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Self maps of spaces

Dedicated to Professor Tsuyoshi Watabe on his sixtieth birthday

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1. Introduction and statements of results.

Given a path-connected space X, we write

$$QH^{n}(X;\boldsymbol{K})=\tilde{H}^{n}(X;\boldsymbol{K})/\left\{\sum_{i}\tilde{H}^{i}(X;\boldsymbol{K})\cdot\tilde{H}^{n-i}(X;\boldsymbol{K})\right\}$$

for $\boldsymbol{K}=\boldsymbol{Z}, \boldsymbol{Q}$.

If G is a connected Lie group, then the k-fold product ${}^{k}id$ of the identity map of G satisfies $({}^{k}id)^{*}(x)=kx$ for all $x\in QH^{*}(G;\boldsymbol{Q})$. This property was important in [**5**]. Apart from extending Haibao's results on H-spaces to more general spaces, the following problem seems interesting in its own sense.

PROBLEM. *Given a function* $\theta:\{1, 2, \cdots\}\rightarrow\boldsymbol{Z}$, *is there a self map* μ_{θ} *of* X *such that*

(1.1) $\qquad \mu_{\theta}^{*}(x)=\theta(\deg(x))x$ *for all homogeneous elements* $x\in QH^{*}(X;\boldsymbol{Q})$?

DEFINITION. We call a path-connected space X an M_{θ}-space if it has a self map μ_{θ}, which is called an M_{θ}-structure on X, satisfying (1.1).

When θ is the constant function to $k\in\boldsymbol{Z}$, we denote M_{θ} and μ_{θ} by M_{k} and μ_{k}, respectively. When there exist an integer k and a function $e:\{1, 2, \cdots\}\rightarrow\{0, 1, 2, \cdots\}$ with $\theta(n)=k^{e(n)}$ for all $n\geqq 1$, we denote M_{θ} and μ_{θ} by $M_{k^{e}}$ and $\mu_{k^{e}}$, respectively. Note that every path-connected space is an M_{0} and M_{1} space.

We shall need some finiteness condition on X. That is, we will frequently assume some of the following:

(1.2) $\qquad H_{n}(X;\boldsymbol{Z})$ is finitely generated for all n;

(1.3) $\qquad \dim H_{n}(X;\boldsymbol{Q})<\infty$ for all n;

(1.4) $\qquad \dim H^{*}(X;\boldsymbol{Q})<\infty$;

(1.5) $\qquad \dim QH^{*}(X;\boldsymbol{Q})<\infty$;

(1.6) $$QH^*(X;\boldsymbol{Z})\otimes\boldsymbol{Q}\cong QH^*(X;\boldsymbol{Q}).$$

Notice that (1.2) implies (1.3) and (1.6), and (1.4) implies (1.5).

We call a space with a base point well-based if the base point is closed and has a contractible open neighbourhood. The unit (resp., co-unit) of an H-space (resp., co-H-space) is always the base point. Given a finite group A, we denote by $|A|$ the exponent of A, that is, $|A|=\min\{k\geqq 1;\ a^k=1 \text{ for all } a\in A\}$. Given $k\in\boldsymbol{Z}$, we denote by $\langle k\rangle$ the self maps of the sphere S^n and the Eilenberg-MacLane space $K(\boldsymbol{Z}, n)$ whose induced homomorphisms on the n-th homotopy groups are the multiplications by k.

EXAMPLE 1. (1) For all θ, each of S^n and $K(\boldsymbol{Z}, n)$ has the unique M_θ-structure $\langle\theta(n)\rangle$ $(=\mu_{\theta(n)})$, but $K(\boldsymbol{Z}\oplus\boldsymbol{Z}/2, n)$ has four M_θ-structures.

(2) Let G be a compact connected semi-simple Lie group and W its Weyl group. If k is prime to $|W|$ and $e(n)=[(n+1)/2]$, then the unstable Adams operation ψ^k [**2**, **12**, **14**] and $\Omega\psi^k$ are M_{k^e}-structures on BG and G, respectively.

(3) A finite product of M_θ-spaces satisfying (1.3) and a finite wedge product of well based M_θ-spaces are M_θ-spaces.

(4) Path-connected H-spaces satisfying (1.3) and path-connected well-based co-H-spaces are M_k-spaces for all $k\geqq 0$.

(5) The following spaces are M_k-spaces for all $k\in\boldsymbol{Z}$: path-connected H-spaces which satisfy (1.3) and have homotopy left or right inverses [**11**], in particular, path-connected H-spaces which satisfy (1.3) and have homotopy types of CW-complexes, connected Lie groups and loop spaces of simply connected spaces satisfying (1.3); path-connected well-based co-H-spaces which have homotopy left or right co-inverses, in particular, suspension spaces of well-based spaces.

(6) (Glover and Homer [**3**]) Let $G_{p,q}(\boldsymbol{F})$ be the Grassmann manifold of p-planes in $\boldsymbol{F}^{p+q}$, where $\boldsymbol{F}$ is one of the fields $\boldsymbol{C}$ (complex) and $\boldsymbol{H}$ (quaternion). If $G_{p,q}(\boldsymbol{F})$ is an M_θ-space and if $p<q$ with $p\leqq 3$ or $2p^2-p-1\leqq q$ with $p>3$, then $\theta=k^e$ for some integer k and $e(n)=\begin{cases}[n/2] & \boldsymbol{F}=\boldsymbol{C}\\ [n/4] & \boldsymbol{F}=\boldsymbol{H}.\end{cases}$

REMARK 1. (1) In [**9**], we prove the following: When $n\geqq 2$, the Stiefel manifold $U(2n+2)/U(2n)$ is an M_θ-space if and only if $\theta(4n+1)\{\theta(4n+3)-1\}\equiv 0$ (mod 8) or $\theta(4n+1)\{\theta(4n+3)-5\}\equiv 0$ (mod 8). In particular, when $n\geqq 2$, $U(2n+2)/U(2n)$ is an M_k-space if and only if $k\equiv 0, 1, 5$ (mod 8).

(2) We have

$$\mu_\theta\circ\mu_\tau=\mu_{\theta\cdot\tau}$$

where $(\theta\cdot\tau)(n)=\theta(n)\tau(n)$. Hence, given e, $\{k\in\boldsymbol{Z};\ X \text{ is an } M_{k^e}\text{-space}\}$ is a multiplicative set, while it is not additive in general by (1).

Our first result is

THEOREM 1. *Let X be a path-connected CW-complex satisfying the following*:

(1) *X satisfies* (1.2) *and $H_n(X; \boldsymbol{Z})=0$ for sufficiently large n*;

(2) *X is nilpotent* [**8**] *and the commutator subgroup of $\pi_1(X)$ is finite*;

(3) *$H^*(X; \boldsymbol{Q})$ is a tensor product of finitely many monogenic algebras.*

Then there exists a positive integer $c(X)$ such that X is an M_θ-space whenever $\theta(n)\equiv 0 \pmod{c(X)}$ for all n with $QH^n(X; \boldsymbol{Q})\neq 0$.

REMARK 2. (1) In [**6, 7**], we prove that any spherical fibre bundle over a sphere satisfies the hypotheses and the conclusion of Theorem 1 except for the condition on nilpotency.

(2) The condition (3) in Theorem 1 can not be removed in general as seen in Example 1 (6).

Given a self homomorphism f of degree 0 of a graded and finite dimensional $\boldsymbol{Q}$-module, we denote by $L(f)$ the Lefschetz number of f. Given a self map f of a space, as usual, we abbreviate $L(f^*)$ to $L(f)$. The next theorem is a generalization of 3.9 in [**5**].

THEOREM 2. *If $e(n)=(b-a)n+2a-b$ with $b\geqq a\geqq 1$ and X is an M_{k^e}-space which satisfies* (1.2) *and* (1.4), *then, for any self map f of X, we have*

$$L(\mu_{k^e}\circ f) = L(f\circ\mu_{k^e}) \equiv 1 \pmod{k}. \tag{1.7}$$

Hence moreover if X is a compact absolute neighbourhood retract and $|k|\geqq 2$, then $\mu_{k^e}\circ f$ and $f\circ\mu_{k^e}$ have fixed points [**1**].

The following is a corollary to the proof of Theorem 2.

COROLLARY 1. *Suppose the following*:

(1) *X is an M_{k^e}-space satisfying* (1.2) *and* (1.4);

(2) *there is a set $\{b_\lambda\}_{\lambda\in\Lambda}$ of homogeneous elements of $\widetilde{H}^*(X; \boldsymbol{Q})$ which represents a basis of $QH^*(X; \boldsymbol{Q})$, and, for all $n\geqq 1$ and $\lambda_i\in\Lambda$ $(1\leqq i\leqq n)$, $\sum_{i=1}^n e(\deg(b_{\lambda_i}))$ is non-zero and depends only on n and $\sum_{i=1}^n \deg(b_{\lambda_i})$.*

Then any self map f of X satisfies (1.7), *and hence moreover if X is a compact absolute neighbourhood retract and $|k|\geqq 2$, then $\mu_{k^e}\circ f$ and $f\circ\mu_{k^e}$ have fixed points.*

EXAMPLE 2. The condition (2) in Corollary 1 is satisfied if $H^*(X; \boldsymbol{Q})=\Lambda(x_1, \cdots, x_n)$ with $\deg(x_i)$ odd and $e(\deg(x_i))=(1/2)(\deg(x_i)+1)$.

We generalize the notion of characteristic polynomial of Haibao [**5**]. Suppose that a path-connected space X satisfies (1.2) and (1.5). Let f, g be self maps of X. Let $R(f)$, $R(g)$ be the matrices representing $f^*, g^*: QH^*(X; \boldsymbol{Q})\to QH^*(X; \boldsymbol{Q})$ with respect to some basis and write

$$\mathrm{ch}(f, g)(t) = \det(tR(f)-R(g)) \in \boldsymbol{Q}[t],$$
$$\deg(f) = \det(R(f)) \in \boldsymbol{Q}.$$

Then we have

THEOREM 3. (1) *The polynomial* $\mathrm{ch}(f, g)(t)$ *and the number* $\deg(f)$ *are independent of the choice of a basis of* $QH^*(X; \boldsymbol{Q})$ *and*

$$\mathrm{ch}(f, g)(t) \in \boldsymbol{Z}[t],$$
$$\deg(f) \in \boldsymbol{Z}$$

where the coefficient of t^i *in* $\mathrm{ch}(f, g)(t)$ *is zero for* $i > \dim QH^*(X; \boldsymbol{Q})$.

(2) *If* $H^*(X; \boldsymbol{Q}) = \Lambda(x_1, \cdots, x_n)$ *with* $\deg(x_i)$ *odd, then*

$$L(f) = (-1)^n \mathrm{ch}(f, id)(1) = \mathrm{ch}(id, f)(1) = \det(E-R(f)),$$
$$L(f\circ g) = L(g\circ f),$$
$$\mathrm{ch}(f, g)(t) \prod_i x_i = \prod_i \{tf^*(x_i)-g^*(x_i)\},$$

where E *is the unit matrix.*

Notice that if G is a compact connected Lie group and f is a self map of G, then $\deg(f)$ is the ordinary degree of f.

Let X be a path-connected H-space whose multiplication is denoted by "$\cdot$". Given a self map f of X and $k\geqq 2$, we denote by ${}^k f$ any k-fold product of f. For example ${}^3 f$ denotes $f\cdot(f\cdot f)$ or $(f\cdot f)\cdot f$. We write ${}^1 f = f$ and denote by ${}^0 f$ the constant map to the unit of X. In case X has a homotopy left or right inverse T, we define ${}^k f = {}^{|k|}(T\circ f)$ for all negative integers k. Notice that

$${}^k f = \begin{cases} ({}^k id)\circ f & k \geqq 0 \\ ({}^{|k|} id)\circ(T\circ f) & k \leqq 0. \end{cases}$$

Let g be also a self map of X. Then we have

THEOREM 4. *Let* X *be a path-connected H-space which satisfies* (1.2) *and* (1.5). *Write* $n = \dim QH^*(X; \boldsymbol{Q})$. *Then the following assertions hold.*

(1) $(-1)^n \mathrm{ch}(f, id)(t)$ *is equal to* $A(f)(t)$ *in* [**5**].

(2) (cf., 3.3 and 3.9 in [**5**]) *Given* $k \geqq 0$ (*or* $k \in \boldsymbol{Z}$ *if* X *has a homotopy left or right inverse*), *we have*

$$\mathrm{ch}({}^k f, g)(t) = \mathrm{ch}(f, g)(kt),$$
$$L({}^k f) \equiv 1 \pmod{k} \quad \textit{if } X \textit{ satisfies } (1.4).$$

Hence if $|k| \geqq 2$ *and* X *is a compact absolute neighbourhood retract, then* ${}^k f$ *has a fixed point.*

(3) (cf., Theorem 1 in [4]) *If X has a homotopy left or right inverse T, then*

$$\mathrm{ch}(f, g)(1) = \deg(f\cdot(T\circ g)) = \deg((T\circ g)\cdot f),$$

$$\deg(f\cdot T)=\deg(T\cdot f)=\mathrm{ch}(f, id)(1)=(-1)^n L(f) \textit{ if } X \textit{ satisfies } (1.4).$$

COROLLARY 2. *Let G be a compact connected Lie group, $k\in \boldsymbol{Z}$, and f, g self maps of G. Then*

(1) $\mathrm{ch}(f, g)(1)=\deg(f\cdot g^{-1})$, *where* $g^{-1}(x)=(g(x))^{-1}$;
(2) *if* $\mathrm{ch}(f, g)(1)\neq 0$, *then there exists* $x\in G$ *with* $f(x)=g(x)$;
(3) *if* $\deg(g)\not\equiv 0 \pmod k$, *then there exists* $x\in G$ *with* ${}^k f(x)=g(x)$.

Proof of Theorem n ($1\leqq n\leqq 4$) shall be given in the section $n+1$.

2. Proof of Theorem 1.

In this section we denote by $\dim X$ the homological dimension of X. That is, if $N=\dim X$ then $H_N(X; \boldsymbol{Z})\neq 0$ and $H_i(X; \boldsymbol{Z})=0$ for all $i>N$. When $\dim X=0$, Theorem 1 is obvious by taking $c(X)=1$. Hence we assume $\dim X>0$. By the hypothesis, we have

$$(2.1)\qquad \begin{aligned} &H^*(X; \boldsymbol{Q}) = \Lambda(x_1, \cdots, x_a)\otimes \boldsymbol{Q}[y_1, \cdots, y_b]/(y_1^{l_1}, \cdots, y_b^{l_b}),\\ &\deg(x_i) = 2n_i+1, \quad \deg(y_j) = 2m_j. \end{aligned}$$

When $a=b=0$, Theorem 1 is obvious by taking $c(X)=1$. Hence we assume $a+b>0$.

The outline of the proof is as follows. We construct a space T_j and a rational equivalence

$$\varphi: X\to K = K(H_1(X; \boldsymbol{Z}), 1)\times \prod_{i;\, n_i>0} K(\boldsymbol{Z}, 2n_i+1)\times \prod_{j=1}^{b} T_j$$

such that $H^*(T_j; \boldsymbol{Q})=\boldsymbol{Q}[y_j]/(y_j^{l_j})$ with $\deg(y_j)=2m_j$ and T_j and hence K are M_θ-spaces for all θ by Example 1 (3). Let the homotopy group of the fibre F of φ be trivial except for the dimensions u_i with $1\leqq u_1<u_2<\cdots$. Let $G_i=\pi_{u_i}(F)=\Gamma^1 G_i\supset \Gamma^2 G_i\supset\cdots$ be the lower central $\pi_1(X)$-series. Set $u_0=0$ and $N_0=1$. Write $N_i=\prod_j|\Gamma^j G_i/\Gamma^{j+1}G_i|^{2^{a+b(u_i+2)}}$ and $c(X)=N_0\cdots N_n$, where n is determined by the inequality $u_n<\dim X\leqq u_{n+1}$. Let θ satisfy $\theta(k)\equiv 0 \pmod{c(X)}$ for $k=2n_i+1$, $2m_j$. Using the step by step construction for the Moore-Postnikov factorization of φ, we lift the map $\mu_\theta\circ\varphi: X\to K$ to $\tilde{\mu}_\theta: X\to X$. Then $\tilde{\mu}_\theta$ is an M_θ-structure on X.

First we will construct T_j. Let $\iota_m\in H^m(K(\boldsymbol{Z}, m); \boldsymbol{Z})$ be the fundamental class. Given positive integers n, l, v, we denote by $T(2n, l, v)$ the homotopy fibre of

$$v\iota_{2n}^l : K(\boldsymbol{Z}, 2n) \longrightarrow K(\boldsymbol{Z}, 2ln)$$

and by

$$\xi : T(2n, l, v) \longrightarrow K(\boldsymbol{Z}, 2n)$$

the inclusion. Using the Serre spectral sequence, we have

$$H^*(T(2n, l, v)\,;\, \boldsymbol{Q}) = \boldsymbol{Q}[\xi]/(\xi^l).$$

By the diagram given below, $T(2n, l, v)$ is an M_θ-space for all θ. From now on we will use only particular M_θ-structures μ_θ on $T(2n, l, v)$ which make the following diagram commutative up to homotopy:

$$\begin{array}{ccccccc}
K(\boldsymbol{Z}, 2ln-1) & \longrightarrow & T(2n, l, v) & \xrightarrow{\xi} & K(\boldsymbol{Z}, 2n) & \xrightarrow{v\iota_{2n}^l} & K(\boldsymbol{Z}, 2ln) \\
\downarrow{\scriptstyle \langle\theta(2n)^l\rangle} & & \downarrow{\scriptstyle \mu_\theta} & & \downarrow{\scriptstyle \langle\theta(2n)\rangle} & & \downarrow{\scriptstyle \langle\theta(2n)^l\rangle} \\
K(\boldsymbol{Z}, 2ln-1) & \longrightarrow & T(2n, l, v) & \xrightarrow{\xi} & K(\boldsymbol{Z}, 2n) & \xrightarrow{v\iota_{2n}^l} & K(\boldsymbol{Z}, 2ln)
\end{array} \tag{2.2}$$

LEMMA 1. *Let A be a finite abelian group. Then*

(1) $\langle k\rangle^*=0$ *on* $\tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, A)$ *if* $k\equiv 0 \pmod{|A|}$;

(2) $(\mu_\theta^w)^*=0$ *on* $\tilde{H}^m(T(2n, l, v)\,;\, A)$ *if* $\theta(2n)\equiv 0 \pmod{|A|}$ *and* $w\geqq m+1$, *where* μ_θ^w *is the w-times iteration of* μ_θ.

PROOF. For (1), it suffices to prove (1) when $A=\boldsymbol{Z}/p^u$ with p a prime. We prove this by the induction on u. The case $u=1$ is true, because $H^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p)$ is generated by $Sq^I\bar{\iota}_n$ $(p=2)$ and $\mathcal{P}_I\bar{\iota}_n$ $(p>2)$, and $\langle k\rangle^*(\bar{\iota}_n)=k\bar{\iota}_n=0\in H^n(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p)$, where $\bar{\iota}_n$ is the mod p reduction of ι_n. Suppose that (1) is true when $A=\boldsymbol{Z}/p^u$. Consider the following commutative diagram:

$$\begin{array}{ccccc}
 & & \tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^{u+1}) & \xrightarrow{\beta_*} & \tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^u) \\
 & & \downarrow{\scriptstyle \langle p^u\rangle^*} & & \downarrow{\scriptstyle \langle p^u\rangle^*} \\
\tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p) & \xrightarrow{\alpha_*} & \tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^{u+1}) & \xrightarrow{\beta_*} & \tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^u) \\
\downarrow{\scriptstyle \langle p\rangle^*} & & \downarrow{\scriptstyle \langle p^*\rangle} & & \\
\tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p) & \xrightarrow{\alpha_*} & \tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^{u+1}) & &
\end{array}$$

Here the middle horizontal sequence is exact and associated to the exact sequence:

$$0 \longrightarrow \boldsymbol{Z}/p \xrightarrow{\alpha} \boldsymbol{Z}/p^{u+1} \xrightarrow{\beta} \boldsymbol{Z}/p^u \longrightarrow 0.$$

Take any $x\in\tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p^{u+1})$. Then $\beta_*\langle p^u\rangle^*(x)=\langle p^u\rangle^*\beta_*(x)=0$ by the inductive hypothesis. Hence there exists $y\in\tilde{H}^*(K(\boldsymbol{Z}, n)\,;\, \boldsymbol{Z}/p)$ such that $\alpha_*(y)$

$=\langle p^u\rangle^*(x)$ so that

$$\langle p^{u+1}\rangle^*(x)=\langle p\rangle^*\langle p^u\rangle^*(x)=\langle p\rangle^*\alpha_*(y)=\alpha_*\langle p\rangle^*(y)=\alpha_*(0)=0.$$

Let $k\equiv 0 \pmod{p^{u+1}}$. Then (1) is true when $A=\boldsymbol{Z}/p^{u+1}$ by the equality $\langle k\rangle=\langle k/p^{u+1}\rangle\circ\langle p^{u+1}\rangle$. This completes the induction.

To prove (2), let $E_r^{p,q}(\mu_\theta): E_r^{p,q}\to E_r^{p,q}$ be the endomorphism of the Serre spectral sequence with coefficients in A induced from the first two squares of (2.2). It follows from (1) that if $p+q\geqq 1$ then $E_2^{p,q}(\mu_\theta)=0$ and hence $E_\infty^{p,q}(\mu_\theta)=0$. Thus $\mu_\theta^* F^{p,q}\subset F^{p+1,q-1}$ if $p+q\geqq 1$, where

$$H^m(T(2n,\,l,\,v)\,;A)=F^{0,m}\supset F^{1,m-1}\supset\cdots\supset F^{m,0}\supset F^{m+1,-1}=0,$$

$$E_\infty^{p,q}=F^{p,q}/F^{p+1,q-1}.$$

Hence

$$(\mu_\theta^*)^{m+1}(F^{0,m})\subset F^{m+1,-1}=0 \quad \text{if } m\geqq 1.$$

This implies (2) and completes the proof of Lemma 1.

LEMMA 2. *Suppose the following diagram of abelian groups and homomorphisms is commutative and the horizontal sequences are exact. Then* $\phi\circ\phi=0$.

$$\begin{array}{ccccc}
B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} \\
B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D \\
{\scriptstyle 0}\downarrow & & \downarrow{\scriptstyle \phi} & & \downarrow{\scriptstyle 0} \\
B & \xrightarrow{\alpha} & C & \xrightarrow{\beta} & D.
\end{array}$$

PROOF. This is trivial.

LEMMA 3. *Let A and B be finite abelian groups, and m a positive integer. Let $K=K(B,\,1)\times\prod_{i=1}^a K(\boldsymbol{Z},\,n_i)\times\prod_{j=1}^b T(2m_j,\,l_j,\,v_j)$ be a finite product and $\theta: \{1,\,2,\,\cdots\}\to\boldsymbol{Z}$ a function such that $\theta(k)\equiv 0 \pmod{|A|^{2^{a+b}(m+1)}}$ for $k=n_i,\,2m_j$. Then K has an M_θ-structure μ_θ satisfying*

(2.3) $$\mu_\theta^*=0 \quad on \quad \tilde{H}^m(K\,;A).$$

PROOF. Let 0 be the constant self map of $K(B,\,1)$. It suffices to prove the assertion when B is trivial, since if f is an M_θ-structure on $\prod_i K(\boldsymbol{Z},\,n_i)\times\prod_j T(2m_j,\,l_j,\,v_j)$ satisfying (2.3), then $0\times f$ is an M_θ-structure on K satisfying (2.3). So we assume $B=0$.

(2.4) If $k=|A|^{2^{a+b}(m+1)}$, then there exists an M_k-structure μ_k on $T(2m_j,\,l_j,\,v_j)$ such that the self map

$$g=\prod_{i=1}^{a}\langle k\rangle\times\prod_{j=1}^{b}\mu_k$$

of K satisfies $g^*=0$ on $\tilde{H}^m(K; A)$.

If this is true, then the map $(\prod_{i=1}^{a}\langle q_i\rangle\times\prod_{j=1}^{b}\mu_{r_j})\circ g$ is a desired M_θ-structure on K, where $q_i=\theta(n_i)/\{|A|^{2^{a+b}(m+1)}\}$ and $r_j=\theta(2m_j)/\{|A|^{2^{a+b}(m+1)}\}$.

We will prove (2.4) by the induction on $a+b$. The case $a+b=1$ is true by Lemma 1. Assume that the case $a+b=l$ is true. Suppose $a+b=l+1$. We treat only the case $a\geqq 1$, because the case $a=0$ can be treated similarly. Write $K_1=K(\boldsymbol{Z}, n_1)$ and $K_2=\prod_{i=2}^{a}K(\boldsymbol{Z}, n_i)\times\prod_{j=1}^{b}T(2m_j, l_j, v_j)$. Taking $B=\sum_i\tilde{H}^i(K_1; A)\otimes\tilde{H}^{m-i}(K_2)$, $C=H^m(K_1\times K_2, K_1\vee K_2; A)$ and $D=\sum_i \mathrm{Tor}(\tilde{H}^i(K_1; A), \tilde{H}^{m+1-i}(K_2))$ in Lemma 2, we have $\{(\langle|A|\rangle\times h)\circ(\langle|A|\rangle\times h')\}^*=0$ on $H^m(K_1\times K_2, K_1\vee K_2; A)$ for any self maps h, h' of K_2. In particular we have $(\langle|A|^2\rangle\times h)^*=0$ on $H^m(K_1\times K_2, K_1\vee K_2; A)$. Taking h to be a map satisfying (2.4) for K_2, and taking $\psi=\phi=(\langle|A|^2\rangle\times h)^*$, $B=H^m(K_1\times K_2, K_1\vee K_2; A)$, $C=H^m(K_1\times K_2; A)$ and $D=H^m(K_1\vee K_2; A)$ in Lemma 2, we have $(\langle|A|^4\rangle\times h\circ h)^*=0$ on $H^m(K_1\times K_2; A)$. Since $4|2^{l+1}$ and $h^2=\prod\mu_t$, where $t=|A|^{2^{l+1}(m+1)}$, it follows that (2.4) is true when $a+b=l+1$. This completes the induction and the proof of Lemma 3.

Now we prove Theorem 1. Suppose (2.1) and $n_1=\cdots=n_{a'}=0<n_{a'+1}\leqq\cdots\leqq n_a$. Choose x_i for $a'<i\leqq a$ and y_j to be integral. This is possible by (1.6). We will shortly choose $\{x_i; 1\leqq i\leqq a'\}$ in a particular way. Denote by v_j the order of $y_j^{l_j}$ in $H^{2l_jm_j}(X; \boldsymbol{Z})$. Write $T_j=T(2m_j, l_j, v_j)$. Then $y_j: X\to K(\boldsymbol{Z}, 2m_j)$ is factored as

$$X\xrightarrow{\tilde{y}_j}T_j\xrightarrow{\xi}K(\boldsymbol{Z}, 2m_j).$$

Write $K=K(H_1(X; \boldsymbol{Z}), 1)\times\prod_{i;\,n_i>0}K(\boldsymbol{Z}, 2n_i+1)\times\prod_{j=1}^{b}T_j$. Let $x_0: X\to K(H_1(X; \boldsymbol{Z}), 1)$ be a map inducing the identity map of $H_1(X; \boldsymbol{Z})$. Let F be the homotopy fibre of

$$\varphi=x_0\times\prod_{i;\,n_i>0}x_i\times\prod\tilde{y}_j: X\to K.$$

Since $\pi_1(\varphi)$ is surjective, F is path-connected. It then follows from the pages 79 and 67 of [8] that, for all $i\geqq 1$, $\pi_i(X)$ is finitely generated and $\pi_1(X)$ operates nilpotently on $\pi_i(F)$. Since $\varphi^*: H^*(K; \boldsymbol{Q})\cong H^*(X; \boldsymbol{Q})$, it follows that φ is a rational equivalence so that $\pi_i(F)$ is finite for all $i\geqq 2$. Also $\pi_1(F)$ is finite by the hypothesis (2) in Theorem 1. Suppose that $\pi_i(F)=0$ if $i\neq u_1, u_2, \cdots$, where $1\leqq u_1<u_2<\cdots$. Set $u_0=0$ and write $G_i=\pi_{u_i}(F)$. Let

$$K(G_i, u_i)\longrightarrow X_i\xrightarrow{p_i}X_{i-1}\quad(i\geqq 1,\ X_0=K)$$

be the i-stage of the Moore-Postnikov factorization of φ (cf., [**10**, **13**]). Under our hypotheses, it admits a principal refinement [8]. That is, every p_i is factored as a product of principal fibrations

$$X_i = X(i,\, w_i) \xrightarrow{q_{w_i}} \cdots \xrightarrow{q_2} X(i,\, 1) \xrightarrow{q_1} X(i,\, 0) = X_{i-1}$$

where q_j is induced by a map $h_j: X(i, j-1) \to K(\Gamma^j G_i/\Gamma^{j+1}G_i,\, u_i+1)$. Here $G_i = \Gamma^1 G_i \supset \Gamma^2 G_i \supset \cdots \supset \Gamma^{w_i+1} G_i = \{1\}$ is the lower central $\pi_1(X)$-series. Set $w_0=0$. Take n with

$$u_n < \dim X \leqq u_{n+1}. \tag{2.5}$$

Write $N(i, j) = |\Gamma^j G_i/\Gamma^{j+1}G_i|^{2a+b(u_i+2)}$ and $N_i = \prod_{j=1}^{w_i} N(i, j)$ for $i \geqq 1$. Set $N_0 \equiv N(0, 0) = 1$ and write $c(X) = N_0 \cdots N_n$. Suppose $\theta(k) \equiv 0 \pmod{c(X)}$ for $k = 2n_i+1$, $2m_j$. Using the maps $\mu_{N(i,j)}$ in Lemma 3, we inductively have maps $g(i, j): K \to X(i, j)$ for $0 \leqq i \leqq n$ and $0 \leqq j \leqq w_i$ such that $g(0, 0)$ is the identity map, $g(i, 0) = g(i-1, w_{i-1})$ and the following diagram is commutative up to homotopy for $i \geqq 1$.

$$\begin{array}{ccccccc}
K & \xrightarrow{\mu_{N(i,w_i)}} \cdots \xrightarrow{\mu_{N(i,2)}} & K & \xrightarrow{\mu_{N(i,1)}} & K \\
{\scriptstyle g(i,\,w_i)}\downarrow & & \downarrow{\scriptstyle g(i,\,1)} & & \downarrow{\scriptstyle g(i,\,0)} \\
X_i = X(i,\, w_i) & \xrightarrow{q_{i,w_i}} \cdots \xrightarrow{q_{i,2}} & X(i,\, 1) & \xrightarrow{q_{i,1}} & X(i,\, 0) = X_{i-1}.
\end{array}$$

Define $\theta'(k)$ to be $\theta(k)/c(X)$ or zero according as $k = 2n_i+1$, $2m_j$ or otherwise. By (2.5), $g(n, w_n) \circ \mu_{\theta'} \circ \varphi: X \to X_n$ can be lifted to $\tilde{\mu}_\theta: X \to X$. Let $\zeta_i: K(H_1(X; \boldsymbol{Z}), 1) \to K(\boldsymbol{Z}, 1)$ $(1 \leqq i \leqq a')$ be a free basis of $H^1(K(H_1(X; \boldsymbol{Z}), 1); \boldsymbol{Z})$. Let $\pi_i: K \to K(\boldsymbol{Z}, 2n_i+1)$ be the projection for $i \geqq a'+1$ and the composition of the projection $K \to K(H_1(X; \boldsymbol{Z}), 1)$ with ζ_i for $i \leqq a'$. We define $x_i = \pi_i \circ \varphi$ for $i \leqq a'$. Let $\pi'_j: K \to T_j \xrightarrow{\xi} K(\boldsymbol{Z}, 2m_j)$ be the composition of the projection with the canonical map ξ. Then $x_i = \pi_i \circ \varphi$ and $y_j = \pi'_j \circ \varphi$ for all i, j, and (2.1) is satisfied. Hence, as is easily seen, we have $\tilde{\mu}_\theta^*(x_i) = \theta(2n_i+1)x_i$ and $\tilde{\mu}_\theta^*(y_j) = \theta(2m_j)y_j$. Therefore $\tilde{\mu}_\theta$ is an M_θ-structure on X. This completes the proof of Theorem 1.

3. Proof of Theorem 2.

Let $\boldsymbol{K} = \boldsymbol{Z}, \boldsymbol{Q}$. We give a decreasing filtration $F_n H^*(X; \boldsymbol{K})$ of $H^*(X; \boldsymbol{K})$ as follows:

$$F_0 H^*(X; \boldsymbol{K}) = H^*(X; \boldsymbol{K}), \quad F_1 H^*(X; \boldsymbol{K}) = \tilde{H}^*(X; \boldsymbol{K}),$$

$$F_n H^m(X; \boldsymbol{K}) = \sum_i F_{n-1} H^i(X; \boldsymbol{K}) \cdot F_1 H^{m-i}(X; \boldsymbol{K}) \quad (n \geqq 2).$$

Write

$$E_n H^m(X; \boldsymbol{K}) = F_n H^m(X; \boldsymbol{K}) / F_{n+1} H^m(X; \boldsymbol{K}).$$

Then

$$E_1H^*(X;\boldsymbol{K}) = QH^*(X;\boldsymbol{K}),$$

$$F_nH^*(X;\boldsymbol{Z})\otimes\boldsymbol{Q} \cong F_nH^*(X;\boldsymbol{Q}), \tag{3.1}$$

$$E_nH^*(X;\boldsymbol{Z})\otimes\boldsymbol{Q} \cong E_nH^*(X;\boldsymbol{Q})$$

and any self map f of X induces endomorphisms

$$F_n^m(f^*): F_nH^m(X;\boldsymbol{K}) \longrightarrow F_nH^m(X;\boldsymbol{K}),$$

$$E_n^m(f^*): E_nH^m(X;\boldsymbol{K}) \longrightarrow E_nH^m(X;\boldsymbol{K}).$$

Take a free basis of $E_nH^m(X;\boldsymbol{Z})/\mathrm{Tor}$ as a basis of $E_nH^m(X;\boldsymbol{Q})$. This is possible by (3.1). With respect to this basis, $E_n^m(f^*)$ is an integral matrix and the trace $\mathrm{Tr}(E_n^m(f^*))$ is an integer.

LEMMA 4. (1) *If a set* $\{b_\lambda\}_{\lambda\in\Lambda}$ *of homogeneous elements of* $\tilde{H}^*(X;\boldsymbol{Q})$ *represents a basis of* $E_1H^*(X;\boldsymbol{Q})$, *then a subset of* $\{b_{\lambda_1}\cdots b_{\lambda_n}\}_{\lambda_j\in\Lambda}$ *represents a basis of* $E_nH^*(X;\boldsymbol{Q})$.

(2) *A function* $e:\{1, 2, \cdots\}\to\{0, 1, \cdots\}$ *satisfies*

$$\sum_{j=1}^n e(i_j) \text{ depends only on } n \text{ and } \sum_{j=1}^n i_j \text{ for every } n\geqq 1 \tag{3.2}$$

if and only if

$$e(n) = (b-a)n+2a-b,\quad b\geqq a\geqq 0 \quad \text{for every } n\geqq 1. \tag{3.3}$$

PROOF. Write $\Lambda_n=\{\lambda\in\Lambda; b_\lambda\in H^n(X;\boldsymbol{Q})\}$ and define

$$\Omega_1 = \{b_\lambda\}_{\lambda\in\Lambda_1},$$

$$\Omega_n = \{b_\lambda\}_{\lambda\in\Lambda_n}\cup\Omega_1\cdot\Omega_{n-1}\cup\cdots\cup\Omega_{[n/2]}\cdot\Omega_{n-[n/2]} \quad (n\geqq 2).$$

Then Ω_n generates $H^n(X;\boldsymbol{Q})$ and $\{b_{\lambda_1}\cdots b_{\lambda_n}\}_{\lambda_j\in\Lambda_{m_j}}$ spans the subspace of $E_nH^*(X;\boldsymbol{Q})$ determined by $H^{m_1}(X;\boldsymbol{Q})\cdots H^{m_n}(X;\boldsymbol{Q})$. Hence (1) follows.

If e is defined by (3.3), then e satisfies (3.2). Conversely suppose that e satisfies (3.2). Then $e(1)+e(n)=e(2)+e(n-1)$ for all $n\geqq 2$. From this, we can show that

$$\begin{aligned} e(n) &= (n-1)e(2)-(n-2)e(1) \\ &= (e(2)-e(1))n+2e(1)-e(2) \end{aligned}$$

so that $e(2)\geqq e(1)$. Hence, setting $e(1)=a$ and $e(2)=b$, we have (3.3). This ends the proof of Lemma 4.

Now we continue the proof of Theorem 2. By the hypotheses $e(n)=(b-a)n+2a-b$ with $b\geqq a\geqq 1$ and $E_1^m(\mu_{k^e}^*)=k^{e(m)}$. Let $\{b_\lambda\}_{\lambda\in\Lambda}\subset\tilde{H}^*(X;\boldsymbol{Q})$ consist of homogeneous elements and represent a basis of $E_1H^*(X;\boldsymbol{Q})$. Take any n

elements $\lambda_j \in \Lambda$ $(1 \leqq j \leqq n)$ and set $m=\sum_{j=1}^{n}\deg(b_{\lambda_j})$. Write $h(m, n)=(2n-m)a+(m-n)b$. Then $h(m, n)\geqq 1$ if $m\geqq n\geqq 1$, and

$$E_n^m(\mu_{k^e}^*)(b_{\lambda_1}\cdots b_{\lambda_n}) = k^{h(m,n)}b_{\lambda_1}\cdots b_{\lambda_n}$$

by Lemma 4 (2). Hence $E_n^m(\mu_{k^e}^*)=k^{h(m,n)}$ by Lemma 4 (1). Thus

$$\begin{aligned} L(\mu_{k^e}\circ f) &= \sum_n L(E_n((\mu_{k^e}\circ f)^*)) \\ &= \sum_n L(E_n(f^*)\circ E_n(\mu_{k^e}^*)) \\ &= \sum_n \sum_m (-1)^m k^{h(m,n)}\mathrm{Tr}(E_n^m(f^*)) \\ &\equiv 1 \pmod{k}. \end{aligned}$$

Similarly $L(f\circ\mu_{k^e})=\sum_n\sum_m(-1)^m k^{h(m,n)}\,\mathrm{Tr}(E_n^m(f^*))$. This ends the proof of Theorem 2.

PROOF OF COROLLARY 1. In the above proof, by setting $h(m, n)=\sum_{j=1}^{n} e(\deg(b_{\lambda_j}))$, we obtain the proof.

4. Proof of Theorem 3.

If we use an other basis of $QH^*(X;\boldsymbol{Q})$, then $R(f)$ changes to $AR(f)A^{-1}$ for some regular matrix A. Hence $\mathrm{ch}(f, g)(t)$ and $\deg(f)$ are independent of the choice of a basis. By (1.6), we can take a free basis of $QH^*(X;\boldsymbol{Z})/\mathrm{Tor}$ as a basis. With respect to this basis, $R(f)$ and $R(g)$ are integral matrices. Hence $\mathrm{ch}(f, g)(t)$ is an integral polynomial and $\deg(f)$ is an integer. This proves (1).

To prove (2), suppose that $H^*(X;\boldsymbol{Q})=\Lambda(x_1, \cdots, x_n)$ with $\deg(x_i)$ odd. Given an ordered sequence $I=(i_1, \cdots, i_k)$ of positive integers, we write

$$l(I) = k\,.$$

We call I n-special if $I=\emptyset$ or $1\leqq i_1<\cdots<i_{l(I)}\leqq n$. When I is n-special, we write

$$x_I = \begin{cases} x_{i_1}\cdots x_{i_{l(I)}} & I\neq\emptyset \\ 1 & I=\emptyset\,, \end{cases}$$

$$|I| = \begin{cases} \sum_j \deg(x_{i_j}) & I\neq\emptyset \\ 0 & I=\emptyset\,. \end{cases}$$

We then have

(4.1) $\qquad |I|\equiv l(I) \pmod 2$ if I is n-special,

$\{x_I\,;\ I$ is n-special$\}$ is a basis of $H^*(X;\boldsymbol{Q})$.

Let

$$f^*(x_i) \equiv \sum_j f_{ij} x_j \pmod{\tilde{H}^*(X; \boldsymbol{Q}) \cdot \tilde{H}^*(X; \boldsymbol{Q})}.$$

That is, $R(f)=(f_{ij})$. If I is n-special and $f^*(x_I)=\alpha_I x_I+\text{other}$, then

$$\alpha_I = \begin{cases} \det(f_{i_p i_q}) & I \neq \emptyset \\ 1 & I = \emptyset . \end{cases}$$

This is proved as follows. Set $k=l(I)$. Since

$$f^*(x_I) = \prod_{p=1}^{k} \sum_j f_{i_p j} x_j = \sum_{l(J)=k} f_{i_1 j_1} \cdots f_{i_k j_k} x_{j_1} \cdots x_{j_k}$$

we have

$$\alpha_I = \sum_{l(J)=k} \operatorname{sgn}\binom{I}{J} f_{i_1 j_1} \cdots f_{i_k j_k} = \det(f_{i_p i_q})$$

where $\operatorname{sgn}\binom{I}{J}$ is the sign of the permutation $\begin{pmatrix} i_1 & \cdots & i_k \\ j_1 & \cdots & j_k \end{pmatrix}$ if $I=J$ as a set and 0 otherwise.

Now

$$\begin{aligned} L(f) &= \sum_{k=0}^{\infty} (-1)^k \sum_{I:\, n\text{-special},\, |I|=k} \alpha_I \\ &= \sum_{I:\, n\text{-special}} (-1)^{|I|} \alpha_I \\ &= \sum_{I:\, n\text{-special}} (-1)^{l(I)} \det(f_{i_p i_q}), \quad \text{by (4.1)} \\ &= \det(E-R(f)) \\ &= \operatorname{ch}(id, f)(1) \\ &= (-1)^n \operatorname{ch}(f, id)(1). \end{aligned}$$

We then have

$$L(f \circ g) = \det(E-R(g)R(f)) = \det(E-R(f)R(g)) = L(g \circ f).$$

We also have

$$\begin{aligned} \prod_i \{t f^*(x_i) - g^*(x_i)\} &= \prod_i \sum_j (t f_{ij} - g_{ij}) x_j \\ &= \sum_{l(J)=n} (t f_{1 j_1} - g_{1 j_1}) \cdots (t f_{n j_n} - g_{n j_n}) x_{j_1} \cdots x_{j_n} \\ &= \sum_{l(J)=n} \operatorname{sgn}(J) (t f_{1 j_1} - g_{1 j_1}) \cdots (t f_{n j_n} - g_{n j_n}) \prod_i x_i \\ &= \det(tR(f) - R(g)) \prod_i x_i \\ &= \operatorname{ch}(f, g)(t) \prod_i x_i \end{aligned}$$

where, according as J is a permutation or not, $\mathrm{sgn}(J)$ denotes the sign of J or 0. This proves (2) and completes the proof of Theorem 3.

5. Proof of Theorem 4.

Take a free basis $\{x_1, \cdots, x_n\}$ of $QH^*(X; \boldsymbol{Z})/\mathrm{Tor}$. Then, as is well-known, $H^*(X; \boldsymbol{Q})=\Lambda(x_1, \cdots, x_n)$ and $\deg(x_i)$ is odd. With respect to this basis, $R(f)$ is an integral matrix for every self map f of X. Haibao [**5**] defined the polynomial $A(f)(t)$ by

$$A(f)(t)\prod_i x_i = \prod_i \{x_i - tf^*(x_i)\}\,.$$

Hence, by Theorem 3(2), $A(f)(t)=(-1)^n\mathrm{ch}(f, id)(t)$. This proves (1).

The following lemma can be proved easily. So we omit it's proof.

LEMMA 5. *Let X be a path-connected H-space satisfying* (1.3). *Let f, g be self maps of X and $k\geqq 0$. Then, for all $x\in\tilde{H}^*(X; \boldsymbol{Q})$, we have*

$$(f\cdot g)^*(x) \equiv f^*(x)+g^*(x) \pmod{\tilde{H}^*(X; \boldsymbol{Q})\cdot\tilde{H}^*(X; \boldsymbol{Q})},$$

$$({}^kf)^*(x) \equiv kf^*(x) \pmod{\tilde{H}^*(X; \boldsymbol{Q})\cdot\tilde{H}(X; \boldsymbol{Q})},$$

$$({}^kf)^*(x) = kf^*(x) \quad \textit{if } x \textit{ is primitive.}$$

If X has a homotopy left or right inverse T, then the above equations hold for all $k\in\boldsymbol{Z}$, and

$$T^*(x) \equiv -x \pmod{\tilde{H}^*(X; \boldsymbol{Q})\cdot\tilde{H}^*(X; \boldsymbol{Q})},$$

$$T^*(x) = -x \quad \textit{if } x \textit{ is primitive.}$$

It follows from Lemma 5 that $R({}^kf)=kR(f)$ so that $\mathrm{ch}({}^kf, g)(t)=\mathrm{ch}(f, g)(kt)$. We then have

$$\begin{aligned} L({}^kf) &= (-1)^n\mathrm{ch}({}^kf, id)(1), \quad \text{by Theorem 3(2)} \\ &= (-1)^n\mathrm{ch}(f, id)(k) = \det(E-kR(f)) \\ &\equiv 1 \pmod{k}. \end{aligned}$$

This proves (2).

It follows from Lemma 5 that

$$\begin{aligned} \deg(f\cdot(T\circ g))\prod_i x_i &= (f\cdot(T\circ g))^*\prod x_i \\ &= \prod (f^*x_i - g^*x_i) \\ &= \mathrm{ch}(f, g)(1)\prod x_i \end{aligned}$$

so that $\deg(f\cdot(T\circ g))=\mathrm{ch}(f, g)(1)$. Similarly $\deg((T\circ g)\cdot f)=\mathrm{ch}(f, g)(1)$. Other relations in (3) then follows immediately. This completes the proof of Theorem 4.

PROOF OF COROLLARY 2. We have (1) by Theorem 4(3). Recall that a self map of G having non-zero degree is a surjection. Hence (2) follows from (1) and so does (3) from the equalities:

$$\begin{aligned}\deg({}^k f\cdot g^{-1}) &= \mathrm{ch}({}^k f, g)(1), \quad \text{by (1)}\\ &= \mathrm{ch}(f, g)(k), \quad \text{by Theorem 4(2)}\\ &= \det(kR(f)-R(g))\\ &\equiv (-1)^n\det(R(g)) \pmod k\\ &\equiv (-1)^n\deg(g) \pmod k\end{aligned}$$

where $n=\mathrm{rank}\, G$.

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