J. Math. Soc. Japan
Vol. 50, No. 1, 1998

On commutativity of diagrams of type II_1 factors

By Jerzy WIERZBICKI*

(Received Nov. 21, 1994)
(Revised Oct. 30, 1995)

1. Introduction

The subject of our study is quadruple of von Neumann algebras with inclusion relations as indicated in the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$. In addition, we assume that $R \not\subset Q$ and $Q \not\subset R$. When the von Neumann algebra K is equipped with a finite trace τ and E_Q^K, E_R^K, and E_S^K are τ-preserving conditional expectations of K onto Q, R and S, respectively, then a special situation may occur:

$$E_Q^K E_R^K = E_R^K E_Q^K = E_S^K.$$

Then the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is called a commuting square. In the special case, when $S = \boldsymbol{C}$, the subalgebras Q and R were called orthogonal by S. Popa ([**P1**]). This case, in the classical probability theory, corresponds to the condition of independence of two σ-fields.

Such diagrams were first introduced and investigated by S. Popa (cf. [**P1**], [**P2**]) and, at present, this concept is linked in a natural way with numerous problems of subfactor theory. See, for example, [**GHD**], [**K**], [**P3**], [**P4**], [**P5**], [**PP1**], [**We**], [**S**], [**Su**], [**SW**], [**WW**], [**Wi**].

Everywhere in this work, the von Neumann algebras, which form such a diagram are type II_1 factors. This situation was studied by S. Popa in [**P3**] and by T. Sano and Y. Watatani in [**SW**].

In Sections 3 and 4 we discuss sufficient conditions for the commutativity of a diagram. It may seem a little surprising that for certain inclusions, say $S \subset Q$, the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ of type II_1 factors with $[K:S] < \infty$ and $S' \cap K = \boldsymbol{C}$ must be a commuting square. We will show several results of this kind.

The notion of commuting square of type II_1 factors is strictly connected to the concept of so called co-commuting square of type II_1 factors, which will be described in

---

1991 *Mathematics Subject Classification*. Primary 46L37.

*Department of Mathematics, Faculty of Science, Hokkaido University, Sapporo 060, Japan.

Section 2. There are plenty of diagrams $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, which are completely irregular in the sense that $Q \vee R = K$, $Q \cap R = S$ and they are neither commuting nor co-commuting squares. We can construct them by "tensoring" degenerate (in the sense of S. Popa's [**P5**]) diagrams. However, when the index $[Q : S]$ is "small" or the second relative commutant of the inclusion "$S \subset Q$" is only 2-dimensional, then the situation is different. For example, we can prove such a result: (See Section 4)

Let $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \boldsymbol{C}$ *and* $[K : S] < \infty$. *If* $[Q : S] = 4\cos^2(\pi/n)$, *for a prime number* n, *then the diagram* $\mathscr{D}$ *is a commuting square.*

To our knowledge this is the first attempt of using primeness of n in the sequence (of index values) $\{\cos^2(\pi/n)\}_{n \geq 3}$. We believe that behind this there must be some interesting connections to the algebraic number theory.

Also, imposing some conditions on the second relative commutant gives the same effect. (See Section 3)

Let $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \boldsymbol{C}$ *and* $[K : S] < \infty$. *If the second relative commutant of the inclusion* $S \subset Q$ *is two dimensional and* $[Q : S] > [R : S]$, *then the diagram* $\mathscr{D}$ *is a commuting square.*

A. Ocneanu's Fourier transform seems to be a very interesting subject to study for its own sake. In this work, however, we use it only as a tool or a language helpful in investigation of the commutativity of a diagram. The crucial observation, stated in Theorem 2.9, is that Ocneanu's convolution $E_Q^K * E_R^K$ is a scalar multiple of a projection. Then, we can use results from [**SW**] or [**P3**] to show that having the "right angle" between two subfactors is not so special in certain situations. T. Sano and Y. Watatani ([**SW**]) showed that the commutativity of a diagram is equivalent to the co-commutativity of another related diagram. Therefore, sufficient conditions for a diagram to be, for example, a co-commuting square can immediately be reformulated in terms of a commuting square. We will try then not to repeat ourselves and we will keep rather to the co-commuting square terminology.

Consequences of the commutativity of a diagram of type II_1 factors were discussed in such papers as [**P3**], [**K**], [**WW**] or [**Wi**]. In Section 5 we will decompose certain von Neumann algebras, which are naturally associated with a commuting square of type II_1 factors.

2. Fourier transform and convolution

We recall here Ocneanu's Fourier transform and convolution in the third relative commutant. We prove also a few properties which will be useful later. We believe that most of them are known to other persons working in this area.

If τ is a faithful normal finite trace on a von Neumann algebra B and A is its von Neumann subalgebra $A \subset B$, then ${}^{\tau}E_A^B$ will be the τ-preserving normal conditional expectation of B onto A. When there is no ambiguity about the trace we will write it simply E_A^B or E_A. It is convenient to keep in mind a lemma which is an immediate consequence of [**P2**, Lemma 1.2.2.] and [**PP1**, Corollary 4.5].

Lemma 2.1. *Let $K \subset A \subset B \subset L$ be all type II_1 factors and let the inclusion $K \subset L$ be extremal with $[L:K] < \infty$. If τ and τ' are the traces on L and on K', respectively, then for any $x \in K' \cap B$,*

$$ {}^{\tau}E_A^B(x) = {}^{\tau}E_{K'\cap A}^{K'\cap B}(x) = {}^{\tau'}E_{K'\cap A}^{K'\cap B}(x) $$

and for any $x \in A' \cap L$,

$$ {}^{\tau'}E_{B'}^{A'}(x) = {}^{\tau'}E_{B'\cap L}^{A'\cap L}(x) = {}^{\tau}E_{B'\cap L}^{A'\cap L}(x). $$

We will often use this lemma without referring to it.

Let $S \subset K$ be an irreducible inclusion of type II_1 factors with $[K:S] = \gamma^{-1} < \infty$. Consider the corresponding Jones tower

$$ S \subset K \subset^{e_S} L \subset^{e_K} L_1. $$

From [**PP2**] the inclusion $S \subset L_1$ is extremal, and so Lemma 2.1 can be applied to any intermediate subfactors between S and L_1.

Definition 2.2. The mapping $\mathscr{F} : K' \cap L_1 \to S' \cap L$ given by

$$ \mathscr{F}(x) = \gamma^{-3/2} E_L(x e_S e_K) = \gamma^{-3/2} E_{S'\cap L}(x e_S e_K) $$

will be called the Fourier transform of $K' \cap L_1$ into $S' \cap L$. By the inverse Fourier transform we mean the mapping $\mathscr{F}^* : S' \cap L \to K' \cap L_1$ defined by

$$ \mathscr{F}^*(y) = \gamma^{-3/2} E_{K'\cap L_1}(y e_K e_S). $$

We will show that the name "inverse Fourier transform" is justified. The algebra $S' \cap L_1$ has the canonical Hilbert space structure determined by the trace $\tau_{L_1} = \tau_{S'\cap L_1}$. Let us consider $S' \cap L$ and $K' \cap L_1$ as its Hilbert subspaces.

Proposition 2.3. *The mappings $\mathscr{F}$ and $\mathscr{F}^*$ are Hilbert space isomorphisms between $S' \cap L$ and $K' \cap L_1$. Moreover,*

$$ \mathscr{F}\mathscr{F}^* = Id_{|S'\cap L} \quad \text{and} \quad \mathscr{F}^*\mathscr{F} = Id_{|K'\cap L_1}. $$

In [**W**] was shown a property of a Pimsner-Popa basis, which is very useful in proof of the above proposition.

LEMMA 2.4. *Let $N \subset M$ be a finite index extremal inclusion of type II_1 factors. If $\{m_i\}_i$ is a Pimsner-Popa basis of N' over M', then the trace preserving conditional expectation E_N^M of M onto N is expressed as follows*:

$$E_N^M(x) = [M:N]^{-1} \sum_i m_i x m_i^*.$$

PROOF OF PROPOSITION 2.3. First, we show that $\mathscr{F}\mathscr{F}^* = Id_{|S'\cap L}$. Let $\{m_i\}_{i=0}^n$ be a Pimsner-Popa basis of K over S such that $m_0 = 1$. The preceding lemma gives: $E_{K'}^{S'}(y) = \gamma \sum_i m_i y m_i^*$ for the trace preserving conditional expectation $E_{K'}^{S'}$. From this and by Lemma 2.1 we have

$${}^{\tau'}E_{K'\cap L_1}^{S'\cap L_1}(ye_K e_S) = E_{K'}^{S'}(ye_K e_S) = \gamma \sum_i m_i y e_K e_S m_i^*,$$

for $y \in S' \cap L$. Hence

$$\begin{aligned}\gamma^3 \mathscr{F}\mathscr{F}^*(y) &= \gamma E_L\left(\sum_i m_i y e_K e_S m_i^* e_S e_K\right) = \gamma E_L\left(\sum_i m_i y e_K E_S(m_i^* m_0) e_S e_K\right)\\ &= \gamma E_L(ye_K e_S e_K) = \gamma^2 E_L(ye_K) = \gamma^3 y.\end{aligned}$$

We show now that $\mathscr{F}^*$ is a contraction. For $y \in S' \cap L$,

$$\begin{aligned}\|\mathscr{F}^*(y)\|_2^2 &= \gamma^{-3}\tau(E_{K'\cap L_1}(ye_K e_S)E_{K'\cap L_1}(ye_K e_S)^*)\\ &= \gamma^{-1}\sum_{i,j}\tau(m_i y e_K e_S m_i^* m_j e_S e_K y^* m_j^*)\\ &= \gamma^{-1}\sum_{i<n}\tau(m_i y e_K e_S e_K y^* m_i^*) + \gamma^{-1}\tau(m_n y e_K p e_S e_K y^* m_n^*),\end{aligned}$$

where $p = E_S(m_n^* m_n)$ is a projection, cf. [**PP1**, Proposition 1.3]. Since $[p, e_K] = 0$,

$$\begin{aligned}\|\mathscr{F}^*(y)\|_2^2 &= \sum_{i<n}\tau(m_i y e_K y^* m_i^*) + \tau(m_n y e_K p e_K y^* m_n^*) \le \sum_{i\le n}\tau(m_i y e_K y^* m_i^*)\\ &= \sum_{i\le n}\tau(E_L(m_i y e_K y^* m_i^*)) = \gamma\tau\left(\sum_{i\le n} m_i y y^* m_i^*\right)\\ &= \tau(E_{K'}^{S'}(yy^*)) = \tau(yy^*) = \|y\|_2^2.\end{aligned}$$

Similarly, using a Pimsner-Popa basis of L' over L_1', we obtain that $\mathscr{F}$ is a contraction too. So $\mathscr{F}$ and $\mathscr{F}^*$ are isometries and the standard polarization argument completes the proof. Q.E.D.

We will always use notation $\langle K, S, e\rangle$, for an algebraic basic construction ([**PP2**]) of a finite index inclusion $S \subset K$ of type II_1 factors, where e is the corresponding Jones projection, i.e. $exe = E_S^K(x)e$, for $x \in K$ and $\langle K, S, e\rangle$ is generated, as a von Neumann algebra, by K and e. If $\langle K, S, e\rangle$ is represented on $L^2(K, \tau)$ and e is just an extension of

E_S^K, then, as in [**J**], we will write simply $\langle K, e\rangle$. Let us consider an intermediate subfactor Q between S and K, with indices $[Q:S] = \alpha^{-1}$ and $[K:Q] = \lambda^{-1}$. By [**J**] we know that $\alpha\lambda = \gamma$. Let $M = \langle K, e_Q\rangle$ and $M_1 = \langle L, e_M\rangle$ be Jones' basic constructions. We have

$$S \subset Q \subset K \subset^{e_Q} M \subset L \subset^{e_M} M_1 \subset L_1,$$

where, as before, $L = \langle K, e_S\rangle$ and $L_1 = \langle L, e_K\rangle$. Obviously, $e_Q \in S' \cap L$ and $e_M \in K' \cap L_1$. Also $e_Q e_S = e_S$ and $e_M e_K = e_K$.

PROPOSITION 2.5.

$$\mathscr{F}(\alpha^{-1/2} e_M) = \lambda^{-1/2} e_Q \quad \textit{and} \quad \mathscr{F}^*(\lambda^{-1/2} e_Q) = \alpha^{-1/2} e_M.$$

PROOF. (See [**B**] for another proof of this property.) We will show only the second equality. First, we remark that $e_Q e_K e_Q = \lambda e_M e_Q$. Indeed, let us represent all these operators on $L^2(L,\tau)$ and let the vector sign denote the canonical embedding of L in $L^2(L,\tau)$. Then for any $\overrightarrow{ae_S b} \in L^2(L,\tau)$, $a, b \in K$ we have:

$$\begin{aligned} e_Q e_K e_Q \overrightarrow{ae_S b} &= \overrightarrow{e_Q E_K(e_Q a e_S b)} = \overrightarrow{e_Q E_K(e_Q a e_Q e_S b)} \\ &= \overrightarrow{e_Q E_Q(a) E_K(e_S) b} = \gamma \overrightarrow{e_Q E_Q(a) b} \\ &= \lambda \overrightarrow{E_Q(a)\alpha e_Q b} = \lambda \overrightarrow{E_M(E_Q(a) e_S b)} = \lambda e_M e_Q \overrightarrow{ae_S b}. \end{aligned}$$

In the last line we used the property $E_M(e_S) = \alpha e_Q$, cf. [**SW**, Lemma 7.1].

Now,

$$\begin{aligned} \mathscr{F}^*(e_Q) &= \gamma^{-3/2} E_{K' \cap L_1}(e_Q e_K e_S) = \gamma^{-3/2} E_{K' \cap L_1}(\lambda e_M e_S) \\ &= \gamma^{-3/2} \lambda e_M E_{K' \cap L_1}(e_S) = \sqrt{\frac{\lambda}{\alpha}} e_M, \end{aligned}$$

where we used [**PP1**, Corollary 4.5]. Q.E.D.

DEFINITION 2.6. For $x, y \in S' \cap L$ we set

$$x * y = \mathscr{F}(\mathscr{F}^*(x)\mathscr{F}^*(y)).$$

Similarly, for $x, y \in K' \cap L_1$

$$x \hat{*} y = \mathscr{F}^*(\mathscr{F}(x)\mathscr{F}(y)).$$

Following [**O**], we call the operation "$*$" or "$\hat{*}$" convolution. The convolution "$\hat{*}$" in $K' \cap L_1$ should not be confused with the convolution "$*$", which we define by building up another basic construction, say $L_2 = \langle L_1, e_L\rangle$, and putting $x * y = \mathscr{F}_1(\mathscr{F}_1^*(x)\mathscr{F}_1^*(y))$, where $\mathscr{F}_1$ and $\mathscr{F}_1^*$ are the shifted Fourier and inverse Fourier transforms: $\mathscr{F}_1^* : K' \cap L_1 \to L' \cap L_2$, $\mathscr{F}_1^*(x) = \gamma^{-3/2} E_{L'}(x e_L e_K)$ and $\mathscr{F}_1 : L' \cap L_2 \to K' \cap L_1$, $\mathscr{F}_1(y) = \gamma^{-3/2} E_{L_1}(y e_K e_L)$. Similarly, we may define "$\hat{*}$" in $S' \cap L$.

PROPOSITION 2.7.

$$\textit{For } x \in S' \cap L, \quad \sqrt{\frac{\alpha}{\lambda}} e_Q * x = E_M(x).$$

$$\textit{For } y \in K' \cap L_1, \quad \sqrt{\frac{\lambda}{\alpha}} e_M \hat{*} y = E_{M'}(y) = E_{M' \cap L_1}(y).$$

PROOF. Let $\tilde{x} = \mathscr{F}^*(x) \in K' \cap L_1$. By [**PP1**, Lemma 1.2] and Proposition 2.5,

$$\begin{aligned} E_L(e_M \tilde{x} e_S e_K) &= \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_K) = \gamma^{-1} E_L(e_M E_L(\tilde{x} e_S e_K) e_M e_K) \\ &= \gamma^{-1} E_L(E_M(\tilde{x} e_S e_K) e_K) = E_M(\tilde{x} e_S e_K) = \gamma^{3/2} E_M(\mathscr{F}(\tilde{x})) = \gamma^{3/2} E_M(x). \end{aligned}$$

Since $S' = \langle K', L', e_S \rangle$ and $Q' = \langle K', M', e_Q \rangle$, proof of the other equality is almost the same. Q.E.D.

LEMMA 2.8. *Let $S \subset Q \subset K$ be type II_1 factors, $[K : S] < \infty$, $M = \langle K, Q, e_Q \rangle$ and $S' \cap Q = \boldsymbol{C}$. Then*

(1) *e_Q is a minimal projection in $S' \cap M$.*

(2) *e_Q is central in $S' \cap M$, iff $S' \cap K = \boldsymbol{C}$.*

PROOF. (1) Suppose $e \leq e_Q$ for some non-zero projection $e \in P(S' \cap M)$. If $f = \lambda^{-1} E_K(e) \in S' \cap K$ $(\lambda = [K : Q]^{-1})$, then

$$e = e e_Q = \lambda^{-1} E_K(e e_Q) e_Q = \lambda^{-1} E_K(e) e_Q = f e_Q.$$

Hence $[f, e_Q] = 0$; therefore $f \in S' \cap Q = \boldsymbol{C}$, which implies $e = e_Q$.

(2) "$\Rightarrow$" If $a \in S' \cap K$, then $a e_Q = e_Q a$ and hence $a \in S' \cap Q = \boldsymbol{C}$.

"$\Leftarrow$" Let $c = C(e_Q, S' \cap M)$ be the central support of e_Q in $S' \cap M$. For any projection $f \in (S' \cap M)_c$ satisfying $\tau(f) = \lambda$, there is a unitary $u \in S' \cap M$ such that $f = u e_Q u^*$. Hence, by [**PP1**, Lemma 1.2] and Lemma 2.1

$$u e_Q = \lambda^{-1} E_K(u e_Q) e_Q = \lambda^{-1} \tau(u e_Q) e_Q.$$

Thus $f = \lambda^{-2} |\tau(u e_Q)|^2 e_Q$. This implies $c = e_Q$. Q.E.D.

From now on, we consider two intermediate subfactors Q, R between S and K: $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$. We always consider such diagrams that $Q \not\subset R$ and $R \not\subset Q$, but we do not assume that always $K = Q \vee R$ and $S = Q \cap R$. Throughout the rest of the paper we keep to the notation preceding Proposition 2.5 and we add the following one: $N = \langle K, e_R \rangle$ and $N_1 = \langle L, e_N \rangle$ will be the Jones basic constructions. For brevity we assume that $\eta = [K : R]^{-1}$ and $\beta = [R : S]^{-1}$. This way we obtain the following diagram of inclusions:

$$\begin{matrix} & & & & M_1 & \subset & L_1 \\ & & & & \cup & & \cup \\ & & N & \subset & L & \subset & N_1 \\ & & \cup & & \cup & & \\ Q & \subset & K & \subset & M & & \\ \cup & & \cup & & & & \\ S & \subset & R & & & & \end{matrix}$$

J_K (resp. J_L) will be modular conjugation on $L^2(K,\tau)$ (resp. $L^2(L,\tau)$). We always assume that the inclusion $S \subset K$ is irreducible.

THEOREM 2.9. *There exists a projection p in $R' \cap M$ such that:*

(1) $e_Q * e_R = \tau(e_Q e_R)\gamma^{-1/2} p = \sqrt{\frac{\lambda}{\alpha}} E_M(e_R) = \sqrt{\frac{\eta}{\beta}} E_{R'}(e_Q)$,

(2) $e_R * e_Q = J_K(e_Q * e_R)J_K = \tau(e_Q e_R)\gamma^{-1/2} J_K p J_K = \sqrt{\frac{\lambda}{\alpha}} E_{Q'}(e_R) = \sqrt{\frac{\eta}{\beta}} E_N(e_Q)$,

(3) *p is a minimal and central projection in $R' \cap M$ and $p \geq e_R \vee e_Q$,*

(4) *$\hat{p} = J_K p J_K$ is a minimal and central projection in $Q' \cap N$ and $\hat{p} \geq e_R \vee e_Q$.*

PROOF. Let $p = \bigwedge\{f \mid f \geq e_Q, f \in P(R' \cap M)\}$. Suppose p_1, $0 \neq p_1 \leq p$, is another projection in $R' \cap M$. By Lemma 2.8, $e_Q p_1 = 0$ or $e_Q p_1 = e_Q$. This implies $e_Q \leq p - p_1 \leq p$ or $e_Q \leq p_1 \leq p$. Hence, p is minimal in $R' \cap M$. Let $E_{R'}(e_Q) = \sum_{1 \leq i \leq n} \alpha_i p_i$, $\alpha_i > 0$ be the spectral decomposition. Then, by $e_Q \leq p$, it follows that $\sum \alpha_i p_i \leq p$ and, in consequence, $\sum p_i \leq p$. Therefore, by minimality of p, we see that $n = 1$ and $E_{R'}(e_Q) = \theta p$, for some scalar θ. From this and Lemma 2.8 we see that p is also central in $R' \cap M$.

Identically, we obtain $E_{Q'}(e_R) = \kappa g$ for a minimal and central projection g in $Q' \cap N$ and a scalar κ. It is easy to check that

$$\forall x \in S' \cap L, \quad E_{N \cap Q'}(x) = J_K E_{R' \cap M}(J_K x J_K) J_K.$$

Using this we see that for $\hat{g} = J_K g J_K$,

$$\kappa \hat{g} = J_K E_{Q' \cap N}(e_R) J_K = E_{R' \cap M}(J_K e_R J_K) = E_M(e_R) = \sqrt{\frac{\alpha}{\lambda}} e_Q * e_R.$$

Also, $\hat{g}$ is a minimal and central projection in $R' \cap M$. Since $e_R \leq g$, we have $e_R \leq \hat{g}$. Hence

$$0 \neq e_S \leq e_Q \wedge e_R \leq p \wedge \hat{g},$$

which implies $p = \hat{g}$. Computation of coefficients θ and κ ends the proof. For example, let us multiply the equality $\kappa p = E_M(e_R)$ by e_Q and take the trace: $\kappa e_Q = E_M(e_R e_Q)$, and so $\kappa = \lambda^{-1}\tau(e_R e_Q)$. Q.E.D.

Let us write $[x]$ for the range projection of an operator x. Similarly as above, we obtain the following corollary.

COROLLARY 2.10. *$[e_M \hat{*} e_N]$ is a minimal and central projection in $M' \cap N_1$, $[e_M \hat{*} e_N] \geq e_M \vee e_N$ and*

$$e_M \hat{*} e_N = J_L(e_N \hat{*} e_M)J_L = \tau(e_M e_N)\gamma^{-1/2}[e_M \hat{*} e_N] = \sqrt{\frac{\alpha}{\lambda}} E_{M'}(e_N) = \sqrt{\frac{\beta}{\eta}} E_{N_1}(e_M).$$

REMARK. From the above theorem and corollary, it is easy to see the following property of the "shifted" convolution "$*$" in $K' \cap L_1 : e_N * e_M = e_M \hat{*} e_N$. Then it is natural to ask, whether this relation is satisfied for arbitrary operators in $K' \cap L_1$. Right now, it is not clear for us.

For later convenience, we carry out some computations. From [**J**] we know that $\lambda/\beta = \eta/\alpha$. We denote this quotient by δ.

PROPOSITION 2.11.

$$\frac{\tau([e_Q * e_R])}{\tau([e_M \hat{*} e_N])} = \frac{\tau(e_Q e_R)}{\tau(e_M e_N)} = \delta.$$

PROOF. From Propositions 2.5 and 2.3 and Theorem 2.9 we have:

$$\begin{aligned}\tau(e_M e_N) &= \|e_M e_N\|_2^2 = \|\mathscr{F}(e_M e_N)\|_2^2 = \frac{\alpha\beta}{\lambda\eta}\|e_Q * e_R\|_2^2 \\ &= \frac{\alpha\beta}{\lambda\eta}\,\tau(e_Q e_R)^2 \gamma^{-1}\,\frac{\lambda\eta}{\tau(e_Q e_R)} = \delta^{-1}\tau(e_Q e_R).\end{aligned}$$ Q.E.D.

T. Sano and Y. Watatani, cf. [**SW**], introduced notion of angles between subfactors Q and R. The diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, with $K = Q \vee R$ and $\text{Op-ang}_K(Q, R) = \{\pi/2\}$ will be called a co-commuting square. (Cf. [**K**], [**WW**], [**Wi**].) The following characterization follows directly from [**SW**, Proposition 4.2, Corollary 4.1 and Lemma 7.2] and it is valid even if the inclusion $S \subset K$ (or equivalently $K \subset L$) is not irreducible.

PROPOSITION 2.12. *The following conditions are equivalent*:

(1) *Diagram* $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *of finite factors is a co-commuting square.*

(2) *Diagram of their commutants* $\begin{matrix} R' & \subset & S' \\ \cup & & \cup \\ K' & \subset & Q' \end{matrix}$ *forms a commuting square.*

(3) *Diagram* $\begin{matrix} M & \subset & L \\ \cup & & \cup \\ K & \subset & N \end{matrix}$ *is a commuting square.*

(4) *Diagram* $\begin{matrix} M_1 & \subset & L_1 \\ \cup & & \cup \\ L & \subset & N_1 \end{matrix}$ *is a co-commuting square.*

(5) $E_M(e_R) \in \boldsymbol{C}$.

As an application of Theorem 2.9 we give a few equivalent conditions on the commutativity. (Remember that $K' \cap L = \boldsymbol{C}$ and $[L : K] = [K : S] = \gamma^{-1} < \infty$.)

COROLLARY 2.13. *The following conditions are equivalent*:

(1) $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$ *is a co-commuting square.*

(2) $N' \cap M_1 = \boldsymbol{C}$.

(3) $\tau(e_N e_M) = \alpha\beta$.

(4) $e_N \hat{*} e_M$ *is a scalar.*

(5) $e_N \hat{*} e_M = e_M \hat{*} e_N$ *and* $N \vee M = L$.

PROOF. "(2) $\Rightarrow$ (1)" follows from [**SW**, Lemma 7.2].

"(1) $\Rightarrow$ (2)" If (1) is satisfied then, from [**SW**, Lemma 7.2], $E_{M_1}(e_N) \in \boldsymbol{C}$. Hence by Theorem 2.9 the identity is minimal in $N' \cap M_1$.

"(1) $\Leftrightarrow$ (3)" Since, by computation in 2.11, (3) is equivalent to $e_Q e_R = e_S$, this is an immediate consequence of [**SW**, Corollary 4.1].

"(2) $\Leftrightarrow$ (4)" is obvious because, by Corollary 2.10, $e_M \hat{*} e_N$ is a scalar multiple of a minimal projection in $M' \cap N_1$.

"(1) $\Leftrightarrow$ (5)" is a direct consequence of Proposition 2.5. Q.E.D.

By the remark following Corollary 2.10, we can replace the symbol "$\hat{*}$" by "$*$", in (4) and (5) of the above corollary.

EXAMPLE. Let μ be an outer action of a finite group G on a type II_1 factor S. If A and B are subgroups of G with trivial intersection $A \cap B = \{e\}$, then we can consider the diagram of crossed products $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, where $Q = S \rtimes_\mu A$, $R = S \rtimes_\mu B$ and $K = S \rtimes_\mu G$. The diagram of corresponding basic constructions, cf. [**SW**, Lemma 5.1], can be identified with the following one $\begin{matrix} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{matrix}$, where $M = (S \otimes L^\infty(G/A)) \rtimes_\mu G$, $N = (S \otimes L^\infty(G/B)) \rtimes_\mu G$, $L = (S \otimes L^\infty(G)) \rtimes_\mu G$. $L^\infty(G/A)$ (resp. $L^\infty(G/B)$) consists of functions constant on left cosets of A (resp. B) and the action μ is extended in an obvious way, cf. [**SW**]. If χ_D denotes the characteristic function of a subset $D \subset G$, then

$$e_Q \approx 1 \otimes \chi_A, \quad e_R \approx 1 \otimes \chi_B \quad \text{and} \quad e_S \approx 1 \otimes \chi_{\{e\}}.$$

One can easily verity that $R' \cap M$ is isomorphic to the subspace of $L^\infty(G)$, which consists of functions constant on double cosets BgA, $g \in G$. Similarly $Q' \cap N \approx L^\infty(A \backslash G/B)$ and we have

$$[e_Q * e_R] \approx \chi_{BA} \quad \text{and} \quad [e_R * e_Q] \approx \chi_{AB}.$$

Also, since $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square, $[e_N \hat{*} e_M] = [e_M \hat{*} e_N] = 1$.

3. Angles for subfactors with "small" second relative commutant

The following lemma may be considered as a generalization of Lemma 5.3 in [**SW**]. As always, we assume that $S' \cap K = \boldsymbol{C}$ and $[K : S] < \infty$.

LEMMA 3.1. *If the diagram* $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *is not a co-commuting square and* $\dim(Q' \cap M) = 2$, *then the only non-zero element of the spectrum* $\mathrm{Sp}(e_M e_N e_M - e_K)$ *is* $(1/(1-\lambda))((\tau(e_N e_M)/\alpha) - \lambda)$.

PROOF. Note that our assumption $\dim(Q' \cap M) = 2$ implies $Q \vee R = N \cap M = K$. Let us denote $E_0 = E_M E_N E_M$ and $E = E_0 - E_K$. It is sufficient to show (see [**SW**, Corollary 3.1]) that $E^2 = (1/(1-\lambda))((\tau(e_N e_M)/\alpha) - \lambda)E$. Since E has the bimodule property, by [**PP1**, Lemma 1.1] we need only to make sure that $E^2(e_S) = (1/(1-\lambda))((\tau(e_N e_M)/\alpha) - \lambda)E(e_S)$. By [**SW**, Lemma 7.1] and Theorem 2.9, for the projection $\hat{p} = [e_R * e_Q] \in Q' \cap N$ we have

$$E_0(e_S) = E_M E_N(\alpha e_Q) = \frac{\tau(e_Q e_R)}{\delta} E_M(\hat{p}).$$

Since $\dim(Q' \cap M) = 2$,

$$E_M(\hat{p}) = \frac{\tau(\hat{p} e_Q)}{\tau(e_Q)} e_Q + \frac{\tau(\hat{p}(1-e_Q))}{\tau(1-e_Q)}(1-e_Q) = e_Q + \frac{\tau(\hat{p}) - \lambda}{1-\lambda}(1-e_Q).$$

Therefore, setting

$$\theta = \tau(e_Q e_R)/\eta \quad \text{and} \quad \kappa = \frac{\lambda}{\eta}\frac{\eta - \tau(e_Q e_R)}{1-\lambda},$$

we get

$$E_0(e_S) = \alpha E_0(e_Q) = \alpha\theta e_Q + \alpha\kappa(1-e_Q).$$

Hence,

$$E_0^2(e_S) = E_0(\alpha\theta e_Q + \alpha\kappa(1-e_Q)) = (\theta-\kappa)E_0(e_S) + \alpha\kappa,$$

which gives

$$\begin{aligned} E^2(e_S) &= (E_0 - E_K)(E_0 - E_K)(e_S) = (E_0^2 - E_K)(e_S) = (\theta-\kappa)E_0(e_S) + \alpha\kappa - \gamma \\ &= (\theta-\kappa)E(e_S) + (\theta-\kappa)\gamma + \alpha\kappa - \gamma = (\theta-\kappa)E(e_S) \end{aligned}$$

and

$$\theta - \kappa = \frac{1}{1-\lambda}\left(\frac{\tau(e_Q e_R)}{\eta} - \lambda\right) = \frac{1}{1-\lambda}\left(\frac{\tau(e_N e_M)}{\alpha} - \lambda\right). \qquad \text{Q.E.D.}$$

By [**SW**, Corollary 4.1], the theorem stated in introduction is a reformulation of the following one.

THEOREM 3.2. *Let* $\mathscr{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be a diagram of type* II_1 *factors* ($[K:S] < \infty$, $S' \cap K = \boldsymbol{C}$) *and let* $[K:R] < [K:Q]$. *If* $\dim Q' \cap M = 2$ ($M = \langle K, Q, e_Q \rangle$), *then* $\mathscr{D}$ *is a co-commuting square.*

PROOF. Suppose that $\mathscr{D}$ is not a co-commuting square. From the preceding lemma, the spectrum $\mathrm{Sp}(e_M e_N e_M - e_K) - \{0\} = \{\sigma\}$, where $\sigma = (1/(1-\lambda))((\tau(e_N e_M)/\alpha) - \lambda)$.

Since

$$\begin{aligned}\mathrm{Sp}(e_N e_M e_N - e_K) - \{0\} &= \mathrm{Sp}(e_N e_M - e_K)(e_N e_M - e_K)^* - \{0\}\\ &= \mathrm{Sp}(e_N e_M - e_K)^*(e_N e_M - e_K) - \{0\} = \{\sigma\},\end{aligned}$$

it follows that

$$e_M e_N e_M = e_K + \sigma g, \quad e_N e_M e_N = e_K + \sigma f,$$

for some projections $g, f \in K' \cap L_1$. Then, since $\tau(e_M e_N) \neq \tau(e_K) = \gamma$ $(= [K:S]^{-1})$, we have $\tau(g) = (\tau(e_M e_N) - \gamma)(1/\sigma) = \alpha - \gamma$, but

$$f \leq e_N - e_K \quad \text{and} \quad \tau(f) = \tau(g),$$

and so $\alpha - \gamma \leq \beta - \gamma$, which gives a contradiction with $\lambda < \eta$. Q.E.D.

COROLLARY 3.3. *If* $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$, $\dim Q' \cap M = \dim R' \cap N = 2$ *and* $[K:Q] \neq [K:R]$, *then* $\mathscr{D}$ *is a co-commuting square.*

4. Angles for subfactors with "small" indices

We use here results of S. Popa, cf. [**P3**], to consider another sufficient conditions which lead to the same effect. Let us outline some of those results. We assume the following notations: $P_n(x)$ will be Jones' polynomials,

$$P_{-1} \equiv P_0 \equiv 1 \quad P_{n+1}(x) = P_n(x) - xP_{n-1}(x).$$

If $x = (4\cos^2(\pi/n))^{-1}$ $(n > 3)$ then

$$\Lambda_0(x) = \left\{ \frac{P_{k-1}(x)}{P_{k-2}(x)} \,\middle|\, 2 \leq k \leq n-2 \right\} = \left\{ x\,\frac{P_{k-1}(x)}{P_k(x)} \,\middle|\, 0 \leq k \leq n-4 \right\}$$

and $\Lambda_1(x) = \emptyset$. If $x \leq 1/4$ then

$$\Lambda_0(x) = \left\{ x\,\frac{P_{k-1}(x)}{P_k(x)} \,\middle|\, 0 \leq k \right\} \cup \left\{ \frac{P_{k+1}(x)}{P_k(x)} \,\middle|\, 0 \leq k \right\}$$

$$\text{and} \quad \Lambda_1(x) = \left[\frac{1 - \sqrt{1-4x}}{2}, \frac{1 + \sqrt{1-4x}}{2} \right].$$

Set $\Lambda(x) = \Lambda_0(x) \cup \Lambda_1(x)$ and $\tilde{\Lambda}(x) = \Lambda(x) \cap (x + \Lambda(x))$. If $B \subset A$ is a finite index inclusion of type II_1 factors, then

$$P(A,B) = \{e \in P(A) \,|\, E_B(e) \text{ is a scalar}\}$$

and we also denote: $\Lambda(A,B) = \{\tau(e) \,|\, e \in P(A,B)\} - \{0,1\}$.

THEOREM 4.1. ([**P3**, Theorems 5.1 and 5.2]) *For a finite index inclusion of type* II_1 *factors* $B \subset A$, $\Lambda(A,B) \subset \Lambda([A:B]^{-1})$. *Moreover, if* $[A:B] \leq 4$, *then* $\Lambda(A,B) = \Lambda([A:B]^{-1})$.

Next proposition is a slight modification of [**P3**, Proposition 4.5]. We will set

$$\mathscr{L}^s(B \subset A) = \{F \mid F \text{ is factor}, B \subset F \subset A \text{ and } [A:F]^{-1} = s\}.$$

PROPOSITION 4.2. *Let $D \subset B \subset A$ be a triple of type Π_1 factors with $[A:D] < \infty$ and $D' \cap B = \boldsymbol{C}$. If $f \in P(D' \cap A)$ and $\tau(f) \in \Lambda_0(t)$ $(t = [A:B]^{-1})$, then there exists $F \in \mathscr{L}^s(D \subset B)$, $s = t^{k+1}/P_k(t)^2$, where $k \geq 0$ is determined by $\tau(f) = tP_{k-1}(t)/P_k(t)$ or by $\tau(f) = P_{k+1}(t)/P_k(t)$.*

PROOF. We may assume that $\tau(f) = tP_{k-1}(t)/P_k(t)$, for some $k \geq 0$. Since evidently $f \in P(A, B)$, exactly as in [**P3**, Lemma 4.3] we consider Jones' tower

$$B \subset A \subset^{e_1} A_1 \subset^{e_2} A_2 \cdots \subset^{e_k} A_k$$

and projections $p_k = (1-e_1) \wedge \cdots \wedge (1-e_k) \in B' \cap A_k$ and $p_{0,k} = f \wedge p_k$. Then, cf. [**P3**, Proposition 4.5], there exists such subfactor $F \subset B$ that $(A_k)_{p_k} = \langle B_{p_k}, F_{p_k}, p_{0,k}\rangle$ and $[B:F]^{-1} = s = t^{k+1}/P_k(t)^2$. We will verify that the additional assumption $f \in D'$ yields $D \subset F$. Take any operator $x \in D$. Since $[x, f] = 0$ and $[x, p_k] = 0$, it follows that $[x, f \wedge p_k] = 0$, and hence $[xp_k, p_{0,k}] = 0$. This implies $xp_k \in F_{p_k}$, so that $x \in F$.

Q.E.D.

Let us now come back to diagrams of type Π_1 factors. We keep to all the notations from the preceding sections as well as to the assumptions $[K:S] < \infty$ and $S' \cap K = \boldsymbol{C}$. In particular, we recall that $[K:R] = \eta^{-1}$, $[R:S] = \beta^{-1}$, $[Q:S] = \alpha^{-1}$ and $[K:Q] = \lambda^{-1}$.

LEMMA 4.3.

$$\tau([e_Q * e_R]) \in [[\tilde{\Lambda}(\lambda) \cap \tilde{\Lambda}(\eta)] \cup \{1\}] \cap \delta[[\tilde{\Lambda}(\alpha) \cap \tilde{\Lambda}(\beta)] \cup \{1\}].$$

PROOF. Suppose that $p = [e_Q * e_R] \neq 1$. Since $E_K(p) = E_K E_{R' \cap M}(p) = \tau(p)$, $p \in P(M, K)$, and so, by Theorem 4.1, $\tau(p) \in \Lambda(M, K) \subset \Lambda(\lambda)$. By Theorem 2.9, $p - e_Q$ is a projection in $S' \cap M$ and the same argument gives $\tau(p) - \lambda \in \Lambda(\lambda)$. Hence $\tau(p) \in \tilde{\Lambda}(\lambda)$. Identically, we obtain $\tau(\hat{p}) = \tau(J_K p J_K) \in \tilde{\Lambda}(\eta)$, but $\tau(p) = \tau(\hat{p})$; therefore,

$$\tau(p) \in \tilde{\Lambda}(\lambda) \cap \tilde{\Lambda}(\eta).$$

Again, the projection $q = [e_M \hat{*} e_N] \in P(N_1, L)$ and analogically we obtain:

$$\tau(q) \in \tilde{\Lambda}(\alpha) \cap \tilde{\Lambda}(\beta)$$

or $\tau(q) = 1$. Now, by Proposition 2.11, $\tau(p) = \delta\tau(q)$, which completes the proof.

Q.E.D.

EXAMPLE. It is easy to see that for $\lambda^{-1} < 2 + \sqrt{5}$, the set $\tilde{\Lambda}(\lambda)$ is finite. Let λ be a transcendental number and η an algebraic number, $1 < \lambda^{-1}$, $\eta^{-1} < 2 + \sqrt{5}$. Then, by the above lemma, the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a co-commuting square.

S. Popa, cf. [**P3**, Theorem 6.1], proved that if a diagram $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ of type II_1 factors is a commuting square, then the indices of the inclusions of this diagram satisfy $\beta \in \Lambda(\lambda)$ and $\alpha \in \Lambda(\eta)$. Note that the condition $\tau(p) \in \Lambda(\lambda) \cup \{1\}$, which follows from Lemma 4.3 and, which is satisfied by any diagram $\mathscr{D}$ (with $S' \cap K = \boldsymbol{C}$) is a generalization (in the case $S' \cap K = \boldsymbol{C}$) of Popa's [**P3**, Theorem 6.1]. Indeed, if the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square, then $\tau(p) = \delta$. Hence

$$\delta = \frac{\lambda}{\beta} \in \Lambda(\lambda) \Leftrightarrow \beta \in \lambda\Lambda(\lambda)^{-1} = \Lambda(\lambda),$$

where the last equality comes directly from the definition of $\Lambda(\lambda)$.

Remark. Incidentally, with a little extra effort, we can make Popa's condition a bit stronger. If the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting square (which does not necessarily satisfy $S' \cap K = \boldsymbol{C}$) and $\beta \leq \alpha$, then either $\beta = \lambda$ or $\beta \in \Lambda_1(\lambda)$ or $\beta = \alpha = \lambda/(1-\lambda)$. In the proof we use Proposition 4.2 (or an obvious modification of it) to obtain a factor in $\mathscr{L}^s(N \subset M_1)$ with "too big" index s^{-1}.

Example. It is interesting to see, how some of the above results reflect in group theory. Let μ be an outer action of a finite group G on a type II_1 factor S. Let, for example, H and F be non-trivial subgroups of G such that $H \cap F = \{1\}$. We can consider the diagram $\begin{matrix} S \rtimes_\mu H & \subset & S \rtimes_\mu G \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu F \end{matrix}$ and apply [**NT**], [**SW**], Proposition 4.2 and Theorem 2.9 to obtain the following property. If $[G:H] = 3$ and $G \neq HF$, then $|F| = 2$ and there is an intermediate subgroup D such that $F \subset D \subset G$, with $[G:D] = 3$ or $[G:D] = 4$.

Theorem 4.4. *Let* $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \boldsymbol{C}$ *and* $[K:S] < \infty$. *If* $[K:Q] = 4\cos^2(\pi/n)$, *for a prime number* n, *then the diagram* $\mathscr{D}$ *is a co-commuting square.*

Proof. Let $n \geq 5$ be a prime number and $[K:Q] = \lambda^{-1} = 4\cos^2(\pi/n)$. By Lemma 4.3, it is sufficient to prove that the set $\tilde{\Lambda}(\lambda)$ is empty. From its definition, it is not empty, iff there exist integers k, s, such that

$$(*) \quad n-2 \geq s > k \geq 2 \quad \text{and} \quad \frac{P_{k-1}(\lambda)}{P_{k-2}(\lambda)} = \frac{P_{s-1}(\lambda)}{P_{s-2}(\lambda)} + \lambda.$$

From [**J**, Lemma 4.2.4], the equation is equivalent to the following one:

$$\frac{1}{\sin(2\pi/n)} = \frac{1}{\tan(k\pi/n)} - \frac{1}{\tan(s\pi/n)}.$$

From this we see that if k, s satisfy $(*)$ then $k' = n - s$, $s' = n - k$ also satisfy $(*)$. Therefore, we are allowed to assume that $k + s \leq n$. Directly from the definition of Jones' polynomials

$$P_{k-1}(\lambda)P_{s-2}(\lambda) - P_{s-1}(\lambda)P_{k-2}(\lambda) = \lambda(P_{k-2}(\lambda)P_{s-3}(\lambda) - P_{s-2}(\lambda)P_{k-3}(\lambda)).$$

Then $(*)$ is equivalent to the following equation

$$(**) \quad P_{k-2}(\lambda)P_{s-2}(\lambda) - P_{k-2}(\lambda)P_{s-3}(\lambda) + P_{s-2}(\lambda)P_{k-3}(\lambda) = 0.$$

By [**J**, Lemma 4.2.4], the degree of the above polynomial does not exceed $[(k-1)/2] + [(s-1)/2]$. We show that the degree of the minimum polynomial of λ is $(n-1)/2$. We use the terminology and results presented in [**ST**]. If we denote $\zeta = \exp(2\pi i/n)$, then $\lambda = (1/4) - (1/4)((1-\zeta)/(1+\zeta))^2$ is an element of the cyclotomic field $Q(\zeta)$ and ζ is a zero of the polynomial $f(t) = t^2 + (2 - \lambda^{-1})t + 1$. By Lemma 3.4 in [**ST**], $f(t)$ is the minimum polynomial of ζ over the field extension $Q(\lambda)$ and the degrees of field extensions satisfy:

$$n - 1 = [Q(\zeta) : Q] = [Q(\zeta) : Q(\lambda)][Q(\lambda) : Q] = 2[Q(\lambda) : Q].$$

Now, the inequality $[(k-1)/2] + [(s-1)/2] \geq (n-1)/2$ clearly contradicts our assumption $k + s \leq n$. Q.E.D.

COROLLARY 4.5. *Let* $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors with* $S' \cap K = \boldsymbol{C}$ *and* $[K : S] < \infty$. *If* $[Q : S] = 4\cos^2(\pi/n)$, *for a prime number* n, *then the diagram* $\mathscr{D}$ *is a commuting square.*

PROOF. This is an immediate consequence from [**SW**, Corollary 4.1] and Theorem 4.4. Q.E.D.

EXAMPLE. T. Teruya, cf. [**Te**], considered characteristic intermediate subfactors. If $S \subset Q \subset K, S' \cap K = \boldsymbol{C}$ is a triple of type II_1 factors then, analogically to group theory, he called Q a characteristic intermediate subfactors, if for any automorphism $\sigma \in \mathrm{Aut}(K)$ such that $\sigma(S) = S$ we have also $\sigma(Q) = Q$. Also, he showed that, if K is an extension of S by a finite group G, then this notion coincides with characteristic subgroups. By Theorem 4.4 we can easily obtain examples of characteristic intermediate subfactors. Let n and $m, n \neq m, n, m > 4$ be any prime numbers and let K be a hyperfinite type II_1 factor. Let $Q \subset K$ and $S \subset Q$ be II_1 subfactors with $[K : Q] = 4\cos^2(\pi/n)$ and $[Q : S] = 4\cos^2(\pi/m)$. Then, no matter how we pick up S, the subfactor Q is characteristic.

Indeed, from Proposition 4.2 and [**J**], we get $S' \cap K = \boldsymbol{C}$. If for some $\sigma \in \mathrm{Aut}(K, S)$, $Q \neq \sigma(Q) = R$ then, by Theorem 4.4, the diagram $\begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ is a commuting and co-commuting square, which, by [**SW**, Theorem 7.1], contradicts the assumption $n \neq m$.

EXAMPLE. Let μ be an outer action of the symmetric group S_3 on a type II_1 factor S. Consider the following diagram:

$$\begin{array}{ccc} S \rtimes_\mu \langle(1,2)\rangle & \subset & S \rtimes_\mu S_3 \\ \cup & & \cup \\ S & \subset & S \rtimes_\mu \langle(2,3)\rangle \end{array}.$$

Clearly, it is not co-commuting square, and so for $n = 6$ the statement of Theorem 4.4 is not satisfied.

The prime numbers n are not the only ones for which the set $\tilde{A}(\lambda), \lambda^{-1} = 4\cos^2(\pi/n)$ is empty. We can easily check that for $n = 4$ or $n = 9$ it is empty too. With a little digital help, we conjecture that it is empty for all odd numbers (≥ 5). For even n $(n > 5)$ the set $\tilde{A}(\lambda)$ is not empty, because it contains $\{1 - \lambda, 2\lambda\}$. We were only able to prove the following

PROPOSITION 4.6. *Let* $[K : Q] = \lambda^{-1} = 4\cos^2(\pi/n)$ *and let the diagram* $\mathscr{D} = \begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ *be not a co-commuting square, where* $S' \cap K = \boldsymbol{C}$ *and* $[K : S] < \infty$. *Then* n *is not a prime number and the following statements hold true*:

(1) *There exists a factor*

$$P \in \bigcup_{0 \leq k \leq n-4} \mathscr{L}^{s_k}(R \subset K), \quad \textit{where} \quad s_k = \frac{\lambda^{k+1}}{P_k(\lambda)^2}.$$

(2) *If* $2\lambda^{-1} \neq [K : R] = \eta^{-1} < (\lambda^{-1} - 1)^2$, *then* n *must be even, the inclusion* "$R \subset K$" *is isomorphic to* "$Q \subset K$" *and*

$$\cos(\text{Op-ang}_K(Q, R)) = \left\{ \left(\frac{\lambda}{1 - \lambda}\right)^2 \right\}.$$

(3) *If* $n = 6$, *then the principal graph of the inclusion* "$Q \subset K$" *is* A_5.

PROOF. (1) If the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is not a co-commuting square, then the projection $p = [e_Q * e_R] \in P(M, K)$ so, by Proposition 4.2, we obtain a factor $P \in \mathscr{L}^{s_k}(R \subset K)$, for some integer $k \in [0, n - 4]$.

(2) Since $(0, (\lambda^{-1} - 1)^2) \cap \{s_k^{-1} | 0 \leq k \leq n - 4\} = \{\lambda^{-1}\}$, the only possibility for the factor P in (1) is $[K : P] = \lambda^{-1}$. Since $[P : R] = \lambda/\eta < \lambda(\lambda^{-1} - 1)^2 < 9/4$, by [**J**], $[P : R] = 1$ or $[P : R] = 2$, with the second possibility eliminated by the assumption $\lambda/\eta \neq 2$. From the construction of the subfactor P (see [**P3**, Proposition 4.5]), in the case of $k = 0$ or $k = n - 4$, we see that it is a downward basic construction for the pair $K \subset M$. Thus $M = \langle K, R, \tilde{e}_R \rangle$, where $\tilde{e}_R$ is a minimal and central projection in $R' \cap M$. If $p\tilde{e}_R \neq 0$ then, by Theorem 2.9, $p = \tilde{e}_R$ and so $\tilde{e}_R = e_Q$ which is impossible because the diagram would collapse. Therefore $p\tilde{e}_R = 0$, and hence $R' \cap M = \boldsymbol{C}_p \oplus \boldsymbol{C}\tilde{e}_R$. Otherwise, there would be another projection, say $f \in R' \cap M$, $f + p$

$+\tilde{e}_R = 1$, $\tau(f) \geq \lambda$, which gives contradiction with $\lambda > 1/4$. By [**PP1**, Lemma 1.8], there is a unitary $u \in K$ such that $\tilde{e}_R = ue_Q u^*$ and $R = uQu^*$, which gives desired isomorphism.

Since $e_Q + \tilde{e}_R$ is a projection and $E_K(e_Q + \tilde{e}_R) = 2\lambda < 1$, by Theorem 4.1, there exists an integer k, $2 \leq k \leq n-3$ such that $P_{k-1}(\lambda)/P_{k-2}(\lambda) = 2\lambda$, hence $P_{k-1}(\lambda) - \lambda P_{k-2}(\lambda) = \lambda P_{k-2}$, and so $P_k(\lambda) = -P_k(\lambda) + P_{k-1}(\lambda)$ and consequently $P_k(\lambda)/P_{k-1}(\lambda) = 1/2$. Therefore n can not be odd. Value of the angle is computed in Lemma 3.1.

(3) By [**P5**] the principal graph of "$Q \subset K$" can only be D_4 or A_5. If it is D_4, then $Q' \cap M = \boldsymbol{C}e_Q \oplus \boldsymbol{C}f \oplus \boldsymbol{C}g$, for some projections f and g such that $\tau(f) = \tau(g) = \lambda (= 1/3)$. If f_1 is a projection in $S' \cap M$, then, since $E_K(f_1)$ is a scalar, $\tau(f_1) \geq \lambda$. Therefore, if $S' \cap M \neq Q' \cap M$ then $S' \cap M = \boldsymbol{C}e_Q \oplus M_2(\boldsymbol{C})$. Since f and g are Jones projections $(E_K(f) = \lambda, E_K(g) = \lambda)$, they are central (see [**PP1**] or Lemma 2.8). Thus we have $S' \cap M = Q' \cap M$ and, in consequence,

$$R' \cap M = (R' \cap M) \cap (Q' \cap M) = R' \cap Q' \cap M = K' \cap M = \boldsymbol{C}.$$

Then, by Corollary 2.13, $\mathscr{D}$ must be a co-commuting square. Q.E.D.

Almost identically as in (3) above we can prove a little more.

COROLLARY 4.7. *If* $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *is a diagram of type* II_1 *factors,* $Q \vee R = K$ $([K:S] < \infty, S' \cap K = \boldsymbol{C})$ *and* K *is a crossed product of* Q *by an action* μ *of a finite group* G, $K = Q \rtimes_\mu G$, *then* $\mathscr{D}$ *is a co-commuting square.*

EXAMPLE. Let $S \subset Q \subset K$ be a triple of type II_1 factors with $S' \cap K = \boldsymbol{C}$. Let Q be a crossed product of S by a finite abelian group and let K be a crossed product of Q by any finite group. From the above corollary and by [**Wi**, Theorem 6], we see that if there is another intermediate subfactor R between S and K such that $R \cap Q = S$ and $R \vee Q = K$, then the inclusion $S \subset K$ has depth two.

If $\lambda^{-1} = 4\cos^2(\pi/n)$ then we set $\Omega(\lambda) = \{P_k(\lambda)^2/\lambda^{k+1} \mid 0 \leq k < n-3\}$. If $\lambda^{-1} \geq 4$ then put $n = \infty$ in the above notation.

COROLLARY 4.8. *Let* $\mathscr{D} = \begin{matrix} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{matrix}$ *be a diagram of type* II_1 *factors* $(S' \cap K = \boldsymbol{C}, [K:S] < \infty, Q \vee R = K)$ *such that* $[K:Q] < 2 + \sqrt{5}$. *If for any intermediate subfactor* $F \neq Q$ *between* S *and* K *(including* S*),* $[K:F] \notin \Omega(\lambda)$, *then* $\mathscr{D}$ *is a co-commuting square.*

PROOF. Suppose that $\mathscr{D}$ is not a co-commuting square. Then $p = [e_Q * e_R] \neq 1$ and the projections p, $p - e_Q$ are elements of $P(M,K)$. Hence $\tau(e_Q)$ is an element of the algebraic difference $\Lambda(\lambda) - \Lambda(\lambda)$ $(\lambda^{-1} = [K : Q])$. Since $\lambda^{-1} < 2 + \sqrt{5}$ and (in case $\lambda^{-1} \geq 4$) $\Lambda_1(\lambda) - \Lambda_1(\lambda) = [-\sqrt{1-4\lambda}, \sqrt{1-4\lambda}]$, we can write a little more:

$$\lambda \in (\Lambda_0(\lambda) - \Lambda_0(\lambda)) \cup (\Lambda_0(\lambda) - \Lambda_1(\lambda)) \cup (\Lambda_1(\lambda) - \Lambda_0(\lambda)).$$

In any case $\tau(1-p) \in 1 - \Lambda_0(\lambda) = \Lambda_0(\lambda)$ or $\tau(p - e_Q) \in \Lambda_0(\lambda)$ (otherwise $\lambda \in \Lambda_1(\lambda) - \Lambda_1(\lambda)$).

If, for $f \in \{1-p, p-e_Q\}$, $\tau(f) \in \Lambda_0(\lambda) - \{\lambda, 1-\lambda\}$, then, by Proposition 4.2, we obtain an intermediate subfactor $F \neq Q$, $S \subset F \subset K$ with index $[K : F] \in \Omega(\lambda)$, which contradicts our assumption. Therefore, there is $f \in \{1-p, p-e_Q\}$ such that $\tau(f) \in \{\lambda, 1-\lambda\}$. Since $\tau(1-p) \neq 1-\lambda$ and $\tau(p-e_Q) \neq 1-\lambda$, we see that for a projection $f \in \{1-p, p-e_Q\}$, $\tau(f) = \lambda$. Hence, there is an intermediate subfactor F, $S \subset F \subset K$, with $[K : F] = \lambda^{-1}$ and $M = \langle K, F, f\rangle$. Since, by our assumption $F = Q$, we obtain $M = \langle K, Q, f\rangle$. Note that $f \perp e_Q$. Let $\mathcal{N}(Q)$ be the normalizer of Q in K. Obviously $\mathcal{N}(Q)''$ is a II_1 factor and $Q \subset \mathcal{N}(Q)'' \subset K$. By [**Sut**] and [**PP1**, Proposition 1.7] (see also remarks preceding [**PP3**, Corollary 1.1.6]) and by Jones' restrictions on the index we see that $[K : \mathcal{N}(Q)''] = [\mathcal{N}(Q)'' : Q] = 2$ or that $K = \mathcal{N}(Q)''$. The second case can be eliminated by Corollary 4.7, because K is then an extension of Q by a finite group (cf. [**Sut**], [**PP3**]). By [**P6**], the principal graph of the inclusion $Q \subset K$ is $D_n^{(1)}$ $(n > 4)$ or D_∞. In any case $Q' \cap M = \boldsymbol{C}e_Q \oplus \boldsymbol{C}f \oplus \boldsymbol{C}g$, where $\tau(g) = 1/2$. Since $S' \cap M \neq Q' \cap M$ (otherwise we proceed like in the proof of Proposition 4.6(3)), it follows that there is a projection $g_1 \leq g$ and $g \neq g_1 \in S' \cap M$. By [**PP1**, Proposition 2.1], $\tau(g_1) = E_K(g_1) \geq (1/4)g_1$, which gives $\tau(g_1) = 1/4 = \lambda$. Now, there exists an intermediate subfactor F, $S \subset F \subset K$ such that $M = \langle K, F, g_1\rangle$. But $[K : F] = 4 \in \Omega(\lambda) = \Omega(1/4)$ and $F \neq Q$, because $g_1 \notin Q' \cap M$. Q.E.D.

5. Associated algebras of a diagram

We assume in this section that the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is a commuting square, or equivalently, that the diagram $\begin{array}{ccc} N & \subset & L \\ \cup & & \cup \\ K & \subset & M \end{array}$ is a co-commuting square. Also, as before, the inclusion $S \subset K$ is irreducible with $[K : S] < \infty$.

PROPOSITION 5.1. *There exists a Jones projection f for the inclusion $M \subset L$ (i.e. $f \in L$ and $E_M(f) = [L : M]^{-1} = \alpha$) such that*

$$e_R = e_R f = [e_Q * e_R] f.$$

PROOF. Let $p = [e_Q * e_R]$. Take any Jones' projection $e_0 \in L$ and the corresponding downward basic construction D, so that $L = \langle M, D, e_0\rangle$. Take a unitary $u \in M$ such that $upu^* \in D$. If we set $e_1 = u^* e_0 u$ and $D_1 = u^* D u$, then, by [**PP1**, Corollary 1.8], $L = \langle M, D_1, e_1\rangle$ and $p \in D_1$. Since $\tau(e_1 p) = \alpha\tau(p) = \eta = \tau(e_R)$, there exists a unitary $v \in M_p$ such that $e_R = v^* e_1 p v$. It is easy to check that $f = (v^* + 1 - p)e_1(v + 1 - p)$ will do the job. Q.E.D.

It is natural to ask what the von Neumann algebras $A = M \cap \{e_R\}'$ and $B = R \vee \{e_Q\}$ look like. Obviously, $R \subset B \subset A \subset M$ and if, in addition, the diagram $\begin{array}{ccc} Q & \subset & K \\ \cup & & \cup \\ S & \subset & R \end{array}$ is also a co-commuting square then $A = B$ is a factor and $\begin{array}{ccc} K & \subset & M \\ \cup & & \cup \\ R & \subset & A \end{array}$ is a commuting and co-commuting square too. (Cf. [**K**], [**P5**], [**Wi**].)

Let f be like in the above proposition and let M_0 be the corresponding downward basic construction, $M_0 = M \cap \{f\}'$ and $L = \langle M, M_0, f\rangle$. For a projection e and a von Neumann algebra F we will denote

$$V(e, F) = \bigvee\{ueu^* \mid u \text{ is a unitary in } F\}.$$

LEMMA 5.2. *Let* $p = [e_Q * e_R]$. *Then*

$$A_p = B_p = (M_0)_p = Ae_QA = Be_QB = Re_QR$$

$$\text{and} \quad p = V(e_Q, R) = V(e_Q, B) = V(e_Q, A).$$

PROOF. First, we show that $Ae_QA = Re_QR$. Let $x \in A$. By [**PP1**, Lemma 1.2], $xe_Q = ye_Q$, for $y \in K$. Multiplying the equality $ye_Qe_R = e_Rye_Q$ by e_S we obtain $ye_S = e_Rye_S = E_R(y)e_S$, which implies $y \in R$. Therefore $Ae_QA = Re_QR = Be_QB$.

Obviously $V(e_Q, R) \in R' \cap M$, and hence (by Theorem 2.9) $V(e_Q, R) \geq p$. The equality follows from $ue_Qu^* \leq p$, which holds for all unitaries $u \in R$. By the first part of the proof, it follows that also $p = V(e_Q, B) = V(e_Q, A)$. Now, it is easy to see that $A_p = B_p = Re_QR$.

Proposition 5.1 implies $(M_0)_p \subset A_p$. Indeed, if $x \in (M_0)_p$ then $xe_R - e_Rx = xpf - pfx = xf - fx = 0$. Since $\{e_R\}' \cap M_p = (\{e_R\}' \cap M)_p = A_p$, we see that $L_p = \langle M_p, A_p, e_R\rangle$; therefore $[M_p : A_p] = \alpha^{-1}$. On the other hand, $[M_p : (M_0)_p] = [M : M_0] = [L : M] = \alpha^{-1}$. We see then that $(M_0)_p = A_p$. Q.E.D.

REMARK. Let $T = Re_QR$ be the set characterized above. We remark that T is an algebraic basic construction for the pair $S_p \subset R_p$ and a downward basic construction for the pair $M_p \subset L_p$. We can write

$$T = \langle R_p, S_p, e_Q\rangle \quad \text{and} \quad L_p = \langle M_p, T, e_R\rangle.$$

Finally, we obtain the following decomposition of the von Neumann algebras A and B.

COROLLARY 5.3. *Let as before* $p = [e_Q * e_R]$. *Then*

$$A = T \oplus M_{1-p} \quad \text{and} \quad B = T \oplus R_{1-p}.$$

PROOF. The elements $x = r + \sum_i a_ie_Qb_i$, with $r, a_i, b_i \in R$ form a dense $*$-subalgebra of B. By Theorem 2.9 we see that $(1-p)x(1-p) = (1-p)r$. Since p is central in B,

$$B = B_p \oplus B_{1-p} = T \oplus R_{1-p}.$$

If we apply this equality to the algebra $Q \vee \{e_R\}$, then, using the modular conjugation J_K, we obtain the remaining one. Q.E.D.

ACKNOWLEDGEMENTS I am greatly indebted to my teacher—Professor Y. Watatani, for introducing me into this subject and for his patience, moral encouragement and many interesting discussions.

I would like to express my gratitude to my former and present supervisors Professors S. Koshi and A. Kishimoto for their continuing interest in my studies and for invaluable help with various administrative procedures in connection with my Monbusho Fellowship.

Also, I would like to thank Professor M. Hayashi for his help and taking care of me during my stay in Sapporo.

Finally, it is a pleasure to thank the staff at the Department of Mathematics of the Hokkaido University for their kindness and help in many situations.

References

[B] D. Bisch, A note on intermediate subfactors, Pacific J. Math. **163** (1994), 201–216.

[GHJ] F. Goodman, P. de la Harpe and V. F. R. Jones, Coxeter Graphs and Towers of Algebras, MSRI Publ. **14**, Springer-Verlag, New York, 1989.

[J] V. F. R. Jones, Index for subfactors, Invent. Math. **72** (1983), 1–25.

[K] Y. Kawahigashi, Classification of paragroup actions on subfactors, Publ. Res. Inst. Math. Sci. **31** (1995), 481–517.

[NT] M. Nakamura and Z. Takeda, On the fundamental theorem of the Galois theory for finite factors, Proc. Japan Acad. **36** (1960), 258–260.

[O] A. Ocneanu, Quantum symmetry, differential geometry of finite graphs and classification of subfactors, Tokyo University Seminary Notes **45**, 1991.

[P1] S. Popa, Orthogonal pairs of $*$-subalgebras in finite von Neumann algebras, J. Operator Theory **9** (1983), 253–268.

[P2] ———, Maximal injective subalgebras in factors associated with free groups, Adv. Math. **50** (1983), 27–48.

[P3] ———, Relative dimension, towers of projections and commuting squares of subfactors, Pacific J. Math. **137** (1989), 181–207.

[P4] ———, Classification of subfactors: the reduction to commuting square, Invent. Math. **101** (1990), 19–43.

[P5] ———, Classification of amenable subfactors of type II, Acta Math. **172** (1994), 163–255.

[P6] ———, Sur la classification des sous-facteurs d'indice fini du facteur hyperfini, C. R. Acad. Sci. Paris **311** (1990), 95–100.

[PP1] M. Pimsner and S. Popa, Entropy and index for subfactors, Ann. Sci. Ec. Norm. Sup. **19** (1986), 57–106.

[PP2] ———, Iterating the basic construction, Trans. Am. Math. Soc. **310** (1988), 127–133.

[PP3] ———, Finite dimensional approximation of pairs of algebras and obstructions for the index, J. Funct. Anal. **98** (1991), 270–291.

[S] T. Sano, Commuting co-commuting squares and finite dimensional Kac algebras, Pacific J. Math. **172** (1996), 243–253.

[Su] V. S. Sunder, On commuting squares and subfactors, J. Funct. Anal. **101** (1991), 286–311.

[Sut] C. Sutherland, Cohomology and extensions of operator algebras II, Publ. Res. Inst. Math. Sci. **16** (1980), 135–174.

[ST] I. N. Steward and D. O. Tall, Chapman and Hall, New York, 1987.

[SW] T. Sano and Y. Watatani, Angles between two subfactors, J. Operator Theory **32** (1994), 209–241.

[SZ] S. Stratila and L. Zsido, Lectures on von Neumann Algebras, Abacus Press/Ed. Academiei, Turnbridge Wells/Bucuresti, 1979.

[T] M. Takesaki, Theory of Operator Algebra I, Springer, Berlin, 1979.

[Te] T. Teruya, Characteristic intermediate subfactors, Preprint, 1995.

[W] Y. Watatani, Index for C^*-subalgebras, Memoirs A.M.S. No. 424, 1990.

[WW] Y. Watatani and J. Wierzbicki, Commuting squares and relative entropy for two subfactors, J. Funct. Anal. **133** (1995), 329–341.

[We] H. Wenzl, Hecke algebras of type A and subfactors, Invent. Math. **92** (1988), 345–383.

[Wi] J. Wierzbicki, An estimate of the depth from an intermediate subfactor, Publ. Res. Inst. Math. Sci. **30** (1994), 1139–1144.

Jerzy WIERZBICKI
04-927 Warszawa
ul. Biernacka 29B
Poland