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Solutions of the Dirichlet problem on a cone with continuous data

Dedicated to Professor Yasuo Okuyama on his 60th birthday

By Hidenobu Yoshida and Ikuko Miyamoto

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1. Introduction

Let $\boldsymbol{R}$ and $\boldsymbol{R}_+$ be the set of all real numbers and all positive real numbers, respectively. The boundary and the closure of a set S in the n-dimensional Euclidean space $\boldsymbol{R}^n (n \geq 2)$ are denoted by ∂S and $\bar{S}$, respectively. We also introduce the spherical coordinates $(r, \Theta), \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})$, in $\boldsymbol{R}^n$ which are related to the cartesian coordinates $(X, y), X = (x_1, x_2, \ldots, x_{n-1})$ by the formulas

$$x_1 = r\left(\prod_{j=1}^{n-1} \sin\theta_j\right) \quad (n \geq 2), \quad y = r\cos\theta_1,$$

and if $n \geq 3$,

$$x_{n+1-k} = r\left(\prod_{j=1}^{k-1} \sin\theta_j\right)\cos\theta_k \quad (2 \leq k \leq n-1),$$

where

$$0 \leq r < \infty, 0 \leq \theta_j \leq \pi (1 \leq j \leq n-2; n \geq 3), \quad -2^{-1}\pi < \theta_{n-1} \leq 2^{-1}3\pi.$$

The unit sphere (the unit circle, if $n = 2$) and the upper half unit sphere $\{(1, \theta_1, \theta_2, \ldots, \theta_{n-1}) \in \boldsymbol{R}^n; 0 \leq \theta_1 < \pi/2\}$ (the upper half unit circle $\{(1, \theta_1) \in \boldsymbol{R}^2; -\pi/2 < \theta_1 < \pi/2\}$, if $n = 2$) in $\boldsymbol{R}^n$ are denoted by $\boldsymbol{S}^{n-1}$ and $\boldsymbol{S}_+^{n-1}$, respectively. The half-space (the half-plane, if $n = 2$)

$$\{(X, y) \in \boldsymbol{R}^n; X \in \boldsymbol{R}^{n-1}, y > 0\} = \{(r, \Theta) \in \boldsymbol{R}^n; \Theta \in \boldsymbol{S}_+^{n-1}, 0 < r < \infty\}$$

is denoted by $\boldsymbol{T}_n$.

Given a domain $D \subset \boldsymbol{R}^n$ and a continuous function g on ∂D, we say that h is a solution of the (classical) Dirichlet problem on D with g, if h is harmonic in D and

$$\lim_{P \in D, P \to Q} h(P) = g(Q)$$

for every $Q \in \partial D$. If D is a smooth bounded domain, then the existence of a solution of the Dirichlet problem and its uniqueness is completely known (see e.g. [**11**, Theorem 5.21]). When D is the typical unbounded domain $\boldsymbol{T}_n$, Helms [**13**, p.42 and p.158] states that even if $g(x)$ is a bounded continuous function on $\partial \boldsymbol{T}_n$, the solution of the Dirichlet problem on T_n with g is not unique and to obtain the unique solution $H(P)(P = (X, y) \in T_n)$ we must specify the behavior of $H(P)$ as $y \to +\infty$.

With respect to particular solutions of the Dirichlet problem on T_n, the following results are known. Let $g(X)$ be a continuous function on $\partial T_n = R^{n-1}$ satisfying (1.1) with a non-negative integer l:

$$\int_{R^{n-1}} \frac{|g(X)|}{1+|X|^{n+l}}\, dX < \infty. \tag{1.1}$$

Then Armitage [**1**, Theorem 2] gave a solution of the Dirichlet problem on T_n with g in an explicit form, which is denoted by $H(T_n, l; g)(P)$ in the following (also see Siegel [**16**, p.1 and p.7]}. Further, for any continuous function $g(X)$ on ∂T_n Finkelstein and Scheinberg [**8**] showed the existence of a solution of the Dirichlet problem on T_n with g and Gardiner [**9**] gave the solution explicitly. These results of the case $n = 2$ had already been obtained by Nevanlinna [**15**].

About general solutions of the Dirichlet problem on T_n, Nevanlinna [**15**] also proved the following result of the case $n = 2$.

Let $g(x)$ be a continuous function on R satisfying

$$\int_{R} \frac{|g(x)|}{1+|x|^{2+l}}\, dx < \infty$$

with a non-negative integer l. If $h(P)$ is a solution of the Dirichlet problem on T_2 with g such that

$$\liminf_{r\to\infty} r^{-(l+1)}\mu(r) = 0, \quad \mu(r) = \sup_{-\pi/2<\theta_1<\pi/2} |h(r,\theta_1)|\cos\theta_1,$$

then $h(P) = H(T_2, l; g)(P) + \chi(P)(P = (r, \theta_1) \in T_2)$, where

$$\chi(P) = \begin{cases} \sum_{k=0}^{l'} A_{2k} r^{2k} \sin 2k\theta_1 + \sum_{k=1}^{l'} A_{2k-1} r^{2k-1} \cos(2k-1)\theta_1 & (l = 2l') \\ \sum_{k=0}^{l'-1} A_{2k} r^{2k} \sin 2k\theta_1 + \sum_{k=1}^{l'} A_{2k-1} r^{2k-1} \cos(2k-1)\theta_1 & (l = 2l'-1) \\ 0 & (l = 0) \end{cases}$$

(l' is a positive integer and all $A_0, A_1, \ldots, A_l$ are constants).

To answer a question of Siegel [**16**, p.8] Yoshida [**19**] recently proved

THEOREM A [**19**, Theorems 1 and 2]. *Let $g(Q)$ be a continuous function on $\partial T_n (n \geq 2)$ satisfying* (1.1) *with a non-negative integer l. Then the solution $H(T_n, l; g)(P)$ of the Dirichlet problem with g satisfies*

$$\lim_{r\to\infty} r^{-l-1} \int_{S_+^{n-1}} H(T_n, l; g)(r, \Theta)\cos\theta_1\, d\sigma_\Theta = 0 \quad (P = (r, \Theta) \in T_n, \Theta = (\theta_1, \theta_2, \ldots, \theta_{n-1})),$$

where $d\sigma_\Theta$ is the surface element of S^{n-1}.

If $h(P)$ is a solution of the Dirichlet problem on T_n with g satisfying

$$\lim_{r\to\infty} r^{-l-1} \int_{S_+^{n-1}} h^+(r, \Theta)\cos\theta_1\, d\sigma_\Theta = 0,$$

then

$$h(P) = H(\boldsymbol{T}_n, l; g)(P) + \Pi(P), \quad \Pi(P) = \begin{cases} y\Pi^*(P) & (l \geq 1) \\ 0 & (l = 0) \end{cases}$$

for every $P = (X, y) \in \boldsymbol{T}_n$, *where* $\Pi^*(P)$ *is a polynomial of* $P = (x_1, x_2, \ldots, x_{n-1}, y) \in \boldsymbol{R}^n$ *of degree at most* $l - 1 (l \geq 1)$ *and even with respect to the variable* y.

A half-space is a special one of more general unbounded domains

$$C(\Omega) = \{(r, \Theta) \in \boldsymbol{R}^n; (1, \Theta) \in \Omega, r \in \boldsymbol{R}_+\} \quad (\Omega \text{ is a domain on } \boldsymbol{S}^{n-1})$$

which are called cones (angular domains, if $n = 2$), i.e. $\boldsymbol{T}_n = C(\boldsymbol{S}_+^{n-1})$. Because of the speciality, it has many advantageous merits which a cone $C(\Omega)$ lacks, e.g., it has a simple Green function and the mirror image to which a harmonic function vanishing on the boundary can be extended, etc.

In this paper, to generalize Theorem A to the conical case and extend Yoshida's results we shall give particular solutions (Theorem 1) and a type of general solutions (Theorem 3) of the Dirichlet problem on a cone by introducing conical generalized Poisson kernels and Poisson integrals. We also generalize the results of Finkelstein and Scheinberg [**8**] and Gardinar [**9**] to the conical case (Theorem 2). Finally a result of Yoshida [**19**, Theorem 3] will be generalized in the conical form (Theorem 4).

2. Preliminaries

Let $\Delta_n (n \geq 2)$ be the Laplace operator and Λ_n the spherical part of the spherical coordinates of Δ_n:

$$\Delta_n = \frac{n-1}{r}\frac{\partial}{\partial r} + \frac{\partial^2}{\partial r^2} + \frac{1}{r^2}\Lambda_n.$$

Given a domain Ω on $\boldsymbol{S}^{n-1} (n \geq 2)$, consider the Dirichlet problem

$$(2.1) \qquad (\Lambda_n + \lambda)F = 0 \text{ in } \Omega$$

$$F = 0 \text{ on } \partial\Omega$$

We denote the non-decreasing sequence of positive eigenvalues of (2.1) by $\{\lambda(\Omega, k)\}_{k=1}^{\infty}$. In this expression we write $\lambda(\Omega, k)$ the same number of times as the dimension of the corresponding eigenspace. When the normalized eigenfunction corresponding $\lambda(\Omega, k)$ is denoted by $f_k^{\Omega}(\Theta)$, the set of sequential eigenfunctions corresponding to the same value of $\lambda(\Omega, k)$ in the sequence $\{f_k^{\Omega}(\Theta)\}_{k=1}^{\infty}$ makes an orthonormal basis for the eigenspace of the eigenvalue $\lambda(\Omega, k)$. Hence for each $\Omega \subset \boldsymbol{S}^{n-1}$ there is a sequence $\{k_i\}$ of positive integers such that $k_1 = 1$, $\lambda(\Omega, k_i) < \lambda(\Omega, k_{i+1})$

$$\lambda(\Omega, k_i) = \lambda(\Omega, k_i + 1) = \lambda(\Omega, k_i + 2) = \cdots = \lambda(\Omega, k_{i+1} - 1)$$

and $\{f_{k_i}^{\Omega}, f_{k_i+1}^{\Omega}, \ldots, f_{k_{i+1}-1}^{\Omega}\}$ is an orthonormal basis for the eigenspace of the eigenvalue

$\lambda(\Omega, k_i)(i = 1, 2, 3, \ldots)$. It is well known that $k_2 = 2$ and $f_1^{\Omega}(\Theta) > 0$ for any $\Theta \in \Omega$ (see Courant and Hilbert [**5**, p.451 and p.458]). With respect to $\{k_i\}$, the following Remark shows that even in the case $\Omega = S_+^{n-1}(n = 2, 3, 4, \ldots)$, not only the simplest case $k_i = i(i = 1, 2, 3, \ldots)$, but also complicated cases can appear.

If Ω is an $(n-1)$-dimensional compact Riemannian manifold with its boundary to be sufficiently regular, we know that

$$\lambda(\Omega, k) \sim A(\Omega, n)k^{2/(n-1)} \quad (k \to \infty)$$

(see e.g. Cheng and Li [**4**]) and

$$\sum_{\lambda(\Omega,k) \leq x} \{f_k^{\Omega}(\Theta)\}^2 \sim B(\Omega, n)x^{(n-1)/2} \quad (x \to \infty)$$

uniformly with respect to Θ (e.g. Minakshisundaram and Pleijel [**14**], and also Essen and Lewis [**7** p.120 and pp.126–128]), where $A(\Omega, n)$ and $B(\Omega, n)$ are both constants depending on Ω and n. Hence there exist two positive constants M_1, M_2 such that

$$M_1 k^{2/(n-1)} \leq \lambda(\Omega, k) \quad (k = 1, 2, 3, \ldots) \tag{2.2}$$

and

$$|f_k^{\Omega}(\Theta)| \leq M_2 k^{1/2} \quad (\Theta \in \Omega, k = 1, 2, 3, \ldots). \tag{2.3}$$

If we denote a positive solution of the equation

$$t^2 + (n-2)t - \lambda(\Omega, k) = 0 \tag{2.4}$$

by $\alpha(\Omega, k)$, then we also have

$$\alpha(\Omega, k) \geq M_3 k^{1/(n-1)} \quad (k = 1, 2, \ldots) \tag{2.5}$$

from (2.2), where M_3 is a positive constant independent of k.

In the following we put the strong assumption relative to Ω on S^{n-1}: if $n \geq 3$, Ω is a $C^{2,a}$-domain $(0 < a < 1)$ on S^{n-1} surrounded by a finite number of mutually disjoint closed hypersurfaces (see e.g. [**10**, pp.88–89] for the definition of $C^{2,a}$-domain). We remark that

$$r^{\alpha(\Omega,k)} f_k^{\Omega}(\Theta) \quad (k = 1, 2, \ldots)$$

is harmonic on $C(\Omega)$ and vanish continuously on $\partial C(\Omega)$. For a domain Ω and the sequence $\{k_i\}$ mentioned above, by $I(\Omega, k_l)$ we denote the set of all positive integers less that $k_l(l = 1, 2, 3, \ldots)$. In spite of the fact $I(\Omega, k_1) = \emptyset$, the summation over $I(\Omega, k_1)$ of a function $S(k)$ of a variable k will be used by promising

$$\sum_{k \in I(\Omega, k_1)} S(k) = 0.$$

Let $G_{C(\Omega)}((r_1, \Theta_1), (r_2, \Theta_2))((r_1, \Theta_1), (r_2, \Theta_2) \in C(\Omega))$ be the Green function of a cone

$C(\Omega)$ and let s_n denote the surface area $2\pi^{n/2}\{\Gamma(n/2)\}^{-1}$ of S^{n-1}. The function

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q),\quad c_n=\begin{cases}2\pi & (n=2)\\ (n-2)s_n & (n\geq 3)\end{cases}$$

of $Q\in\partial C(\Omega)-\{O\}$ (O is the origin of $\boldsymbol{R}^n$) for any fixed $P\in C(\Omega)$ is an ordinary Poisson kernel, where $\partial/\partial\nu$ denotes the differentiation at Q along the inward normal into $C(\Omega)$.

REMARK 1. Suppose $\Omega=\boldsymbol{S}_+^{n-1}(n\geq 2)$. Then

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{\boldsymbol{T}_n}((r,\Omega),(t,\Xi))=2s_n^{-1}\sum_{k=0}^{\infty}c_{k,n+2}r^{k+1}t^{-k-n}\cos\theta_1 L_{k,n+2}(\cos\gamma)\tag{2.6}$$

for any $(X,y)=(r,\Theta)\in\boldsymbol{T}_n$ and any $(Z,0)=(t,\Xi)\in\partial\boldsymbol{T}_n$ satisfying $r<t$, where

$$c_{k,n+2}=\binom{k+n-1}{k},$$

$L_{k,n+2}$ is the $(n+2)$-dimensional Legendre polynomial of degree k and γ is the angle between $M=(X,0)$ and $N=(Z,0)$ defined by

$$\cos\gamma=\frac{(M,N)}{|M|\,|N|}$$

(see Armitage [**1**, Theorem E]). On the other hand, Remark 3 in Section 5 applied to $\Omega=\boldsymbol{S}_+^{n-1}$ gives the Fourier series expansion of the function

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{\boldsymbol{T}_n}((r,\Theta),(t,\Xi))\quad(r<t)$$

of Θ with respect to the sequence of eigenfunctions of (2.1). Hence, in comparison with (2.6) we obtain

$$\alpha(\boldsymbol{S}_+^{n-1},k_i)=i,\quad(i=1,2,3,\ldots;n=2,3,4,\ldots).\tag{2.7}$$

Consider the simplest case $n=2$ i.e. $\Omega=\boldsymbol{S}_+^1$. For $(r,\theta_1)\in\boldsymbol{T}_2$ and $(|t|,\Xi)=t\in\boldsymbol{R}$, we see $\cos\gamma=|t|^{-1}t\sin\theta_1$ and hence

$$k_i=i\quad(i=1,2,3,\ldots)$$

$$f_k^{\Omega}(\theta_1)=\rho_k\cos\theta_1 L_{k-1,4}(\sin\theta_1)\quad(k=1,2,\ldots),$$

where ρ_k is a constant such that

$$\int_{-\pi/2}^{\pi/2}\{f_k^{\Omega}(\theta_1)\}^2\,d\theta_1=1.$$

Next, suppose $n=3$ i.e. $\Omega=\boldsymbol{S}_+^2$. Then for $(r,\Theta)=(X,y)\in\boldsymbol{T}_3,\Theta=(\theta_1,\theta_2)$ and $(t,\Xi)\in\partial\boldsymbol{T}_3=\boldsymbol{R}^2,\Xi=(2^{-1}\pi,\xi_2)$, we see

$$\cos\gamma=\sin\theta_1\sin\theta_2\sin\xi_2+\sin\theta_1\cos\theta_2\cos\xi_2.$$

If we put

$$L_{0,5} = \Phi_{0,0} = 1$$

and

$$\begin{aligned} &L_{k,5}(\sin\theta_1 \sin\theta_2 \sin\xi_2 + \sin\theta_1 \cos\theta_2 \cos\xi_2) \\ &\quad = \Phi_{k,0}(\theta_1,\theta_2)\cos^k \xi_2 + \Phi_{k,1}(\theta_1,\theta_2)\cos^{k-2}\xi_2 + \cdots + \Phi_{k,[k/2]}(\theta_1,\theta_2)\cos^{k-2[k/2]}\xi_2 \\ &\qquad + \Psi_{k,0}(\theta_1,\theta_2)\cos^{k-1}\xi_2 \sin\xi_2 + \Psi_{k,1}(\theta_1,\theta_2)\cos^{k-3}\xi_2\sin\xi_2 + \cdots \\ &\qquad + \Psi_{k,[(k-1)/2]}(\theta_1,\theta_2)\cos^{k-1-2[(k-1)/2]}\xi_2 \sin\xi_2 \quad (k = 1,2,3,\dots), \end{aligned}$$

then

$$k_i = 1 + \frac{(i-1)i}{2} \quad (i = 1,2,3,\dots)$$

and

$$\begin{aligned} &f^{\Omega}_{k_i+j}(\Theta) \\ &= \begin{cases} \rho_{k_i+j}\Phi_{i-1,j}(\theta_1,\theta_2)\cos\theta_1 & (j = 0,1,\dots,[\frac{i-1}{2}]; i = 1,2,\dots) \\ \rho_{k_i+j}\Psi_{i-1,j-[(i-1)/2]-1}(\theta_1,\theta_2)\cos\theta_1 & (j = [\frac{i-1}{2}]+1,\dots,[\frac{i-1}{2}]+[\frac{i-2}{2}]+1; i = 2,3,\dots), \end{cases} \end{aligned}$$

where ρ_{k_i+j} is a constant such that

$$\int_{S^2_+} \{f^{\Omega}_{k_i+j}(\Theta)\}^2 \, d\sigma_\Theta = 1.$$

3. Existence and properties of particular solutions

The Fourier coefficient

$$\int_\Omega F(\Theta) f^{\Omega}_k(\Theta)\, d\sigma_\Theta$$

of a function $F(\Theta)$ on Ω with respect to the orthonormal sequence $\{f^{\Omega}_k(\Theta)\}$ is denoted by $c(F,k)$, if it exists. We also denote the set $\partial C(\Omega) - \{O\}$ by $S(\Omega)$. Now we shall define generalized Poisson kernels of the conical type. For a non-negative integer l and two points $P = (r,\Theta) \in C(\Omega), Q = (t,\Xi) \in S(\Omega)$, we put

$$(3.1) \qquad V(C(\Omega),l)(P,Q) = \sum_{k \in I(\Omega,k_{l+1})} 2^{\alpha(\Omega,k)+n-1} c((H_\Xi)_1,k) t^{-\alpha(\Omega,k)-n+1} r^{\alpha(\Omega,k)} f^{\Omega}_k(\Theta),$$

where

$$(H_\Xi)_1(\Theta) = c_n^{-1} \frac{\partial}{\partial v} G_{C(\Omega)}((1,\Theta),(2,\Xi)).$$

We introduce another function of $P \in C(\Omega)$ and $Q = (t,\Xi) \in S(\Omega)$

$$W(C(\Omega),l)(P,Q) = \begin{cases} V(C(\Omega),l)(P,Q) & (1 \le t < \infty) \\ 0 & (0 < t < 1). \end{cases}$$

The generalized Poisson kernel $K(C(\Omega),l)(P,Q)(P \in C(\Omega), Q \in S(\Omega))$ with respect to $C(\Omega)$ is defined by

$$K(C(\Omega),l)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q) - W(C(\Omega),l)(P,Q) \quad (P \in C(\Omega), Q \in S(\Omega)).$$

In fact

$$K(C(\Omega),0)(P,Q) = c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q).$$

REMARK 2. We shall show that the kernel $K(\boldsymbol{T}_n,l)(P,Q)(l \geq 1)$ coincides with ones in Armitage [**1**], Siegel [**16**] and Yoshida [**19**]. Put $\Omega = S_+^{n-1}$ and $r_2 = 1$ in Remark 3 of Section 5. Then from (2.7) we have

$$c_n^{-1}\frac{\partial}{\partial\nu}G_{\boldsymbol{T}_n}((r,\Theta),(t,\Xi)) = \sum_{i=1}^{\infty} 2^{n-1+i} r^i t^{1-n-i}\left(\sum_{k=k_i}^{k_{i+1}-1} c((H_\Xi)_1,k)f_k^\Omega(\Theta)\right)$$

for any $(r,\Theta) \in \boldsymbol{T}_n$ and any $(t,\Xi) \in \partial\boldsymbol{T}_n(r < t)$, which is (2.6). Hence we obtain

$$2^{n+i}\left(\sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_\Xi)_1,k)f_k^\Omega(\Theta)\right) = 2s_n^{-1}c_{i,n+2}\cos\theta_1 L_{i,n+2}(\cos\gamma) \quad (i = 0,1,2,\dots).$$

Since

$$V(\boldsymbol{T}_n,l)(P,Q) = \sum_{i=0}^{l-1} 2^{n+i} r^{i+1} t^{-n-i}\left(\sum_{k=k_{i+1}}^{k_{i+2}-1} c((H_\Xi)_1,k)f_k^\Omega(\Theta)\right)$$

from (2.7), we finally have

$$V(\boldsymbol{T}_n,l)(P,Q) = 2s_n^{-1}\sum_{i=0}^{l-1} c_{i,n+2}r^{i+1}t^{-n-i}\cos\theta_1\, L_{i,n+2}(\cos\gamma).$$

Let $F(P) = F(r,\Theta)$ be a function on $C(\Omega)$ and put

$$N(F)(r) = \int_\Omega F(r,\Theta)f_1^\Omega(\Theta)\,d\sigma_\Theta.$$

For a non-negative integer p we write

$$\mu_p(F) = \lim_{r\to\infty} r^{-\alpha(\Omega,k_{p+1})}N(F)(r),$$

if it exists.

The following theorem is a generalization of the first part of Yoshida [**18**, Theorem 3] and Yoshida [**18**, Lemma 3] which are the case $l = 0$ of Theorem 1.

THEOREM 1. *Let l be a non-negative integer and let $g(Q) = g(t,\Xi)$ be a continuous function on $\partial C(\Omega)$ satisfying*

$$\int_1^\infty t^{-\alpha(\Omega,k_{l+1})-1}\left(\int_{\partial\Omega}|g(t,\Xi)|\,d\sigma_\Xi\right)dt < \infty. \tag{3.2}$$

Then

$$H(C(\Omega), l; g)(P) = \int_{S(\Omega)} g(Q)K(C(\Omega), l)(P, Q)d\sigma_Q$$

is a solution of the classical Dirichlet problem on $C(\Omega)$ *with* g *and satisfies*

$$\mu_l(|H(C(\Omega), l; g)|) = 0. \tag{3.3}$$

By taking $\Omega = S_+^{n-1}$, we obtain from (2.6) and Remark 2

Corollary 1 (Yoshida [**19**], Theorem 1]). *Let* $g(X)$ *be a continuous function on* $\partial T_n = R^{n-1}$ *satisfying* (1.1) *with a non-negative integer* l. *Then* $H(T_n, l; g)(P)$ *is a solution of the Dirichlet problem on* T_n *with* g *such that*

$$\mu_l(|H(T_n, l; g)|) = 0.$$

To solve the Dirichlet problem on $C(\Omega)$ with any continuous function $g(Q)$, we shall define another Poisson kernel. Let $\varphi(t)$ be a positive continuous function of $t \geq 1$ satisfying

$$\varphi(1) = 2^{-\alpha(\Omega,1)}.$$

Denote the set

$$\{t \geq 1; -\alpha(\Omega, k_i) = (\log 2)^{-1}(\log(t^{n-1}\varphi(t)))\}$$

by $S(\Omega, \varphi, i)$. Then $1 \in S(\Omega, \varphi, 1)$. When there is an integer N such that $S(\Omega, \varphi, N) \neq \varnothing$ and $S(\Omega, \varphi, N+1) = \varnothing$, denote the set $\{i; 1 \leq i \leq N\}$ of integers by $J(\Omega, \varphi)$. Otherwise, denote the set of all positive integers by $J(\Omega, \varphi)$. Let $t(i) = t(\Omega, \varphi, i)$ be the minimum of elements t in $S(\Omega, \varphi, i)$ for each $i \in J(\Omega, \varphi)$. In the former case, we put $t(N+1) = \infty$. Then $t(1) = 1$. We define $W(C(\Omega), \varphi)(P, Q)(P \in C(\Omega), Q = (t, \Xi) \in S(\Omega))$ by

$$W(C(\Omega), \varphi)(P, Q) = \begin{cases} 0 & (0 < t < 1) \\ V(C(\Omega), i)(P, Q) & (t(i) \leq t < t(i+1); i \in J(\Omega, \varphi)). \end{cases}$$

The Poisson kernel $K(C(\Omega), \varphi)(P, Q)(P \in C(\Omega), Q \in S(\Omega))$ is defined by

$$K(C(\Omega), \varphi)(P, Q) = c_n^{-1}\frac{\partial}{\partial \nu}G_{C(\Omega)}(P, Q) - W(C(\Omega), \varphi)(P, Q).$$

Now we have

Theorem 2. *Let* $g(Q)$ *be a continuous function on* $\partial C(\Omega)$. *Then there is a positive continuous function* $\varphi_g(t)$ *of* $t \geq 1$ *depending on* g *such that*

$$H(C(\Omega), \varphi_g)(P) = \int_{S(\Omega)} g(Q)K(C(\Omega), \varphi_g)(P, Q)\, d\sigma_Q$$

is a solution of the Dirichlet problem on $C(\Omega)$ *with* g.

If we take $\Omega = S_+^{n-1}$ in Theorem 2, Then we have

COROLLARY 2 (Finkelstein and Scheinberg [8] and Gardiner [9]). *Let $g(Q)$ be a continuous function on $\partial \boldsymbol{T}_n$. Then there is a positive continuous function $\varphi_g(t)$ of $t \geq 1$ depending on g such that*

$$H(\boldsymbol{T}_n, \varphi_g)(P) = \int_{\partial \boldsymbol{T}_n} g(Q) K(\boldsymbol{T}_n, \varphi_g)(P, Q)\, d\sigma_Q$$

is a solution of the Dirichlet problem on $\boldsymbol{T}_n$ with g.

4. A type of general solutions

To obtain a type of general solutions, the following is essential.

THEOREM B (Yoshida and Miyamoto [20, Theorem 3]). *Let $h(r, \Theta)$ be a harmonic function in $C(\Omega)$ vanishing continuously on $\partial C(\Omega)$ and let p be a positive integer. If h satisfies*

$$\mu_p(h^+) = 0,$$

then

$$h(r, \Theta) = \sum_{k \in I(\Omega, k_{p+1})} A_k r^{\alpha(\Omega, k)} f_k^{\Omega}(\Theta)$$

for every $(r, \Theta) \in C(\Omega)$, where $A_k (k = 1, 2, 3, \dots, k_{p+1} - 1)$ is a constant.

By using Theorem 1 and Theorem B, we can prove the following Theorem 3.

THEOREM 3. *Let l be a non-negative integer and p be a positive integer satisfying $p \geq 1$. Let $g(t, \Xi)$ be a continuous function on $\partial C(\Omega)$ satisfying (3.2) with l. If $h(r, \Theta)$ is a solution of the Dirichlet problem on $C(\Omega)$ with g satisfying*

$$\mu_p(h^+) = 0, \tag{4.1}$$

then

$$h(r, \Theta) = H(C(\Omega), l; g)(P) + \sum_{k \in I(\Omega, k_{p+1})} A_k r^{\alpha(\Omega, k)} f_k^{\Omega}(\Theta)$$

for every $P = (r, \Theta) \in C(\Omega)$, where $A_k (k = 1, 2, \dots, k_{p+1} - 1)$ is a constant.

If we take $l = 0$ and $p = 1$ in Theorem 3, then we have the following result which is the second part of Yoshida [18, Theorem 3].

COROLLARY 3. *Let $g(Q)$ be a continuous function on $\partial C(\Omega)$ satisfying*

$$\int_1^{\infty} t^{-\alpha(\Omega, 1) - 1} \left(\int_{\partial \Omega} |g(t, \Xi)|\, d\sigma_{\Xi} \right) dt < \infty.$$

If $h(r, \Theta)$ is a solution of the Dirichlet problem on $C(\Omega)$ with g satisfying

$$\mu_1(h^+) = 0,$$

then

$$h(r,\Theta)=c_n^{-1}\int_{S(\Omega)} g(Q)\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q)\,d\sigma_Q+\mu_0(h)r^{\alpha(\Omega,1)}f_1^{\Omega}(\Theta)$$

for every $P=(r,\Theta)\in C(\Omega)$.

If we put $\Omega=\boldsymbol{S}_+^{n-1}, l=\rho$ and $p=\rho$ (resp. $l=\rho-1$ and $p=\rho$) (ρ is a positive integer) in Theorem 3, we obtain

COROLLARY 4 (Yoshida [**19**, Theorem 2 (resp. Corollary 2)]). *Let ρ be a positive integer and $g(X)$ be a continuous function on $\partial\boldsymbol{T}_n=\boldsymbol{R}^{n-1}$ satisfying* (1.1) *with ρ (resp.* (1.1) *with $\rho-1$). If $h(P)$ is a solution of the Dirichlet problem on $\boldsymbol{T}_n$ with g such that*

$$\lim_{r\to\infty} r^{-(\rho+1)}N(h^+)(r)=0,$$

then

$$h(P)=H(\boldsymbol{T}_n,\rho;g)(P)+y\Pi(P)$$

$$(resp.\ h(P)=H(\boldsymbol{T}_n,\rho-1;g)(P)+y\Pi(P)),$$

where $\Pi(P)$ is a harmonic polynomial (of $P=(x_1,x_2,\dots,x_{n-1},y)\in\boldsymbol{R}^n$) of at most degree $\rho-1$ and even with respect to the variable y.

The following Theorem 4 also generalizes a result of Yoshida [**19**, Theorem 3].

THEOREM 4. *If $h(r,\Theta)$ is a harmonic function on $C(\Omega)$ and is continuous on $\overline{C(\Omega)}$ such that the restriction $h=h|_{\partial C(\Omega)}$ of h to $\partial C(\Omega)$ satisfies*

$$\int_1^{\infty} t^{-\alpha(\Omega,k_{l+1})-1}\left(\int_{\partial\Omega}|h(t,\Xi)|\,d\sigma_{\Xi}\right)dt<\infty$$

for some non-negative integer l and

$$\limsup_{r\to\infty}\frac{\log N(h^+)(r)}{\log r}<\infty,$$

then for some positive integer p

$$h(r,\Theta)=H(C(\Omega),l;h)(P)+\sum_{k\in I(\Omega,k_{p+1})}A_k r^{\alpha(\Omega,k)}f_k^{\Omega}(\Theta)$$

at every $P=(r,\Theta)\in C(\Omega)$, where $A_k (k=1,2,\dots,k_{p+1}-1)$ is a constant.

5. Proof of Theorems 1,2,3,4, and Corollary 4

Given a domain Ω on $\boldsymbol{S}^{n-1}$ and an interval $I\subset\boldsymbol{R}_+$, the sets $\{(r,\Theta)\in\boldsymbol{R}^n;(1,\Theta)\in\Omega, r\in I\}$ and $\{(t,\Xi)\in\boldsymbol{R}^n;(1,\Xi)\in\partial\Omega, t\in I\}$ are denoted by $C(\Omega;I)$ and $S(\Omega;I)$, respectively.

LEMMA 1. *Let $h(r,\Theta)$ be a harmonic function in $C(\Omega;(a,b)), 0 \le a < b \le \infty$, which vanishes continuously on $S(\Omega;(a,b))$. For any fixed $r(a<r<b)$, define the function $h_r(\Theta)$ on Ω by $h_r(\Theta) = h(r,\Theta)$. Then*

$$c(h_r,k) = \{(r_1 r^{-1})^{\beta(\Omega,k)} c(h_{r_1},k)(r_2^{\delta(\Omega,k)} - r^{\delta(\Omega,k)}) + (r_2 r^{-1})^{\beta(\Omega,k)} c(h_{r_2},k)(r^{\delta(\Omega,k)} - r_1^{\delta(\Omega,k)})\}(r_2^{\delta(\Omega,k)} - r_1^{\delta(\Omega,k)})^{-1}$$

for any given $r_1, r_2 (0 \le a < r_1 < r_2 < b \le \infty)$, where $-\beta(\Omega,k)$ is a negative solution of (2.4) *and $\delta(\Omega,k) = \alpha(\Omega,k) + \beta(\Omega,k)$.*

PROOF. First of all, we note that $h(r,\Theta)$ is continuously differentiable twice on $\{(r,\Theta); \Theta \in \bar{\Omega}, a<r<b\}$ (see [**10**, pp.101–102]). Now, by differentiating twice under the integral sign,

$$\frac{\partial^2 c(h_r,k)}{\partial r^2} = \int_\Omega \frac{\partial^2 h(r,\theta)}{\partial r^2} f_k^\Omega(\Theta)\, d\sigma_\Theta = -(n-1)r^{-1}\int_\Omega \frac{\partial h}{\partial r} f_k^\Omega\, d\sigma_\Theta - r^{-2}\int_\Omega (\Lambda_n h) f_k^\Omega\, d\sigma_\Theta.$$

Hence, if we see from the formula of Green (see e.g. Helgason [**12**, p.387]) that

$$\int_\Omega (\Lambda_n h) f_k^\Omega\, d\sigma_\Theta = \int_\Omega h(\Lambda_n f_k^\Omega)\, d\sigma_\Theta,$$

we have that

$$\frac{\partial^2}{\partial r^2} c(h_r,k) + (n-1)r^{-1}\frac{\partial}{\partial r} c(h_r,k) - \lambda(\Omega,k) r^{-2} c(h_r,k) = 0$$

for any r, $a<r<b$. This gives that

$$c(h_r,k) = A_k r^{\alpha(\Omega,k)} + B_k r^{-\beta(\Omega,k)} \quad (a<r<b),$$

A_k and B_k being constants independent of r. Since $c(h_r,k)$ takes a value $c(h_{r_j},k)$ at a point $r_j (j=1,2)$, the conclusion of Lemma 1 follows immediately.

LEMMA 2. *Let $H(r,\Theta)$ be a harmonic function in $C(\Omega;(0,2))$ such that $H(r,\Theta)$ vanishes continuously on $S(\Omega;(0,2))$ and is uniformly bounded as $r \to 0$. Then for any non-negative integer l we have*

$$\Big|H(r,\Theta) - \sum_{k \in I(\Omega,k_{l+1})} c(H_1,k) r^{\alpha(\Omega,k)} f_k^\Omega(\Theta)\Big| \le L_1(H) r^{\alpha(\Omega,k_{l+1})} \quad (0<r<1),$$

where $H_1(\Theta) = H(1,\Theta)$ and $L_1(H)$ is a constant dependent only on H.

PROOF. Put $H_r(\Theta) = H(r,\Theta)$. For any fixed r, $0<r<2$, we see from Lemma 1 that

$$c(H_r,k) = \{(r_1 r^{-1})^{\beta(\Omega,k)} c(H_{r_1},k)(r_2^{\delta(\Omega,k)} - r^{\delta(\Omega,k)}) + (r_2 r^{-1})^{\beta(\Omega,k)} c(H_{r_2},k)(r^{\delta(\Omega,k)} - r_1^{\delta(\Omega,k)})\}(r_2^{\delta(\Omega,k)} - r_1^{\delta(\Omega,k)})^{-1}$$

for any r_1 and $r_2, 0 < r_1 < r_2 < 2$. Since $c(H_{r_1}, k)$ is also uniformly bounded as $r_1 \to 0$, we obtain

$$c(H_r, k) = (r/r_2)^{\alpha(\Omega,k)} c(H_{r_2}, k) \quad (0 < r_2 < 2). \tag{5.1}$$

Now, take a number r_2^* satisfying $r < r_2^* < 2$. Then we have from (2.3) that

$$|c(H_{r_2^*}, k)| \leq s_n M_2 k^{1/2} \times \max_{\Theta \in \Omega} |H(r_2^*, \Theta)|. \tag{5.2}$$

It follows from (2.3), (2.5), (5.1) and (5.2) that

$$\sum_{k=1}^{\infty} |c(H_r, k)|\, |f_k^{\Omega}(\Theta)| \leq s_n M_2^2 \times \max_{\Theta \in \Omega} |H(r_2^*, \Theta)| \times \sum_{k=1}^{\infty} k (r/r_2^*)^{M_3 k^{1/(n-1)}}. \tag{5.3}$$

Hence we know from the completeness of the orthonormal sequence $\{f_k^{\Omega}(\Theta)\}$ that

$$\sum_{k=1}^{\infty} c(H_r, k) f_k^{\Omega}(\Theta) = H(r, \Theta) \tag{5.4}$$

for any $\Theta \in \Omega$.

If we take $r = 1, r_2^* = 3/2$ in (5.3) and put

$$L_1(H) = s_n M_2^2 \times \max_{\Theta \in \Omega} \left| H\left(\frac{3}{2}, \Theta\right) \right| \times \sum_{k=1}^{\infty} k \left(\frac{2}{3}\right)^{M_3 k^{1/(n-1)}},$$

then we obtain that

$$\sum_{k=1}^{\infty} |c(H_1, k)|\, |f_k^{\Omega}(\Theta)| \leq L_1(H).$$

If $0 < r < 1$, then by taking $r_2 = 1$ in (5.1) we have from (5.3) and (5.4) that

$$\begin{aligned} |H(r, \Theta) - \sum_{k \in I(\Omega, k_{l+1})} c(H_1, k) r^{\alpha(\Omega,k)} f_k^{\Omega}(\Theta)| &\leq \sum_{k=k_{l+1}}^{\infty} |c(H_r, k)|\, |f_k^{\Omega}(\Theta)| \\ &= \sum_{k=k_{l+1}}^{\infty} |c(H_1, k)|\, |f_k^{\Omega}(\Theta)| r^{\alpha(\Omega,k)} \leq L_1(H) r^{\alpha(\Omega, k_{l+1})}, \end{aligned}$$

which gives the conclusion.

LEMMA 3. *For a non-negative integer l we have*

$$\left| c_n^{-1} \frac{\partial}{\partial \nu} G_{C(\Omega)}(P, Q) - V(C(\Omega), l)(P, Q) \right| \leq L_2 (2r)^{\alpha(\Omega, k_{l+1})} t^{-\alpha(\Omega, k_{l+1}) - n + 1}$$

for any $P = (r, \Theta) \in C(\Omega)$ and any $Q = (t, \Xi) \in S(\Omega)$ satisfying $0 < (2r/t) < 1$, where L_2 is a constant independent of P, Q and l.

PROOF. Take any $P = (r, \Theta) \in C(\Omega)$ and any $Q = (t, \Xi) \in S(\Omega)$. Put $R_1 = (2r/t)$, $u = (t/2)$ and $\Theta_1 = \Theta$ in

$$u^{n-2} G_{C(\Omega)}((uR_1, \Theta_1), (uR_2, \Theta_2)) = G_{C(\Omega)}((R_1, \Theta_1), (R_2, \Theta_2))$$

$$((R_1, \Theta_1) \in C(\Omega), (R_2, \Theta_2) \in C(\Omega), 0 < u < \infty).$$

When (R_2, Θ_2) approaches to $(2, \Xi) \in S(\Omega)$ along the inward normal, we obtain

$$(5.5) \qquad \left(\frac{1}{2}t\right)^{n-1} \frac{\partial}{\partial \nu} G_{C(\Omega)}((r,\Theta),(t,\Xi)) = \frac{\partial}{\partial \nu} G_{C(\Omega)}\left(\left(\frac{2r}{t},\Theta\right),(2,\Xi)\right) \quad (\Xi \in \partial\Omega).$$

we remark that

$$H_\Xi(R,\Theta) = c_n^{-1} \frac{\partial}{\partial \nu} G_{C(\Omega)}((R,\Theta),(2,\Xi))$$

is a harmonic function of $(R,\Theta) \in C(\Omega)$ such that $H_\Xi(R,\Theta)$ vanishes continuously on $\partial C(\Omega) - \{(2,\Xi)\}$ and tends uniformly to zero as $R \to 0$ (see Azarin [**2**, Lemma 1]). If we apply Lemma 2 to $H_\Xi(2r/t,\Theta)$ and put

$$L_2 = 2^{n-1} \max_{\Xi \in \partial\Omega} L_1(H_\Xi),$$

then we obtain the conclusion from (5.5).

REMARK 3. Take any $(r,\Theta) \in C(\Omega)$ and any $(t,\Xi) \in S(\Omega)$ satisfying $r < t$. Then the proof of Lemma 2 gives the expansion

$$H_\Xi\left(\frac{2r}{t},\Theta\right) = \sum_{k=1}^{\infty} c((H_\Xi)_{r_2},k)\left(\frac{2r}{tr_2}\right)^{\alpha(\Omega,k)} f_k^\Omega(\Theta)$$

for any $r_2, 0 < r_2 < 2$. Hence it follows from (5.5) that

$$\begin{aligned} c_n^{-1} \frac{\partial}{\partial \nu} G_{C(\Omega)}((r,\Theta),(t,\Xi)) &= \sum_{k=1}^{\infty} 2^{\alpha(\Omega,k)+n-1} r_2^{-\alpha(\Omega,k)} c((H_\Xi)_{r_2},k) r^{\alpha(\Omega,k)} t^{1-n-\alpha(\Omega,k)} f_k^\Omega(\Theta) \\ &= \sum_{i=1}^{\infty} 2^{\alpha(\Omega,k_i)+n-1} r_2^{-\alpha(\Omega,k_i)} r^{\alpha(\Omega,k_i)} t^{1-n-\alpha(\Omega,k_i)} \times \left(\sum_{k=k_i}^{k_{i+1}-1} c((H_\Xi)_{r_2},k) f_k^\Omega(\Theta)\right). \end{aligned}$$

LEMMA 4. *Let $\varphi(t)$ be a positive continuous function of $t \geq 1$ satisfying*

$$\varphi(1) = 2^{-\alpha(\Omega,1)}$$

Then

$$\left| c_n^{-1} \frac{\partial}{\partial \nu} G_{C(\Omega)}(P,Q) - W(C(\Omega),\varphi)(P,Q) \right| < L_2 \varphi(t)$$

for any $P = (r,\Theta) \in C(\Omega)$ and any $Q = (t,\Xi) \in S(\Omega)$ satisfying

$$(5.6) \qquad t > \max(1, 4r).$$

PROOF. Take any $P = (r,\Theta) \in C(\Omega)$ and any $Q = (t,\Xi) \in S(\Omega)$ satisfying (5.6). Choose an integer $i = i(P,Q) \in J(\Omega,\varphi)$ such that

$$(5.7) \qquad t(i-1) \leq t < t(i).$$

Then

$$W(C(\Omega),\varphi)(P,Q)=V(C(\Omega),i-1)(P,Q).$$

Hence we have from Lemma 3, (5.6) and (5.7) that

$$\left|c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q)-W(C(\Omega),\varphi)(P,Q)\right|<L_2 2^{-\alpha(\Omega,k_i)}t^{-n+1}<L_2\varphi(t),$$

which is the conclusion.

LEMMA 5. *Let $g(Q)$ be locally integrable and upper semicontinuous on $\partial C(\Omega)$. Let $W(P,Q)$ be a function of $P\in C(\Omega), Q\in\partial C(\Omega)$ such that for any fixed $P\in C(\Omega)$ the function $W(P,Q)$ of $Q\in\partial C(\Omega)$ is a locally integrable function on $\partial C(\Omega)$. Put*

$$K(P,Q)=c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q)-W(P,Q)\quad(P\in C(\Omega),Q\in\partial C(\Omega)).$$

Suppose that the following (I) *and* (II) *are satisfied:*

(I) *For any $Q^*\in\partial C(\Omega)$ and any $\varepsilon>0$, there exist a neighbourhood $U(Q^*)$ of Q^* in $\boldsymbol{R}^n$ and a number $R(0<R<\infty)$ such that*

$$\int_{S(\Omega;[R,\infty))}|g(Q)K(P,Q)|\,d\sigma_Q<\varepsilon$$

for any $P=(r,\Theta)\in C(\Omega)\cap U(Q^*)$.

(II) *For any $Q^*\in\partial C(\Omega)$ and any number $R(0<R<\infty)$,*

$$\limsup_{P\to Q^*,P\in C(\Omega)}\int_{S(\Omega;(0,R))}|g(Q)W(P,Q)|\,d\sigma_Q=0.$$

Then

$$\limsup_{P\to Q^*,P\in C(\Omega)}\int_{S(\Omega)}g(Q)K(P,Q)\,d\sigma_Q\le g(Q^*)$$

for any $Q^*\in\partial C(\Omega)$.

PROOF. Let $Q^*=(t^*,\Theta^*)$ be any fixed point of $\partial C(\Omega)$ and let ε be any positive number. From (I), we can choose a number $R^*(0<R^*<\infty)$ such that

$$\int_{S(\Omega;[R^*,\infty))}|g(Q)K(P,Q)|\,d\sigma_Q<\frac{\varepsilon}{2}\tag{5.8}$$

for any $P=(r,\Theta)\in C(\Omega)\cap U(Q^*)$. Let Φ be a continuous function on $\partial C(\Omega)$ such that $0\le\Phi\le1$ and

$$\Phi=\begin{cases}1 & \text{on } S(\Omega;(0,R^*])\cup\{O\}\\ 0 & \text{on } S(\Omega;(2R^*,\infty)).\end{cases}$$

Let $G^j_{C(\Omega)}(P,Q)$ be the Green function of $C(\Omega;(0,j))$ (j is a positive integer) and put

$\Gamma_j(P,Q) = G_{C(\Omega)}(P,Q) - G^j_{C(\Omega)}(P,Q)$. Then we can find an integer $j^*, j^* > 2R^*$ such that

$$c_n^{-1}\int_{S(\Omega;(0,2R^*))} |\Phi(Q)g(Q)|\left|\frac{\partial}{\partial\nu}\Gamma_{j^*}(P,Q)\right| d\sigma_Q < \frac{\varepsilon}{4} \tag{5.9}$$

for any $P=(r,\Theta)\in C(\Omega)\cap U(Q^*)$. Thus we have from (5.8) and (5.9) that

$$\begin{aligned}\int_{\partial C(\Omega)} g(Q)K(P,Q)\,d\sigma_Q &\le c_n^{-1}\int_{S(\Omega;(0,2R^*))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{C(\Omega)}(P,Q)\,d\sigma_Q \\ &\quad + c_n^{-1}\int_{S(\Omega;(0,2R^*))}\left|\Phi(Q)g(Q)\frac{\partial}{\partial\nu}\Gamma_{j^*}(P,Q)\right| d\sigma_Q \\ &\quad + \int_{S(\Omega;(0,2R^*))}|g(Q)W(P,Q)|\,d\sigma_Q + 2\int_{S(\Omega;(R^*,\infty))}|g(Q)K(P,Q)|\,d\sigma_Q \\ &\le c_n^{-1}\int_{S(\Omega;(0,2R^*))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{C(\Omega)}(P,Q)\,d\sigma_Q \\ &\quad + \int_{S(\Omega;(0,2R^*))}|g(Q)W(P,Q)|\,d\sigma_Q + \frac{5}{4}\varepsilon\end{aligned} \tag{5.10}$$

for any $P=(r,\Theta)\in C(\Omega)\cap U(Q^*)$. Consider an upper semicontinuous function

$$V(Q) = \begin{cases}\Phi(Q)g(Q) & \text{on } S(\Omega;(0,2R^*])\cup\{O\} \\ 0 & \text{on } \partial C(\Omega;(0,j^*)) - S(\Omega;(0,2R_2^*]) - \{O\}\end{cases}$$

on $\partial C(\Omega;[0,j^*))$ and denote the Perron-Wiener-Brelot solution of the Dirichlet problem on $C(\Omega;(0,j^*))$ by $H_V(P;C(\Omega;(0,j^*)))$ (see, e.g., [**13**]). We know that

$$c_n^{-1}\int_{S(\Omega;(0,2R^*))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{C(\Omega)}(P,Q)\,d\sigma_Q = H_V(P;C(\Omega;(0,j^*))) \tag{5.11}$$

(see Dahlberg [**6**, Theorem 3]). If $C(\Omega;(0,j^*))$ is not a Lipschitz domain at O, we can prove (5.11) by considering a sequence of the Lipschitz domains $C(\Omega;(1/m,j^*))$ which converges to $C(\Omega;(0,j^*))$ as $m\to\infty$. We also have that

$$\limsup_{P\in C(\Omega),P\to Q^*} H_V(P;C(\Omega;(0,j^*))) \le \limsup_{Q\in S(\Omega),Q\to Q^*} V(Q) = g(Q^*)$$

(see, e.g., Helms [**13**, Lemma 8.20]). Hence we obtain

$$\limsup_{P\in C(\Omega),P\to Q^*} c_n^{-1}\int_{S(\Omega;(0,2R^*))}\Phi(Q)g(Q)\frac{\partial}{\partial\nu}G^{j^*}_{C(\Omega)}(P,Q)\,d\sigma_Q \le g(Q^*).$$

With (5.10) and (II) this gives the conclusion.

Proof of Theorem 1. First of all, we shall show that $H(C(\Omega),l;g)(P)$ is a harmonic function on $C(\Omega)$. For any fixed $P=(r,\Theta)\in C(\Omega)$, take a number R

satisfying $R > \max(1, 2r)$. Then

$$
\begin{aligned}
(5.12)\qquad & \int_{S(\Omega;(R,\infty))} |g(Q)|\,|K(C(\Omega),l)(P,Q)|\,d\sigma_Q \\
&= \int_{S(\Omega;(R,\infty))} |g(Q)|\left|c_n^{-1}\frac{\partial}{\partial\nu}G_{C(\Omega)}(P,Q) - V(C(\Omega),l)(P,Q)\right| d\sigma_Q \\
&\le L_2(2r)^{\alpha(\Omega,k_{l+1})}\int_R^\infty t^{-\alpha(\Omega,k_{l+1})-1}\left(\int_{\partial\Omega}|g(t,\Xi)|\,d\sigma_\Xi\right)dt < \infty
\end{aligned}
$$

from Lemma 3 and (3.2). Thus $H(C(\Omega), l; g)(P)$ is finite for any $P \in C(\Omega)$. Since $K(C(\Omega), l)(P, Q)$ is a harmonic function of $P \in C(\Omega)$ for any $Q \in S(\Omega), H(C(\Omega), l; g)(P)$ is also a harmonic function of $P \in C(\Omega)$.

To prove that

$$\lim_{P \in C(\Omega), P \to Q^*} H(C(\Omega), l; g)(P) = g(Q^*)$$

for any $Q^* \in \partial C(\Omega)$, apply Lemma 5 to $g(Q)$ and $-g(Q)$ by putting

$$W(P, Q) = W(C(\Omega), l)(P, Q),$$

which is locally integrable on $\partial C(\Omega)$ for any fixed $P \in C(\Omega)$. Then we shall see that (I) and (II) hold. Take any $Q^* = (t^*, \Xi^*) \in \partial C(\Omega)$ and any $\varepsilon > 0$. Let δ be a positive number. Then from (3.2) and (5.12) we can choose a number $R, R > \max\{1, 2(t^* + \delta)\}$ such that for any $P \in C(\Omega) \cap U_\delta(Q^*), U_\delta(Q^*) = \{X \in \boldsymbol{R}^n; |X - Q^*| < \delta\}$,

$$\int_{S(\Omega;[R,\infty))} |g(Q)K(C(\Omega), l)(P, Q)|\,d\sigma_Q < \varepsilon,$$

which is (I) in Lemma 5. To see (II), we only need to observe from (3.1) that for any $Q^* \in \partial C(\Omega)$ and any $Q \in S(\Omega)$

$$\lim_{P \in C(\Omega), P \to Q^*} W(C(\Omega), l)(P, Q) = 0,$$

because

$$\lim_{\Theta \to \Xi^*} f_k(\Theta) = 0 \quad (k = 1, 2, \dots)$$

as $P = (r, \Theta) \to Q^* = (t^*, \Xi^*) \in S(\Omega)$.

We shall prove (3.3). To simplify expression in the proceeding part, we use the following notation. When $I(r)$ is a function on $\boldsymbol{R}_+$ and l is a non-negative integer, we denote

$$\lim_{r \to \infty} r^{-\alpha(\Omega, k_{l+1})} I(r)$$

by $\mu_l^*(I)$, if it exists. Hence for a function $F(r, \Theta)$ on $C(\Omega)$, we see

$$\mu_l(F) = \mu_l^*(N(F)).$$

Consider the inequality

$$N(|H(C(\Omega), l : g^+)|)(r) \le I_1(r) + I_2(r), \tag{5.13}$$

where

$$I_1(r) = \int_\Omega \left(\int_{S(\Omega;(2r,\infty))} g^+(Q)|K(C(\Omega),l)(P,Q)|\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta$$

and

$$I_2(r) = \int_\Omega \left(\int_{S(\Omega;(0,2r])} g^+(Q)|K(C(\Omega),l)(P,Q)|\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta,$$

$$(P = (r,\Theta), 0 < r < \infty).$$

Let ε be any positive number. From (3.2) we can take a sufficiently large number r_0 such that

$$\int_{2r}^\infty t^{-\alpha(\Omega,k_{l+1})-1} \left(\int_{\partial\Omega} |g(t,\Xi)|\, d\sigma_\Xi \right) dt < \frac{\varepsilon}{2^{\alpha(\Omega,k_{l+1})+1} L L_2} \quad (r > r_0),$$

where L_2 is the constant in Lemma 3 and

$$L = \int_\Omega f_1^\Omega(\Theta)\, d\sigma_\Theta.$$

Then from Lemma 3 we have

$$\begin{aligned} 0 \le I_1(r) &\le L L_2 (2r)^{\alpha(\Omega,k_{l+1})} \int_{2r}^\infty t^{-\alpha(\Omega,k_{l+1})-1} \left(\int_{\partial\Omega} g^+(t,\Xi)\, d\sigma_\Xi \right) dt \\ &< \frac{\varepsilon}{2} r^{\alpha(\Omega,k_{l+1})} \quad (r > r_0), \end{aligned}$$

which gives

$$\mu_l^*(I_1) = 0. \tag{5.14}$$

To estimate $I_2(r)$, we use the inequality

$$I_2(r) \le I_{2,1}(r) + I_{2,2}(r), \tag{5.15}$$

where

$$I_{2,1}(r) = c_n^{-1} \int_\Omega \left(\int_{S(\Omega;(0,2r])} g^+(Q) \frac{\partial}{\partial \nu} G_{C(\Omega)}(P,Q)\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta$$

and

$$I_{2,2}(r) = \int_\Omega \left(\int_{S(\Omega;(0,2r])} g^+(Q)|V(C(\Omega),l)(P,Q)|\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta$$

$$\left(P = (r,\Theta), r > \frac{1}{2} \right).$$

First we have from (2.3) and (3.1) that if $l \geq 1$, then

$$I_{2,2}(r) \leq s_n BLM_2^2 \sum_{k \in I(\Omega, k_{l+1})} k 2^{\alpha(\Omega,k)+n-1} r^{\alpha(\Omega,k)} \Psi_k(r) \quad \left(r > \frac{1}{2}\right),$$

where

$$B = c_n^{-1} \max_{\Theta \in \Omega, \Xi \in \partial\Omega} \frac{\partial}{\partial \nu} G_{C(\Omega)}((1, \Theta), (2, \Xi))$$

and

$$\Psi_k(r) = \int_1^{2r} t^{-\alpha(\Omega,k)-1} \left(\int_{\partial\Omega} g^+(t, \Xi)\, d\sigma_\Xi \right) dt \quad \left(r > \frac{1}{2}, k \in I(\Omega, k_{l+1})\right).$$

We shall later show that

$$\text{(5.16)} \qquad \Psi_k(r) = o(r^{\alpha(\Omega,k_{l+1})-\alpha(\Omega,k)}) \quad (r \to \infty) \quad (l \geq 1, k \in I(\Omega, k_{l+1})).$$

Hence we can conclude that if $l \geq 1$, then

$$\text{(5.17)} \qquad \mu_l^*(I_{2,2}) = 0.$$

This also holds in the case $l = 0$, because $I_{2,2}(r) \equiv 0$ then. Further we can obtain

$$\text{(5.18)} \qquad \mu_l^*(I_{2,1}) = 0,$$

which will be proved at the end of this proof. We thus obtain from (5.15), (5.17) and (5.18) that

$$\text{(5.19)} \qquad \mu_l^*(I_2) = 0.$$

We can finally conclude from (5.13), (5.14) and (5.19) that

$$\mu_l(|H(C(\Omega), l; g^+)|) = 0.$$

In the completely same way applied to g^- we also have that

$$\mu_l(|H(C(\Omega), l; g^-)|) = 0.$$

Since

$$|H(C(\Omega), l; g)(P)| \leq |H(C(\Omega), l; g^+)(P)| + |H(C(\Omega), l; g^-)(P)|,$$

these give the conclusion (3.3).

We shall prove (5.16). We note that $\Psi_k(r)$ is increasing,

$$\int_1^\infty \Psi_k'(r) r^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)}\, dr$$
$$= 2^{\alpha(\Omega,k_{l+1})-\alpha(\Omega,k)} \int_2^\infty t^{-\alpha(\Omega,k_{l+1})-1} \left(\int_{\partial\Omega} g^+(t, \Xi)\, d\sigma_\Xi \right) dt$$

and

$$\Psi_k(r) r^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)} \le 2^{\alpha(\Omega,k_{l+1})-\alpha(\Omega,k)} \int_1^{2r} t^{-\alpha(\Omega,k_{l+1})-1} \left(\int_{\partial\Omega} g^+(t,\Xi)\, d\sigma_\Xi \right) dt$$
$$\le L_3 2^{\alpha(\Omega,k_{l+1})-\alpha(\Omega,k)} \quad \left(r > \frac{1}{2} \right),$$

where

$$L_3 = \int_1^\infty t^{-\alpha(\Omega,k_{l+1})-1} \left(\int_{\partial\Omega} g^+(t,\Xi)\, d\sigma_\Xi \right) dt.$$

From these we see

$$\int_1^\infty \Psi_k(r) r^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)-1}\, dr < \infty \tag{5.20}$$

by integration by parts. Since

$$\Psi_k(r) r^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)}$$
$$= (\alpha(\Omega,k_{l+1}) - \alpha(\Omega,k)) \Psi_k(r) \int_r^\infty t^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)-1}\, dt$$
$$\le (\alpha(\Omega,k_{l+1}) - \alpha(\Omega,k)) \int_r^\infty \Psi_k(t) t^{-\alpha(\Omega,k_{l+1})+\alpha(\Omega,k)-1}\, dt \quad (1 \le k < k_{l+1}),$$

(5.20) gives (5.16).

At the end we shall show (5.18). First we note that

$$0 \le I_{2,1}(r) = N(H(C(\Omega), l; g^+))(r) - I_1^*(r) + I_{2,2}^*(r) \quad \left(r > \frac{1}{2} \right), \tag{5.21}$$

where

$$I_1^*(r) = \int_\Omega \left(\int_{S(\Omega;(2r,\infty))} g^+(Q) K(C(\Omega), l)(P,Q)\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta,$$

and

$$I_{2,2}^*(r) = \int_\Omega \left(\int_{S(\Omega;(1,2r])} g^+(Q) V(C(\Omega), l)(P,Q)\, d\sigma_Q \right) f_1^\Omega(\Theta)\, d\sigma_\Theta \quad \left(r > \frac{1}{2} \right).$$

Since

$$|I_1^*(r)| \le I_1(r) \quad \text{and} \quad |I_{2,2}^*(r)| \le I_{2,2}(r) \quad \left(r > \frac{1}{2} \right),$$

we easily see from (5.14) and (5.17) that

$$\mu_l^*(|I_1^*|) = \mu_l^*(|I_{2,2}^*|) = 0 \tag{5.22}$$

If we can show that

$$\limsup_{r\to\infty} r^{-\alpha(\Omega,k_{l+1})} N(H(C(\Omega),l;g^+))(r) \le 0, \tag{5.23}$$

then we finally conclude from (5.21) and (5.22) that

$$\limsup_{r\to\infty} r^{-\alpha(\Omega,k_{l+1})} I_{2,1}(r) \le 0,$$

which give (5.18). To prove (5.23), remember that $-H(C(\Omega),l;g^+)(P)$ is also a harmonic function on $C(\Omega)$ satisfying

$$\lim_{P\in C(\Omega_j), P\to Q^*} -H(C(\Omega),l;g^+)(P) = -g^+(Q^*) \le 0$$

for every $Q^* \in \partial C(\Omega)$. Hence from Yoshida [**17**, Theorem 3.3] we know that

$$-\infty < \mu_0(-H(C(\Omega),l;g^+)) \le \infty$$

and hence

$$-\infty \le \mu_0(H(C(\Omega),l;g^+)) < \infty.$$

Thus we obtain that if $l \ge 1$, then

$$\limsup_{r\to\infty} r^{-\alpha(\Omega,k_{l+1})} N(H(C(\Omega),l;g^+))(r) \le 0. \tag{5.24}$$

Since

$$\mu_0(H(C(\Omega),0;g^+)) = 0$$

(see [**18**, Lemma 3]), this and (5.24) also give (5.23) for any non-negative integer l.

PROOF OF THEOREM 2. Take a positive continuous function $\varphi(t)(t \ge 1)$ such that

$$\varphi(1) = 2^{-\alpha(\Omega,1)} \tag{5.25}$$

and

$$\varphi(t)\int_{\partial\Omega} |g(t,\Xi)|\, d\sigma_\Xi \le L_4 t^{-n} \quad (t > 1),$$

where

$$L_4 = 2^{-\alpha(\Omega,1)} \int_{\partial\Omega} |g(1,\Xi)|\, d\sigma_\Xi.$$

For any fixed $P = (r,\Theta) \in C(\Omega)$, choose a number R, $R > \max(1,4r)$. Then we see from Lemma 4 that

$$\begin{aligned}\int_{S(\Omega;(R,\infty))} &|g(Q)K(C(\Omega),\varphi)(P,Q)|\, d\sigma_Q \\ &\le L_2 \int_R^\infty \left(\int_{\partial\Omega} |g(t,\Xi)|\, d\sigma_\Xi\right)\varphi(t)t^{n-2}\, dt < L_2L_4 \int_R^\infty t^{-2}\, dt < \infty.\end{aligned} \tag{5.26}$$

It is evident that

$$\int_{S(\Omega;(0,R))} |g(Q)K(C(\Omega),\varphi)(P,Q)|\,d\sigma_Q < \infty.$$

These give that

$$\int_{S(\Omega)} |g(Q)K(C(\Omega),\varphi)(P,Q)|\,d\sigma_Q < \infty.$$

To see that $H(C(\Omega),\varphi;g)(P)$ is harmonic in $C(\Omega)$, we remark that $H(C(\Omega),\varphi;g)(P)$ satisfies the locally mean-valued property by Fubini's theorem.

Finally we shall show

$$\lim_{P\in C(\Omega),P\to Q^*} H(C(\Omega),\varphi;g)(P) = g(Q^*) \tag{5.27}$$

for any $Q^* \in \partial C(\Omega)$. Put

$$W(P,Q) = W(C(\Omega),\varphi)(P,Q)$$

in Lemma 5, which is a locally integrable function of $\partial C(\Omega)$ for any fixed $P \in C(\Omega)$. Then we can see from (5.26) in the same way as in the proof of Theorem 1 that both (I) and (II) are satisfied. Thus Lemma 5 applied to $g(Q)$ and $-g(Q)$ gives (5.27).

Proof of Theorem 3. From Theorem 1, we have the solution $H(C(\Omega),l;g)(P)$ of the Dirichlet problem on $C(\Omega)$ with g satisfying (3.3). Consider the function $h - H(C(\Omega),l;g)$. Then it follows that this is harmonic in $C(\Omega)$ and vanishes continuously on $\partial C(\Omega)$. Since

$$0 \le \{h - H(C(\Omega),l;g)\}^+(P) \le h^+(P) + \{H(C(\Omega),l;g)\}^-(P)$$

for any $P \in C(\Omega)$ and

$$\mu_l(\{H(C(\Omega),l;g)\}^-) = 0$$

from (3.3), (4.1) gives that

$$\mu_p(\{h - H(C(\Omega),l;g)\}^+) = 0.$$

From Theorem B we see that

$$h(P) - H(C(\Omega),l;g)(P) = \sum_{k\in I(\Omega,k_{p+1})} A_k r^{\alpha(\Omega,k)} f_k^{\Omega}(\Theta)$$

for every $P = (r,\Theta) \in \Omega$, where $A_k (k = 1,2,3,\dots,k_{p+1}-1)$ is a constant. Thus we obtain the conclusion of Theorem 3.

Proof of Corollary 4. From Theorem 3, we obtain

$$h(P) = H(\boldsymbol{T}_n,\rho;g)(P) \quad (\text{resp. } H(\boldsymbol{T}_n,\rho-1;g)(P)) + \textstyle\prod_1(r,\Theta) \quad (P = (r,\Theta) \in \boldsymbol{T}_n),$$

where

$$\prod_1(r,\Theta)=\textstyle\sum_{k\in I(\Omega,k_{p+1})} A_k r^{\alpha(\Omega,k)} f_k^{\Omega}(\Theta) \quad (\Omega=S_+^{n-1}).$$

If we extend $\prod_1$ to a harmonic function $\prod_2$ on $\boldsymbol{R}^n$ by defining

$$\prod_2(r,\Theta)=\begin{cases}\prod_1(r,\Theta) & ((r,\Theta)\in \boldsymbol{T}_n)\\ -\prod_1(r,-\Theta) & ((r,\Theta)\in -\boldsymbol{T}_n=\{(X,-y)\in\boldsymbol{R}^n;(X,y)\in\boldsymbol{T}_n\})\end{cases}$$

and observe

$$r^{-\rho-1}M(\textstyle\prod_2^+)(r)\to 0(r\to\infty),\quad M(\textstyle\prod_2^+)(r)=\displaystyle\int_{\boldsymbol{S}^{n-1}}\textstyle\prod_2^+(r,\Theta)\,d\sigma_\Theta,$$

from (2.7), we know from a result of Brelot [**3**, Appendix, § 26] that $\prod_2$ is a harmonic polynomial on $\boldsymbol{R}^n$ of degree less than $\rho+1$. From the fact $\prod_2(r,\Theta)=-\prod_2(r,-\Theta)$, we can write $\prod_2=y\prod$, where $\prod$ is a polynomial of degree less than ρ and even with respect to y.

Proof of Theorem 4. Put

$$\limsup_{r\to\infty}\frac{\log N(h^+)(r)}{\log r}=\gamma.$$

Take a positive integer p_0 satisfying $\alpha(\Omega,k_{p_0+1})>\gamma$ and put $p=\max(l,p_0)$. Since

$$0\le\{h-H(C(\Omega),l;h)\}^+(P)\le h^+(P)+\{H(C(\Omega),l;h)\}^-(P),$$

we have $\mu_p(\{h-H(C(\Omega),l;h)\}^+)=0$, which with Theorem 3 gives the conclusion.

References

[1] D. H. Armitage, Representations of harmonic functions in half-spaces, Proc. London Math. Soc. (3) **38** (1979), 53–71.
[2] V. S. Azarin, Generalization of a theorem of Hayman on subharmonic functions in an *m*-dimensional cone, Mat. Sb. **66** (108) (1965), 248–264; Amer. Math. Soc. Translation (2) **80** (1969), 119–138.
[3] M. Brelot, Elements de la théorie classique du potentiel, Centre de Documentation Universitaire, Paris, 1965
[4] S. Y. Cheng and P. Li, Heat kernel estimates and lower bound of eigenvalues, Comment. Math. Helvetici **56** (1981), 327–338.
[5] R. Courant and D. Hilbert, Methods of mathematical physics, Vol. **1** Interscience Publishers, New York, 1953.
[6] B. E. J. Dahlberg, Estimates of harmonic measure, Arch. Rational Mech. Anal. **65** (1977), 257–288.
[7] M. Essén and J. L. Lewis, The generalized Ahlfors-Heins theorems in certain d-dimensional cones, Math. Scand. **33** (1973), 111–129.
[8] M. Finkelstein and S. Scheinberg, Kernels for solving problems of Dirichlet type in a half-plane, Advances in Math. **18** (1975), 108–113.
[9] S. J. Gardiner, The Dirichlet and Neumann problems for harmonic functions in half-spaces, J. London Math. Soc.(2) **24** (1981), 502–512.
[10] D. Gilbarg and N. S. Trudinger, Elliptic partial differential equations of second order, Springer Verlag, Berlin, 1977.
[11] W. K. Hayman and P. B. Kennedy, Subharmonic Functions, Vol. **1** Academic Press, London, 1976.
[12] S. Helgason, Differential geometry and symmetric spaces, Academic Press, London, 1962.
[13] L. L. Helms, Introduction to Potential Theory, Wiley-Interscience, New York, 1969.

[14] S. Minakshisundaram and A. Pleijel, Some properties of the eigenfunctions of the Laplace-operator on Riemannian manifolds, Canad. J. Math. **1** (1949), 242–256.

[15] R. Nevanlinna, Über eine Erweiterung des Poissonschen Integrals, Ann. Acad. Sci. Fenn. Ser. A **24** (4) (1925), 1–15.

[16] D. Siegel, The Dirichlet problem in a half-spaces and a new Phragmén-Lindelöf principle, Maximum Principles and Eigenvalue Problems in Partial Differential Equations, e.d. P. W. Schaefer, Pitman, New York, 1988.

[17] H. Yoshida, Nevanlinna norm of a subharmonic function on a cone or on a cylinder, Proc. London Math. Soc. (3) **54** (1987), 267–299.

[18] H. Yoshida, Harmonic majorization of a subharmonic function on a cone or on a cylinder, Pacific J. Math. **148** (1991), 369–395.

[19] H. Yoshida, A type of uniqueness for the Dirichlet problem on a half-space with continuous data, Pacific J. Math. **172** (1996), 591–609.

[20] H. Yoshida and I. Miyamoto, Harmonic functions in a cone which vanish on the boundary, Math. Nachr. (to appear).

Hidenobu Yoshida and Ikuko Miyamoto

Department of Mathematics and Informatics
Faculty of Science
Chiba University
Chiba 263
Japan