

Intersection of harmonics and Capelli identities for symmetric pairs

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Abstract. We consider a see-saw pair consisting of a Hermitian symmetric pair $(G_{\mathbf{R}}, K_{\mathbf{R}})$ and a compact symmetric pair $(M_{\mathbf{R}}, H_{\mathbf{R}})$, where $(G_{\mathbf{R}}, H_{\mathbf{R}})$ and $(K_{\mathbf{R}}, M_{\mathbf{R}})$ form a real reductive dual pair in a large symplectic group. In this setting, we get Capelli identities which explicitly represent certain $K_{\mathbf{C}}$ -invariant elements in $U(\mathfrak{g}_{\mathbf{C}})$ in terms of $H_{\mathbf{C}}$ -invariant elements in $U(\mathfrak{m}_{\mathbf{C}})$. The corresponding $H_{\mathbf{C}}$ -invariant elements are called Capelli elements.

We also give a decomposition of the intersection of O_{2n} -harmonics and Sp_{2n} -harmonics as a module of $GL_n = O_{2n} \cap Sp_{2n}$, and construct a basis for the GL_n highest weight vectors. This intersection is in the kernel of our Capelli elements.

Introduction.

Consider a see-saw pair of real reductive Lie groups in the real symplectic group $Sp_{2N}(\mathbf{R})$,

$$\begin{array}{ccc} G_0 & & M_0 \\ \cup & \times & \cup \\ K_0 & & H_0, \end{array}$$

where both (G_0, H_0) and (K_0, M_0) are dual pairs (see [How89b, (5.1)] and [How83, Section 3]). Recall that the pair (G_0, H_0) is called a *dual pair* if G_0 and H_0 are the mutual commutants of each other in $Sp_{2N}(\mathbf{R})$. Let \mathfrak{g}_0 be the Lie algebra of G_0 and \mathfrak{g} its complexification. We use the same notation for the other Lie groups.

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The symplectic group $Sp_{2N}(\mathbf{R})$ has the unique non-trivial double cover $Mp_{2N}(\mathbf{R})$, which is called the metaplectic group. It has a unique unitary representation ω called the Weil representation (or the oscillator representation; see [How89b] or [KV78], for example). We use the same symbol ω to denote the differentiated action on the Harish-Chandra module of ω . Thus ω is a representation of the complexified Lie algebra \mathfrak{sp}_{2N} of $Sp_{2N}(\mathbf{R})$. The Harish-Chandra module can be explicitly realized as the polynomial ring on the N -dimensional vector space $V \simeq \mathbf{C}^N$ (see [How89b, p. 537]). In this realization, elements of \mathfrak{sp}_{2N} act by polynomial coefficient differential operators on V . According to the result of Howe ([How89a]), we have the following equation of the algebras of differential operators:

$$\omega(U(\mathfrak{g})^K) = \omega(U(\mathfrak{m})^H), \tag{0.1}$$

where K and H denote the complexifications of K_0 and H_0 respectively, and $U(\mathfrak{g})^K$ denotes the set of K -invariants in the universal enveloping algebra $U(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. The subalgebra $S(\mathfrak{p})^K$ of the K -invariants in the symmetric algebra $S(\mathfrak{p})$ is isomorphic to a polynomial ring (see [KR69]), and let X_1, X_2, \dots, X_r be a set of homogeneous generators of $S(\mathfrak{p})^K$. If $\iota : S(\mathfrak{p}) \rightarrow U(\mathfrak{g})$ is a K -linear pseudo-symmetrization map (for the definition, see Section 1.1), then the image $\iota(X_d)$ is K -invariant and hence $\omega(\iota(X_d))$ can be expressed in terms of $\omega(U(\mathfrak{m})^H)$ thanks to (0.1):

$$\omega(\iota(X_d)) = \omega(C_d) \quad (C_d \in U(\mathfrak{m})^H).$$

We call this formula a *Capelli identity for a symmetric pair* and C_d the *Capelli element* corresponding to X_d . In fact, the identity gives a relation between differential operators in various kind of determinantal form, which is similar to the original Capelli identity ([Cap90]). However, our Capelli elements are *not* central in the enveloping algebra, and they depend on the choice of the pseudo-symmetrization map ι . Despite this, the Capelli elements in this paper resemble the original Capelli element ([Cap90]) or its variants (see [HU91], [Ito04], [Naz00], [MN99], [Osh96] and [Wac06], for example).

We give explicit forms of the Capelli elements for the see-saw pair, where G_0/K_0 is an irreducible Hermitian symmetric space, and M_0 is compact. The main results are summarized in Theorems 2.1, 2.6 and 2.7 respectively.

Our Capelli identities are closely related to the (pluri-)Harmonic polynomials for compact classical Lie groups. In fact, if we consider the orthogonal group $O_{2n}(\mathbf{R})$ and the unitary symplectic group USp_{2n} in U_{2n} , it turns out that the harmonic polynomials are annihilated by Capelli elements (Section 3.5). Thus the

joint harmonics for O_{2n} and USp_{2n} are in the kernel of different kinds of Capelli elements, which are written as an element of the enveloping algebra of U_{2n} in determinantal form.

We are interested in the kernel of the Capelli elements. However, we have not found a good characterization via representation theory. Instead, we study the intersection of harmonics for O_{2n} and Sp_{2n} . Note that the joint harmonics carries a representation of $GL_n = O_{2n} \cap Sp_{2n}$. In Theorem 3.1, we give a decomposition of the joint harmonics as a GL_n -module in terms of the Littlewood-Richardson coefficients. We also indicate how a basis for the space of GL_n highest weight vectors can be obtained in Section 4. This is related to the tensor product algebras constructed in [HTW05] (see also [HL07]).

As a result of our study of joint harmonics, we can answer the following problem, which is of independent interest. Let ρ_{2n}^λ be an irreducible finite dimensional representation of GL_{2n} with the highest weight $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k, 0, \dots, 0)$ ($k \leq n/2$). Let us consider the subrepresentation of O_{2n} (respectively Sp_{2n}) generated by the highest weight vector of ρ_{2n}^λ , and denote it by σ_{2n}^λ (respectively by τ_{2n}^λ). Then we can identify the intersection $\sigma_{2n}^\lambda \cap \tau_{2n}^\lambda$ as a subspace of joint harmonics, and we get a decomposition as a representation of $GL_n = O_{2n} \cap Sp_{2n}$:

$$\sigma_{2n}^\lambda \cap \tau_{2n}^\lambda \simeq \bigoplus_{\mu, \nu \in \mathcal{P}_k} c_{\mu, \nu}^\lambda \rho_n^{\mu \odot \nu},$$

where $c_{\mu, \nu}^\lambda$ is the Littlewood-Richardson coefficients, \mathcal{P}_k is the set of partitions with length $\leq k$, and $\mu \odot \nu$ is given in (0.3) below. From this, we conclude that the intersection is in the kernel of Capelli elements, which is not easy to see a priori.

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NOTATION FOR FINITE DIMENSIONAL IRREDUCIBLE REPRESENTATIONS.

Let us fix the notation for irreducible finite dimensional representations of the complex matrix groups GL_n , O_{2n} and Sp_{2n} .

Let $\lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}^n$ be a dominant integral weight for GL_n , i.e., $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$. We denote by ρ_n^λ the irreducible finite dimensional representation of GL_n with highest weight λ . Similarly, if $\lambda_n \geq 0$, we denote by σ_{2n}^λ (respectively τ_{2n}^λ) the irreducible finite dimensional representation of O_{2n} (respectively Sp_{2n}) with highest weight λ . We define $\lambda^* = -(\lambda_n, \lambda_{n-1}, \dots, \lambda_1)$, which is the highest weight for the contragredient representation $(\rho_n^\lambda)^*$. Let

$$\mathcal{P}_n = \{ \lambda = (\lambda_1, \dots, \lambda_n) \in \mathbf{Z}_{\geq 0}^n \mid \lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n \geq 0 \} \tag{0.2}$$

be the set of partitions of length $\leq n$. We denote the length of a partition λ by $\ell(\lambda)$. Take two partitions $\mu \in \mathcal{P}_p$ and $\nu \in \mathcal{P}_q$ ($p + q \leq n$). Then we denote

$$\begin{aligned} \mu \odot \nu &= (\mu, 0, \dots, 0, \nu^*) \\ &= (\mu_1, \mu_2, \dots, \mu_p, 0, \dots, 0, -\nu_q, -\nu_{q-1}, \dots, -\nu_1) \in \mathbf{Z}^n, \end{aligned} \tag{0.3}$$

which is a dominant integral weight for GL_n .

1. Capelli identities for symmetric pairs.

1.1. Pseudo-symmetrization map.

Let G_0 be a real reductive Lie group, and K_0 its maximal compact subgroup. We denote by \mathfrak{g} and \mathfrak{k} the complexifications of the Lie algebras of G_0 and K_0 respectively.

Let $U(\mathfrak{g})$ be the enveloping algebra of \mathfrak{g} , and denote by $F_i U(\mathfrak{g})$ the standard filtration of $U(\mathfrak{g})$. For the symmetric algebra $S(\mathfrak{g})$, we have a direct sum decomposition by the standard grading $S(\mathfrak{g}) = \bigoplus_{i \geq 0} S_i(\mathfrak{g})$. Then there is a natural grading map $\text{gr}_i : F_i U(\mathfrak{g}) \rightarrow S_i(\mathfrak{g})$ with the kernel $F_{i-1} U(\mathfrak{g})$.

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. If a K -linear map $\iota : S(\mathfrak{p}) \rightarrow U(\mathfrak{g})$ brings $S_i(\mathfrak{g})$ into $F_i U(\mathfrak{g})$ and satisfies $\text{gr}_i(\iota(u)) = u$ for every homogeneous element $u \in S_i(\mathfrak{p})$, we call ι a *pseudo-symmetrization map*.

There are two basic examples of pseudo-symmetrization maps. First, the full symmetrization map

$$Y_1 Y_2 \cdots Y_d \mapsto \left(\frac{1}{d!} \right) \sum_{\sigma \in \mathfrak{S}_d} Y_{\sigma(1)} Y_{\sigma(2)} \cdots Y_{\sigma(d)} \quad (Y_i \in \mathfrak{p}).$$

Second, when $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian symmetric type, we have the irreducible decomposition $\mathfrak{p} = \mathfrak{p}^+ \oplus \mathfrak{p}^-$ as K -modules. In this case,

$$\begin{aligned} \iota : S(\mathfrak{p}) &\rightarrow U(\mathfrak{g}) \\ u_1 u_2 &\mapsto u_1 u_2 \quad (u_1 \in U(\mathfrak{p}^+), u_2 \in U(\mathfrak{p}^-)) \end{aligned} \tag{1.1}$$

is a pseudo-symmetrization map, since both \mathfrak{p}^+ and \mathfrak{p}^- are K -stable abelian Lie algebras.

1.2. Abstract Capelli identity for see-saw pairs.

As in the introduction, consider a see-saw pair of real reductive Lie groups in the ambient real symplectic group $S_0 = Sp_{2N}(\mathbf{R})$,

$$\begin{array}{ccc} G_0 & & M_0 \\ \cup & \times & \cup \\ K_0 & & H_0, \end{array}$$

where both (G_0, H_0) and (K_0, M_0) are dual pairs (see [How83, Section 3]). Let ω be the Harish-Chandra module of the Weil representation, realized in the space of all polynomial coefficient differential operators on the N -dimensional vector space $V \simeq \mathbf{C}^N$. Thus ω is a (\mathfrak{s}, U_0) -module, where $\mathfrak{s} = \mathfrak{sp}_{2N}(\mathbf{C})$ is the complexification of the Lie algebra of $S_0 = Sp_{2N}(\mathbf{R})$, and U_0 denotes a maximal compact subgroup of S_0 , which is isomorphic to the unitary group U_N .

Let H be the complexification of the subgroup H_0 . Then, due to Howe [How89a, Theorem 7], H -invariants in the algebra of representation operators $\omega(U(\mathfrak{s}))$ coincides with $\omega(U(\mathfrak{g}))$. Similarly the space of K -invariants $\omega(U(\mathfrak{s}))^K$ is an algebra $\omega(U(\mathfrak{m}))$ generated by the representation image of $\mathfrak{m} = \text{Lie}(M_0)_{\mathbf{C}}$. Thus we have the following identity of algebras of polynomial differential operators.

$$\omega(U(\mathfrak{g})^K) = \omega(U(\mathfrak{s}))^{K \times H} = \omega(U(\mathfrak{m})^H) \tag{1.2}$$

Let $\mathfrak{g} = \mathfrak{k} \oplus \mathfrak{p}$ be the complexified Cartan decomposition. The subalgebra $S(\mathfrak{p})^K$ of the K -invariants in the symmetric algebra $S(\mathfrak{p})$ is isomorphic to a polynomial ring, and let X_1, X_2, \dots, X_r be a set of homogeneous generators of $S(\mathfrak{p})^K$. Choose a K -equivariant pseudo-symmetrization map $\iota : S(\mathfrak{p}) \rightarrow U(\mathfrak{g})$. Then the image $\iota(X_d)$ is K -invariant and hence $\omega(\iota(X_d))$ can be expressed in terms of $\omega(U(\mathfrak{m})^H)$ thanks to Equation (1.2):

$$\omega(\iota(X_d)) = \omega(C_d) \quad (C_d \in U(\mathfrak{m})^H). \tag{1.3}$$

We call this formula a *Capelli identity for a symmetric pair* and C_d the *Capelli element* corresponding to X_d . Note that our Capelli elements are *not* central and they depend on the choice of the pseudo-symmetrization map ι . Even if we fix the map ι , the Capelli element C_d is *not* uniquely determined in general because the Weil representation ω has a non-trivial kernel. However, if M is relatively small by comparison with K , it is uniquely determined.

2. Capelli identities for symmetric pairs of Hermitian type.

Let us assume that $(\mathfrak{g}, \mathfrak{k})$ is of Hermitian symmetric type, and take a pseudo-symmetrization map ι as in (1.1). There are three of such (irreducible) see-saw pairs given in Table 1 (see [How89b, Eq.(5.2) & Table (5.3)]). Note that (M_0, H_0) is also a symmetric pair in all the three cases.

Table 1. see-saw pairs with G_0 Hermitian type, M_0 compact.

| | $Sp_{2N}(\mathbf{R})$ | G_0 | K_0 | M_0 | H_0 |
|---------------|----------------------------|-----------------------|------------------|------------------|---------|
| Case R | $Sp_{2kn}(\mathbf{R})$ | $Sp_{2k}(\mathbf{R})$ | U_k | U_n | O_n |
| Case C | $Sp_{2(p+q)n}(\mathbf{R})$ | $U_{p,q}$ | $U_p \times U_q$ | $U_n \times U_n$ | U_n |
| Case H | $Sp_{4kn}(\mathbf{R})$ | O_{2k}^* | U_k | U_{2n} | USp_n |

In the subsequent subsections, we will give explicit form of the Capelli elements for these three cases. Thus we have identities of differential operators in the expression of various minor determinants, which is of independent interest.

We note that we can give the Capelli identities also for the cases where M_0 is non-compact by using the Fourier transform, although we do not describe them in this note. We thank Hiroyuki Ochiai and Jiro Sekiguchi for pointing out it to us.

2.1. Case R.

In this subsection, we give the Capelli identity for the symmetric pair of Case **R**. We first fix the notation and describe the generators of $S(\mathfrak{p})^K$ to state the main theorem. For Case **R**, the complex Lie algebra \mathfrak{g} and its subalgebras \mathfrak{k} and \mathfrak{p}^\pm are explicitly given as follows.

$$\mathfrak{g} = \mathfrak{sp}_{2k} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \mid \begin{array}{l} H \in \mathfrak{gl}_k, \\ G, F \in \text{Sym}_k \end{array} \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_k, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$H_{ij} = E_{ij} - E_{k+j, k+i} \in \mathfrak{k}, \quad G_{ij} = E_{i, k+j} + E_{j, k+i} \in \mathfrak{p}^+, \quad F_{ij} = E_{k+i, j} + E_{k+j, i} \in \mathfrak{p}^-,$$

where E_{ij} denotes the matrix unit with 1 at its (i, j) -th position and 0 elsewhere, and $\text{Sym}_k = \text{Sym}_k(\mathbf{C})$ the set of the complex symmetric $k \times k$ matrices. The complex Lie algebra \mathfrak{m} and its subalgebra \mathfrak{h} are given by

$$\mathfrak{m} = \mathfrak{gl}_n, \quad \mathfrak{h} = \mathfrak{o}_n = \{X \in \mathfrak{gl}_n; {}^tX + X = O_n\}.$$

Let $V = M_{n,k}(\mathbf{C})$ be the space of $n \times k$ matrices and denote the linear coordinate functions on V and the corresponding differential operators by

$$x_{si}, \quad \partial_{si} = \frac{\partial}{\partial x_{si}} \quad (1 \leq s \leq n, 1 \leq i \leq k).$$

Let $\mathfrak{s} = \mathfrak{sp}_{2kn}$ be the complex symplectic Lie algebra, in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. We have (the Harish-Chandra module of) the Weil representation ω of \mathfrak{s} on the space $\mathbf{C}[V]$ of polynomial functions on V . The explicit representation operators of \mathfrak{g} and \mathfrak{m} are given as follows (for this, see [NOT01, Section 4] for example):

$$\begin{aligned} \omega(G_{ij}) &= \sqrt{-1} \sum_{s=1}^n x_{si} x_{sj}, & \omega(F_{ij}) &= \sqrt{-1} \sum_{s=1}^n \partial_{si} \partial_{sj}, \\ \omega(H_{ij}) &= \sum_{s=1}^n x_{si} \partial_{sj} + \frac{n}{2} \delta_{ij}, & \omega(E_{st}) &= \sum_{i=1}^k x_{si} \partial_{ti} + \frac{k}{2} \delta_{st}. \end{aligned} \tag{2.1}$$

We now recall the structure of $S(\mathfrak{p})^K$. Since \mathfrak{g}_0 is of Hermitian type, $K \simeq GL_k(\mathbf{C})$ acts multiplicity-freely both on the symmetric algebra $S(\mathfrak{p}^+)$ and on $S(\mathfrak{p}^-)$.

$$S(\mathfrak{p}^+) = \bigoplus_{\mu \in \mathcal{P}_k} W_\mu, \quad S(\mathfrak{p}^-) = \bigoplus_{\mu \in \mathcal{P}_k} W_\mu^*; \quad W_\mu \simeq \rho_k^\mu.$$

Thus we have the expression of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = (S(\mathfrak{p}^+) \otimes_{\mathbf{C}} S(\mathfrak{p}^-))^K = \bigoplus_{\mu \in \mathcal{P}_k} (W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K,$$

which is isomorphic to a polynomial ring with k algebraically independent generators. For $d = 1, 2, \dots, k$, the d -th generator is the basis vector of the one-dimensional vector space $(W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K$ for $\mu = (2, \dots, 2, 0, \dots, 0) = (2^d, 0^{k-d})$. The explicit form of the generators are

$$X_d^R = \sum_{I, J \in \mathcal{I}_d^k} \det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI} \in S(\mathfrak{p})^K \quad (d = 1, 2, \dots, k; k = r), \tag{2.2}$$

where \mathcal{I}_d^k is the index set defined by

$$\mathcal{I}_d^k = \{I \subset \{1, 2, \dots, k\} \mid \#I = d\}, \tag{2.3}$$

and \mathbf{G}_{IJ} denotes the $d \times d$ submatrix of the $k \times k$ matrix (G_{ij}) with the rows and the columns chosen from I and J respectively. Note that the generators above belong to the symmetric algebra $S(\mathfrak{p})$, and that G_{ij} and $F_{i'j'}$ appearing in the generators commute with each other in this context.

We use ι defined by (1.1) for the pseudo-symmetrization map. The images $\iota(X_d^{\mathbf{R}})$ of the generators (2.2) of $S(\mathfrak{p})^K$ look the same as $X_d^{\mathbf{R}}$ themselves, except that the images are in $U(\mathfrak{g})$. In the following theorem, we use the column-determinant for the determinant of a matrix with non-commutative entries, defined by

$$\det(Z_{ij}) = \sum_{\sigma \in \mathfrak{S}_d} \text{sgn}(\sigma) Z_{\sigma(1)1} Z_{\sigma(2)2} \cdots Z_{\sigma(d)d}.$$

THEOREM 2.1. *For $1 \leq d \leq \min(k, n)$, we have the Capelli identities for the symmetric pair of Case \mathbf{R} :*

$$\omega(\iota(X_d^{\mathbf{R}})) = \omega(C_d^{\mathbf{R}}), \tag{2.4}$$

where $C_d^{\mathbf{R}} \in U(\mathfrak{m})^H$ ($d = 1, 2, \dots, n$) are the Capelli elements defined by sums of products of two $d \times d$ minors with entries in $U(\mathfrak{m})$:

$$\begin{aligned} C_d^{\mathbf{R}} &= (-1)^d \sum_{S, T \in \mathcal{I}_d^n} \det \left(E_{s_a, t_b} + \left(d - b - 1 - \frac{k}{2} \right) \delta_{s_a, t_b} \right)_{a,b} \\ &\quad \times \det \left(E_{s_a, t_b} + \left(d - b - \frac{k}{2} \right) \delta_{s_a, t_b} \right)_{a,b}, \end{aligned} \tag{2.5}$$

where s_a denotes an element of the index set S with $s_1 < s_2 < \cdots < s_d$, and the similar rule applies to t_b . Note that the identity is trivial when $n < d \leq k$ since the right hand side becomes an empty sum. \square

Note that (2.4) is in fact an identity of differential operators expressed explicitly by the formula (2.1).

To prove Theorem 2.1, we demonstrate the computation for the commutative principal symbols. We first recall a basic lemma.

LEMMA 2.2 (Cauchy-Binet). *Let R be a commutative ring and $d \leq N$. For a $d \times N$ matrix $A \in M_{d,N}(R)$ with coefficients in R and $N \times d$ matrix $B \in M_{N,d}(R)$, we have*

$$\det AB = \sum_{S \in \mathcal{I}_d^N} \det A_{\bullet, S} \det B_{S, \bullet},$$

where $A_{\bullet, S}$ is the $d \times d$ submatrix of A in which all the rows are chosen and the

columns are chosen by S . □

For the proof of this lemma, see [Sat75, Section 5, Theorem 9] for example. See also [IOW96, Corollary 2.2].

Define $n \times k$ matrices X and ∂ by

$$X = (x_{si})_{1 \leq s \leq n, 1 \leq i \leq k}, \quad \partial = (\partial_{si})_{1 \leq s \leq n, 1 \leq i \leq k}.$$

In the following computation we take the principal symbols, and we write the principal symbol of ∂_{si} by the same letter ∂_{si} . For $I, J \in \mathcal{J}_d^k$, the lemma above yields

$$\begin{aligned} \omega(\det(\mathbf{G}_{IJ})) &= \det \left(\sqrt{-1} \sum_{s=1}^n x_{s,i_a} x_{s,j_b} \right)_{1 \leq a, b \leq d} \\ &= (\sqrt{-1})^d \det({}^t(X_{\bullet, I})X_{\bullet, J}) = (\sqrt{-1})^d \sum_{S \in \mathcal{J}_d^n} \det({}^t(X_{SI}) \det X_{SJ}. \end{aligned}$$

Similarly we have

$$\omega(\det(\mathbf{F}_{JI})) = (\sqrt{-1})^d \sum_{T \in \mathcal{J}_d^n} \det({}^t(\partial_{TJ}) \det \partial_{TI},$$

and the equation of matrices

$$\omega(\mathbf{E}_{ST}) = (X {}^t\partial)_{ST} \quad (S, T \in \mathcal{J}_d^n),$$

where $\mathbf{E}_{ST} = (E_{s_a, t_b})_{1 \leq a, b \leq d}$ is a matrix with coefficients in \mathfrak{m} . Note that the diagonal shift appearing in (2.1) vanishes here, since we are considering only the principal symbols. Note also that elements in the expressions above commute with each other for the same reason, and we have

$$\begin{aligned} &\sum_{I, J \in \mathcal{J}_d^k} \omega(\det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}) \\ &= (-1)^d \sum_{I, J} \sum_{S, T \in \mathcal{J}_d^n} \det({}^t(X_{SI}) \det X_{SJ} \det({}^t(\partial_{TJ}) \det \partial_{TI} \\ &\stackrel{(*)}{=} (-1)^d \sum_{I, S, T} \det({}^t(X_{SI}) \det(X {}^t\partial)_{ST} \det \partial_{TI} \\ &\stackrel{(**)}{=} (-1)^d \sum_{I, S, T} \det(X {}^t\partial)_{ST} \det({}^t(X_{SI}) \det \partial_{TI} \end{aligned}$$

$$\begin{aligned}
 & \stackrel{(*)}{=} (-1)^d \sum_{S,T} \det(X {}^t\partial)_{ST} \det(X {}^t\partial)_{ST} \\
 & = (-1)^d \sum_{S,T} \omega(\det \mathbf{E}_{ST} \det \mathbf{E}_{ST}). \tag{2.6}
 \end{aligned}$$

This is nothing but our desired formula in Theorem 2.1 except that there are no diagonal shifts in the last expression. The equalities with (*) and (**) do *not* hold when we do *not* take the principal symbols. We prove the non-commutative analogue of these two equalities with diagonal shifts in the following two lemmas. So from now on, ∂ denotes the usual differential operator.

LEMMA 2.3. *We have the following equation:*

$$\sum_{J \in \mathcal{J}_d^k} \det X_{SJ} \cdot \det \partial_{TJ} = \det \left(\omega \left(E_{s_a, t_b} + \left(d - b - \frac{k}{2} \right) \delta_{s_a, t_b} \right) \right)_{1 \leq a, b \leq d}.$$

PROOF. Since the right-hand side equals $\det(\sum_{i=1}^k x_{s_a, i} \partial_{t_b, i} + (d - b) \delta_{s_a, t_b})_{1 \leq a, b \leq d}$, this lemma reduces to the original Capelli identity ([**Cap90**]). \square

LEMMA 2.4. *We have the following formula for $u_1, u_2, \dots, u_d \in \mathbf{C}$:*

$$\begin{aligned}
 & \det X_{SI} \cdot \det \left(\sum_{i=1}^k x_{s_a, i} \partial_{t_b, i} + u_b \delta_{s_a, t_b} \right)_{1 \leq a, b \leq d} \\
 & = \det \left(\sum_{i=1}^k x_{s_a, i} \partial_{t_b, i} + (u_b - 1) \delta_{s_a, t_b} \right)_{1 \leq a, b \leq d} \cdot \det X_{SI}.
 \end{aligned}$$

PROOF. For $s \in S$, it is easy to see the commutator $[\sum_{i=1}^k x_{s_i} \partial_{t_i}, \det X_{SI}]$ is equal to $\delta_{st} \det X_{SI}$. Since the determinant is multi-linear (in its column), we have the lemma. \square

PROOF OF THEOREM 2.1. From the first equality of (2.6), we have

$$\begin{aligned}
 & \sum_{I, J \in \mathcal{J}_d^k} \omega(\det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}) \\
 & = (-1)^d \sum_{I, J} \sum_{S, T \in \mathcal{J}_d^{2n}} \det {}^t(X_{SI}) \det X_{SJ} \det {}^t(\partial_{TJ}) \det \partial_{TI}.
 \end{aligned}$$

It follows from Lemma 2.3 that this is equal to

$$(-1)^d \sum_{I,S,T} \det {}^t(X_{SI}) \det \left(\omega \left(E_{s_a,t_b} + \left(d - b - \frac{k}{2} \right) \delta_{s_a,t_b} \right) \right)_{1 \leq a,b \leq d} \det \partial_{TI}.$$

By Lemma 2.4, it turns out that the expression above equals

$$(-1)^d \sum_{I,S,T} \det \left(\omega \left(E_{s_a,t_b} + \left(d - b - 1 - \frac{k}{2} \right) \delta_{s_a,t_b} \right) \right)_{1 \leq a,b \leq d} \cdot \det {}^t(X_{SI}) \det \partial_{TI}.$$

By using Lemma 2.3 again, this is equal to

$$\begin{aligned} & (-1)^d \sum_{S,T} \det \left(\omega \left(E_{s_a,t_b} + \left(d - b - 1 - \frac{k}{2} \right) \delta_{s_a,t_b} \right) \right)_{1 \leq a,b \leq d} \\ & \times \det \left(\omega \left(E_{s_a,t_b} + \left(d - b - \frac{k}{2} \right) \delta_{s_a,t_b} \right) \right)_{1 \leq a,b \leq d}. \end{aligned}$$

We have thus proved Theorem 2.1. □

We conclude this subsection by proving the H -invariance of the Capelli element $C_d^{\mathbf{R}}$ defined in (2.5).

PROPOSITION 2.5. *The Capelli element is H -invariant, that is, $C_d^{\mathbf{R}} \in U(\mathfrak{m})^H$, where $H = O_n$ is the complex orthogonal group.*

PROOF. Consider an algebra of tensor products $W = \bigwedge \mathbf{C}^n \otimes_{\mathbf{C}} \bigwedge \mathbf{C}^n \otimes_{\mathbf{C}} U(\mathfrak{m})$, which has a natural M -module structure; $M = GL_n$ acts on the both \mathbf{C}^n by the dual of the natural representation (hence on its exterior product), and acts on $U(\mathfrak{m})$ by the adjoint action. Denote the standard basis of \mathbf{C}^n in the first and the second factor by $\{e_t\}_t$ and $\{e'_t\}_t$ respectively. Define $\eta_t(u)$ and $\eta'_t(u)$ in W by

$$\eta_t(u) = \sum_{s=1}^n e_s(E_{st} + u\delta_{st}), \quad \eta'_t(u) = \sum_{s=1}^n e'_s(E_{st} + u\delta_{st}).$$

Note that we have for $u, v \in \mathbf{C}$ and $T \in \mathcal{I}_d^n$,

$$\eta_T(u) = \sum_{S \in \mathcal{I}_d^n} e_S \det \mathbf{E}_{ST}(u), \quad \eta'_T(v) = \sum_{S \in \mathcal{I}_d^n} e'_S \det \mathbf{E}_{ST}(v),$$

where

$$\eta_T(u) = \eta_{t_1}(u-1)\eta_{t_2}(u-2)\cdots\eta_{t_d}(u-d), \quad e_S = e_{s_1}e_{s_2}\cdots e_{s_d}$$

(the product is taken in the exterior power) and $E_{ST}(u)$ denotes the $d \times d$ matrix whose (a, b) -entry is $E_{s_a, t_b} + (u - b)\delta_{s_a, t_b}$. Let W_d be the GL_n -submodule of W spanned by $\{e_T e'_{T'} \mid T, T' \in \mathcal{I}_d^n\}$. Then it is easy to check that the mapping

$$\Delta : W_d \rightarrow W, \quad e_T e'_{T'} \mapsto \eta_T(u) \eta'_{T'}(v) \quad (T, T' \in \mathcal{I}_d^n)$$

is a GL_n -homomorphism for $u, v \in \mathbf{C}$, hence an O_n -homomorphism in particular. Since the element $\sum_T e_T e'_{T'} \in W_d$ is O_n -invariant, $\sum_T \eta_T(u) \eta'_{T'}(v)$ is also O_n -invariant.

We define the contraction map ε on W_d by $\varepsilon(e_T e'_{T'}) = \delta_{T, T'}$, which is naturally extended to $W_d \otimes_{\mathbf{C}} U(\mathfrak{m})$. Note that ε is O_n -equivariant. Now we conclude that

$$\begin{aligned} \varepsilon \left(\sum_T \eta_T(u) \eta'_{T'}(v) \right) &= \varepsilon \left(\sum_{T, S, S'} e_S \det E_{ST}(u) e'_{S'} \det E_{S'T}(v) \right) \\ &= \sum_{S, T} \det E_{ST}(u) \det E_{ST}(v) \end{aligned}$$

is O_n -invariant. □

2.2. Case C.

In this subsection, we treat Case C. We realize Lie algebras \mathfrak{g} , \mathfrak{k} and \mathfrak{p}^\pm as given below, and define elements $H_{ij}^{(x)}, H_{ij}^{(y)}, F_{ij}, G_{ij}$ of these algebras by

$$\begin{aligned} \mathfrak{g} &= \mathfrak{gl}_{p+q} = \left\{ \begin{pmatrix} H^{(x)} & G \\ F & H^{(y)} \end{pmatrix} \mid \begin{array}{l} H^{(x)} \in \mathfrak{gl}_p, \quad G \in M_{p,q}(\mathbf{C}) \\ H^{(y)} \in \mathfrak{gl}_q, \quad F \in M_{q,p}(\mathbf{C}) \end{array} \right\}, \\ \mathfrak{k} &= \left\{ \begin{pmatrix} H^{(x)} & 0 \\ 0 & H^{(y)} \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_p \oplus \mathfrak{gl}_q, \\ \mathfrak{p}^+ &= \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \end{aligned}$$

$$\begin{aligned} H_{ij}^{(x)} &= E_{ij} \in \mathfrak{k} & (1 \leq i, j \leq p), & \quad G_{ij} = E_{i, p+j} \in \mathfrak{p}^+ & (1 \leq i \leq p, 1 \leq j \leq q), \\ H_{ij}^{(y)} &= E_{p+i, p+j} \in \mathfrak{k} & (1 \leq i, j \leq q), & \quad F_{ij} = E_{p+i, j} \in \mathfrak{p}^- & (1 \leq i \leq q, 1 \leq j \leq p). \end{aligned}$$

The Lie algebra \mathfrak{m} , its subalgebra \mathfrak{h} , and elements of \mathfrak{m} are given by

$$\begin{aligned} \mathfrak{m} &= \mathfrak{gl}_n \oplus \mathfrak{gl}_n, & \mathfrak{h} &= \{(X, -{}^tX) \in \mathfrak{m}\} \simeq \mathfrak{gl}_n, \\ E_{st}^{(x)} &= (E_{st}, 0) \in \mathfrak{m}, & E_{st}^{(y)} &= (0, E_{st}) \in \mathfrak{m} & (1 \leq s, t \leq n). \end{aligned}$$

Set $V = M_{n,p}(\mathbf{C}) \oplus M_{n,q}(\mathbf{C})$ and denote the linear coordinate functions on each component of V by x_{si}, y_{sj} ($1 \leq s \leq n, 1 \leq i \leq p, 1 \leq j \leq q$), respectively. Put $\mathfrak{s} = \mathfrak{sp}_{2(p+q)n}$, in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. The Weil representation ω of \mathfrak{s} is realized on $\mathbf{C}[V]$, and its explicit forms are given as follows (cf. [NOT01, Section 3]):

$$\begin{aligned} \omega(H_{ij}^{(x)}) &= \sum_{s=1}^n x_{si} \frac{\partial}{\partial x_{sj}} + \frac{n}{2} \delta_{ij}, & \omega(H_{ij}^{(y)}) &= -\sum_{s=1}^n y_{sj} \frac{\partial}{\partial y_{si}} - \frac{n}{2} \delta_{ij}, \\ \omega(G_{ij}) &= \sqrt{-1} \sum_{s=1}^n x_{si} y_{sj}, & \omega(F_{ji}) &= \sqrt{-1} \sum_{s=1}^n \frac{\partial}{\partial x_{si}} \frac{\partial}{\partial y_{sj}}, \\ \omega(E_{st}^{(x)}) &= \sum_{i=1}^p x_{si} \frac{\partial}{\partial x_{ti}} + \frac{p}{2} \delta_{st}, & \omega(E_{st}^{(y)}) &= \sum_{j=1}^q y_{sj} \frac{\partial}{\partial y_{tj}} + \frac{q}{2} \delta_{st}. \end{aligned} \tag{2.7}$$

In the similar notation to Case **R**, we have the decomposition of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = \bigoplus_{\mu \in \mathcal{P}_r} (W_\mu \otimes_{\mathbf{C}} W_\mu^*)^K, \quad W_\mu \simeq \rho_p^\mu \otimes \rho_q^{\mu*},$$

where $r = \min(p, q)$. In fact, $S(\mathfrak{p})^K$ is isomorphic to a polynomial ring with r algebraically independent generators, and their explicit forms are

$$X_d^{\mathbf{C}} = \sum_{I \in \mathcal{I}_d^p, J \in \mathcal{I}_d^q} \det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI} \quad (d = 1, 2, \dots, r; \quad r = \min(p, q)). \tag{2.8}$$

THEOREM 2.6. *For $1 \leq d \leq \min(p, q, n)$, we have the Capelli identities for the symmetric pair of Case **C**.*

$$\omega(\iota(X_d^{\mathbf{C}})) = \omega(C_d^{\mathbf{C}}),$$

where $C_d^{\mathbf{C}} \in U(\mathfrak{m})^H$ ($d = 1, 2, \dots, n$) are the Capelli elements defined by sums of products of two $d \times d$ minors with entries in $U(\mathfrak{m})$:

$$\begin{aligned} C_d^{\mathbf{C}} &= (-1)^d \sum_{S, T \in \mathcal{I}_d^n} \det \left(E_{s_a, t_b}^{(x)} + \left(d - b - \frac{p}{2} \right) \delta_{s_a, t_b} \right)_{a,b} \\ &\quad \times \det \left(E_{s_a, t_b}^{(y)} + \left(d - b - \frac{q}{2} \right) \delta_{s_a, t_b} \right)_{a,b}. \end{aligned} \tag{2.9}$$

Note that the Capelli identity is trivial when $n < d \leq \min(p, q)$.

PROOF. As in Case **R**, we define the matrices

$$\begin{aligned}
 X &= (x_{si})_{1 \leq s \leq n, 1 \leq i \leq p}, & Y &= (y_{sj})_{1 \leq s \leq n, 1 \leq j \leq q}, \\
 \partial^X &= \left(\frac{\partial}{\partial x_{si}} \right)_{1 \leq s \leq n, 1 \leq i \leq p}, & \partial^Y &= \left(\frac{\partial}{\partial y_{sj}} \right)_{1 \leq s \leq n, 1 \leq j \leq q},
 \end{aligned}$$

and we thus have

$$\begin{aligned}
 \omega(\det(\mathbf{G}_{IJ})) &= (\sqrt{-1})^d \sum_{S \in \mathcal{S}_d^n} \det {}^t(X_{SI}) \cdot \det Y_{SJ}, \\
 \omega(\det(\mathbf{F}_{JI})) &= (\sqrt{-1})^d \sum_{T \in \mathcal{S}_d^n} \det {}^t(\partial_{TJ}^Y) \cdot \det \partial_{TI}^X,
 \end{aligned}$$

for $I \in \mathcal{S}_d^p, J \in \mathcal{S}_d^q$. Using these formulas, we can prove the identity as follows:

$$\begin{aligned}
 &\sum_{I \in \mathcal{S}_d^p, J \in \mathcal{S}_d^q} \omega(\det \mathbf{G}_{IJ} \cdot \det \mathbf{F}_{JI}) \\
 &= (-1)^d \sum_{I, J} \sum_{S, T \in \mathcal{S}_d^n} \det {}^t(X_{SI}) \det Y_{SJ} \det {}^t(\partial_{TJ}^Y) \det \partial_{TI}^X \\
 &= (-1)^d \sum_{I, J, S, T} \det X_{SI} \det \partial_{TI}^X \cdot \det Y_{SJ} \det \partial_{TJ}^Y \\
 &= (-1)^d \sum_{S, T \in \mathcal{S}_d^n} \omega \left(\det \left(E_{s_a, t_b}^{(x)} + \left(d - b - \frac{p}{2} \right) \delta_{s_a, t_b} \right)_{a, b} \right. \\
 &\quad \left. \times \det \left(E_{s_a, t_b}^{(y)} + \left(d - b - \frac{q}{2} \right) \delta_{s_a, t_b} \right)_{a, b} \right).
 \end{aligned}$$

The last equality follows from Lemma 2.3 by replacing k with p or q . The H -invariance of the Capelli element can be proved similarly to Proposition 2.5 of Case **R**. □

2.3. Case H.

For Case **H**, we need Lemma 2.8 on the identity of minor Pfaffians due to Ishikawa-Wakayama [IW00]. Define complex Lie algebras $\mathfrak{g}, \mathfrak{k}$ and \mathfrak{p}^\pm and elements H_{ij}, F_{ij}, G_{ij} of these algebras by

$$\mathfrak{g} = \mathfrak{o}_{2k} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \mid H \in \mathfrak{gl}_k, G, F \in \text{Alt}_k \right\},$$

$$\mathfrak{k} = \left\{ \begin{pmatrix} H & 0 \\ 0 & -{}^tH \end{pmatrix} \in \mathfrak{g} \right\} \simeq \mathfrak{gl}_k, \quad \mathfrak{p}^+ = \left\{ \begin{pmatrix} 0 & G \\ 0 & 0 \end{pmatrix} \in \mathfrak{g} \right\}, \quad \mathfrak{p}^- = \left\{ \begin{pmatrix} 0 & 0 \\ F & 0 \end{pmatrix} \in \mathfrak{g} \right\},$$

$$H_{ij} = E_{ij} - E_{k+j, k+i} \in \mathfrak{k},$$

$$G_{ij} = E_{i, k+j} - E_{j, k+i} \in \mathfrak{p}^+, \quad F_{ij} = E_{k+i, j} - E_{k+j, i} \in \mathfrak{p}^- \quad (1 \leq i, j \leq k),$$

where $\text{Alt}_k = \text{Alt}_k(\mathbf{C})$ denotes the set of the complex alternating $k \times k$ matrices. The complex Lie algebra \mathfrak{m} and its subalgebra \mathfrak{h} are given by

$$\mathfrak{m} = \mathfrak{gl}_{2n}, \quad \mathfrak{h} = \left\{ \begin{pmatrix} H & G \\ F & -{}^tH \end{pmatrix} \mid \begin{matrix} H \in \mathfrak{gl}_n, \\ G, F \in \text{Sym}_n \end{matrix} \right\} \simeq \mathfrak{sp}_{2n}.$$

We put $V = M_{2n, k}(\mathbf{C})$ and denote the linear coordinate functions on V and the corresponding differential operators by

$$x_{si}, \quad \partial_{si} \quad (1 \leq s \leq 2n, 1 \leq i \leq k),$$

respectively. Let $\mathfrak{s} = \mathfrak{sp}_{4kn}$ in which both $(\mathfrak{g}, \mathfrak{h})$ and $(\mathfrak{k}, \mathfrak{m})$ form dual pairs. The explicit representation operators of the Weil representation ω are given as follows.

$$\begin{aligned} \omega(H_{ij}) &= \sum_{s=1}^{2n} x_{si} \partial_{sj} + n \delta_{ij}, & \omega(G_{ij}) &= \sqrt{-1} \sum_{s=1}^n (x_{si} x_{\bar{s}j} - x_{\bar{s}i} x_{sj}), \\ \omega(F_{ji}) &= \sqrt{-1} \sum_{s=1}^n (\partial_{si} \partial_{\bar{s}j} - \partial_{\bar{s}i} \partial_{sj}), & \omega(E_{st}) &= \sum_{i=1}^k x_{si} \partial_{ti} + \frac{k}{2} \delta_{st}, \end{aligned} \tag{2.10}$$

where $\bar{s} = s + n$. For these formulas, we refer to [NOT01, Section 3] for example. The structure of $S(\mathfrak{p})^K$ is completely similar to Cases **R** and **C**; we have the decomposition of $S(\mathfrak{p})^K$,

$$S(\mathfrak{p})^K = \bigoplus_{\mu} (W_{\mu} \otimes_{\mathbf{C}} W_{\mu}^*)^K, \quad W_{\mu} \simeq \rho_k^{\mu},$$

where μ runs over the set of all the partitions of the form $(\mu_1, \mu_1, \mu_2, \mu_2, \dots) \in \mathcal{P}_k$. The polynomial ring $S(\mathfrak{p})^K$ has $r = \lfloor k/2 \rfloor$ algebraically independent generators

$$X_d^H = \sum_{I \in \mathcal{I}_{2d}^k} \text{Pf } \mathbf{G}_{II} \cdot \text{Pf } \mathbf{F}_{II} \quad \left(d = 1, 2, \dots, r; \quad r = \left\lfloor \frac{k}{2} \right\rfloor \right), \tag{2.11}$$

where Pf denotes the Pfaffian of an alternating matrix.

THEOREM 2.7. *For $1 \leq d \leq \min(\lfloor k/2 \rfloor, n)$, we have the Capelli identities for the symmetric pair of Case **H**:*

$$\omega(\iota(X_d^H)) = \omega(C_d^H),$$

where $C_d^H \in U(\mathfrak{m})^H$ ($d = 1, 2, \dots, n$) are the Capelli elements defined by sums of $2d \times 2d$ minors with entries in $U(\mathfrak{m})$:

$$C_d^H = \sum_{S_0, T_0 \in \mathcal{I}_d^n} \det \left(E_{s_a, t_b} + \left(2d - b - \frac{k}{2} \right) \delta_{s_a, t_b} \right)_{1 \leq a, b \leq 2d}, \tag{2.12}$$

where $S \in \mathcal{I}_{2d}^{2n}$ is defined as $S = \{s_1, s_2, \dots, s_d; n + s_1, n + s_2, \dots, n + s_d\}$ in terms of $S_0 = \{s_1, s_2, \dots, s_d\} \in \mathcal{I}_d^n$. Note that the identity is trivial when $n < d \leq \lfloor k/2 \rfloor$.

To prove the theorem, we use the following lemma to compute Pfaffians.

LEMMA 2.8 (Ishikawa-Wakayama [IW00]). *Let R be a commutative ring and $d \leq n$. For $A, B \in M_{n, 2d}(R)$, $X \in \text{Sym}_n(R)$, we have*

$$\text{Pf}({}^tAXB - {}^tBXA) = \sum_{S \in \mathcal{I}_{2d}^{2n}} \text{Pf} \begin{pmatrix} 0 & X \\ -X & 0 \end{pmatrix}_{SS} \det \begin{pmatrix} A \\ B \end{pmatrix}_{S, \bullet}.$$

In particular, when $X = I_n$ we have

$$\text{Pf}({}^tAB - {}^tBA) = (-1)^{\frac{d(d-1)}{2}} \sum_{S_0 \in \mathcal{I}_d^n} \det \begin{pmatrix} A \\ B \end{pmatrix}_{S, \bullet},$$

where $S \in \mathcal{I}_{2d}^{2n}$ is determined by S_0 as in Theorem 2.7.

PROOF. The first formula is due to Ishikawa-Wakayama [IW00, Corollary 2.1]. For the second formula we use two facts. The condition $\text{Pf} \begin{pmatrix} 0 & I_n \\ -I_n & 0 \end{pmatrix}_{SS} \neq 0$ implies $s_{d+i} = n + s_i$ ($1 \leq i \leq d$), and that $\text{Pf} \begin{pmatrix} 0 & I_d \\ -I_d & 0 \end{pmatrix} = (-1)^{d(d-1)/2}$. These facts give the second formula. \square

PROOF OF THEOREM 2.7. Define matrices by

$$\begin{aligned} X &= (x_{si})_{1 \leq s \leq n, 1 \leq i \leq k}, & \bar{X} &= (x_{\bar{s}i})_{1 \leq s \leq n, 1 \leq i \leq k}, \\ \partial &= (\partial_{si})_{1 \leq s \leq n, 1 \leq i \leq k}, & \bar{\partial} &= (\partial_{\bar{s}i})_{1 \leq s \leq n, 1 \leq i \leq k}. \end{aligned}$$

Then we have the equations of matrices

$$\begin{aligned} \omega(\mathbf{G}_{II}) &= \sqrt{-1}({}^t(X_{\bullet, I})\bar{X}_{\bullet, I} - {}^t(\bar{X}_{\bullet, I})X_{\bullet, I}), \\ \omega(\mathbf{F}_{II}) &= -\sqrt{-1}({}^t(\partial_{\bullet, I})\bar{\partial}_{\bullet, I} - {}^t(\bar{\partial}_{\bullet, I})\partial_{\bullet, I}), \end{aligned}$$

for $I \in \mathcal{I}_{2d}^k$. Using these equations we prove the identity as follows:

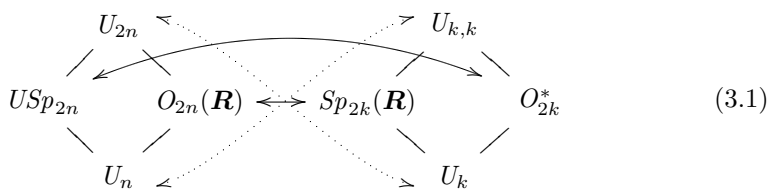
$$\begin{aligned} &\sum_{I \in \mathcal{I}_{2d}^k} \omega(\text{Pf } \mathbf{G}_{II} \cdot \text{Pf } \mathbf{F}_{II}) \\ &= \sum_{I \in \mathcal{I}_{2d}^k} \text{Pf}({}^t(X_{\bullet, I})\bar{X}_{\bullet, I} - {}^t(\bar{X}_{\bullet, I})X_{\bullet, I}) \cdot \text{Pf}({}^t(\partial_{\bullet, I})\bar{\partial}_{\bullet, I} - {}^t(\bar{\partial}_{\bullet, I})\partial_{\bullet, I}) \\ &= \sum_I \sum_{S_0 \in \mathcal{I}_d^n} (-1)^{\frac{d(d-1)}{2}} \det \left(\begin{matrix} X_{\bullet, I} \\ \bar{X}_{\bullet, I} \end{matrix} \right)_{S, \bullet} \cdot \sum_{T_0 \in \mathcal{I}_d^n} (-1)^{\frac{d(d-1)}{2}} \det \left(\begin{matrix} \partial_{\bullet, I} \\ \bar{\partial}_{\bullet, I} \end{matrix} \right)_{T, \bullet} \\ &\hspace{15em} \text{(by Lemma 2.8)} \\ &= \sum_{I, S_0, T_0} \det \left(\begin{matrix} X \\ \bar{X} \end{matrix} \right)_{SI} \cdot \det \left(\begin{matrix} \partial \\ \bar{\partial} \end{matrix} \right)_{TI} \\ &= \sum_{S_0, T_0 \in \mathcal{I}_d^n} \omega \left(\det \left(E_{s_a, t_b} + \left(2d - b - \frac{k}{2} \right) \delta_{s_a, t_b} \right)_{1 \leq a, b \leq 2d} \right). \text{ (by Lemma 2.3)} \end{aligned}$$

The H -invariance of C_d^H can be shown similarly as in the proof of Proposition 2.5, which we omit. □

3. The intersection of harmonics.

3.1. Diamond pair.

The three see-saw pairs considered above is a part of the diamond pair listed below (cf. [How89b, Eq.(3.3)]).



All the groups which appear in the diagram are subgroups of the ambient symplectic group $S_0 = Sp_{4nk}(\mathbf{R})$, and four pairs (U_{2n}, U_k) , $(O_{2n}(\mathbf{R}), Sp_{2k}(\mathbf{R}))$, (USp_{2n}, O_{2k}^*) and $(U_n, U_{k,k})$ connected by arrows (solid or dotted) are dual pairs in S_0 . Lines connecting two subgroups represents inclusions. Note that every pair connected by a line is a symmetric pair.

We give an explicit realization of each group here because the realization itself is important in the sequel.

$$\begin{aligned}
 U_{2n} &= \{g \in GL_{2n}(\mathbf{C}) \mid {}^t\bar{g} = g^{-1}\} \\
 USp_{2n} &= \{g \in U_{2n} \mid {}^t g J_n g = J_n\}, \quad J_n = \begin{pmatrix} 0 & -1_n \\ 1_n & 0 \end{pmatrix} \\
 O_{2n}(\mathbf{R}) &= \{g \in GL_{2n}(\mathbf{R}) \mid {}^t g = g^{-1}\} = U_{2n} \cap GL_{2n}(\mathbf{R}) \\
 U_n &= USp_{2n} \cap O_{2n}(\mathbf{R})
 \end{aligned}$$

Note that the unitary group U_n above is not realized in the standard way.

$$\begin{aligned}
 U_{k,k} &= \{g \in GL_{2k}(\mathbf{C}) \mid {}^t\bar{g} I_{k,k} g = I_{k,k}\}, \quad I_{k,k} = \text{diag}(1_k, -1_k) \\
 Sp_{2k}(\mathbf{R}) &= \{g \in U_{k,k} \mid {}^t g J_k g = J_k\} \\
 O_{2k}^* &= \{g \in U_{k,k} \mid {}^t g = g^{-1}\} \\
 U_k &= Sp_{2k}(\mathbf{R}) \cap O_{2k}^*
 \end{aligned}$$

These are almost standard realizations except U_k in the bottom. For $Sp_{2k}(\mathbf{R})$, see [Kna02, Section VII.10, Problem 30]. In the book [Kna02, Section I.17, Eq.(1.141)], O_{2k}^* is defined precisely as here. Also refer [Kna02, Section VII.10, Problems 32 & 35] for U_k .

In the following, we complexified the compact subgroups (the left hand side of the diamond pair) and consider the complex groups GL_{2n}, Sp_{2n}, O_{2n} and GL_n .

3.2. A problem.

Consider an irreducible finite dimensional representation ρ_{2n}^λ of GL_{2n} with the highest weight $\lambda \in \mathcal{P}_{2n}$. Let $B_{2n} = A_{2n}N_{2n}$ be the standard Borel subgroup

of upper triangular matrices in GL_{2n} , where A_{2n} is the diagonal torus and N_{2n} is the maximal unipotent subgroup consisting of all the upper triangular matrices with 1's on the diagonal. We take a nonzero highest weight vector v_λ in ρ_{2n}^λ with respect to B_{2n} . Let O_{2n} (respectively Sp_{2n}) be the orthogonal group (respectively symplectic group) realized as a subgroup of GL_{2n} as above. Let us consider

$$\begin{cases} U_\lambda = \text{irreducible } O_{2n} \text{ submodule in } \rho_{2n}^\lambda \text{ generated by } v_\lambda, \\ V_\lambda = \text{irreducible } Sp_{2n} \text{ submodule in } \rho_{2n}^\lambda \text{ generated by } v_\lambda. \end{cases} \tag{3.2}$$

Here, if λ is a partition of length $\leq n$, we have isomorphisms $U_\lambda \cong \sigma_{2n}^\lambda$ and $V_\lambda \cong \tau_{2n}^\lambda$.

Note that $O_{2n} \cap Sp_{2n} \cong GL_n$ (see Equation (3.3) below). We shall denote this explicitly realized subgroup of GL_{2n} again by GL_n . Note that the intersection $U_\lambda \cap V_\lambda$ is a GL_n module. We shall study the following problems.

- (i) What is the GL_n module structure of $U_\lambda \cap V_\lambda$?
- (ii) What are the GL_n highest weight vectors in $U_\lambda \cap V_\lambda$?
- (iii) How to characterize the subspace $U_\lambda \cap V_\lambda$ inside the representation of GL_{2n} ?

The problems (i) and (ii) are closely related to the theory of joint harmonics, and we will give a satisfactory answer to these problems. On the other hand, the problem (iii) seems to be related to the Capelli elements for symmetric pairs. For this, we do not have a complete answer yet, but we will prove that the subspace $U_\lambda \cap V_\lambda$ is in a joint eigenspace of the Capelli elements.

3.3. The intersection of harmonics.

From now on, we assume $n \geq k$ throughout the rest of this section.

Choose a joint complete polarization of \mathbf{C}^{2n} with respect to the standard symmetric and symplectic bilinear forms. So we have $\mathbf{C}^{2n} = L^+ \oplus L^-$, where L^\pm is a maximal totally isotropic space for the both bilinear forms. If we choose a basis of L^+ and then L^- , then they together form a basis for \mathbf{C}^{2n} . We can assume that O_{2n} is a subgroup of GL_{2n} which preserves the symmetric bilinear form

$$((x_1, \dots, x_n, y_1, \dots, y_n), (x'_1, \dots, x'_n, y'_1, \dots, y'_n)) = \sum_{j=1}^n (x_j y'_j + y_j x'_j)$$

in the coordinate of \mathbf{C}^{2n} with respect to the basis specified above. Also Sp_{2n} is an isometry group of the symplectic form $\langle \cdot, \cdot \rangle$ on \mathbf{C}^{2n} given by

$$\langle (x_1, \dots, x_n, y_1, \dots, y_n), (x'_1, \dots, x'_n, y'_1, \dots, y'_n) \rangle = \sum_{j=1}^n (x_j y'_j - y_j x'_j)$$

with respect to the same coordinate of \mathbf{C}^{2n} . In the following, we fix this specific basis and use the coordinate expression freely. Then, in matrix form, the intersection of O_{2n} and Sp_{2n} is given by

$$O_{2n} \cap Sp_{2n} = \left\{ \begin{pmatrix} g & 0 \\ 0 & {}^t g^{-1} \end{pmatrix} : g \in GL_n \right\}, \tag{3.3}$$

which we will denote simply by GL_n . Let $M_{2n,k} = M_{2n,k}(\mathbf{C})$ be the space of $2n \times k$ complex matrices, and let

$$\begin{pmatrix} X \\ Y \end{pmatrix} \in M_{2n,k}; \quad X = (x_{ij})_{1 \leq i \leq n, 1 \leq j \leq k}, \quad Y = (y_{ij})_{1 \leq i \leq n, 1 \leq j \leq k}$$

be the system of standard coordinates on $M_{2n,k}$. For $1 \leq i, j \leq n$, put

$$\Delta_{ij} = \sum_{a=1}^n \frac{\partial^2}{\partial x_{ai} \partial y_{aj}}, \quad \begin{cases} \Delta_{ij}^O = \Delta_{ij} + \Delta_{ji}, \\ \Delta_{ij}^{Sp} = \Delta_{ij} - \Delta_{ji}. \end{cases} \tag{3.4}$$

Notice that the Laplace operators above coincide with $\omega(F_{ij})$'s in Equations (2.1), (2.7) and (2.10) up to constant multiple. We now define the following spaces of harmonic polynomials:

$$\begin{aligned} \mathcal{H}(GL_n) &= \{f \in \mathbf{C}[M_{2n,k}] : \Delta_{ij}(f) = 0 \ \forall 1 \leq i, j \leq n\}, \\ \mathcal{H}(O_{2n}) &= \{f \in \mathbf{C}[M_{2n,k}] : \Delta_{ij}^O(f) = 0 \ \forall 1 \leq i, j \leq n\}, \\ \mathcal{H}(Sp_{2n}) &= \{f \in \mathbf{C}[M_{2n,k}] : \Delta_{ij}^{Sp}(f) = 0 \ \forall 1 \leq i, j \leq n\}. \end{aligned}$$

It is clear from the definitions that $\mathcal{H}(GL_n) = \mathcal{H}(O_{2n}) \cap \mathcal{H}(Sp_{2n})$.

Let $GL_{2n} \times GL_k$ act on $M_{2n,k}$ by

$$(g, h).T = {}^t g^{-1} \cdot T \cdot h^{-1}; \quad (g, h) \in GL_{2n} \times GL_k, \ T \in M_{2n,k}.$$

This action induces an action of $GL_{2n} \times GL_k$ on the algebra $\mathbf{C}[M_{2n,k}]$ of polynomial functions on $M_{2n,k}$. By the (GL_{2n}, GL_k) -duality, we have

$$\mathbf{C}[M_{2n,k}] \cong \bigoplus_{\lambda \in \mathcal{P}_k} \rho_{2n}^\lambda \otimes \rho_k^\lambda.$$

Let $B_k = A_k N_k$ be the standard Borel subgroup of upper triangular matrices in GL_k similarly defined as $B_{2n} \subset GL_{2n}$. By taking N_k -invariants, we obtain

$$\mathbf{C}[M_{2n,k}]^{N_k} \cong \bigoplus_{\lambda \in \mathcal{P}_k} \rho_{2n}^\lambda \otimes (\rho_k^\lambda)^{N_k}.$$

Note that $\mathbf{C}[M_{2n,k}]^{N_k}$ is a module for $GL_{2n} \times A_k$. Let W_λ denote the the submodule $\rho_{2n}^\lambda \otimes (\rho_k^\lambda)^{N_k}$ of $\mathbf{C}[M_{2n,k}]^{N_k}$. We denote by ψ_k^λ the character of A_k defined by

$$\psi_k^\lambda(a) = a^\lambda = a_1^{\lambda_1} \cdots a_k^{\lambda_k} \quad (a = \text{diag}(a_1, \dots, a_k) \in A_k).$$

Then W_λ is the ψ_k^λ -eigenspace for A_k in $\mathbf{C}[M_{2n,k}]^{N_k}$. Since $\dim(\rho_k^\lambda)^{N_k} = 1$, if we ignore the action of A_k and consider only the action of GL_{2n} on W_λ , then W_λ is a copy of the representation ρ_{2n}^λ .

We now let ξ_λ be a $GL_{2n} \times GL_k$ joint highest weight vector in $\mathbf{C}[M_{2n,k}]$ of weight $\psi_{2n}^\lambda \times \psi_k^\lambda$ (which is unique up to scalar multiples). Then ξ_λ is a GL_{2n} highest weight vector in W_λ . Now O_{2n} and Sp_{2n} act on $\mathbf{C}[M_{2n,k}]$ as subgroups of GL_{2n} . In this realization, we can identify the subspaces U_λ and V_λ in Equation (3.2) with

$$\begin{cases} U_\lambda = \text{irreducible } O_{2n} \text{ submodule in } \rho_{2n}^\lambda \text{ generated by } \xi_\lambda, \\ V_\lambda = \text{irreducible } Sp_{2n} \text{ submodule in } \rho_{2n}^\lambda \text{ generated by } \xi_\lambda. \end{cases}$$

The space $\mathcal{H}(O_{2n})$ of O_{2n} harmonic polynomials is a module for $O_{2n} \times GL_k$ with its structure given by

$$\mathcal{H}(O_{2n}) \cong \bigoplus_{\mu \in \mathcal{P}_k} \sigma_{2n}^\mu \otimes \rho_k^\mu.$$

In addition, the joint $O_{2n} \times GL_k$ highest weight vector in $\sigma_{2n}^\mu \otimes \rho_k^\mu$ coincides with the $GL_{2n} \times GL_k$ highest weight vector in $\rho_{2n}^\mu \otimes \rho_k^\mu$. By taking N_k invariants, we obtain

$$\mathcal{H}(O_{2n})^{N_k} \cong \bigoplus_{\mu \in \mathcal{P}_k} \sigma_{2n}^\mu \otimes (\rho_k^\mu)^{N_k}$$

which is now a module for $O_{2n} \times A_k$. From here, we see that the subspace U_λ coincides with the submodule $\sigma_{2n}^\lambda \otimes (\rho_k^F)^\lambda$ of $\mathcal{H}(O_{2n})$, and it is precisely the ψ_k^λ -eigenspace $\mathcal{H}(O_{2n})_{\psi_k^\lambda}^{N_k}$ of A_k in $\mathcal{H}(O_{2n})^{N_k}$. On the other hand, the space $\mathcal{H}(Sp_{2n})$ is a module for $Sp_{2n} \times GL_k$ with its structure given by

$$\mathcal{H}(Sp_{2n}) \cong \bigoplus_{\mu \in \mathcal{P}_k} \tau_{2n}^\mu \otimes \rho_k^\mu.$$

In a similar way as the O_{2n} harmonics, the subspace V_λ is the ψ_k^λ -eigenspace $\mathcal{H}(Sp_{2n})_{\psi_k^\lambda}^{N_k}$ of A_k in $\mathcal{H}(Sp_{2n})^{N_k}$. Consequently,

$$U_\lambda \cap V_\lambda = \mathcal{H}(O_{2n})_{\psi_k^\lambda}^{N_k} \cap \mathcal{H}(Sp_{2n})_{\psi_k^\lambda}^{N_k}. \tag{3.5}$$

Next we consider the space $\mathcal{H}(GL_n)$ of GL_n harmonics. It is a $GL_n \times GL_k$ module. Let $\mathcal{H}(GL_n)_{\psi_k^\lambda}^{N_k}$ denotes the ψ_k^λ -eigenspace of A_k in the space $\mathcal{H}(GL_n)^{N_k}$ of N_k invariants in $\mathcal{H}(GL_n)$. Then since $\mathcal{H}(GL_n) = \mathcal{H}(O_{2n}) \cap \mathcal{H}(Sp_{2n})$,

$$\mathcal{H}(GL_n)_{\psi_k^\lambda}^{N_k} = \mathcal{H}(O_{2n})_{\psi_k^\lambda}^{N_k} \cap \mathcal{H}(Sp_{2n})_{\psi_k^\lambda}^{N_k}.$$

It follows from this and Equation (3.5),

$$U_\lambda \cap V_\lambda = \mathcal{H}(GL_n)_{\psi_k^\lambda}^{N_k}. \tag{3.6}$$

3.4. Decomposition of $U_\lambda \cap V_\lambda$ as a GL_n -module.

We now determine the GL_n module structure of $\mathcal{H}(GL_n)_{\psi_k^\lambda}^{N_k}$. Put $\mathcal{P}_{(k,n)} = \{(\mu, \nu) \in \mathcal{P}_k \times \mathcal{P}_k \mid \ell(\mu) + \ell(\nu) \leq n\}$. Then, under the action of $GL_n \times GL_k$, the GL_n harmonics decomposes as follows.

$$\begin{aligned} \mathcal{H}(GL_n) &\cong \bigoplus_{(\mu, \nu) \in \mathcal{P}_{(k,n)}} \rho_n^{\mu \odot \nu} \otimes (\rho_k^\mu \otimes \rho_k^\nu) \\ &\cong \bigoplus_{(\mu, \nu) \in \mathcal{P}_{(k,n)}} \rho_n^{\mu \odot \nu} \otimes \left[\bigoplus_{\eta \in \mathcal{P}_k} c_{\mu, \nu}^\eta \rho_k^\eta \right] \\ &\cong \bigoplus_{\eta \in \mathcal{P}_k} \left[\bigoplus_{(\mu, \nu) \in \mathcal{P}_{(k,n)}} c_{\mu, \nu}^\eta \rho_n^{\mu \odot \nu} \right] \otimes \rho_k^\eta. \end{aligned}$$

In the above formula, we use the Littlewood-Richardson rule and $c_{\mu, \nu}^\eta$ denotes the Littlewood-Richardson coefficient which counts the number of the Littlewood-Richardson tableaux of shape η/μ and content ν (see [Ful97, Section 5.2, Proposition 3]). By taking N_k invariants, we obtain

$$\mathcal{H}(GL_n)^{N_k} \cong \bigoplus_{\eta \in \mathcal{P}_k} \left[\bigoplus_{(\mu, \nu) \in \mathcal{P}_{(k,n)}} c_{\mu, \nu}^\eta \rho_n^{\mu \odot \nu} \right] \otimes (\rho_k^\eta)^{N_k}. \tag{3.7}$$

It follows that under GL_n ,

$$\mathcal{H}(GL_n)_{\psi_k^\lambda}^{N_k} \cong \bigoplus_{(\mu, \nu) \in \mathcal{P}_{(k, n)}} c_{\mu, \nu}^\lambda \rho_n^{\mu \odot \nu}.$$

Thus we have proved:

THEOREM 3.1. *Assume that $k \leq n$, and take $\lambda \in \mathcal{P}_k$. We define the subspaces $U_\lambda \simeq \sigma_{2n}^\lambda$ and $V_\lambda \simeq \tau_{2n}^\lambda$ in ρ_{2n}^λ as in Equation (3.2). Then we have a decomposition under the action of GL_n ,*

$$U_\lambda \cap V_\lambda \cong \bigoplus_{\mu, \nu \in \mathcal{P}_k} c_{\mu, \nu}^\lambda \rho_n^{\mu \odot \nu},$$

where the summation is taken over the pair (μ, ν) such that $\ell(\mu) + \ell(\nu) \leq n$.

3.5. Annihilator of Capelli elements.

In this subsection, we clarify the relation between the intersection of harmonics and Capelli elements which we have constructed in Section 2.

Since the harmonics is by definition annihilated by the (generalized) Laplace operators (3.4), and those Laplace operators are precisely those $\omega(F_{ij})$'s, we conclude that they are annihilated by the Capelli elements $\omega(\iota(X_d^K)) = \omega(C_d^K)$. Here, the symbol \mathbf{K} stands for \mathbf{R}, \mathbf{C} or \mathbf{H} respectively. Note that C_d^K 's ($\mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}$) all belong to the enveloping algebra $U(\mathfrak{gl}_n)$. Thus, the intersection of the harmonics $\mathcal{H}(GL_n) = \mathcal{H}(O_{2n}) \cap \mathcal{H}(Sp_{2n})$ are annihilated by all the operators $\{C_d^K \mid 1 \leq d \leq r_K; \mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}\}$, where $r_K = k$ for $\mathbf{K} = \mathbf{R}, \mathbf{C}$ and $r_H = \lfloor k/2 \rfloor$.

However, note that $\omega(C_d^K)$ does not coincide with the above action of GL_{2n} on $\mathbf{C}[M_{2n, k}]$ given by the left multiplication. Instead, it differs by a character (of the two fold covering group) of GL_{2n} which originates from the definition of the Weil representation. In the present situation, the character is $\chi_k = \det^{k/2}$ whose highest weight is $d\chi_k = (k/2)(1, 1, \dots, 1)$.

PROPOSITION 3.2. *Assume that $k \leq n$, and let U_λ and V_λ be as in Theorem 3.1. Then the Capelli elements annihilate $U_\lambda \cap V_\lambda$ up to the shift of the central character χ_k . Namely we have*

$$\tilde{C}_d^K(U_\lambda \cap V_\lambda) = 0, \quad \text{where } \tilde{C}_d^K = d(\rho_{2n}^\lambda \otimes \chi_k)(C_d^K) \quad (1 \leq d \leq r_K; \mathbf{K} = \mathbf{R}, \mathbf{C}, \mathbf{H}).$$

Here, $d\rho$ stands for the differentiated representation of \mathfrak{gl}_{2n} .

It is interesting to study the kernel $\bigcap_{d, \mathbf{K}} \text{Ker } \tilde{C}_d^K$ in the representation space of ρ_{2n}^λ . However, up to now, no complete characterization is known.

4. The GL_k tensor product algebra.

In this section, we assume that $2k \leq n$, which is stronger than the one in Section 3. Under the assumption, we shall prove that the space $\mathcal{H}(GL_n)^{N_n \times N_k}$ of $N_n \times N_k$ -invariants in $\mathcal{H}(GL_n)$ is an algebra isomorphic to the GL_k tensor product algebra constructed in [HTW05]. Using their results, one can obtain a basis for $\mathcal{H}(GL_n)^{N_n \times N_k}$.

By taking N_n -invariants in Equation (3.7), we obtain

$$\mathcal{H}(GL_n)^{N_n \times N_k} \cong \bigoplus_{\lambda, \mu, \nu \in \mathcal{P}_k} c_{\mu, \nu}^\lambda [(\rho_n^{\mu \odot \nu})^{N_n} \otimes (\rho_k^\lambda)^{N_k}]. \tag{4.1}$$

Let $\mathcal{H}_{(\mu, \nu), \lambda}$ be the $\psi_n^{\mu \odot \nu} \times \psi_k^\lambda$ -eigenspace for $A_n \times A_k$ in $\mathcal{H}(GL_n)^{N_n \times N_k}$. Then

$$\mathcal{H}(GL_n)^{N_n \times N_k} = \bigoplus_{\lambda, \mu, \nu \in \mathcal{P}_k} \mathcal{H}_{(\mu, \nu), \lambda}, \tag{4.2}$$

and by Equation (4.1), $\dim \mathcal{H}_{(\mu, \nu), \lambda} = c_{\mu, \nu}^\lambda$.

For convenience, we shall replace the coordinates $Y = (y_{ij})$ by $\hat{Y} = (\hat{y}_{ij})$, where

$$\hat{y}_{ij} = y_{n-i+1, j}, \quad (1 \leq i \leq n, 1 \leq j \leq k).$$

We now let \mathfrak{A}_X , $\mathfrak{A}_{\hat{Y}}$ and \mathfrak{A} be the subalgebras of $\mathcal{C}[M_{2n, k}]$ generated by the following coordinates:

$$\begin{aligned} \mathfrak{A}_X &= \mathcal{C}[x_{ij} : 1 \leq i, j \leq k] \cong \mathcal{C}[M_k] \\ \mathfrak{A}_Y &= \mathcal{C}[\hat{y}_{ij} : 1 \leq i, j \leq k] \cong \mathcal{C}[M_k] \\ \mathfrak{A} &= \mathcal{C}[x_{ij}, \hat{y}_{ij} : 1 \leq i, j \leq k] \cong \mathfrak{A}_X \otimes \mathfrak{A}_Y \cong \mathcal{C}[M_k] \otimes \mathcal{C}[M_k], \end{aligned}$$

where $M_k = M_k(\mathcal{C})$ is the space of $k \times k$ complex matrices. By examining the definition of $\mathcal{H}(GL_n)$, we note that

OBSERVATION 4.1. *The algebra \mathfrak{A} is contained in the space $\mathcal{H}(GL_n)$ of GL_n harmonics.*

Now two copies of GL_k are acting on M_k : one by left multiplication and the other by right multiplication. Specifically, for $(g, h) \in GL_k \times GL_k$ and $T \in M_k$,

$$(g, h) \cdot T = {}^t g^{-1} \cdot T \cdot h^{-1}.$$

We shall distinguish these two copies of GL_k by writing them as $L(GL_k)$ and $R(GL_k)$ respectively. More generally, if H is a subgroup of GL_k , we shall write the corresponding subgroups of $L(GL_k)$ and $R(GL_k)$ as $L(H)$ and $R(H)$ respectively. In particular, $L(N_k)$ is the maximal unipotent subgroup of $L(GL_k)$.

LEMMA 4.2. *We have the isomorphism*

$$\mathcal{H}(GL_n)^{N_n} \cong \mathfrak{A}_X^{L(N_k)} \otimes \mathfrak{A}_Y^{L(N_k)}.$$

In particular, $\mathcal{H}(GL_n)^{N_n}$ is a subalgebra of $\mathbf{C}[M_{2n,k}]$.

In fact,

$$\mathcal{H}(GL_n)^{N_n} \cong \bigoplus_{\mu, \nu} (\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu \tag{4.3}$$

as an $A_n \times GL_k \times GL_k$ module. In particular, the isotypic component $(\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu$ can be characterized as the $\psi_n^{\mu \odot \nu}$ -eigenspace for A_n in $\mathcal{H}(GL_n)^{N_n}$. We shall first describe a spanning set for this isotypic component.

For $1 \leq j \leq k$ and $I = \{i_1, \dots, i_j\} \in \mathcal{I}_j^n$ where $1 \leq i_1 < \dots < i_j \leq n$, let

$$\begin{aligned} \gamma_j(I) &= \det X_{JI}, & \text{where } J = \{1, 2, \dots, j\} \text{ and } X = (x_{s,t})_{1 \leq s \leq n, 1 \leq t \leq k}, \\ \eta_j(I) &= \det \widehat{Y}_{JI}, & \text{where } J = \{1, 2, \dots, j\} \text{ and } \widehat{Y} = (\widehat{y}_{s,t})_{1 \leq s \leq n, 1 \leq t \leq k}. \end{aligned}$$

We also write

$$\gamma_j = \gamma_j(\{1, 2, \dots, j\}), \quad \text{and} \quad \eta_j = \eta_j(\{1, 2, \dots, j\}).$$

For $\mu = (\mu_1, \dots, \mu_k) \in \mathcal{P}_k$ and $\nu = (\nu_1, \dots, \nu_k) \in \mathcal{P}_k$, define

$$\begin{aligned} a_j &= \mu_{j-1} - \mu_j, & b_j &= \nu_{j-1} - \nu_j & \text{for } 1 \leq j \leq k-1, \\ a_k &= \mu_k, & b_k &= \nu_k. \end{aligned} \tag{4.4}$$

Then $\xi_{\mu, \nu} = \gamma_1^{a_1} \gamma_2^{a_2} \dots \gamma_k^{a_k} \eta_1^{b_1} \eta_2^{b_2} \dots \eta_k^{b_k}$ is the unique (up to scalar multiple) $GL_n \times GL_k \times GL_k$ highest weight vector in $\mathcal{H}(GL_n)$ of weight $\psi_n^{\mu \odot \nu} \times \psi_k^\mu \times \psi_k^\nu$.

LEMMA 4.3. *For $\mu, \nu \in \mathcal{P}_k$, the set of polynomials of the form*

$$\prod_{j=1}^k [\gamma_j(I_j)]^{a_j} \prod_{m=1}^k [\eta_m(J_m)]^{b_m} \tag{4.5}$$

spans the isotypic component $(\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu$ in $\mathcal{H}(GL_n)$. Here $I_j \in \mathcal{I}_j^k$ and $J_m \in \mathcal{J}_m^k$ are subsets of $\{1, 2, \dots, k\}$ with j and m elements respectively, and a_j and b_m are given in Equation (4.4).

PROOF. First we note that all elements of the form (4.5) are GL_n highest weight vectors of weight $\psi_n^{\mu \odot \nu}$. Let W be the subspace of $\mathcal{H}(GL_n)$ spanned by them. Then W is contained in the isotypic component $(\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu$ in $\mathcal{H}(GL_n)$.

On the other hand, we note that the operators which arise from the infinitesimal action of $GL_k \times GL_k$ are derivations and affect only the second index j of the variables x_{ij} and y_{ij} . Thus W is invariant under $GL_k \times GL_k$. Since the space $(\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu$ is an irreducible $GL_k \times GL_k$ module, we have $W = (\rho_n^{\mu \odot \nu})^{N_n} \otimes \rho_k^\mu \otimes \rho_k^\nu$. \square

COROLLARY 4.4. Assume that $2k \leq n$. Then the space $\mathcal{H}(GL_n)^{N_n}$ is the subalgebra of $\mathcal{C}[M_{2n,k}]$ generated by all polynomials of the form $\gamma_j(I_j)$ and $\eta_m(J_m)$, where $1 \leq j, m \leq k$ and I_j and J_m are subsets of $\{1, 2, \dots, k\}$ with j and m elements respectively.

PROOF OF LEMMA 4.2. By similar reasonings as Lemma 4.3 (see [How95, p. 83]), the algebra $\mathfrak{A}_X^{L(N_k)}$ is generated by the polynomials $\gamma_j(I_j)$ where $1 \leq j \leq k$ and $I_j \subseteq \{1, \dots, k\}$ has j elements. Similarly $\mathfrak{A}_Y^{L(N_k)}$ is generated by $\eta_m(J_m)$ where $1 \leq m \leq k$ and $J_m \subseteq \{1, \dots, k\}$ has m elements. It follows that the algebra $\mathfrak{A}_X^{L(N_k)} \otimes \mathfrak{A}_Y^{L(N_k)}$ and $\mathcal{H}(GL_n)^{N_n}$ have the same generators. Hence they are isomorphic. \square

The GL_k tensor product algebra is defined as

$$\text{TA}_k := [\mathcal{C}[M_k]^{R(N_k)} \otimes \mathcal{C}[M_k]^{R(N_k)}]^\Delta(L(N_k))$$

in [HTW05]. A basis for this algebra was also given in the same paper.

COROLLARY 4.5. Under the assumption $2k \leq n$, we have the isomorphism

$$\mathcal{H}(GL_n)^{N_n \times N_k} \cong [\mathfrak{A}_X^{L(N_k)} \otimes \mathfrak{A}_Y^{L(N_k)}]^\Delta(R(N_k)).$$

In particular, $\mathcal{H}(GL_n)^{N_n \times N_k}$ is isomorphic to the GL_k tensor product algebra

TA_k .

Note that by switching the roles of $L(GL_k)$ and $R(GL_k)$ in TA_k , we obtain the algebra in $\mathcal{H}(GL_n)^{N_n \times N_k}$. It follows that to construct a basis for $\mathcal{H}(GL_n)^{N_n \times N_k}$, we only need to take the basis constructed in [HTW05] and replace their coordinates by ours. We leave the details to the interested readers.

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