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On the classification of the third reduction with a spectral value condition

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0. Introduction.

This paper is a continuation of [**BS1**] and [**BS2**], and gives a new method to investigate the structure of the third reduction, by adjusting the spectral value (see Definition 1.7).

The first and second reductions (see Definitions 2.1) are studied in [**BS1**] and [**Fj2**]. When we face the third reduction, or equivalently, consider a reduction series $X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_3} X_3$, severe difficulties are caused by

(1) the isolated, 2-factorial, terminal singularities of X_2,
(2) the ray contractions of flipping type, through which the third reduction ϕ_3 factors, and
(3) that L_2 (see Definitions 2.1) is not necessarily ample, spanned or even a line bundle.

However if we put some condition on the spectral value $u(X_0, L_0)$, we can exclude these bad situations. More precisely, we will use the main theorem, [**BS2,** Theorem (3.1.4)], which says that if the *third spectral condition*: $u(X_0, L_0) > 2\lceil(n-1)/3\rceil$ is satisfied, then ϕ_1 and ϕ_2 are isomorphisms. Hence X_2 is smooth, and L_2 is a very ample line bundle, which settle (1) and (3). At the same time, this condition kills the ray contractions of flipping type in (2). By means of it, we will have Propositions 2.5, 2.7 which express nice properties of the third reductions chosen by this condition. The third reduction $\phi_3: X_2 \to X_3$ with the above third spectral condition is called the *spectral third reduction* and is denoted by $\phi: X \to Y$ (see Definition 2.2). Applying [**Na1,** Theorem 1.3, Propositions 2.1, 2.3], the classification of the third adjoint contractions, we will obtain the structure theorem on the positive dimensional fibers of the spectral third reduction, the main Theorem 2.3, where it will also be shown that Y has factorial, terminal singularities.

To classify the next reduction by the same method, it is important to ask what spectral condition implies that the third reduction is trivial. This condition, the fourth spectral condition, is given in Definition 3.1. To prove this, we make essential use of [**Na2**, Proposition (1.1), Corollary (1.3)], a criterion for a non-trivial section of adjoint systems to exist. See Theorem 3.2.

As an application of the structure theorem of the spectral third reduction, we will give an interesting example of a birational canonical morphism of a minimal smooth threefold in Corollary 3.4.

From [**KMM**, Theorem 3-2-1], there is a unique (up to isomorphisms) ample line bundle $\mathcal{L}$ on Y such that $K_X+(n-3)L\cong\phi^*\mathcal{L}$ (see Proposition 2.5). The third spectral condition will make it quite easy to prove that the nef value $\tau(\mathcal{L})$ of $\mathcal{L}$ (see Definition 1.4) satisfies $\tau(\mathcal{L})\leqq n-4$, and to classify the pair $(Y, \mathcal{L})$ when $K_X+(n-4)\mathcal{L}$ is not big, in Theorem 4.1. (cf. [**BS1**])

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1. Background material.

We use [**KMM**] and [**Ha**] as our main sources of information. A *variety* means a separated integral scheme of finite type over the complex number field $\boldsymbol{C}$. A *subvariety* means an integral subscheme of a variety. A *point* means a closed point. A normal variety is said to have *m-factorial singularities* if for every Weil divisor Z, mZ is a Cartier divisor. Note that for a normal, locally noetherian scheme, 1-factoriality is equivalent to local factoriality with respect to the Zariski topology, as shown in [**AK**, p. 139, Proposition (3.10)].

On Cone Theory, our extremal face and extremal ray do not exclude 0, i.e. always $0\in F$ and $0\in R$ in [**KMM**, Definition 3-2-3]. Let us point out that our definition of an extremal ray is equivalent to the one defined in [**Mo**], if the variety is a smooth projective variety.

DEFINITION 1.1. Let X be an n-dimensional projective variety with terminal singularities. Assume that X has an extremal ray R. Let $\varphi_R: X\to W$ be the ray contraction of R (see [**KMM**, Definition 3-2-3]) and define the exceptional locus:

$$\tilde{E}:=\{x\in X\,|\,\varphi_R \text{ is not isomorphic at } x\}.$$

Then we say that the ray contraction φ_R is *of fiber type, of divisorial type,* or *of flipping type* (or equivalently, a *small* contraction) if $\dim\tilde{E}=n$, $=n-1$, or $\leqq n-2$, respectively, where 'dim' denotes the maximum of the dimensions of irreducible components.

In the rest of this section, we will repeat to use the following set-up:

(SU) Let X be an n-dimensional projective variety with terminal singularities, and $\pi: X \to S$ a morphism onto a projective variety S. Also, let L be a π-ample line bundle on X.

The following is another expression of the discreteness of extremal rays in the cone theorem (see [**KMM**, Chap. 4] and [**Na3**, Section 1.3]).

LEMMA 1.2. *Under (SU), if F is a non-zero extremal face of $\overline{NE}(X/S)$, then there exists a finite number of distinct, unique and non-zero extremal rays $R_1, \cdots, R_r$ $(r \geqq 1)$ of $\overline{NE}(X/S)$ such that $F = R_1 + \cdots + R_r$.*

To define the nef value, we need the following

LEMMA 1.3 (Rationality theorem, [**KMM**, Theorem 4-1-1], [**BS1**, Theorem (0.8.1)]). *Let X be a normal variety with terminal singularities and $\pi: X \to S$ a projective morphism onto a variety S. Let L be a π-ample line bundle on X. If K_X is not π-nef, then*

$$\tau := \min\{t \in \boldsymbol{R};\, K_X + tL \text{ is } \pi\text{-nef}\}$$

is a positive rational number. Furthermore, defining coprime positive integers u, v by $e\tau = u/v$, we have

$$u \leqq e(\delta+1),$$

where $e := \min\{m \in \boldsymbol{N};\, mK_X \text{ is Cartier}\}$ and $\delta := \max_{s \in S}\{\dim \pi^{-1}(s)\}$.

In the following definition, we will also define the nef value morphisms (cf. [**BS1**, (0.8)]).

DEFINITION 1.4. With the notations as in Lemma 1.3, we call the rational number τ the *π-nef value* of (X, L). If S is a point, it is called simply the *nef value* of (X, L). Let us define $F := (K_X + \tau L)^{\perp} \cap \overline{NE}(X/S)$. Then from the Kleiman's criterion on ampleness, for any $\gamma \in F \setminus \{0\}$, we have $K_X \cdot \gamma < 0$. Hence F is an extremal face on X, and thus from [**KMM**, Theorem 3-2-1], there exist a unique (up to isomorphisms) morphism $\phi: X \to Y$ onto a normal projective variety Y and a surjective morphism $\alpha: Y \to S$ with $\pi = \alpha \circ \phi$. The morphism $\phi: X \to Y$ is called the *π-nef value morphism* of (X, L) over S, and if S is a point, it is called simply the *nef value morphism* of (X, L).

The following lemma can be proven by exactly the same method in [**BS1**, Lemma (0.8.3)].

LEMMA 1.5. *Under (SU), assume that K_X is not π-nef. Then a rational number τ is the π-nef value of (X, L) if and only if $K_X + \tau L$ is π-nef but not*

π-ample.

An adjoint system $K_X+\tau L$ can be localized in the following sense.

LEMMA 1.6. *Under (SU) with $S=\{a\ point\}$, assume that K_X is not nef. Let $\tau:=\tau(L)$ be the nef value of L. From the extremal face $F=(K_X+\tau L)^{\perp}\cap\overline{NE}(X)$, take any extremal ray R, and let $\varphi_R: X\to W$ be its ray contraction. Assume that* $\dim W>0$. *Then for any hyperplane section V on W and any positive integer t, we have $R=(K_X+\tau L+t\varphi_R{}^*V)^{\perp}\cap\overline{NE}(X)$.*

PROOF. From Lemma 1.2, we have $F=R_1+\cdots+R_r$ and $R_j=\boldsymbol{R}_+[C_j]$, $j=1,\cdots,r$, where R_j's are distinct extremal rays and C_j is an irreducible curve for each R_j. Say $R=R_1$. Take any $\gamma\in(K_X+\tau L+t\varphi_R{}^*V)^{\perp}\cap\overline{NE}(X)$. Since both $K_X+\tau L$ and $\varphi_R{}^*V$ are nef, it follows that $(K_X+\tau L)\cdot\gamma=0$ and $\varphi_R{}^*V\cdot\gamma=0$. Hence the first equation implies $\gamma\in F$. Thus we have $\gamma=a_1C_1+\cdots+a_rC_r$, for some real numbers $a_j\geqq 0$ $(j=1,\cdots,r)$. By the second equation, $0=\varphi_R{}^*V\cdot(a_1C_1)+\cdots+\varphi_R{}^*V\cdot(a_rC_r)=a_1V\cdot\varphi_R(C_1)+\cdots+a_rV\cdot\varphi_R(C_r)=a_2V\cdot\varphi_R(C_2)+\cdots+a_rV\cdot\varphi_R(C_r)$, since $\varphi_R(C_1)$ is a point. Because R_j's are all distinct, $\dim\varphi_R(C_2)=\cdots=\dim\varphi_R(C_r)=1$. Thus the ampleness of V implies $a_2=\cdots=a_r=0$. And thus $\gamma=a_1C_1\in R$.

On the other hand, for any $\gamma\in R\setminus\{0\}$, there exists a real number $a_1>0$ such that $\gamma=a_1C_1$. Since $R\subset F=(K_X+\tau L)^{\perp}\cap\overline{NE}(X)$ and $\varphi_R(C_1)$ is a point, $\gamma\cdot(K_X+\tau L+t\varphi_R{}^*V)=\gamma\cdot(K_X+\tau L)+ta_1V\cdot\varphi_R(C_1)=0$. Thus $\gamma\in(K_X+\tau L+t\varphi_R{}^*V)^{\perp}\cap\overline{NE}(X)$. Q.E.D.

The spectral value is the key concept in this paper.

DEFINITION 1.7 ([**BS2**, (0.4)]). Let X be an n-dimensional normal projective variety with $\boldsymbol{Q}$-Gorenstein singularities. Let D be a $\boldsymbol{Q}$-Cartier divisor on X. Then we define $u(X, D):=\sup\{t\in\boldsymbol{Q}\,;\, h^0(X, N(K_X+tD))=0$ for all integer $N>0$ such that $N(K_X+tD)$ is Cartier$\}$ and call $u(X, D)$ the *unnormalized spectral value* of (X, D). In this paper, we simply refer to it as the *spectral value* of (X, D).

The relation between the spectral value and the nef value will be established in the following lemma, the relative version of [**BS2**, Lemma (0.4.3)]. The same proof in [ibid.] works also for this.

LEMMA 1.8. *Under (SU), assume that K_X is not π-nef. Let $\theta(L)$ be the π-nef value of L, and $\phi: X\to Y$ its π-nef value morphism. Then the spectral value $u(X, L)$ satisfies that $u(X, L)\leqq\theta(L)$. Moreover, $u(X, L)=\theta(L)$ if and only if ϕ is of fiber type.*

Let us introduce special varieties in order to classify nef value morphisms of fiber type from the adjunction theoretic point of view in Section 4.1.

DEFINITION 1.9. Let Y be a normal projective variety with factorial singularities. We say that Y is *Fano* if $-K_Y$ is ample. Let $\mathcal{L}$ be an ample line bundle on Y and $n:=\dim Y$. Assume that there exists a morphism $\mu: Y \to Z$ onto a normal projective variety Z of dimension m with only connected fibers such that

$$K_Y + k\mathcal{L} \cong \mu^* Q$$

for some ample line bundle Q on Z and some positive integer k. We call $(Y, \mathcal{L})$ a *scroll* if $k=n-m+1$, a *quadric fibration* if $k=n-m$, a *Del Pezzo fibration* if $k=n-m-1$, a *Mukai fibration* if $k=n-m-2$, or a *Fano fibration of co-index four* if $k=n-m-3$.

The following lemma is a technical device from Scheme Theory, which will be necessary in Proofs for Proposition 2.7 and Theorem 3.2. For a proof, see [**Na3**, Section 1.2].

LEMMA 1.10. *Let X be a locally noetherian scheme of equidimension, and $D_1, \cdots, D_r$ effective Cartier divisors on X. Let $Y=D_1 \cap \cdots \cap D_r$ be the scheme theoretic intersection of $D_1, \cdots, D_r$. If Y is equicodimension r and located in the Cohen-Macaulay locus of X, we have*

$$\mathcal{I}_Y/\mathcal{I}_Y^2 \cong [\mathcal{I}_{D_1}/\mathcal{I}_{D_1}^2 \otimes \mathcal{O}_Y] \oplus \cdots \oplus [\mathcal{I}_{D_r}/\mathcal{I}_{D_r}^2 \otimes \mathcal{O}_Y],$$

or

$$\mathcal{N}_{Y/X} \cong \mathcal{N}_{D_1/X}|_Y \oplus \cdots \oplus \mathcal{N}_{D_r/X}|_Y.$$

2. The structure theorem of the spectral third reduction.

Definitions and some basic properties of reductions are given in the following

DEFINITION 2.1. 1) Let X_0 be an n-dimensional smooth projective variety ($n \geqq 2$), and L_0 a very ample line bundle on X_0. We assume that $K_{X_0}+(n-1)L_0$ is nef and big. Then from [**KMM**, Theorem 3-2-1], there is a unique (up to isomorphisms) birational contraction

$$\phi_1: X_0 \to X_1$$

onto a normal projective variety X_1. ϕ_1 is called the *first reduction*. From [**BS1**, Theorem (3.1)], X_1 is smooth, and if we define $L_1=(\phi_{1*}L_0)^{**}$ (the double dual), then L_1 is an ample line bundle on X_1 such that $K_{X_1}+(n-1)L_1$ is ample, and that $K_{X_0}+(n-1)L_0 \cong \phi_1^*(K_{X_1}+(n-1)L_1)$.

2) Let $n \geqq 4$. Assume that $K_{X_1}+(n-2)L_1$ is nef and big, then from [**KMM**, Theorem 3-2-1], there is a unique (up to isomorphisms) birational contraction

$$\phi_2: X_1 \to X_2$$

onto a normal projective variety X_2, and there is an ample line bundle $\mathcal{K}$ on X_2 such that $K_{X_1}+(n-2)L_1\cong\phi_2{}^*\mathcal{K}$. ϕ_2 is called the *second reduction*. We define $L_2=(\phi_{2*}L_1)^{**}$ (the double dual). From [**BS1**, Theorem (4.1), Lemma (4.4)] and [**Fj2**, (2.5)], X_2 has 2-factorial, terminal singularities, and $K_{X_2}+(n-2)L_2\cong\mathcal{K}$.

3) Let $n\geqq 4$. Assume that $2[K_{X_2}+(n-3)L_2]$ is nef and big, then as above from [**KMM**, Theorem 3-2-1], there is a unique (up to isomorphisms) birational contraction

$$\phi_3: X_2\rightarrow X_3$$

onto a normal projective variety X_3. ϕ_3 is called the *third reduction*.

We introduce the third reduction with a spectral value condition.

DEFINITION 2.2. The n-dimensional third reduction $\phi_3: X_2\rightarrow X_3$ is called the *spectral third reduction* if the spectral value $u(X_0, L_0)$ (see Definition 1.7) satisfies

$$(*SP)\quad u(X_0, L_0)>2\lceil(n-1)/3\rceil.$$

We denote the spectral third reduction by $\phi: X\rightarrow Y$ and set $L=L_2$. The condition $(*SP)$ is called the *third spectral condition*.

Now we will state the main theorem of this paper.

THEOREM 2.3 (The structure theorem of the spectral third reduction). *Let $\phi: X\rightarrow Y$ be the n-dimensional spectral third reduction. Then*

a: *Y has factorial, terminal singularities,*

b: *$n\geqq 9$,*

c: *the exceptional locus $\tilde{E}$ of ϕ is a disjoint union of a finite number of prime divisors, and such a prime divisor E is classified as follows.*

(1) $\dim\phi(E)=0$, $L_E{}^{n-1}=3$, $\omega_E^0\cong(2-n)L_E$, *and*
$$(E, L_E, \mathcal{N}_{E/X})\cong(\boldsymbol{Hc}, \mathcal{O}(1), \mathcal{O}(-1)),$$
where $\boldsymbol{Hc}$ is a (not necessarily normal) hypercubic in $\boldsymbol{P}^n$.

(2) $\dim\phi(E)=0$, $L_E{}^{n-1}=4$, $\omega_E^0\cong(2-n)L_E$, *and*
$$(E, L_E, \mathcal{N}_{E/X})\cong(\boldsymbol{I}(2, 2), \mathcal{O}(1), \mathcal{O}(-1)),$$
where $\boldsymbol{I}(2, 2)$ is a (not necessarily normal) complete intersection of two hyperquadrics in $\boldsymbol{P}^{n+1}$.

(3) $\dim\phi(E)=1$, *and for a general fiber F of $E\rightarrow\phi(E)$,*
$$(F, L_F, \mathcal{N}_{E/X}|_F)\cong(\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)),$$
where $\boldsymbol{Q}$ is a normal hyperquadric in $\boldsymbol{P}^{n-1}$.

REMARK 2.4. In the cases c-(1) and c-(2) in the above theorem, (E, L_E) is a Del Pezzo variety in the sense of [**Fj1**].

To prove the above theorem, we will make an essential use of the classification of the third adjoint contractions, [**Na1**, Theorem 1.3, Proposition 2.1, 2.3]. Let us start with the following proposition which establishes the basic properties of the spectral third reduction.

PROPOSITION 2.5. *If $\phi: X \rightarrow Y$ is the n-dimensional spectral third reduction, then it has the four properties*:

(1) *X is smooth, and L is a very ample line bundle on X.*

(2) *$K_X+(n-3)L$ is a nef and big line bundle such that*

$$K_X+(n-3)L \cong \phi^*\mathcal{L}$$

for a unique (up to isomorphisms) ample line bundle $\mathcal{L}$ on Y.

(3) $n \geqq 9$.

(4) *If we define $F:=(K_X+(n-3)L)^{\perp} \cap \overline{NE}(X)$, then F is an extremal face, and $F=R_1+\cdots+R_r$ for a unique set of distinct extremal rays $R_1, \cdots, R_r$. Moreover, if we take any R among $R_1, \cdots, R_r$, then R defines birational ray contraction*

$$\varphi_R: X \rightarrow W$$

whose supporting divisor is $K_X+(n-3)L'$ for some ample line bundle L', and ϕ factors through φ_R.

PROOF. Since ϕ satisfies the condition $(*SP)$, from [**BS2**, Theorem (3.1.4)], the first and the second reductions preceding ϕ are trivial. Hence the assertion (1) is clear.

Definition 2.1 and [**KMM**, Theorem 3-2-1] imply (2). Hence from Definition 1.4, the nef value of L is less than or equal to $n-3$, and thus from Lemma 1.8, we have $u(X, L) \leqq n-3$. Combining it with $(*SP)$, we get $n \geqq 9$, the assertion (3).

To prove (4), we assume that $F \neq \{0\}$. Then from Lemma 1.6, the nef value of L is equal to $n-3$. Let $R=\boldsymbol{R}_+[C]$. Since $\phi(C)$ is a point, ϕ factors through the ray contraction $\varphi_R: X \rightarrow W$, i.e. $\phi=\alpha \circ \varphi_R$ for some morphism $\alpha: W \rightarrow Y$. Note that since $\phi: X \rightarrow Y$ is birational, φ_R is also birational. In Lemma 1.6, $\tau=n-3$, and thus if we take $t=n-3$ and define $L'=L+\varphi_R{}^*V$, then $K_X+(n-3)L'$ will be a supporting divisor for R. Q.E.D.

Let us clarify the connection of the spectral third reduction with the birational third adjoint contractions ([**Na1**, Definition 1.1]). Since $\varphi_R: X \rightarrow W$ satisfies all conditions in [ibid., Proposition 2.1], we can use the classification lists in [ibid., Theorem 1.3 and Proposition 2.1]. Note that L' defined in the above proof is very ample, $L'_F \cong L_F$ for any fiber F of φ_R, and $n=\dim X \geqq 9$. Hence we can also apply [ibid., Proposition 2.3] to $\varphi_R: X \rightarrow W$. As a result we will obtain the following

LEMMA 2.6. *Let* $\varphi_R: X \to W$ *be any ray contraction in* (4) *of Proposition* 2.5, *and let* $\tilde{E}$, E *be the exceptional locus of* φ_R, *an irreducible component of* $\tilde{E}$, *respectively.*

a) *If* φ_R *is of divisorial type, then* $\tilde{E}$ *is a prime divisor* E, W *has* $\boldsymbol{Q}$*-factorial, terminal singularities, and* $\dim f(E)=0$, 1 *or* 2. *Furthermore, one of the following seven cases possibly occurs*:

(1) $\dim \varphi_R(E)=0$:

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(-3)), \text{ and } W \text{ is } 3\text{-factorial.}$$

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(-2)), \text{ and } W \text{ is } 2\text{-factorial.}$$

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Hc}, \mathcal{O}(1), \mathcal{O}(-1)), \text{ and } W \text{ is } 1\text{-factorial.}$$

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{I}(\boldsymbol{2}, \boldsymbol{2}), \mathcal{O}(1), \mathcal{O}(-1)), \text{ and } W \text{ is } 1\text{-factorial.}$$

(2) $\dim \varphi_R(E)=1$: *For any fiber* F *of* $E \to \varphi_R(E)$ *over a smooth point of a curve* $\varphi_R(E)$,

$$(F, L_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-2)), \text{ and } W \text{ is } 2\text{-factorial.}$$

For a general fiber F *of* $E \to \varphi_R(E)$,

$$(F, L_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)), \text{ and } W \text{ is } 1\text{-factorial.}$$

(3) $\dim \varphi_R(E)=2$: *For a general fiber* F *of* $E \to \varphi_R(E)$,

$$(F, L_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{P}^{n-3}, \mathcal{O}(1), \mathcal{O}(-1)), \text{ and } W \text{ is } 1\text{-factorial.}$$

b) *If* φ_R *is of flipping type, then* $\tilde{E}$ *is a disjoint union of its irreducible components* E, *which satisfy* $\dim \varphi_R(E)=0$ *and*

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)\oplus\mathcal{O}(-1)).$$

The third spectral condition excludes several cases among the ray contractions in Lemma 2.6.

PROPOSITION 2.7. *Let* $\varphi_R: X \to W$ *be any ray contraction in* (4) *of Proposition* 2.5. *Then* W *has factorial, terminal singularities, and* φ_R *is of divisorial type whose exceptional divisor* E *has the following classification.*

(1) *If* $\dim \varphi_R(E)=1$, *then for a general fiber* F *of* $E \to \varphi_R(E)$,

$$(F, L_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1)).$$

(2) *If* $\dim \varphi_R(E)=0$, *then*

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Hc}, \mathcal{O}(1), \mathcal{O}(-1)), \text{ or}$$

$$(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{I}(\boldsymbol{2}, \boldsymbol{2}), \mathcal{O}(1), \mathcal{O}(-1)).$$

PROOF. 1. We will exclude the case of flipping type in b) of Lemma 2.6. Assuming the existence of this case, we would like to have a contradiction. To apply [**Na2**, Corollary (1.3)], we use the same notation there: $(Z, L_Z, \mathcal{N}_{Z/X}{}^*)$ $\cong(\boldsymbol{P}^{n-2}, \mathcal{O}(1), \mathcal{O}(1)\oplus\mathcal{O}(1))$. Hence we would have $r=2$, $e=1$ and $a_1=a_2=1$. Then from [ibid., (1)], we would have $h^0(X, K_X+\lceil n/2\rceil L)>0$, which contradicts the third spectral condition $(*SP)$ which asserts $h^0(X, K_X+\lceil n/2\rceil L)=0$ since $2\lceil(n-1)/3\rceil\geqq\lceil n/2\rceil$ $(n\geqq 9)$.

2. We will exclude the two projective space cases in a-(2) and a-(3) of Lemma 2.6. As above assume the existence of a-(2), $(Z, L_Z)\cong(\boldsymbol{P}^{n-2}, \mathcal{O}(1))$. Then since $Z=F=E\cap\varphi_R{}^*V$ for some hyperplane section V on W, Lemma 1.10 would deduce $\mathcal{N}_{Z/X}\cong\mathcal{O}\oplus\mathcal{O}(-2)$. Hence $r=2$, $e=1$, $a_1=0$ and $a_2=2$. Thus from [ibid., (2)], we would have $h^0(X, K_X+\lceil(n+3)/3\rceil L)>0$, which contradicts $(*SP)$ since $2\lceil(n-1)/3\rceil\geqq\lceil(n+3)/3\rceil$ $(n\geqq 9)$.

Assuming the existence of the other case in a-(3), similarly, Lemma 1.10 would imply $(Z, L_Z, \mathcal{N}_{Z/X}{}^*)\cong(\boldsymbol{P}^{n-3}, \mathcal{O}(1), \mathcal{O}\oplus\mathcal{O}\oplus\mathcal{O}(1))$. Hence $r=3$, $e=1$, $a_1=a_2=0$ and $a_3=1$. Thus from [ibid., (1)], we would have $h^0(X, K_X+\lceil(n+1)/2\rceil L)>0$, which contradicts $(*SP)$ since $2\lceil(n-1)/3\rceil\geqq\lceil(n+1)/2\rceil$ $(n\geqq 9)$.

3. We will exclude the projective space case and the hyperquadric case in a-(1) of Lemma 2.6. From $(Z, L_Z, \mathcal{N}_{Z/X}{}^*)\cong(\boldsymbol{P}^{n-1}, \mathcal{O}(1), \mathcal{O}(3))$, we would have $r=1$, $e=1$ and $a_1=3$. Thus from [ibid., (1)], we would have $h^0(K_X+\lceil(n+3)/4\rceil L)>0$, which contradicts $(*SP)$ since $2\lceil(n-1)/3\rceil\geqq\lceil(n+3)/4\rceil$ $(n\geqq 9)$.

From $(Z, L_Z, \mathcal{N}_{Z/X}{}^*)\cong(\boldsymbol{Q}^{n-1}, \mathcal{O}(1), \mathcal{O}(2))$, we would have $r=1$ and $a_1=2$. Thus from [ibid., (2)], we would have $h^0(K_X+(2\lceil(n-2)/4\rceil+1)L)>0$, which contradicts $(*SP)$ since $2\lceil(n-1)/3\rceil\geqq 2\lceil(n-2)/4\rceil+1$ $(n\geqq 9)$. The proof of Proposition 2.7 is completed. Q.E.D.

PROOF OF THEOREM 2.3. From Propositions 2.5 and 2.7, it follows that $n\geqq 9$ and there exist only the ray contractions of divisorial type whose image W has factorial, terminal singularities. Therefore, the same argument as in the proof of [**BS1**, (3.1.4)] implies our assertion. Q.E.D.

3. The fourth spectral condition and an application of the structure theorem.

It is natural to ask when the spectral third reduction is trivial. As a matter of fact, for the (not necessarily spectral) third reduction we will have a sufficient condition, which is given by the following

DEFINITION 3.1. A condition:

$$u(X_0, L_0) > \max\{3\lceil(n-2)/4\rceil, 4\lceil(n-2)/5\rceil\}$$

is called the *fourth spectral condition*.

Now we will show that the fourth spectral condition is sufficient for the third reduction to be an isomorphism.

THEOREM 3.2. *Assume that there exists the third reduction* $\phi_3: X_2 \to X_3$, *or the reduction series,* $X_0 \xrightarrow{\phi_1} X_1 \xrightarrow{\phi_2} X_2 \xrightarrow{\phi_3} X_3$. *Then this series is trivial, and* $n \geqq 9$, *provided that the fourth spectral condition holds true.*

PROOF. 0) Note that since $u(X_0, L_0) > 4\lceil(n-2)/5\rceil \geqq 2\lceil(n-1)/3\rceil$ for $n \geqq 4$, $\phi_3: X_2 \to X_3$ turns out to be the spectral third reduction $\phi: X \to Y$, and thus from Theorem 2.3, X_3 is factorial, and $n \geqq 9$. If ϕ_3 were not an isomorphism, then from Theorem 2.3 we would have the three distinct types of non-trivial fibers of ϕ_3.

1) $(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{Hc}, \mathcal{O}(1), \mathcal{O}(-1))$, and $L_E{}^{n-1}=3$. In the notations of [**Na2**, Proposition (1.1)], since $d=3$ and $m=1$, we would have $\omega_E^0 + t(d+m)L_E = [-(n-2)+4t]L_E$ and d would not divide m. Hence applying the last statement in [ibid.], it would follow that $h^0(X, K_X + 3\lceil(n-2)/4\rceil L) > 0$. However it contradicts the fourth spectral condition above.

2) $(E, L_E, \mathcal{N}_{E/X}) \cong (\boldsymbol{I}(2, 2), \mathcal{O}(1), \mathcal{O}(-1))$, and $L_E{}^{n-1}=4$. As above, since we would have $d=4$ and $m=1$, from [ibid.], $h^0(X, K_X + 4\lceil(n-2)/5\rceil L) > 0$. Hence, contradiction.

3) $(F, L_F, \mathcal{N}_{E/X}|_F) \cong (\boldsymbol{Q}^{n-2}, \mathcal{O}(1), \mathcal{O}(-1))$. Note that $\boldsymbol{Q}$ is of higher codimension and possibly singular. From Lemma 1.10, $\mathcal{N}_{F/X} \cong \mathcal{O} \oplus \mathcal{O}(-1)$. Here we apply [**Na2**, Corollary (1.3)-(2)] with $r=2$, $a_1=0$ and $a_2=1$ so that we would have $h^0(X, K_X + (2\lceil(n-2)/3\rceil + 1)L) > 0$. But it contradicts the fourth spectral condition since $2\lceil(n-2)/3\rceil + 1 \leqq 4\lceil(n-2)/5\rceil$ for $n \geqq 9$. Thus ϕ_3 must be an isomorphism.

Q.E.D.

After cutting out X, $(n-3)$-times, by general hyperplane sections of L, we will obtain a birational canonical morphism $f: V \to Z$ from a smooth threefold V. Although K_V is nef, the structure of a positive dimensional fiber of f will remain the same as the one of $\phi: X \to Y$, and thus f gives an interesting example of a birational canonical morphism for a minimal smooth threefold. Let us recall first the definition of a canonical morphism.

DEFINITION 3.3 (cf. [**KMM**, Definition 0-4-1]). A morphism $f: V \to Z$ from a normal complete variety V with canonical singularities onto a variety Z is called *canonical* if f is defined by the linear system $|mK_V|$ for some integer $m \gg 0$. Note that in this case, V is a minimal variety and Z is a normal variety.

We will state precisely the structure of the three-dimensional canonical morphism $f: V \to Z$ which comes from the spectral third reduction.

COROLLARY 3.4. *Let* $\phi: X \to Y$ *be the n-dimensional spectral third reduction, and* V *the intersection of* $(n-3)$*-general hyperplane sections of* $|L|$. *We denote the* *restriction* $\phi|_V: V\to\phi(V)$ *by* $f: V\to Z$. *Then*

a: *f is a canonical morphism from a three dimensional smooth minimal projective variety V onto a three dimensional normal projective variety Z,*

b: *Z has factorial, canonical singularities (which is not terminal),*

c: *the exceptional locus $\tilde{D}$ of f is a disjoint union of a finite number of prime divisors, and such a prime divisor D is classified as follows.*

(1) $\dim f(D)=0$, $L_D{}^2=3$, $\omega_D^0\cong -L_D$, *and*

$$(D, L_D, \mathcal{N}_{D/X})\cong(\boldsymbol{Hc}, \mathcal{O}(1), \mathcal{O}(-1)),$$

where $\boldsymbol{Hc}$ *is a hypercubic in* $\boldsymbol{P}^3$.

(2) $\dim f(D)=0$, $L_D{}^2=4$, $\omega_D^0\cong -L_D$, *and*

$$(D, L_D, \mathcal{N}_{D/X})\cong(\boldsymbol{I}(2, 2), \mathcal{O}(1), \mathcal{O}(-1)),$$

where $\boldsymbol{I}(2, 2)$ *is a complete intersection of two hyperquadrics in* $\boldsymbol{P}^4$.

(3) $\dim f(D)=1$, *and for a general fiber G of $D\to f(D)$,*

$$(G, L_G, \mathcal{N}_{D/X}|_G)\cong(\boldsymbol{Q}, \mathcal{O}(1), \mathcal{O}(-1)),$$

where $\boldsymbol{Q}$ *is a smooth conic in* $\boldsymbol{P}^2$.

PROOF. From the assumption, it follows by induction that $K_V\cong(K_X+(n-3)L)|_V$, and thus f is canonical. Here, an essential use is made of Base Point Free Theorem, [**KMM**, Theorem 3-1-1, Remark 3-1-2]. Furthermore, since $L_E\cong\mathcal{O}_{Hc}(1)$ or $\mathcal{O}_{I(2,2)}(1)$, and $L_F\cong\mathcal{O}_Q(1)$, we can recover the same classification as the one of Theorem 2.3 for $f: V\to Z$. For a full detail, see [**Na3**, Section 4.2]. Q.E.D.

4. The classification of the polarized variety $(Y, \mathcal{L})$.

Let $\phi: X\to Y$ be the n-dimensional spectral third reduction and let $\mathcal{L}$ be the unique ample line bundle defined in (2) of Proposition 2.5. When K_Y is not nef, it will be shown quite easily by virtue of the third spectral condition that the nef value $\tau(\mathcal{L})$ of the polarized variety $(Y, \mathcal{L})$ is no greater than $n-4$. Furthermore, we will classify $(Y, \mathcal{L})$ in the case that $K_Y+(n-4)\mathcal{L}$ is not big.

THEOREM 4.1. *Let $\phi: X\to Y$ be the n-dimensional spectral third reduction with $K_X+(n-3)L\cong\phi^*\mathcal{L}$ for a unique (up to isomorphisms) ample line bundle $\mathcal{L}$ on Y. We define $\mathcal{L}=(\phi_*L)^{**}$ (the double dual). Assume that K_Y is not nef. Let*

$$\mu: Y\to Z$$

be the nef value morphism of $(Y, \mathcal{L})$. *Then* a: *$\mathcal{L}$ is a μ-ample line bundle, and the nef value $\tau(\mathcal{P})$ of $\mathcal{P}$ and the μ-nef value $\theta(\mathcal{L})$ of $\mathcal{L}$ satisfy $\tau(\mathcal{P})\leqq n-4$, $\theta(\mathcal{L})\leqq n-4$ and $\theta(\mathcal{L})\in\boldsymbol{N}$,*

b: *$n=13$ or $\geqq 15$, and*

c: *$K_Y+(n-4)\mathcal{P}$ is big unless one of the following cases occurs:*

(1) *Y is a Fano n-fold with $K_Y\cong-(n-4)\mathcal{P}\cong-(n-4)\mathcal{L}$, and $\mathcal{L}$ is an ample line bundle on Y.*

(2) *$(Y, \mathcal{P})$ is a Fano fibration of co-index four over a smooth curve Z.*

(3) *$(Y, \mathcal{P})$ is a Mukai fibration over a normal surface Z.*

(4) *$(Y, \mathcal{P})$ is a Del Pezzo fibration over a normal three fold Z such that for a general fiber F of μ, $(F, \mathcal{P}_F)$ is an $(n-3)$-dimensional smooth Del Pezzo variety in the sense of* [**Fj1**].

(5) *$(Y, \mathcal{P})$ is a quadric fibration over a normal four fold Z such that for a general fiber F of μ, $(F, \mathcal{P}_F)\cong(\boldsymbol{Q}^{n-4}, \mathcal{O}(1))$, where $\boldsymbol{Q}$ is a smooth hyperquadric in $\boldsymbol{P}^{n-3}$.*

(6) *$(Y, \mathcal{P})$ is a scroll over a normal five fold Z such that for a general fiber F of μ, $(F, \mathcal{P}_F)\cong(\boldsymbol{P}^{n-5}, \mathcal{O}(1))$.*

Under the same assumption of the above theorem with $\tau:=\tau(\mathcal{P})$ and $\theta:=\theta(\mathcal{L})$, we will prepare Lemmas 4.2 and 4.3.

LEMMA 4.2. 1) *$\mathcal{L}=(\phi_*L)^{**}$ is a μ-ample line bundle on Y.*
2) *$K_Y+\tau\mathcal{P}\cong(1+\tau)(K_Y+[(n-3)\tau/(1+\tau)]\mathcal{L})$.*

PROOF. From Theorem 2.3, Y is factorial, and thus $\mathcal{L}$ is a line bundle on Y. Since $K_X+(n-3)L\cong\phi^*\mathcal{P}$, $[K_Y+(n-3)\mathcal{L}]|_{Y_{\mathbf{reg}}}\cong[K_X+(n-3)L]|_{\phi^{-1}(Y_{\mathbf{reg}})}\cong\mathcal{P}|_{Y_{\mathbf{reg}}}$. Hence $K_Y+(n-3)\mathcal{L}\cong\mathcal{P}$. Define $F:=(K_Y+\tau\mathcal{P})^{\perp}\cap\overline{NE}(Y)$, and from [**KMM**, Lemma 3-2-4], we have $\overline{NE}(Y/Z)=F$. Take any $\gamma\in\overline{NE}(Y/Z)\setminus\{0\}$, and $(K_Y+\tau\mathcal{P})\cdot\gamma=0$. Then from $K_Y+(n-3)\mathcal{L}\cong\mathcal{P}$, we get $\mathcal{L}\cdot\gamma=(\mathcal{P}-K_Y)\cdot\gamma/(n-3)=(1+\tau)\mathcal{P}\cdot\gamma/(n-3)>0$, which shows the first assertion.

Again $K_Y+(n-3)\mathcal{L}\cong\mathcal{P}$ implies $K_Y+\tau\mathcal{P}\cong(1+\tau)(K_Y+[(n-3)\tau/(1+\tau)]\mathcal{L})$.

Q.E.D.

LEMMA 4.3. *Define $u, v, a, b\in\boldsymbol{N}$ by $u/v=\tau$, $(u, v)=1$ and $a/b=\theta$, $(a, b)=1$. Then*

1) *$u, a\leqq n+1$ and*
2) *$2\lceil(n-1)/3\rceil<\theta=a/b=(n-3)u/(u+v)$.*

PROOF. The first assertion is just the rationality theorem, Lemma 1.3.

For the second assertion we will first show that the spectral value $u(Y, \mathcal{L})$ of $(Y, \mathcal{L})$ satisfies $\theta\geqq u(Y, \mathcal{L})=u(X, L)>2\lceil(n-1)/3\rceil$. Since from Lemma 4.2-1) $\mathcal{L}$ is μ-ample, the first inequality is straightforward from Lemma 1.8 if we take $\pi=\mu$ there. Since Y has terminal singularities, and $\phi: X\rightarrow Y$ is a desingularization of Y, by [**KMM**, Definition 0-2-6], we have $K_X \cong \phi^* K_Y + \sum a_i E_i$ for some $0 < a_i \in \boldsymbol{Q}$. [**BS2**, Lemma (0.4.4)] thus implies $u(X, L) = u(Y, \mathcal{L})$, and thus we are done by the third spectral value condition (Definition 2.2).

From Lemma 4.2-2), $K_Y + \tau \mathcal{P} \cong (1+\tau)(K_Y + [(n-3)u/(u+v)]\mathcal{L})$, which is nef but not μ-ample. Hence from Lemma 1.5, we conclude that $\theta = (n-3)u/(u+v)$. Q.E.D.

PROOF OF THEOREM 4.1. Note that from Proposition 2.5, we have $n \geqq 9$. From Lemma 4.3 we have $2(n-1)/3 < (n+1)/b$ and thus $b < 3(n+1)/2(n-1) < 2$. Hence $b=1$ and $\theta = a \in \boldsymbol{N}$. Again from Lemma 4.3-2) we have $a/\mathbf{1} = (n-3)u/(u+v)$ and thus $av = (n-3-a)u$, which implies $n-3-a \geqq 1$. Thus we have $\theta = a \leqq n-4$.

If $\tau > n-4$, then $u > (n-4)v$. From $av = (n-3-a)u > (n-3-a)(n-4)v$, $a > (n-3-a)(n-4) = (n-3)(n-4) - (n-4)a$, and thus $(n-3)a > (n-3)(n-4)$. Hence $a > n-4$, which contradicts the previous fact: $a \leqq n-4$. Therefore, we have proven that $\tau \leqq n-4$.

From $2\lceil (n-1)/3 \rceil < a \leqq n-4$, putting $n = 3k+i$ $(i=0, 1, 2)$, we have $2\lceil (i-1)/3 \rceil \leqq k+i-5$. It results in $k \geqq 4$ $(i=1)$ or $k \geqq 5$ $(i=0, 2)$. If $k=4$, then $i=1$, and thus $n=13$. If $k \geqq 5$, then $i=0, 1, 2$, and thus $n \geqq 15$. Thus we have $n=13$ or $\geqq 15$.

To prove the last part, we can assume that $\tau = n-4$. Note that $K_Y + (n-4)\mathcal{P} \cong \mu^* Q$ for some ample line bundle Q on Z.

For (1). Assume that $\dim Z = 0$. Since from Lemma 4.2-1) $\mathcal{L}$ is μ-ample, $\mathcal{L}$ is really ample. From $(n-3)[K_Y + (n-4)\mathcal{L}] \cong K_Y + (n-4)\mathcal{P} \cong \mathcal{O}_Y$ (Lemma 4.2-2)), the nef line bundle $K_Y + (n-4)\mathcal{L}$ defines the same extremal face as $K_Y + (n-4)\mathcal{P}$ does, and thus from [**KMM**, Theorem 3-2-1], we have $K_Y + (n-4)\mathcal{L} \cong \mathcal{O}_Y$. That is, we are in the case 1).

For (2) through (6). Assume that $m := \dim Z \geqq 2$. From Theorem 2.3, $\dim Y_{\text{sing}} \leqq 1$. Hence by means of the generic smoothness ([**Ha**, Chapter 3, Corollary 10.7 and Theorem 10.2]), a general fiber F of μ is a smooth projective variety and $K_F \cong K_Y|_F$ (the adjunction formula). It thus follows that $K_F + (n-4)\mathcal{P}_F \cong \mathcal{O}_F$. It shows that $n-4$ is the nef value of $\mathcal{P}_F$, and thus from the rationality theorem, Lemma 1.3, $n-4 \leqq \dim F + 1$. Thus $n-5 \leqq \dim F$, or $m \leqq 5$. If $m=4, 5$, from [**BS1**, Theorem (1.3)], we have $(F, \mathcal{P}_F) \cong (\boldsymbol{Q}^{n-4}, \mathcal{O}(1))$, $(\boldsymbol{P}^{n-5}, \mathcal{O}(1))$, respectively. If $m=3$, from [**Fjl**, (6.4)], $(F, \mathcal{P}_F)$ is asmooth Del Pezzo variety.

Since $K_Y + (n-m-(4-m))\mathcal{P} \cong \mu^* Q$, from Definition 1.9 of the special varieties, according as $m = 1, 2, 3, 4$ and 5, we are in the case (2), (3), (4), (5) and (6). Q.E.D.

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