1}\exp(-\delta'|x-y|a^{1/2m})a\,d\theta \\
&\leqq C\exp(-\delta'|x-y|a^{1/2m})\int_a^{\infty}s^{n/2m-1}e^{-ts/2}ds \\
&\quad+Ct^{-n/2m}(\eta\rho)^{n/2m}\exp(\eta\rho-\delta'\eta^{1/2m}\rho) \\
&\leqq Ct^{-n/2m}\exp(-\delta'\eta^{1/2m}\rho)+Ct^{-n/2m}\exp((2\eta-\delta'\eta^{1/2m})\rho).
\end{aligned}$$

Taking η so that $2\eta=2^{-1}\delta'\eta^{1/2m}$ or $\eta=(\delta'/4)^{2m/(2m-1)}$, we get the lemma. □

We set

$$a^{\varepsilon}_k(x, \xi)=\sum_{|\alpha|=k}a^{\varepsilon}_{\alpha}(x)\xi^{\alpha}\quad(0\leqq k\leqq 2m),$$

$$a_{\varepsilon}(x, \xi)=a^{\varepsilon}_{2m}(x, \xi)=\sum_{|\alpha|=|\beta|=m}a^{\varepsilon}_{\alpha\beta}(x)\xi^{\alpha+\beta}.$$

From (6.3) we have

$$a_{\varepsilon}(x, \xi)\geqq(\delta/3)|\xi|^{2m}\quad\text{for } x\in\boldsymbol{R}^n \text{ and } \xi\in\boldsymbol{R}^n. \tag{6.5}$$

LEMMA 6.3. *Let $0<\varepsilon<\varepsilon_0$. For a multi-index α there are constants $C=C_{n,m,\alpha,\delta,\tau,M_\tau}$ and $\delta'=C_{n,m,\delta,\tau,M_\tau}$ such that*

$$\left|\int_{\boldsymbol{R}^n}e^{i(x-y)\xi}\xi^{\alpha}e^{-ta_{\varepsilon}(x,\xi)}d\xi\right|\leqq Ct^{-(n+|\alpha|)/2m}E_m(t, x-y, \delta')$$

for $x, y\in\boldsymbol{R}^n$ and $t>0$.

PROOF. See [**5**]. □

LEMMA 6.4. *Let $a>-1$, $b>-1$ and $\delta'>0$. There is a constant $C=C_{n,m,a,b,\delta'}$ such that*

$$\int_0^t (t-s)^{-n/2m+a} s^{-n/2m+b} ds \int_{R^n} E_m(t-s,\ x-z,\ \delta') E_m(s,\ z-y,\ \delta') dz$$
$$\leq C t^{-n/2m+a+b+1} E_m(t,\ x-y,\ \delta'/2)$$

for $x,\ y \in \boldsymbol{R}^n$ *and* $t>0$.

PROOF. See [**5**]. □

LEMMA 6.5. *Let* $0<\varepsilon<\varepsilon_0$. *There is a constant* $C=C_{n,m,\delta,\tau,M_\tau}$ *such that*

$$|U_\varepsilon(t,\ x,\ x)-c_{n,m}\mu_{A_\varepsilon}(x)t^{-n/2m}| \leq C t^{(\tau-n)/2m}$$

for $x \in \Omega$ *and* $0<t\leq \varepsilon^{2m}$.

PROOF. Since the proof is similar to that of [**11**, Proposition 7.1], we give only the outline of the proof. From the theory of pseudo-differential operators and the theory of semigroups we have

$$U_\varepsilon(t,\ x,\ y)=H_\varepsilon(t,\ x,\ y)-\int_0^t ds \int_{R^n} U_\varepsilon(t-s,\ x,\ z) R_\varepsilon(s,\ z,\ y) dz, \tag{6.6}$$

where

$$H_\varepsilon(t,\ x,\ y)=(2\pi)^{-n}\int_{R^n} e^{i(x-y)\xi} e^{-ta_\varepsilon(x,\xi)} d\xi,$$

$$R_\varepsilon(t,\ x,\ y)=(2\pi)^{-n}\int_{R^n} e^{i(x-y)\xi} \sum_{k,\alpha} \frac{1}{\alpha !} \partial_\xi^\alpha a_{2m-k}^\varepsilon D_x^\alpha \{e^{-ta_\varepsilon(x,\xi)}\} d\xi$$

with the sum taken over $0\leq k\leq 2m$, $0\leq|\alpha|\leq 2m-k$ and $k+|\alpha|\geq 1$. From Lemma 6.3 and (6.4) we have

$$|R_\varepsilon(t,\ x,\ y)| \leq C \sum_{k,\alpha} \varepsilon^{\tau-k-|\alpha|} t^{-(n+2m-k-|\alpha|)/2m} E_m(t,\ x-y,\ \delta') \tag{6.7}$$
$$\leq C t^{(\tau-n)/2m-1} E_m(t,\ x-y,\ \delta').$$

Combining Lemma 6.2, 6.4, (6.6) and (6.7), we get the lemma. □

We are now ready to prove Theorem A. Consider first the case of $0<t<\min\{\delta(x)^{2m},\ \varepsilon_0^{2m}\}$. Let us apply Lemma 5.1 with $\Omega_1=\boldsymbol{R}^n$, $V_1=H^m(\boldsymbol{R}^n)$, $A_1=A_\varepsilon$, $N_1=C_{n,m,\delta,\tau,M_\tau}\varepsilon^\tau$, $N_2=C_{n,m,\delta,\tau,M_\tau}$ and $r=\varepsilon$. Then we have

$$|U(t,\ x,\ x)-U_\varepsilon(t,\ x,\ x)| \leq C(\varepsilon^\tau t^{-n/2m}+\delta(x)^{-1} t^{(1-n)/2m})$$

when $0<t\leq\varepsilon^{2m}$ and $0<\varepsilon<\varepsilon_0$. This combined with Lemma 6.5 gives

$$|U(t,\ x,\ x)-c_{n,m}\mu_A(x)t^{-n/2m}|$$
$$\leq |U(t,\ x,\ x)-U_\varepsilon(t,\ x,\ x)|+|U_\varepsilon(t,\ x,\ x)-c_{n,m}\mu_{A_\varepsilon}(x)t^{-n/2m}|$$
$$+c_{n,m}|\mu_{A_\varepsilon}(x)-\mu_A(x)|t^{-n/2m}$$
$$\leq C(\varepsilon^\tau t^{-n/2m}+\delta(x)^{-1}t^{(1-n)/2m}+t^{(\tau-n)/2m}),$$

where we used $|\mu_{A_\varepsilon}(x)-\mu_A(x)|\leqq C\varepsilon^\tau$ (see [9]). Setting $\varepsilon=t^{1/2m}$, we get the desired estimate.

When $t\geqq\min\{\delta(x)^{2m}, \varepsilon_0^{2m}\}$, we easily obtain the desired estimate from Lemma 5.2. Thus we complete the proof of Theorem A.

7. Estimates for the resolvent kernels—2.

In order to derive a sharper estimate for $G_\lambda(x, y)-G_\lambda^1(x, y)$ than that in Lemma 4.6 we shall evaluate $\|G_\lambda^1(\cdot, y)\|_{j,\Omega_1}$ and $\|G_\lambda(\cdot, y)\|_j$ for each $y\in\Omega$ and each integer $j\in[0, m]$ by using the information on the asymptotic behavior of the spectral function $e_1(t, x, y)$ of A_1. To this end, we suppose (HS) below in addition to (H0)-(H3) and $(\mathrm{H0})_1$-$(\mathrm{H3})_1$.

(HS) There are constants $\sigma\in(0, 1]$ and $N_3>0$ such that for $x\in\Omega_1$ and $t\geqq0$

$$\begin{cases} e_1(t, x, x) = \mu_{A_1}(x)t^{n/2m}+\gamma_1(t, x) \\ |\gamma_1(t, x)| \leqq N_3 t^{(n-\sigma)/2m}. \end{cases}$$

We note that $e_1(t, x, x)=0$ for $t<\delta$ because of $(\mathrm{H2})_1$.

LEMMA 7.1. *Suppose* (i) $a\geqq0$ *and* $k>a+1$, *or* (ii) $-1<a<1$ *and* $k=2$. *Then we have*

$$\int_0^\infty \frac{s^a}{|s-\lambda|^k}ds \leqq C_{a,k}\frac{|\lambda|^a}{d(\lambda)^{k-1}}. \tag{7.1}$$

PROOF. Case (i). We denote by I the left hand side of (7.1). Set $\lambda=t+iu$. First let $t\leqq0$. Since $|s-\lambda|^2\geqq s^2+|\lambda|^2$, we have

$$I\leqq\int_0^\infty\frac{s^a}{(s^2+|\lambda|^2)^{k/2}}ds = |\lambda|^{a+1-k}\int_0^\infty\frac{s^a}{(s^2+1)^{k/2}}ds.$$

Next let $t>0$. Since $||u|s+t|\leqq|u||s|+|t|\leqq|\lambda|(|s|+1)$, we have

$$\begin{aligned} I &= \int_0^\infty\frac{s^a}{\{(s-t)^2+u^2\}^{k/2}}ds = \int_{-t/|u|}^\infty\frac{(|u|s+t)^a|u|}{|u|^k(s^2+1)^{k/2}}ds \\ &\leqq \frac{|\lambda|^a}{d(\lambda)^{k-1}}\int_{-\infty}^\infty\frac{(|s|+1)^a}{(s^2+1)^{k/2}}ds. \end{aligned}$$

Therefore (7.1) follows.

Case (ii). We set $\lambda=|\lambda|e^{i\theta}$ with $0<\theta<2\pi$. Then by complex integration we have

$$I=\frac{\pi|\lambda|^{a-1}\sin(\pi-\theta)a}{\sin\theta\,\sin(\pi a)},$$

from which (7.1) follows. □

LEMMA 7.2. *Let* $0\leqq j\leqq m$. *There is a constant* $C=C_{n,m,\delta,\sigma,\Omega}$ *such that*

$$\|G_\lambda^1(\cdot, y)\|_{j,\Omega_1}\leqq C\Delta_\lambda|\lambda|^{(j-m)/2m}$$

for $y\in\Omega_1$, *where*

$$\Delta_\lambda=\left(\frac{|\lambda|^{n/2m}}{d(\lambda)}+N_3\frac{|\lambda|^{(n-\sigma)/2m+1}}{d(\lambda)^2}\right)^{1/2}.$$

PROOF. By virtue of Lemma 4.1 we have only to prove the lemma when $j=0$ or $j=m$.

Since $\overline{G_\lambda^1(x, y)}=G_{\bar\lambda}^1(y, x)$, we have

$$\|G_\lambda^1(\cdot, y)\|_{0,\Omega_1}^2=\int_{\Omega_1}G_{\bar\lambda}^1(y, x)G_\lambda^1(x, y)dx=\mathcal{K}[G_{\bar\lambda}^1G_\lambda^1](y, y). \tag{7.2}$$

By the definition of A_1 we have for $f\in L^2(\Omega_1)$

$$\begin{aligned}\delta\|G_\lambda^1 f\|_{m,\Omega_1}^2&\leqq B_1[G_\lambda^1 f, G_\lambda^1 f]\\&=(A_1G_\lambda^1 f, G_\lambda^1 f)_{L^2(\Omega_1)}=(G_{\bar\lambda}^1A_1G_\lambda^1 f, f)_{L^2(\Omega_1)}.\end{aligned}\tag{7.3}$$

Because of $\overline{G_\lambda^1(x, y)}=G_{\bar\lambda}^1(y, x)$ and Lemma 4.5, we find that $\|G_\lambda^1(\cdot, y)\|_{m,\Omega_1}$ is a continuous function of y in Ω_1 and that $\mathcal{K}[G_{\bar\lambda}^1A_1G_\lambda^1]$ is continuous in $\Omega_1\times\Omega_1$. Let ϕ_ε be the function defined in Section 6. Then setting $f=\phi_\varepsilon(\cdot-y)$ in (7.3) for fixed $y\in\Omega_1$, and letting $\varepsilon\to 0$, we get

$$\|G_\lambda^1(\cdot, y)\|_{m,\Omega_1}^2\leqq\delta^{-1}\mathcal{K}[G_{\bar\lambda}^1A_1G_\lambda^1](y, y). \tag{7.4}$$

In view of (7.2) and (7.4) we find that the proof of the lemma is reduced to the estimate for $\mathcal{K}[G_{\bar\lambda}^1A_1^aG_\lambda^1](y, y)$ with $a=0$ or $a=1$.

Using the spectral resolution of A_1, integration by parts, Lemma 7.1 and (HS), we have

$$\begin{aligned}\mathcal{K}[G_{\bar\lambda}^1A_1^aG_\lambda^1](y, y)&=\int_0^\infty\frac{s^a}{|s-\lambda|^2}d_se_1(s, y, y)\\&=\int_0^\infty\frac{s^a}{|s-\lambda|^2}d_s(\mu_{A_1}(y)s^{n/2m}+\gamma_1(s, y))\\&\leqq\frac{n}{2m}\mu_{A_1}(y)\int_0^\infty\frac{s^{n/2m+a-1}}{|s-\lambda|^2}ds+\int_0^\infty\left(\frac{a}{|s-\lambda|^2}+\frac{2s^a}{|s-\lambda|^3}\right)N_3s^{(n-\sigma)/2m}ds\\&\leqq C\Delta_\lambda^2|\lambda|^{a-1}.\end{aligned}$$

This completes the proof. □

LEMMA 7.3. *Let* $0\leqq j\leqq m$. *There is a constant* $C=C_{n,m,\delta,\Omega}$ *such that the following hold.*

(i) $V\subset D(A^{j/2m})\subset H^{j}(\Omega)$ *and*

$$\|u\|_{j}\leqq C\|u\|_{D(A^{j/2m})} \quad for\ u\in D(A^{j/2m}).$$

(ii) $H^{j}(\Omega)^{*}\subset D(A^{j/2m})^{*}\subset V^{*}$ *and*

$$\|u\|_{D(A^{j/2m})^{*}}\leqq C\|u\|_{H^{j}(\Omega)^{*}} \quad for\ u\in H^{j}(\Omega)^{*}.$$

REMARK 5. When we treat the fractional power of A, we consider A as a self-adjoint operator in $L^{2}(\Omega)$ (not the extended operator on V^{*} to V).

PROOF. (i) The assertion is clear when $j=0$. It is known that

$$\text{(7.5)} \qquad \begin{cases} D(A^{1/2})=V\subset H^{m}(\Omega) \\ \delta\|u\|_{m}^{2}\leqq B[u, u]=\|A^{1/2}u\|_{0}^{2}\leqq\|u\|_{D(A^{1/2})}^{2}, \end{cases}$$

which is the assertion for $j=m$.

Let $0<j<m$. Since $A\geqq\delta$, the complex interpolation method gives

$$\text{(7.6)} \qquad \begin{cases} D(A^{j/2m})=[L^{2}(\Omega), D(A^{1/2})]_{j/m} \\ \|u\|_{[L^{2}(\Omega),D(A^{1/2})]_{j/m}}\leqq(\delta^{-1/2}+1)\|u\|_{D(A^{j/2m})}. \end{cases}$$

From (7.5) we have

$$\text{(7.7)} \qquad \begin{cases} [L^{2}(\Omega), D(A^{1/2})]_{j/m}\subset[L^{2}(\Omega), H^{m}(\Omega)]_{j/m} \\ \|u\|_{[L^{2}(\Omega),H^{m}(\Omega)]_{j/m}}\leqq\delta^{-j/2m}\|u\|_{[L^{2}(\Omega),D(A^{1/2})]_{j/m}}. \end{cases}$$

By virtue of (H0) we have

$$\text{(7.8)} \qquad \begin{cases} [L^{2}(\Omega), H^{m}(\Omega)]_{j/m}=H^{j}(\Omega) \\ \|u\|_{j}\leqq C_{n,m,\Omega}\|u\|_{[L^{2}(\Omega),H^{m}(\Omega)]_{j/m}}. \end{cases}$$

Combining (7.6), (7.7) and (7.8), and noting $V=D(A^{1/2})\subset D(A^{j/2m})$, we get the assertion for $0<j<m$.

(ii) easily follows from (i). □

LEMMA 7.4. *Let* $0\leqq a\leqq 1$.

(i) $\|A^{a}u\|_{0}\leqq\|u\|_{D(A^{a})}\leqq(\delta^{-a}+1)\|A^{a}u\|_{0}$ *for* $u\in D(A^{a})$.

(ii) $(\delta^{-a}+1)^{-1}\|A^{-a}u\|_{0}\leqq\|u\|_{D(A^{a})^{*}}\leqq\|A^{-a}u\|_{0}$ *for* $u\in L^{2}(\Omega)$.

PROOF. Since $\|u\|_{D(A^{a})}=\|u\|_{0}+\|A^{a}u\|_{0}$ and $A\geqq\delta$, we easily get (i).

Setting $v=A^{-a}f$ for any $f\in L^{2}(\Omega)$, we have

$$\text{(7.9)} \qquad \|u\|_{D(A^{a})^{*}}=\sup_{v\in D(A^{a})}\frac{|(u, v)|}{\|v\|_{D(A^{a})}}=\sup_{f\in L^{2}(\Omega)}\frac{|(A^{-a}u, f)|}{\|A^{-a}f\|_{0}+\|f\|_{0}}.$$

Since (i) implies

$$\|f\|_0 \leqq \|A^{-a}f\|_0 + \|f\|_0 \leqq (\delta^{-a}+1)\|f\|_0,$$

(ii) immediately follows from (7.9). □

LEMMA 7.5. *For* $0 \leqq a \leqq 1$ *we have*

$$\|A^a G_\lambda\|_{L^2(\Omega)\to L^2(\Omega)} \leqq \frac{2|\lambda|^a}{d(\lambda)}.$$

PROOF. From the spectral theory we have

$$\begin{aligned}\|A^a G_\lambda\|_{L^2(\Omega)\to L^2(\Omega)} &= \sup_{s>0}\left|\frac{s^a}{s-\lambda}\right| = \sup_{s>0}\left|1+\frac{\lambda}{s-\lambda}\right|^a \frac{1}{|s-\lambda|^{1-a}} \\ &\leqq \left(\frac{2|\lambda|}{d(\lambda)}\right)^a \frac{1}{d(\lambda)^{1-a}} \leqq \frac{2|\lambda|^a}{d(\lambda)},\end{aligned}$$

which is the desired result. □

LEMMA 7.6. *Let* $0 \leqq j \leqq m$ *and* $0 \leqq k \leqq m$. *There is a constant* $C = C_{m,\delta}$ *such that*

$$\|G_\lambda\|_{D(A^{j/2m})^* \to D(A^{k/2m})} \leqq C \frac{|\lambda|^{(k+j)/2m}}{d(\lambda)}.$$

PROOF. From Lemma 7.4 and 7.5 we have for any $u \in L^2(\Omega)$

$$\begin{aligned}\|G_\lambda u\|_{D(A^{k/2m})} &\leqq (\delta^{-k/2m}+1)\|A^{k/2m}G_\lambda u\|_0 \\ &\leqq (\delta^{-k/2m}+1)\|A^{(k+j)/2m}G_\lambda\|_{L^2(\Omega)\to L^2(\Omega)}\|A^{-j/2m}u\|_0 \\ &\leqq (\delta^{-k/2m}+1)2|\lambda|^{(k+j)/2m} d(\lambda)^{-1}(\delta^{-j/2m}+1)\|u\|_{D(A^{j/2m})^*}.\end{aligned}$$

Since $L^2(\Omega)$ is dense in $D(A^{j/2m})^*$, we get the lemma. □

LEMMA 7.7. *Let* $0 \leqq i \leqq m$, $0 \leqq j \leqq m$ *and* $0 \leqq k \leqq m$. *Let* P *be a bounded linear operator on* $H^i(\Omega_1)$ *to* $H^j(\Omega)^*$. *Then the following estimates hold.*

(i) *There is a constant* $C = C_{n,m,\delta,\Omega}$ *such that for* $y \in \Omega_1$

$$\begin{aligned}&\|\mathcal{K}[G_\lambda P G_\lambda^1](\cdot, y)\|_k \\ &\leqq C\|G_\lambda\|_{D(A^{j/2m})^* \to D(A^{k/2m})}\|P\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1(\cdot, y)\|_{i,\Omega_1}.\end{aligned}$$

(ii) *For* $x \in \Omega$ *and* $y \in \Omega_1$

$$|\mathcal{K}[G_\lambda P G_\lambda^1](x, y)| \leqq \|G_\lambda(x, \cdot)\|_j \|P\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1(\cdot, y)\|_{i,\Omega_1}.$$

PROOF. By Lemma 7.3 we have

$$\|G_\lambda P G_\lambda^1 f\|_k \leqq C\|G_\lambda\|_{D(A^{j/2m})^* \to D(A^{k/2m})}\|P\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1 f\|_{i,\Omega_1}. \tag{7.10}$$

Since $(G_\lambda w, g)=\langle w, G_{\bar\lambda} g\rangle$ holds for any $w\in V^*$ and any $g\in L^2(\Omega)$, we have for any $f, g\in L^2(\Omega)$

$$(G_\lambda P G_\lambda^1 f, g)=\langle PG_\lambda^1 f, G_{\bar\lambda} g\rangle,$$

and therefore

$$|(G_\lambda PG_\lambda^1 f, g)|\leqq \|P\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1 f\|_{i,\Omega_1}\|G_{\bar\lambda} g\|_j. \tag{7.11}$$

Then the lemma follows from (7.10) and (7.11) by setting $f=\phi_\varepsilon(\cdot-y)$ and $g=\phi_\varepsilon(\cdot-x)$ for fixed x, y and letting $\varepsilon\to 0$ (see the proof of (7.4)). □

LEMMA 7.8. *Let* $0\leqq k\leqq m$. *There is a constant* $C=C_{n,m,\delta,\sigma,\Omega,\Omega_1}$ *such that*

$$\|G_\lambda(x, \cdot)\|_k\leqq C\Delta_\lambda J_\lambda(x)|\lambda|^{(k-m)/2m}$$

for $x\in\Omega$ *and* $|\lambda|\geqq\max\{r^{-2m}, \delta(x)^{-2m}\}$, *where*

$$J_\lambda(x)=1+N_1\frac{|\lambda|}{d(\lambda)}+N_2\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda)}.$$

PROOF. Let $|\lambda|\geqq\max\{r^{-2m}, d^{-2m}\}$. From Lemma 7.7, 7.2, 7.6 and (3.4) we have

$$\begin{aligned}(7.12)\qquad &\|\mathcal{K}[G_\lambda P_{ij}G_\lambda^1](\cdot, y)\|_k\\ &\leqq C\|G_\lambda\|_{D(A^{j/2m})^*\to D(A^{k/2m})}\|P_{ij}\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1(\cdot, y)\|_{i,\Omega_1}\\ &\leqq C\frac{|\lambda|^{(k+j)/2m}}{d(\lambda)}N_1 r^{i+j-2m}\Delta_\lambda|\lambda|^{(i-m)/2m}\\ &\leqq CN_1\Delta_\lambda\frac{|\lambda|^{(k+m)/2m}}{d(\lambda)}.\end{aligned}$$

In the same way we have

$$\begin{aligned}(7.13)\qquad \|\mathcal{K}[G_\lambda Q_{ijl}G_\lambda^1](\cdot, y)\|_k&\leqq C\frac{|\lambda|^{(k+j)/2m}}{d(\lambda)}N_2 r^{i+j+l-2m}d^{-l}\Delta_\lambda|\lambda|^{(i-m)/2m}\\ &\leqq CN_2\Delta_\lambda\frac{|\lambda|^{(k+m-1)/2m}}{d(\lambda)d},\end{aligned}$$

where we used $l\geqq 1$. It is easily seen that

$$\|\varphi G_\lambda^1(\cdot, y)\|_k\leqq C\Delta_\lambda|\lambda|^{(k-m)/2m} \tag{7.14}$$

by using Lemma 7.2 and (2.5). Combining Lemma 3.1, (7.12), (7.13) and (7.14), we have

$$\|G_\lambda(\cdot, y)\varphi(y)\|_k\leqq C\Delta_\lambda|\lambda|^{(k-m)/2m}\left(1+N_1\frac{|\lambda|}{d(\lambda)}+N_2\frac{|\lambda|^{1-1/2m}}{d(\lambda)d}\right).$$

Recalling the definition of φ and d, setting $y=x_0$, and noting $\overline{G_\lambda(x, y)}=G_{\bar\lambda}(y, x)$, we obtain the lemma for $x=x_0$. □

LEMMA 7.9. *There is a constant* $C=C_{n,m,\delta,\sigma,\Omega,\Omega_1}$ *such that*

$$|G_\lambda(x, x)-G_\lambda^1(x, x)| \leqq C(N_1+N_2\delta(x)^{-1}|\lambda|^{-1/2m})\Delta_\lambda^2 J_\lambda(x)$$

for $x\in\Omega$ *and* $|\lambda|\geqq\max\{r^{-2m}, \delta(x)^{-2m}\}$.

PROOF. Let $|\lambda|\geqq\max\{r^{-2m}, d^{-2m}\}$. Combining Lemma 3.1, 7.7, 7.2, 7.8, (3.4) and (3.7), we have

$$\begin{aligned}
&|G_\lambda(x, y)\varphi(y)-\varphi(x)G_\lambda^1(x, y)| \\
&\leqq \sum_{i,j}|\mathcal{K}[G_\lambda P_{ij}G_\lambda^1](x, y)|+\sum_{i,j,l}|\mathcal{K}[G_\lambda Q_{ijl}G_\lambda^1](x, y)| \\
&\leqq \sum_{i,j}\|G_\lambda(x, \cdot)\|_j\|P_{ij}\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1(\cdot, y)\|_{i,\Omega_1} \\
&\quad +\sum_{i,j,l}\|G_\lambda(x, \cdot)\|_j\|Q_{ijl}\|_{H^i(\Omega_1)\to H^j(\Omega)^*}\|G_\lambda^1(\cdot, y)\|_{i,\Omega_1} \\
&\leqq \sum_{i,j}C\Delta_\lambda J_\lambda(x)|\lambda|^{(j-m)/2m}N_1 r^{i+j-2m}\Delta_\lambda|\lambda|^{(i-m)/2m} \\
&\quad +\sum_{i,j,l}C\Delta_\lambda J_\lambda(x)|\lambda|^{(j-m)/2m}N_2 r^{i+j+l-2m}d^{-l}\Delta_\lambda|\lambda|^{(i-m)/2m} \\
&\leqq C(N_1+N_2 d^{-1}|\lambda|^{-1/2m})\Delta_\lambda^2 J_\lambda(x),
\end{aligned}$$

where we used $l\geqq 1$. Recalling the definition of φ and d, and setting $x=y=x_0$, we get the lemma for $x=x_0$. □

8. Estimates for the spectral functions.

LEMMA 8.1. *Let* $0<\theta\leqq 1$. *There is a constant* $C=C_{n,m,\delta,\sigma,\Omega,\Omega_1}$ *such that*

$$\begin{aligned}
(8.1)\qquad |e(t, x, x)-e_1(t, x, x)| &\leqq C(\delta(x)^{-\theta}t^{(n-\theta)/2m}+N_3 t^{(n-\sigma)/2m}) \\
&\quad +Ct^{n/2m}(N_1+N_2\delta(x)^{-1}t^{-1/2m})K_\theta(t, x)
\end{aligned}$$

for $x\in\Omega$ *and* $t\geqq\max\{r^{-2m}, \delta(x)^{-2m}\}$, *where*

$$K_\theta(t, x)=(1+N_3 t^{(\theta-\sigma)/2m})(1+N_1 t^{\theta/2m}+N_2)+\log(\delta(x)t^{1/2m}).$$

PROOF. We take λ so that $\mathrm{Re}\,\lambda=t$ and $0<\mathrm{Im}\,\lambda\leqq t$. Then we see that

$$(8.2)\qquad t\leqq|\lambda|\leqq\sqrt{2}t,\quad d(\lambda)=\mathrm{Im}\,\lambda.$$

Let $L(\lambda)$ be the oriented curve from $\bar\lambda$ to λ:

$$L(\lambda)=\{z:\mathrm{Re}\,z=t,\ d(\lambda)\leqq|\mathrm{Im}\,z|\leqq t\}\cup\{z:\ |z|=\sqrt{2}t,\ \mathrm{Re}\,z\leqq t\}.$$

Setting

$$I_1 = d(\lambda)|G_\lambda^1(x, x)|, \quad I_2 = d(\lambda)|G_\lambda(x, x) - G_\lambda^1(x, x)|,$$
$$I_3 = \left|\frac{1}{2\pi i}\int_{L(\lambda)} \{G_z(x, x) - G_z^1(x, x)\}\, dz\right|,$$

and using Pleijel's formula (see [2])

$$\left|e(t, x, x) - \frac{1}{2\pi i}\int_{L(\lambda)} G_z(x, x)dz\right| \leqq 2d(\lambda)|G_\lambda(x, x)|,$$

we have

$$(8.3) \qquad |e(t, x, x) - e_1(t, x, x)| \leqq 2d(\lambda)|G_\lambda(x, x)| + 2d(\lambda)|G_\lambda^1(x, x)| + I_3$$
$$\leqq 4I_1 + 2I_2 + I_3.$$

By integration by parts we have

$$G_\lambda^1(x, x) = \int_0^\infty \frac{1}{s-\lambda} d_s e_1(s, x, x)$$
$$= C_{n,m}\mu_{A_1}(x)(-\lambda)^{n/2m-1} + \int_0^\infty \frac{\gamma_1(s, x)}{(s-\lambda)^2} ds.$$

This combined with Lemma 7.1 and (HS) gives

$$(8.4) \qquad I_1 \leqq C(d(\lambda)|\lambda|^{n/2m-1} + N_3|\lambda|^{(n-\sigma)/2m}).$$

Noting that $z \in L(\lambda)$ implies $\sqrt{2}|\lambda| \geqq |z| \geqq t \geqq \max\{r^{-2m}, \delta(x)^{-2m}\}$, and using Lemma 7.9 and the inequality

$$\int_{L(\lambda)} \frac{|z|^a}{d(z)^k}|dz| \leqq \begin{cases} C_{a,k}|\lambda|^a d(\lambda)^{1-k} & (k>1) \\ C_a|\lambda|^a\{1+\log(|\lambda|/d(\lambda))\} & (k=1), \end{cases}$$

where $a \geqq 0$, we have

$$(8.5) \qquad 2I_2 + I_3 \leqq C(N_1 + N_2\delta(x)^{-1}|\lambda|^{-1/2m})\Big(|\lambda|^{n/2m} + N_3\frac{|\lambda|^{(n-\sigma)/2m+1}}{d(\lambda)}\Big)$$
$$\times\Big(1 + N_1\frac{|\lambda|}{d(\lambda)} + N_2\frac{|\lambda|^{1-1/2m}}{\delta(x)d(\lambda)}\Big)$$
$$+ C(N_1 + N_2\delta(x)^{-1}|\lambda|^{-1/2m})|\lambda|^{n/2m}\{1+\log(|\lambda|/d(\lambda))\}.$$

Combining (8.2), (8.3), (8.4) and (8.5), and setting $\lambda = t + it(t^{1/2m}\delta(x))^{-\theta}$ or $d(\lambda) = t(t^{1/2m}\delta(x))^{-\theta}$, we obtain the lemma. □

LEMMA 8.2. *Let* $0 < \theta \leqq 1$. *Instead of* (HS) *we assume that there is a constant* $b > 0$ *such that*

$$e_1(t, x, x) = \mu_{A_1}(x)(t-b)_+^{n/2m} \quad \textit{for } x \in \Omega_1 \textit{ and } t \in \boldsymbol{R},$$

where $t_+ = \max\{t, 0\}$. *Then there is a constant* $C = C_{n,m,\delta,\Omega,\Omega_1}$ *such that*

$$|e(t, x, x)-e_1(t, x, x)|$$
$$\leqq C\delta(x)^{-\theta}t^{(n-\theta)/2m}+Ct^{n/2m}(N_1+N_2\delta(x)^{-1}t^{-1/2m})\tilde{K}_\theta(t, x)$$

for $x\in\Omega$ *and* $t\geqq\max\{r^{-2m}, \delta(x)^{-2m}, 2b\}$, *where*

$$\tilde{K}_\theta(t, x)=1+N_1t^{\theta/2m}+N_2+\log(\delta(x)t^{1/2m}).$$

PROOF. Let us again evaluate $\|G_\lambda^1(\cdot, y)\|_{j,\Omega_1}$ and $G_\lambda^1(x, x)$ under the condition $|\lambda|\geqq 2b$.

Using Lemma 7.1 and the inequalities $d(\lambda-b)\geqq d(\lambda)$, $|\lambda|/2\leqq|\lambda-b|\leqq 3|\lambda|/2$, we have for $a=0$ or $a=1$

$$\int_0^\infty\frac{s^a}{|s-\lambda|^2}d((s-b)_+^{n/2m})=\int_0^\infty\frac{(s+b)^a}{|s-(\lambda-b)|^2}d(s^{n/2m})$$
$$\leqq\int_0^\infty\frac{s^a+b^a}{|s-(\lambda-b)|^2}d(s^{n/2m})\leqq C\frac{|\lambda-b|^{n/2m+a-1}+b^a|\lambda-b|^{n/2m-1}}{d(\lambda-b)}$$
$$\leqq C\frac{|\lambda|^{n/2m+a-1}}{d(\lambda)}.$$

Hence as for $\|G_\lambda^1(\cdot, y)\|_{j,\Omega_1}$, Lemma 7.2 remains valid if we reset $\Delta_\lambda=(|\lambda|^{n/2m}d(\lambda)^{-1})^{1/2}$.

As for $G_\lambda^1(x, x)$, we have

$$|G_\lambda^1(x, x)|\leqq C|\lambda|^{n/2m-1},$$

since $G_\lambda^1(x, x)=C_{n,m}\mu_{A_1}(x)(b-\lambda)^{n/2m-1}$.

Therefore the proof of Lemma 8.1 remains valid if we replace N_3 with 0 and add the condition $t\geqq 2b$. □

9. Proof of Theorem B.

We are now ready to prove Theorem B. Let A_ε be the operator defined in Section 6, and let $e_\varepsilon(t, x, y)$ be the spectral function of A_ε. We note that A_ε satisfies the following conditions:

(i) the ellipticity condition (6.5);

(ii) the estimate for $G_\lambda^\varepsilon(x, y)$ as in Lemma 6.1;

(iii) $|D^\gamma a_\alpha^\varepsilon|_{0,R^n}\leqq C_{n,m,\gamma,\delta,\tau,M_\tau}\varepsilon^{-2m+|\alpha|-|\gamma|}$ (see (6.4)).

According to Tsujimoto [17], these conditions guarantee that there is a constant C_2 independent of x, t and ε such that for $x\in\boldsymbol{R}^n$ and $t\geqq 0$

$$\begin{cases} e_\varepsilon(t, x, x)=\mu_{A_\varepsilon}(x)t^{n/2m}+\gamma_\varepsilon(t, x) \\ |\gamma_\varepsilon(t, x)|\leqq C_2\varepsilon^{-1}t^{(n-1)/2m}. \end{cases}$$

Consider first the case of $t>\max\{\delta(x)^{-2m}, \varepsilon_0^{-2m(\tau+1)}\}$. Let us apply Lemma 8.1 with $\Omega_1=\boldsymbol{R}^n$, $V_1=H^m(\boldsymbol{R}^n)$, $A_1=A_\varepsilon$, $N_1=C_{n,m,\delta,\tau,M_\tau}\varepsilon^\tau$, $N_2=C_{n,m,\delta,\tau,M_\tau}$, $N_3=C_2\varepsilon^{-1}$, $r=\varepsilon^\tau$, $\sigma=1$ and $\theta=\tau/(\tau+1)$. We note that $\varepsilon=t^{(\theta-1)/2m}$ implies $\varepsilon^\tau t^{\theta/2m}=1$, $t=\varepsilon^{-2m(\tau+1)}\geqq r^{-2m}$ and $0<\varepsilon<\varepsilon_0$. Hence setting $\varepsilon=t^{(\theta-1)/2m}$, we have

$$\begin{aligned}
&|e(t, x, x)-\mu_A(x)t^{n/2m}|\\
&\leqq |e(t, x, x)-e_\varepsilon(t, x, x)|+|\gamma_\varepsilon(t, x)|+|\mu_{A_\varepsilon}(x)t^{n/2m}-\mu_A(x)t^{n/2m}|\\
&\leqq C(\delta(x)^{-\theta}t^{(n-\theta)/2m}+\varepsilon^{-1}t^{(n-1)/2m})\\
&\quad+Ct^{n/2m}(\varepsilon^\tau+\delta(x)^{-1}t^{-1/2m})\\
&\qquad\times\{(1+\varepsilon^{-1}t^{(\theta-1)/2m})(1+\varepsilon^\tau t^{\theta/2m})+\log(\delta(x)t^{1/2m})\}\\
&\quad+C_2\varepsilon^{-1}t^{(n-1)/2m}+C\varepsilon^\tau t^{n/2m}\\
&\leqq C\{\delta(x)^{-\theta}+\log(\delta(x)t^{1/2m})\}t^{(n-\theta)/2m},
\end{aligned}$$

which is the desired estimate.

Consider next the case of $0<t\leqq\max\{\delta(x)^{-2m}, \varepsilon_0^{-2m(\tau+1)}\}$. Using (5.2), we have

$$\begin{aligned}
\frac{e(t, x, x)}{2t}\leqq\int_0^t\frac{d_s e(s, x, x)}{s+t}\leqq\int_0^\infty\frac{d_s e(s, x, x)}{s+t}\\
=G_{-t}(x, x)\leqq Ct^{-1+n/2m},
\end{aligned}$$

which yields

$$0\leqq e(t, x, x)\leqq Ct^{n/2m}. \tag{9.1}$$

Then the desired estimate easily follows from (9.1). Thus we complete the proof of Theorem B.

10. Proof of Theorem C.

In the same way as in Remark 3, we find that there exist $\tilde{p}_j\in\mathscr{B}^\tau(\boldsymbol{R}^n)$ $(1\leqq j\leqq n)$ and $C_3=C_{n,m,\delta,\tau,\tilde{M}_\tau}$ such that $\tilde{p}_j$ is a function only of x_j for each j and

$$\tilde{p}_j|_\Omega=p_j,\quad |\tilde{p}_j|_{\tau,\boldsymbol{R}^n}\leqq C_{n,\delta,\tau,\tilde{M}_\tau},$$

$$\int_{\boldsymbol{R}^n}\sum_{|\alpha|=|\beta|=m}b_{\alpha\beta}\tilde{p}(x)^{\alpha+\beta}D^\alpha u\,\overline{D^\alpha u}\,dx\geqq\frac{\delta}{2}\|u\|_{m,\boldsymbol{R}^n}^2-C_3\|u\|_{0,\boldsymbol{R}^n}^2$$
$$\text{for } u\in H^m(\boldsymbol{R}^n),$$

where $\tilde{p}(x)=(\tilde{p}_1(x), \cdots, \tilde{p}_n(x))$.

We set $p_j^\varepsilon(x)=\phi_\varepsilon*\tilde{p}_j(x)$, where ϕ_ε was defined in Section 6. We define L^ε, $B_\varepsilon[u, v]$ and $a_{\alpha\beta}^\varepsilon(x)$ by

$$L^\varepsilon(x, D) = (L_1^\varepsilon(x, D), \cdots, L_n^\varepsilon(x, D)),$$

$$L_j^\varepsilon(x, D) = p_j^\varepsilon(x)D_j + \frac{1}{2}D_j p_j^\varepsilon(x),$$

$$B_\varepsilon[u, v] = \int_{\boldsymbol{R}^n} \sum_{|\alpha|=|\beta|=m} b_{\alpha\beta} L^\varepsilon(x, D)^\alpha u \overline{L^\varepsilon(x, D)^\beta v}\, dx$$

$$= \int_{\boldsymbol{R}^n} \sum_{|\alpha|, |\beta| \leq m} a_{\alpha\beta}^\varepsilon(x) D^\alpha u \overline{D^\beta v}\, dx.$$

It is seen that

$$|a_{\alpha\beta}^\varepsilon - b_{\alpha\beta}\tilde{p}^{\alpha+\beta}|_{0,\boldsymbol{R}^n} \leqq C_{n,m,\delta,\tau,\tilde{M}_\tau}\varepsilon^\tau \qquad (|\alpha|=|\beta|=m),$$

$$|a_{\alpha\beta}^\varepsilon|_{0,\boldsymbol{R}^n} \leqq C_{n,m,\delta,\tau,\tilde{M}_\tau}\varepsilon^{\min\{0,\tau-2m+|\alpha|+|\beta|\}} \qquad (|\alpha| \leqq m,\ |\beta| \leqq m)$$

(see [**10**]). Using the interpolation inequality, we find that there are $\varepsilon_0 = C_{n,m,\delta,\tau,\tilde{M}_\tau} \in (0, 1]$ and $C_4 = C_{n,m,\delta,\tau,\tilde{M}_\tau} > 0$ such that if $0 < \varepsilon < \varepsilon_0$

$$B_\varepsilon[u, u] \geqq \frac{\delta}{3}\|u\|_{m,\boldsymbol{R}^n}^2 - C_4\varepsilon^{\tau-2m}\|u\|_{0,\boldsymbol{R}^n}^2 \quad \text{for } u \in H^m(\boldsymbol{R}^n).$$

Let A_ε be the operator associated with $\{B_\varepsilon + C_4\varepsilon^{\tau-2m}, H^m(\boldsymbol{R}^n), L^2(\boldsymbol{R}^n)\}$. By virtue of [**11**, Lemma 8.1] the spectral function of A_ε is given by

$$e_\varepsilon(t, x, x) = \mu_{A_\varepsilon}(x)(t - C_4\varepsilon^{\tau-2m})_+^{n/2m} \quad \text{for } x \in \boldsymbol{R}^n \text{ and } t \in \boldsymbol{R}.$$

Consider first the case of $t > \max\{\delta(x)^{-2m}, \varepsilon_0^{-2m}, (2C_4)^{2m/\tau}\}$. Let us apply Lemma 8.2 with $\Omega_1 = \boldsymbol{R}^n$, $V_1 = H^m(\boldsymbol{R}^n)$, $A_1 = A_\varepsilon$, $N_1 = C_{n,m,\delta,\tau,\tilde{M}_\tau}\varepsilon^\tau$, $N_2 = C_{n,m,\delta,\tau,\tilde{M}_\tau}$, $r = \varepsilon$, $b = C_4\varepsilon^{\tau-2m}$ and $\theta = \tau$. We note that $\varepsilon = t^{-1/2m}$ implies $t \geqq r^{-2m}$, $t \geqq 2C_4\varepsilon^{\tau-2m} = 2b$ and $0 < \varepsilon < \varepsilon_0$. Hence setting $\varepsilon = t^{-1/2m}$, we have

$$\begin{aligned}
&|e(t, x, x) - \mu_A(x)t^{n/2m}| \\
&\leqq |e(t, x, x) - e_\varepsilon(t, x, x)| + |(\mu_{A_\varepsilon}(x) - \mu_A(x))(t - C_4\varepsilon^{\tau-2m})_+^{n/2m}| \\
&\quad + |\mu_A(x)(t^{n/2m} - (t - C_4\varepsilon^{\tau-2m})_+^{n/2m})| \\
&\leqq C\delta(x)^{-\tau}t^{(n-\tau)/2m} + Ct^{n/2m}(\varepsilon^\tau + \delta(x)^{-1}t^{-1/2m}) \\
&\qquad \times (1 + \varepsilon^\tau t^{\tau/2m} + \log(\delta(x)t^{1/2m})) + C\varepsilon^\tau t^{n/2m} + Ct^{n/2m}t^{-1}\varepsilon^{\tau-2m} \\
&\leqq C\delta(x)^{-\tau}t^{(n-\tau)/2m} + Ct^{n/2m}(t^{-\tau/2m} + \delta(x)^{-1}t^{-1/2m})(1 + \log(\delta(x)t^{1/2m})),
\end{aligned}$$

from which we get the desired estimate.

When $0 < t \leqq \max\{\delta(x)^{-2m}, \varepsilon_0^{-2m}, (2C_4)^{2m/\tau}\}$, the desired estimate easily follows from (9.1). Thus we complete the proof of Theorem C.

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