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Noncommutative 3-sphere: A model of noncommutative contact algebras

Dedicated to Professor Masafumi Okumura for his sixtieth birthday

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Introduction

The notion of deformation quantization, which is a deformation of Poisson algebras, has been formulated beautifully by Bayen et al [**B**] to describe quantum mechanics from the algebraic point of view. The fundamental idea of deforming Poisson algebras is roughly to quantize commutative algebras in the spirit of microlocal analysis (cf. [**BG**]). According to the remarkable description by Guillemin Sternberg [**GS**], a crucial point in microlocal analysis is a selection of the asymptotics usually expressed in terms of a parameter.

Let us recall that there are different frameworks for microlocal analysis which are almost, but not entirely equivalent (cf. [**GS**]). The first framework is the "asymptotic theory", in which the parameter $\hbar$ plays an essential role, and all results are phrased in terms of the asymptotics as $\hbar$ tends to zero. The symbol calculus for this theory lives in the algebra $C^{\infty}(T^*M)[[\hbar]]$ of the cotangent bundle T^*M of the underlying manifold M. In the algebraic counter part, this algebra corresponds to the deformation quantization of symplectic manifolds. In contrast to the above, another framework is the "homogeneous" theory of pseudodifferential operators, in which the asymptotics in $\hbar$ is replaced by asymptotics on the symplectic manifold $\mathring{T}^*M$, where $\mathring{T}^*M$ is now the cotangent bundle minus the zero section, provided with a conformally symplectic action of $\boldsymbol{R}$ defining the asymptotics.

The aim of this paper is to provide an algebraic theory which does for homogeneous symplectic manifolds, without referring to $\hbar$, what deformation quantization does for general symplectic manifold. It is well known that such homogeneous symplectic manifolds, i.e., symplectic manifolds with a free conformally symplectic action of $\boldsymbol{R}$, are in a natural 1-1 correspondence with contact manifolds. From this point of view, using the example of noncommutative 3-sphere we present a notion of noncommutative algebras corresponding to deformations of contact algebras.

The construction of noncommutative algebras provides another motivation for the notion of noncommutative manifold. A commutative algebra is realized as an algebra of (polynomial) functions on a certain space. Even a noncommutative algebra may be considered perhaps as an algebra of "functions" on a "noncommutative space", which is the object of study in noncommutative geometry. In particular, in order to obtain the

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deformation quantization of Poisson algebras on symplectic manifolds, we introduced the notion of Weyl manifold, which can be viewed as a "space" for noncommutative algebras (cf. [**OMY1**], [**OMY2**]).

We will show that the noncommutative algebra derived from the contact algebra of the 3-sphere still has the notion of an associated "space". Deformation quantization suggests a mode of deforming various geometric structures. From this point of view, we will discuss the deformations of (contact) algebras arisen from contact structures as a typical example of geometric structures similar to symplectic structures. In particular, it is shown that the deformation parameter of a contact algebra is not in the center; in contrast, deformation quantization of Poisson algebras is given by algebras of formal power series of functions on manifolds with the deformation parameter a central element. We will call this deformation algebra a *noncommutative contact algebra*. This notion is closely related to the early work of Boutet de Monvel and Guillemin [**BG**], who used microlocal analysis to define the notion of quantized contact structures on any compact oriented contact manifold, and who constructed the operator algebra which quantizes the classical observables.

Since our aim is to exhibit a concrete noncommutative 3-sphere as an example of noncommutative contact algebras, this paper is not intended to go into the details of noncommutative contact algebras, which are treated in papers [**OMMY1**], [**OMMY2**].

The noncommutative 3-sphere we present here has a hierarchy structure corresponding to the Hopf fibration of the 3-sphere over the 2-sphere. By "noncommutizing" of the Hopf fibration, we derive a deformation quantization of the Riemann sphere (cf. [**CGR**], [**B**]). We also arrive at a notion of noncommutative Kähler manifolds, which is a Weyl manifold with a complex structure. Although it is essentially the same notion as in Karabegov [**K**], we work only in the noncommutative algebra setting.

This paper is organized as follows. We start with a noncommutative algebra of matrices of infinite rank given by the Fock space representation of the Wick algebra. By choosing a transcendental element, called the radial element, of the Wick algebra, we obtain an algebra $\mathscr{A}$ as a reduction of the Wick algebra W (see Definition 1.4 in §1). In §2, we give another approach to the deformation quantization of $\boldsymbol{C}^2 - \{0\}$. The radial element also gives a reduction of the deformation quantization of the Poisson algebra $C^\infty(\boldsymbol{C}^2 - \{0\})$, and the algebra $\mathscr{A}^\infty$ obtained by the reduction contains the algebra $\mathscr{A}$ densely, where $\mathscr{A}^\infty$ is endowed canonically with the C^∞-topology. We will show that the algebra $\mathscr{A}$ gives a noncommutative algebra corresponding to the standard 3-sphere, which we call the *noncommutative 3-sphere*. Introducing the notion of localization in §4, we give the geometric picture of the noncommutative 3-sphere. The localization process is crucial for the algebra $\mathscr{A}^\infty$.

For a geometric description of the noncommutative algebras we treat, we set up a class of noncommutative algebras, called *regulated smooth algebras*, which include deformation quantizations and noncommutative contact algebras. In §3, we give a rigorous meaning to certain delicate computations in noncommutative algebras. In particular, we establish a useful formula, called the *bumping lemma* (cf. Lemma 3.7) that simplifies our computations. Although these are abstract versions of symbol calculus of pseudodifferential operators, we have to establish this calculus without using any operator representations in order to treat localized algebras. Through these computations, we find several interesting transcendental formulas coming from the non-commutativity.

§5 is devoted to the explicit description of the noncommutative Riemann sphere whose generators can be expressed as matrices of infinite rank. In §6, we study the representations of noncommutative Riemann sphere. This agrees with the well known work on geometric quantization for Kähler manifolds by Berezin [**Be**] and Cahen-Gutt-Rawnsley [**CGR**]. It should be remarked that our approach gives the representation not only for functions but for automorphisms of the noncommutative Riemann sphere. In particular, we show that a class of infinitesimal automorphisms of the noncommutative Riemann sphere is obtained by the projective limit of finite dimensional Lie groups (cf. Theorem 6.2).

We should note that there are interesting works by Šťovíček [**St**] and in particular Chu, Ho and Zumino [**CHZ**], who obtain a non-commutative Riemann sphere as a factor space of quantum group $SU_q(2)$, whose results are close to ours. However, the noncommutative sphere given by [**CHZ**] has a different Poisson structure in the classical level from that obtained above. The Poisson structure of quantum homogeneous space seems to have always some singular points (cf. [**MO**]).

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1. Wick Algebra, matrix representation and reduction

Consider an associative algebra W over $\boldsymbol{C}$ generated by $\{\hbar, \zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2\}$ with the relations:

$$(1.1)\qquad [\zeta_i, \bar{\zeta}_j] = -2\hbar\delta_{ij}, \quad [\zeta_i, \zeta_j] = [\bar{\zeta}_i, \bar{\zeta}_j] = 0 \quad \hbar \in \text{center}.$$

The algebra W is called the *Wick algebra* (cf. [**M**] and [**D**]). W has a canonical involutive anti-automorphism $a \to \bar{a}$. We denote its product by $*$.

Let us note the following elementary relation used below:

$$(1.2)\qquad \bar{\zeta}_1 * \zeta_1 + \zeta_2 * \bar{\zeta}_2 = \zeta_1 * \bar{\zeta}_1 + \bar{\zeta}_2 * \zeta_2.$$

It is known that $W/\hbar W$ is isomorphic to the polynomial algebra of 4-variables and that any maximal 2-sided ideal of W corresponds to a point of $\boldsymbol{C}^4$.

We recall the Fock space representation of W (cf. Messiah [**M**], Dirac [**D**]): first of all, we extend the algebra W by adjoining $\sqrt{2\hbar}$, $\sqrt{2\hbar}^{-1}$. These adjoined elements remain in the center. Define a left ideal $\mathscr{L}$ generated by $\bar{\zeta}_i$ for $i = 1, 2$, i.e. $\bar{\zeta}_i|0\rangle = 0$, and consider the factor space $V^{(\infty)} = W/\mathscr{L}$. We see that $V^{(\infty)} = \sum \oplus V^{(m)}$, where

$$(1.3)\qquad V^{(m)} = span \frac{1}{\sqrt{2\hbar}^m}\left\{\frac{1}{\sqrt{m!}}\zeta_1^m, \cdots, \frac{1}{\sqrt{k!l!}}\zeta_1^k * \zeta_2^l, \cdots, \frac{1}{\sqrt{m!}}\zeta_2^m\right\} \quad (m = k + l)$$

in the extended algebra.

The left action of $w \in W$ on $V^{(\infty)}$ gives a representation of W. This action can be expressed as a matrix of infinite rank and it gives the representation of the Wick algebra.

For $w \in W$, $\hat{w}$ denotes the matrix representation of w on $V^{(\infty)}$. Namely, $\hat{w}$ has the following blockwise form:

$$\hat{w} = \begin{bmatrix} B_{1,1} & B_{1,2} & \cdot & \cdots & \cdots & \cdots \\ B_{2,1} & B_{2,2} & B_{2,3} & \cdot & \cdots & \cdots \\ \cdot & B_{3,2} & B_{3,3} & B_{3,4} & \cdot & \cdots \\ \cdots & \cdot & B_{4,3} & B_{4,4} & B_{4,5} & \cdot \\ \cdots & \cdots & \cdot & B_{5,4} & B_{5,5} & B_{5,6} \\ \vdots & \vdots & \vdots & \ddots & \ddots & \ddots \end{bmatrix}, \tag{1.4}$$

where $B_{i,j}$ is an $i \times j$-matrix satisfying $B_{i,j} = 0$ for $|i-j| \gg 1$. A matrix is said to be *blockwise diagonal*, if it has the blockwise form of (1.4) such that $B_{i,j} = 0$ for $i \neq j$. In fact, the generators of W satisfies $B_{i,j} = 0$ for $|i-j| \geq 2$ and have the following form:

$$\text{For } \hat{\zeta}_1, \quad B_{s+1,s} = \sqrt{2\hbar} \begin{bmatrix} \sqrt{s} & \cdot & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ \cdot & \cdot & \sqrt{2} & \\ & & 0 & \sqrt{1} \\ 0 & \cdots & \cdots & 0 \end{bmatrix} \quad \text{other blocks are } 0.$$

$$\text{For } \hat{\zeta}_2, \quad B_{s+1,s} = \sqrt{2\hbar} \begin{bmatrix} 0 & \cdots & \cdots & 0 \\ \sqrt{1} & \cdot & \cdot & \cdot \\ \cdot & \sqrt{2} & \cdot & \cdot \\ \vdots & \vdots & \ddots & \\ & & & \sqrt{s} \end{bmatrix} \quad \text{other blocks are } 0. \tag{1.5}$$

We also have $\hat{\bar{\zeta}}_i = {}^t\hat{\zeta}_i$. Notice that almost all elements in W are represented as unbounded operators in general.

In what follows, w will be substituted for the matrix representation $\hat{w}$, whenever it creates no confusion. We will still denote by $a * b$ the product of the matrix a and b. Note that there are elements which are not well defined as elements of W but have rigorous meanings as matrices of infinite rank. Such an element will be called a *transcendental element*. As a typical example, we consider a matrix given by

$$r = {}^{*}\!\sqrt{\bar{\zeta}_1 * \zeta_1 + \zeta_2 * \bar{\zeta}_2} \tag{1.6}$$

where ${}_{*}\!\sqrt{}$ denotes the square root of the matrix. It is easily seen that r is given as a diagonal matrix:

$$r = diag\{B_{1,1}, B_{2,2}, \cdots B_{k,k}, \cdots\},$$

where

$$B_{k,k} = \sqrt{2\hbar} \begin{bmatrix} \sqrt{k} & & & \\ & \sqrt{k} & & \\ & & \ddots & \\ & & & \ddots & \\ & & & & \sqrt{k} \end{bmatrix} \tag{1.7}$$

We easily see that $\bar{r} = r$ and r is invertible. By (1.2), we note that

$$(1.8)\qquad \begin{aligned} &r * r = \bar{\zeta}_1 * \zeta_1 + \zeta_2 * \bar{\zeta}_2 = \zeta_1 * \bar{\zeta}_1 + \bar{\zeta}_2 * \zeta_2, \\ &r^{-1} * r = 1. \end{aligned}$$

We call r the *radial element*. Using the matrix representation, we define invertible matrices r_i by

$$(1.9)\qquad r_i = {}^*\sqrt{\bar{\zeta}_i * \zeta_i} \quad \text{for } i = 1, 2.$$

These are also transcendental elements, although ${}^*\sqrt{\zeta_i * \bar{\zeta}_i}$ has no inverse in the matrix representation.

The matrix representation gives a transcendental method for extending algebras. Namely, postulating r^{-1}, we obtain a new algebra in the space of matrices whose relations represent a noncommutative 3-sphere given in §2. We set

$$(1.10)\qquad \begin{aligned} &\mu = -2\hbar r^{-2}, \quad \Xi_1 = r^{-1} * \zeta_1, \quad \Xi_2 = r^{-1} * \zeta_2, \\ &\qquad\qquad \Xi_1^* = \bar{\zeta}_1 * r^{-1}, \quad \Xi_2^* = \bar{\zeta}_2 * r^{-1}, \end{aligned}$$

where u^* stands for the complex conjugate of u.

${}^*\sqrt{1+\mu}$ and ${}^*\sqrt{1-\mu}$ are given as diagonal matrices:

$$(1.11)\qquad \begin{aligned} &{}^*\sqrt{1-\mu} = diag\{\sqrt{2}I_1, \ldots, \sqrt{1+k^{-1}}I_k, \ldots\}, \\ &{}^*\sqrt{1+\mu} = diag\{0I_1, \ldots, \sqrt{1-k^{-1}}I_k, \ldots\} \end{aligned}$$

where I_k is the $k \times k$ identity matrix.

Although we have not mentioned the topology of matrices of infinite rank, we have to make it explicit in order to treat transcendental elements. Recall again that any matrix in W can be considered as a densely defined unbounded operator acting on l_2. We say that a series $\{u_n\}$ of elements of W *converges* to a matrix u if it converges to u in the weak topology (cf. [Y]). We denote this by $u = w\text{-}\lim_{n\to\infty} u_n$. This is equivalent to convergence with respect to any (i,j)-component of matrices.

Let $\{p_k(t)\}$ be a sequence of polynomials which converges to a function ϕ on a certain domain I. For $u \in W$, we consider the matrix $p_k(u)_*$ by replacing t with u in p_k, where the lower index $*$ denotes the product of matrices. Assuming moreover that $\{p_k(u)_*\}$ converges to a matrix v, we denote it by $v = \phi(u)_*$, i.e.

$$\phi(u)_* = w\text{-}\lim_{k\to\infty} p_k(u)_*.$$

In computations it is very useful to remark that $v * (u * v)^m = (v * u)^m * v$ holds for any m. Hence $v * p_k(u * v)_* = p_k(v * u)_* * v$ for any polynomial p_k. Therefore, we can expect that the following formula holds for various cases (cf. Lemma 3.7):

$$(1.12)\qquad v * \phi_*(u * v) = \phi_*(v * u) * v.$$

Suggested by (1.12), we have

$$(1.13)\qquad \begin{aligned} &\zeta_i * r = {}^*\sqrt{1 - 2\hbar r^{-2}} * r * \zeta_i, \quad (i = 1, 2) \\ &r * \zeta_i = \zeta_i * r * {}^*\sqrt{1 + 2\hbar r^{-2}}, \quad (i = 1, 2). \end{aligned}$$

Note that (1.13) can be derived directly by composing the matrices for ζ_i and r. Since $[\mu, r] = 0$, we have

$$\zeta_i * r = {}^*\sqrt{1+\mu} * r * \zeta_i, \quad r * \zeta_i = \zeta_i * {}^*\sqrt{1-\mu} * r. \tag{1.14}$$

Thus, we get

$$\Xi_i * r = {}^*\sqrt{1+\mu} * r * \Xi_i, \quad r * \Xi_i = \Xi_i * r * {}^*\sqrt{1-\mu}. \tag{1.15}$$

By (1.11), for any integer k, ${}^*\sqrt{1-k\mu}$ is defined and

$$\Xi_i * r * {}^*\sqrt{1-k\mu} = r * {}^*\sqrt{1-(k-1)\mu} * \Xi_i. \tag{1.16}$$

LEMMA 1.1. *The elements in* (1.10) *have the following relations*:

$$[\mu^{-1}, \Xi_i] = -\Xi_i, \quad [\mu^{-1}, \Xi_i^*] = \Xi_i^*. \tag{1.17}$$

$$[\Xi_1, \Xi_2] = 0. \tag{1.18}$$

$$\Xi_i * \Xi_j^* - (1-\mu)\Xi_j^* * \Xi_i = \mu\delta_{ij} \quad \textit{for } i, j = 1, 2. \tag{1.19}$$

$$\Xi_1^* * \Xi_1 + \Xi_2^* * \Xi_2 = 1. \tag{1.20}$$

PROOF. By (1.15), we get

$$\Xi_i * r^2 = (1+\mu) * r^2 * \Xi.$$

Since $2\hbar = -\mu * r^2$, we have

$$[r^2, \Xi_i] = 2\hbar\Xi_i \qquad (i = 0, 1),$$

which implies (1.17). (1.18) is obvious by a direct computation. By (1.8) and the relation $[r, \zeta_i * \bar{\zeta}_i] = 0$, (1.19) is obtained by (1.1). (1.20) follows directly from (1.10) and (1.13). □

REMARK 1.2. Since the use of μ^{-1} is an inconvenient form of an algebraic relation, the relation (1.17) should read as the following equivalent relation:

$$[\mu, \Xi_i] = \mu * \Xi_i * \mu, \quad [\mu, \Xi_i^*] = -\mu * \Xi_i^* * \mu. \tag{1.21}$$

We show later that (1.17) is preferable to (1.21).

By (1.19–20), we also see that

$$\Xi_1 * \Xi_1^* + \Xi_2 * \Xi_2^* = 1 + \mu. \tag{1.22}$$

DEFINITION 1.3. We denote by $\mathscr{A}$ the algebra generated by $\{\mu, \Xi_1, \Xi_2, \Xi_1^*, \Xi_2^*\}$ with relations (1.17–1.20).

2. Smooth algebras

Using deformation quantization, we give another approach to extending the algebra $\mathscr{A}$. Let $\zeta_1, \bar{\zeta}_1, \zeta_2, \bar{\zeta}_2$ be complex coordinates on $\boldsymbol{C}^2$ and $\boldsymbol{C}[\zeta, \bar{\zeta}, \hbar]$ the space of all polynomials on $\boldsymbol{C}^2$ with coefficients in the polynomials of $\hbar$. The Wick algebra W is linearly isomorphic to $\boldsymbol{C}[\zeta, \bar{\zeta}, \hbar]$ and its associative product $*$ is given by the Moyal product formula:

(2.1) $$a * b = a \exp \hbar \{ \overleftarrow{\partial_\zeta} \cdot \overrightarrow{\partial_{\bar\zeta}} - \overleftarrow{\partial_{\bar\zeta}} \cdot \overrightarrow{\partial_\zeta} \} b,$$

where

$$a(\overleftarrow{\partial_\zeta} \cdot \overrightarrow{\partial_{\bar\zeta}} - \overleftarrow{\partial_{\bar\zeta}} \cdot \overrightarrow{\partial_\zeta})b = \sum_i (\partial_{\zeta_i} a\, \partial_{\bar\zeta_i} b - \partial_{\bar\zeta_i} a\, \partial_{\zeta_i} b).$$

The formula (2.1) extends naturally to the associative product on $C^\infty(\boldsymbol{C}^2)[[\hbar]]$. The associative algebra $(C^\infty(\boldsymbol{C}^2)[[\hbar]], *)$ is called the *deformation quantization* of $C^\infty(\boldsymbol{C}^2)$. As a general reference for deformation quantizations, we refer to e.g. [**B**], [**OMY2**], [**A**]. Here, $C^\infty(\boldsymbol{C}^2)[[\hbar]]$ is the set of formal power series with values in $C^\infty(\boldsymbol{C}^2)$ with the formal parameter $\hbar$. We endow $C^\infty(\boldsymbol{C}^2)$ and $C^\infty(\boldsymbol{C}^2)[[\hbar]]$ the C^∞ topology and the direct product topology, respectively. Then the Wick algebra W is a dense subalgebra of $(C^\infty(\boldsymbol{C}^2)[[\hbar]], *)$.

It is easy to see that

(2.2) $$r^2 = \bar\zeta_1 * \zeta_1 + \zeta_2 * \bar\zeta_2 = \bar\zeta_1 \cdot \zeta_1 + \zeta_2 \cdot \bar\zeta_2.$$

As each $\hbar^k$-term in the Moyal product formula (2.1) is expressed as a bidifferential operator, the star-product $*$ has locality. Hence, for any open subset U of $\boldsymbol{C}^2$, we can define the star-product $*$ of the deformation quantization $C^\infty(U)[[\hbar]]$ by the same formula (2.1). Note that any maximal 2-sided ideals (classical points) of $C^\infty(U)[[\hbar]]$ correspond to points of U. In the following, we will work mainly on $\boldsymbol{C}^2_* = \boldsymbol{C}^2 - \{0\}$.

We consider a function $r_\cdot$ as the square root of r^2 with respect to the ordinary commutative product $\cdot$ on the space $\boldsymbol{C}^2_*$. The function $r_\cdot$ is an element of $C^\infty(\boldsymbol{C}^2_*)$ satisfying

(2.3) $$r_\cdot * r_\cdot = r_\cdot \cdot r_\cdot = r^2.$$

We regard $r_\cdot$ as the radial element defined in §1, (1.5) and we will denote $r_\cdot$ by r alone.

We define a one parameter group of automorphisms

(2.4) $$R(e^t) : C^\infty(\boldsymbol{C}^2_*)[[\hbar]] \to C^\infty(\boldsymbol{C}^2_*)[[\hbar]]$$

as follows:

(2.5) $$R(e^t)\zeta_i = e^t \zeta_i, \ R(e^t)\bar\zeta_i = e^t \bar\zeta_i, \ R(e^t)\hbar = e^{2t}\hbar.$$

Define a closed subalgebra $\mathscr{A}^\infty$ of $C^\infty(\boldsymbol{C}^2_*)[[\hbar]]$ by

(2.6) $$\mathscr{A}^\infty = \{ f \in C^\infty(\boldsymbol{C}^2_*)[[\hbar]]; R(e^t)f = f \}.$$

Under the relative topology from $C^\infty(\boldsymbol{C}^2_*)[[\hbar]]$, $\mathscr{A}^\infty$ is a complete topological associative algebra. As in (1.10), we set

$$\mu = -2\hbar r^{-2}, \quad \Xi_1 = r^{-1} * \zeta_1, \quad \Xi_2 = r^{-1} * \zeta_2,$$
$$\Xi_1^* = \bar\zeta_1 * r^{-1}, \quad \Xi_2^* = \bar\zeta_2 * r^{-1}.$$

Since the above elements have the same relations as in Lemma 1.1, the algebra $\mathscr{A}$ is densely embedded in $\mathscr{A}^\infty$. Note that $\mu+1$ is not invertible in $\mathscr{A}$ but it is in $\mathscr{A}^\infty$. The following is easy to see:

LEMMA 2.1. *For any nonnegative integer l, we have*

$$\mathscr{A}^\infty \cap \hbar^l * C^\infty(\boldsymbol{C}_*^2)[[\hbar]] = \mu^l * \mathscr{A}^\infty.$$

Using Lemma 2.1, we see the following:

THEOREM 2.2. *Set $B = \mathscr{A}^\infty \cap C^\infty(\boldsymbol{C}_*^2)$.*

(A.1) $[\mu, \mathscr{A}^\infty] \subset \mu * \mathscr{A}^\infty * \mu$.

(A.2) $[\mathscr{A}^\infty, \mathscr{A}^\infty] \subset \mu * \mathscr{A}^\infty$, *where $[a,b] = a*b - b*a$ is the commutator bracket.*

(A.3) $\mathscr{A}^\infty = B \oplus \mu * \mathscr{A}^\infty$ *(topological direct sum).*

(A.4) *(Self-similarity.) The mappings $\mu* : \mathscr{A}^\infty \to \mu * \mathscr{A}^\infty$, $*\mu : \mathscr{A}^\infty \to \mathscr{A}^\infty * \mu$ defined by $a \to \mu * a$, $a \to a * \mu$ respectively are linear isomorphisms.*

(A.5) *$a \to \bar{a}$ is an involutive anti-automorphism such that $\bar{\mu} = \mu$.*

By property (A.3), we see for any positive integer N, $\mathscr{A}^\infty$ decomposes as follows:

$$\mathscr{A}^\infty = B \oplus \mu * B \oplus \cdots \oplus \mu^{N-1} * B \oplus \mu^N * \mathscr{A}^\infty. \tag{2.7}$$

$\mathscr{A}^\infty$ satisfies

(A.6) $\bigcap_k \mu^k * \mathscr{A}^\infty = \{0\}$.

The algebra $\mathscr{A}^\infty$ is called the *non-commutative contact algebra* on S^3. The reason for the terminology is as follows. Let us first introduce some general notation for complete topological algebras.

We call a complete topological associative algebra $\tilde{\mathscr{A}}$ a *regulated algebra*, (or more explicitly *μ-regulated algebra*) if there exists an element μ and a closed subspace $\tilde{B}$ satisfying (A.1)–(A.5). μ is called the *regulator* of $\tilde{\mathscr{A}}$. Note that (2.7) holds for any μ-regulated algebra. A μ-regulated algebra $\tilde{\mathscr{A}}$ is called *formal* if it satisfies (A.6). By (2.7), a formal μ-regulated algebra $\tilde{\mathscr{A}}$ may be denoted by $\tilde{\mathscr{A}} = \tilde{B}[[\mu]]$.

On any formal μ-regulated algebra $\tilde{\mathscr{A}}$, the axioms (A.1) and (A.4) permit us to introduce the formal symbol μ^{-1} such that $\mu^{-1} * \mu = \mu * \mu^{-1} = 1$. It gives a derivation $[\mu^{-1}, a]$ of $\tilde{\mathscr{A}}$ defined by

$$[\mu^{-1}, a] = -\mu^{-1} * [\mu, a] * \mu^{-1}. \tag{2.8}$$

It is easy to see that

$$[\mu^{-1} * \tilde{\mathscr{A}}, \tilde{\mathscr{A}}] \subset \tilde{\mathscr{A}}, \quad [\mu^{-1} * \tilde{\mathscr{A}}, \mu^{-1} * \tilde{\mathscr{A}}] \subset \mu^{-1} * \tilde{\mathscr{A}}. \tag{2.9}$$

Let (m_0, m_1) be the maximal integers such that

$$[\mu^{-1}, \tilde{\mathscr{A}}] \subset \mu^{m_0} * \tilde{\mathscr{A}}, \quad [\tilde{\mathscr{A}}, \tilde{\mathscr{A}}] \subset \mu^{m_1} * \tilde{\mathscr{A}}.$$

If $[\mu^{-1}, \tilde{\mathscr{A}}] = \{0\}$, we put $m_0 = \infty$. We call (m_0, m_1) the *weight* of $\tilde{\mathscr{A}}$. A formal μ-regulated algebra $(\tilde{\mathscr{A}}, *)$ of weight $(\infty, 1)$ will be called a *quantum Poisson algebra*. In particular, the deformation quantization $C^\infty(\boldsymbol{C}^2)[[\hbar]]$ of $C^\infty(\boldsymbol{C}^2)$ is a formal $\hbar$-regulated algebra of weight $(\infty, 1)$, and $\mathscr{A}^\infty$ in Theorem 2.2 is a formal μ-regulated algebra of weight $(0, 1)$ respectively.

For any formal μ-regulated algebra $\tilde{\mathscr{A}} = \tilde{B}[[\mu]]$, its associative product $*$ is determined by giving $a * b$ for $a, b \in \tilde{B}$: Set

$$a * b = \sum_{k \geq 0} \mu^k * \pi_k(a, b), \quad \pi_k(a, b) \in \tilde{B}. \tag{2.10}$$

We put

$$[\mu^{-1}, a] = \xi_0(a) + \cdots + \mu^k * \xi_k(a) + \cdots. \tag{2.11}$$

(2.11) is used for computing the following:

$$a * \mu = \mu * a + \mu^2 * \xi_0(a) + \mu^3 * (\xi_1(a) - \xi_0^2(a)) + \cdots. \tag{2.12}$$

A commutative associative product $\cdot$ on $\tilde{\mathscr{A}}/\mu\tilde{\mathscr{A}}$ induces one on $\tilde{B}$ by the identification $\tilde{B}$ with $\tilde{\mathscr{A}}/\mu\tilde{\mathscr{A}}$.

It is easy to see that π_1 in (2.10) is a biderivation of $(\tilde{B}, \cdot)$ and ξ_0 in (2.11) is a derivation of $(\tilde{B}, \cdot)$. We remark here that one can change the filtration by a linear isomorphism $a \to a + \mu * L(a)$ of $\tilde{B}[[\mu]]$ defined by any continuous linear operator $L : \tilde{B} \to \tilde{B}$.

Definition 2.3. A formal (μ-)regulated algebra $\tilde{\mathscr{A}}$ will be called a (μ-)regulated *smooth algebra* if there exists a filtration $\tilde{\mathscr{A}} = \tilde{B}[[\mu]]$ satisfying the following:

(1) $\tilde{B}$ in (A.3) is isomorphic to a subalgebra of the commutative algebra $C^\infty(M)$ of all C^∞ functions on a finite dimensional manifold M, and $\tilde{B} \supset C_0^\infty(M)$ the space of all support compact functions.

(2) With $\tilde{B}$ considered as a subalgebra of $C^\infty(M)$, ξ_k in (2.11) is a linear operator of $\tilde{B}$ into $\tilde{B}$ expressed as a differential operator on M for any $k \geq 0$.

(3) For any $k \geq 0$, π_k in (2.10) is a bilinear operator of $\tilde{B} \times \tilde{B}$ into $\tilde{B}$ expressed as a bidifferential operator on M.

In any smooth algebra, ξ_0 in (2.11) is a C^∞ vector field on M, called the characteristic vector field, and π_1 in (2.10) is a C^∞ biderivation on M.

Remark 2.4. If $B = C^\infty(M)$, the localization theorem in [**OMY2**] shows that the properties (2) and (3) are automatically satisfied.

Definition 2.5. Let π_1^- be the skew symmetric part of π_1. A smooth algebra of weight $(0, 1)$ is called a *noncommutative contact algebra* if the rank of π_1^- in (2.10) is maximal at each point of M.

Remark 2.6. Here we give a little general set up and remarks. Suppose there are a derivation ξ_0 and a skew biderivation π^-. We set $\{f, g\} = 2\pi^-(f, g)$. $(C^\infty(M), \xi_0, \{\ ,\ \})$ is called a *Jacobi algebra* (cf. [**L**]), if the following are satisfied:

(1) $\xi_0(\{f, g\}) = \{\xi_0(f), g\} + \{f, \xi_0(g)\}$,

(2) $\{f, g\}_L = f\xi_0(g) - \xi_0(f)g + \{f, g\}$ defines a Lie algebra structure on $C^\infty(M)$.

The notion of Jacobi algebras can be obtained by considering the first term (μ^0-term) and the second term (μ^1-term) of the product (2.10) and (2.11) in μ-regulated algebras together with the fact that $\mu^{-1}\tilde{\mathscr{A}}$ forms a Lie algebra under the commutator bracket.

A Jacobi algebra with $\xi_0 = 0$ is called a *Poisson algebra*. A Poisson algebra is called a *symplectic algebra*, if the rank of $\{\ ,\ \}$ is $\dim M$ at every point.

$(C^\infty(M), \xi_0, \{\ ,\ \})$ is a *contact algebra*, if ξ_0 vanishes nowhere and $\mathrm{rank}\{\ ,\ \} = \dim M - 1$ everywhere.

The space of C^∞ functions on a contact manifold naturally forms a contact algebra. Moreover, any contact algebra extends to a noncommutative contact algebra, that is *any contact algebra is quantizable.*

Although this fact seems to be a restatement of the result seen in Appendix of Boutet de Monvel and Guillemin [**BG**], this can be proved within the deformation theory without using operator representations. Namely, this is proved by remarking first that any contact algebra is obtained from a symplectic algebra as the subalgebra of invariant functions of free conformally symplectic action of $\boldsymbol{R}$, and next that the symplectic algebra can be deformation quantizable so that the free conformally symplectic action is lifted to a one parameter automorphism group. We gave the short proof in [**OMMY1**].

As for noncommutative contact algebras, let us only note their local property. By properties (2) and (3), the product $*$ and the action of $\mathrm{ad}(\mu^{-1})$ of a regulated smooth algebra $\tilde{\mathscr{A}}$ extend to $C^\infty(M)[[\mu]] = \prod \mu^k * C^\infty(M)$ by the same product formulas. Moreover, they have locality: that is, for any open subset U of M one can make $C^\infty(U)[[\mu]]$ a smooth algebra by the same formulae, where we give the C^∞ topology on $C^\infty(U)$ and the direct product topology on $C^\infty(U)[[\mu]]$. We will refer to this topology as the *direct C^∞ topology*. Moreover, the associative product $*$ of a regulated smooth algebra can be defined on $C^\infty(\tilde{M})[[\mu]]$ for the universal covering space $\tilde{M}$ of M.

Recall the algebra $\mathscr{A}^\infty$ in Theorem 2.2 is a closed subalgebra of $C^\infty(\boldsymbol{C}_*^2)[[\hbar]]$. By (1.20), the following is easy to see:

PROPOSITION 2.7. *The noncommutative contact algebra $\mathscr{A}^\infty = B[[\mu]]$ given in Theorem 2.2 is a μ-regulated smooth algebra with $B = C^\infty(S^3)$.*

Although, as seen in §1, $\mu, \Xi_1, \Xi_2, \Xi_1^*, \Xi_2^*$ are represented as matrices, we regard them as elements of $C^\infty(S^3)[[\mu]]$ without matrix representations.

LEMMA 2.8. *For any formal power series $f(t)$ of t, we have*

$$\Xi_i * f(\mu) = f\left(\frac{\mu}{1+\mu}\right) * \Xi_i, \quad f(\mu) * \Xi_i = \Xi_i * f\left(\frac{\mu}{1-\mu}\right).$$

PROOF. By

$$\zeta_i * r^2 = (r^2 - 2\hbar) * \zeta_i, \quad r^2 * \zeta_i = \zeta_i * (r^2 + 2\hbar),$$

we see that $\mu * \Xi_i = (1+\mu) * \Xi_i * \mu$ and $\mu * \Xi_i = \Xi_i * \mu/(1-\mu)$. The desired equalities follow easily by polynomial approximation of $f(\mu)$. □

REMARK 2.9. Note that r^{-1} is not equal to $1/r.$ defined by $(1/r.)r = 1$. We shall distinguish these inverses by the notation r^{-1} and $1/r.$. Similar remarks will occur for a general $f \in C^\infty(\boldsymbol{C}_*^2)[[\hbar]]$ in §3.

REMARK 2.10. One might try to construct the noncommutative 3-sphere by "restricting" the algebras $C^\infty(\boldsymbol{C}_*^2)[[\hbar]]$ to the *energy surface* $r^2 = 1$. However, this is problematic, because $r^2 - 1 = 0$ implies $[\zeta_i, r^2] = 0$ and hence $\hbar\zeta_i = 0$ for $i = 1, 2$.

To consider the *energy surface* $f=1$ for given function f, we have to *localize* the algebra to an open neighborhood U of the subset $f=1$, on which the element f^{-1} can be formally defined. Then, we consider the algebra generated by $f^{-1} * \{\text{generators}\}$.

3. Calculus on smooth algebras

In a matrix algebra or an operator algebra it is possible to define $f(u)$ for a matrix or an operator u for a suitable class of smooth function $f(t)$, which we will call a *transcendental computation*. Since operator calculus is sometimes too heavy to manage, we will extend transcendental computations to the algebra $\mathscr{A}^\infty$ without operator representations. Through such abstract treatment, we can consider localized algebras.

Let $\tilde{\mathscr{A}}$ be a regulated smooth algebra with regulator μ. Note that, by (3) in Definition 2.3, the product $*$ on $\tilde{\mathscr{A}}$ extends naturally to $C^\infty(M)[[\mu]]$. Thus, in this section, we may assume the following:

$$\tilde{\mathscr{A}} = C^\infty(M)[[\mu]] \quad \text{and} \quad \tilde{B} = C^\infty(M).$$

As we do not have operator representations in a smooth algebra, it is not trivial that ${}^*\!\sqrt{a * a^*}$ is an element of $\tilde{\mathscr{A}}$.

To manage this safely, we remark that the Cauchy's integral theorem holds for any $C^\infty(M)[[\mu]]$-valued holomorphic functions. To apply the Cauchy's integral theorem, we first remark the following:

LEMMA 3.1. *Let $\tilde{f} = \sum_{k\geq 0} \mu^k * f_k \in C^\infty(M)[[\mu]]$. If the* 0*-th term* ($\mu^0$*-component*) $f_0 \in C^\infty(M)$ *of $\tilde{f}$ is non-zero on M, then $\tilde{f}$ is invertible, i.e., there exists $\tilde{f}^{-1} \in C^\infty(M)[[\mu]]$ such that $\tilde{f} * \tilde{f}^{-1} = \tilde{f}^{-1} * \tilde{f} = 1$.*

PROOF. Set $\tilde{f}^{-1} = 1/f_0 + \mu * g_1/f_0 + \mu^2 * g_2/f_0 + \cdots$ and consider the equation

$$\left\{\frac{1}{f_0} + \mu * \frac{g_1}{f_0} + \mu^2 * \frac{g_2}{f_0} + \cdots\right\} * \tilde{f} = 1 \tag{3.1}$$

at each k-th term in μ^k. Using (2.10), and (2.11), we compute the k-th term in μ, which produces g_k inductively. □

LEMMA 3.2. *Let $\tilde{f} = \sum_{k\geq 0} \mu^k * f_k \in C^\infty(M)[[\mu]]$. Assume that the closure $\bar{R}(f_0)$ of the range of f_0 is bounded in $\mathbf{C}$. Then, for any holomorphic function $\phi(z)$ on an open subset U containing $\bar{R}(f_0)$, an element $\phi_*(\tilde{f})$ is defined as an element of $C^\infty(M)[[\mu]]$ by the following formula*:

$$\phi_*(\tilde{f}) = \frac{1}{2\pi i}\int_C \phi(z)(z-\tilde{f})_*^{-1}\,dz \tag{3.2}$$

where C is a simple closed curve in U containing $\bar{R}(f_0)$ in the interior.

In particular, if $\phi(z)$ is approximated by a series of polynomials $\{p_n(z)\}$, then $\phi_*(\tilde{f})$ is approximated by a series $\{p_n(\tilde{f})\}$ in $C^\infty(M)[[\mu]]$.

PROOF. $z-\tilde{f}$ is invertible in this algebra whenever z moves on the contour C. As in the above lemma, we see that $(z-\tilde{f})_*^{-1}$ is written in the form

$$(z-\tilde{f})_*^{-1} = \frac{1}{z-f_0} + \mu * \frac{g_1}{z-f_0} + \mu^2 * \frac{g_2}{z-f_0} + \cdots \tag{3.3}$$

where g_i depends on z continuously. It follows that the right hand side of (3.2) is well defined.

Expanding $(z-\tilde{f})_*^{-1}$ in a Neumann series, we see easily that

$$\frac{1}{2\pi i}\int_C z^m(z-\tilde{f})_*^{-1}dz = \tilde{f}_*^m \quad (m \geq 0).$$

Thus the second assertion follows directly. □

For the computation of the value of (3.2) at each point $p \in M$, it is enough to know the value $(z-\tilde{f})_*^{-1}(p)$ by choosing C to be a small circle with center at $f_0(p)$, or only to know the jet $j^\infty\tilde{f}(p)$.

The above remark is useful on a noncompact manifold M, since there are unbounded elements in $C^\infty(M)$. Even if the contour C is so small that C is covered by $\bar{R}(f_0)$, (3.2) still defines a smooth function $\phi_*(f)$ on the open subset $f_0^{-1}(D)$ by locality, where D is the interior of C.

Taking a small circle C_p around $f_0(p)$ gives the following:

THEOREM 3.3. *For any holomorphic function $\phi(z)$ on an open subset U containing $R(f_0)$, an element $\phi_*(\tilde{f})$ is defined by*

$$\phi_*(\tilde{f})(p) = \frac{1}{2\pi i}\int_{C_p} \phi(z)(z-\tilde{f})_*^{-1}(p)\,dz.$$

By Theorem 3.3, we have

COROLLARY 3.4. *Let ϕ be an entire function on $\boldsymbol{C}$. Then for any $\tilde{f} \in C^\infty(M)[[\mu]]$ $\phi_*(\tilde{f})$ is defined as an element of $C^\infty(M)[[\mu]]$.*

By a direct computation we have also the following formula:
Let $\tilde{f} = \sum_{k\geq 0}\mu^k * f_k \in C^\infty(M)[[\mu]]$. For a point $p \in M$, take a contour C_p around $f_0(p)$. If $z, w \in \boldsymbol{C}$ are outside C_p, then multiplying $(z-\tilde{f}) * (w-\tilde{f})(p)$ to both sides we have

$$(z-\tilde{f})_*^{-1} * (w-\tilde{f})_*^{-1}(p) = \frac{1}{2\pi i}\int_{C_p} \frac{1}{(z-\eta)(w-\eta)}(\eta-\tilde{f})_*^{-1}(p)\,d\eta. \tag{3.4}$$

Using (3.4), we get the following formulas:

LEMMA 3.5. (i) *Let ϕ, ψ be holomorphic functions on a domain containing the range $R(f_0)$. It holds $\phi_*(\tilde{f}) * \psi_*(\tilde{f}) = (\phi\psi)_*(\tilde{f})$.*

(ii) *Let Φ be a holomorphic function on a domain containing the range $R(\phi)$. For the compositions, it holds $\Phi_*(\phi_*(\tilde{f})) = (\Phi \circ \phi)_*(\tilde{f})$ where $\circ$ indicates the composition of functions.*

Corollary 3.4 shows in particular that $e_*^{t\tilde{f}} = \exp_*(t\tilde{f})$ is defined for any element $\tilde{f} \in C^\infty(M)[[\mu]]$. Recall that $\tilde{g}(t) = e_*^{t\tilde{f}}$ satisfies the differential equation

$$\frac{d}{dt}\tilde{g}(t) = \tilde{f} * \tilde{g}(t), \quad g(0) = 1. \tag{3.5}$$

LEMMA 3.6. *The solution has the form*

$$e_*^{t\tilde{f}} = e^{tf_0} + \mu * e^{tf_0} g_1(t) + \cdots + \mu^k * e^{tf_0} g_k(t) + \cdots \tag{3.6}$$

with $g_k(t) = g_k(t, f_0, f_1, \cdots, f_k)$ *a polynomial in* t *whose coefficients involve derivatives of* $f_0, f_1, \cdots, f_k$.

PROOF. Assume that $\tilde{g}(t) = \sum_k \mu^k * h_k(t)$. Substituting this into (3.5) and using the formulae (2.10) and (2.11), we have the following infinite system of differential equations:

$$\frac{d}{dt}h_0 = f_0 \cdot h_0,$$

$$\frac{d}{dt}h_1 = f_0 \cdot h_1 + f_1 \cdot h_0 + \pi_1(f_0, h_0),$$

$$\frac{d}{dt}h_2 = f_0 \cdot h_2 + f_1 \cdot h_1 + f_2 \cdot h_0 + \pi_1(f_0, h_1) + \pi_1(f_1, h_0) + \pi_2(f_0, h_0) + \xi_0(f_0)h_1,$$

$\cdots\cdots$

$\cdots\cdots$

where $h_0(0) = 1$ and $h_k(0) = 0$ for $k = 1, 2, \cdots$.

We remark that the solution of the differential equation

$$\frac{d}{dt}h(t) = f_0 \cdot h(t) + H(t), \quad h(0) = 0$$

is given by

$$h(t) = \int_0^t e^{(t-\tau)f_0} \cdot H(\tau)\, d\tau. \tag{3.7}$$

If $H(t) = H(t,p)$ is given as $e^{tf_0(p)}P_l(t,p)$ for a polynomial P_l of order l in t, then (3.7) becomes

$$h(t) = e^{tf_0} \int_0^t P_l(\tau, p)\, d\tau = e^{tf_0} P_{l+1}.$$

The lemma follows. □

The above idea is applicable to various C^∞ functions. Let us give some examples of C^∞ *functions* ϕ on the real line $\boldsymbol{R}$ such that $\phi_*(\tilde{f})$ is defined. Although we do not use these properties in this paper, these materials will be useful in elsewhere.

Case 1. Let $\phi(t)$ be a rapidly decreasing function on the real line $\boldsymbol{R}$ and $\hat{\phi}(\xi)$ the Fourier transform of $\phi(t)$. Let $\tilde{f} = \sum_{k \geq 0} \mu^k f_k$ be an element of $C^\infty(M)[[\mu]]$ such that f_0 is real valued. Using the Fourier inversion formula, we define

$$\phi_*(\tilde{f}) = \frac{1}{2\pi} \int_{-\infty}^{\infty} \hat{\phi}(\xi) e_*^{i\xi\tilde{f}}\, d\xi. \tag{3.8}$$

Since $\hat{\phi}(\xi)$ is rapidly decreasing in ξ, Lemma 3.6 shows that the integral (3.8) defines an element of $C^\infty(M)[[\mu]]$. As in Theorem 3.3, the value $\phi_*(\tilde{f})(p)$ is given by evaluating $e_*^{i\xi\tilde{f}}$ at $p \in M$.

It is not hard to see that

$$\phi_*(\tilde{f}) * \phi'_*(\tilde{f}) = (\phi\phi')_*(\tilde{f}). \tag{3.9}$$

Case 2. We consider a real valued function with the exponential growth with slightly strong properties: let $\phi(t)$ be a C^∞ function $\phi(t)$ on the real line $\boldsymbol{R}$ satisfying the following; (a) $\phi(t) = 0$ on $t \leq -K$ for some positive constant K, and (b) there is a constant α such that $\phi(t)e^{-\alpha t}$ is rapidly decreasing. For a real valued function $\phi(t)$ with the above properties (a) and (b), we consider an integral similar to the Laplace transform

$$V(p) = \int_{-K}^{\infty} \phi(t)e^{-pt}\,dt.$$

For a constant $c > \alpha$, $V(c + \xi i)$ is the Fourier transform of $\phi(t)e^{-ct}$, and hence $V(c + \xi i)$ is rapidly decreasing in ξ. Therefore the Fourier inversion formula shows that ϕ is recaptured in the shape of the Bromwich integral

$$\phi(t) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} V(p)e^{pt}\,dp.$$

Let $\tilde{f} = \sum_{k\geq 0}\mu^k * f_k$ be an element of $C^\infty(M)[[\mu]]$ such that f_0 is real valued. Let $\phi(t)$ be a C^∞ function on $\boldsymbol{R}$ with properties (a) and (b). We consider the following quantity:

$$\phi_*(\tilde{f}) = \frac{1}{2\pi i}\int_{c-i\infty}^{c+i\infty} V(p)e_*^{p\tilde{f}}\,dp. \tag{3.10}$$

By Lemma 3.6, $e_*^{(c+\xi i)\tilde{f}}$ is written in the shape (3.6). Since $V(c + \xi i)$ is rapidly decreasing in ξ, the integral (3.10) defines an element of $C^\infty(M)[[\mu]]$. Similar to the cases of Theorem 3.3 and the Case 1, the value $\phi_*(\tilde{f})(p)$ at $p \in M$ is given by

$$\phi_*(\tilde{f})(p) = \frac{1}{2\pi}\int_{-\infty}^{\infty} V(c + \xi i)e_*^{(c+\xi i)\tilde{f}}(p)\,d\xi. \tag{3.11}$$

The following is an important lemma for computations:

LEMMA 3.7 (Bumping lemma). *Let* $a = \sum \mu^k * a_k$, $b = \sum \mu^k * b_k \in C^\infty(M)[[\mu]]$ *and* $\phi(s)$ *be a smooth function on the range* $R(a_0 \cdot b_0)$. *If* $\phi(s)$ *satisfies any of the assumptions in Theorem* 3.3, *Case* 1 *or Case* 2, *then we have the following identity*:

$$a * \phi_*(b * a) = \phi_*(a * b) * a. \tag{3.12}$$

PROOF. It is obvious that $a * (b * a)^k = (a * b)^k * a$. Thus, (3.12) holds for any polynomial ϕ. Suppose ϕ is a holomorphic function on a neighborhood of $R(a_0 \cdot b_0)$. For $c = a * b$, $c' = b * a$, we define $\phi_*(c)$, $\phi_*(c')$ by Theorem 3.3. The desired identity follows from the polynomial approximation theorem.

By the above proof, we see in particular that

$$a * e_*^{zb*a} = e_*^{za*b} * a. \tag{3.13}$$

The desired identity follows directly from (3.8) and (3.10). □

The radial element r given in §1 is redefined as an element of $C^\infty(\boldsymbol{C}^2 - \{0\})[[\mu]]$ as follows: Remark that $\sqrt{z}$ is a holomorphic function on an appropriate sector in $\boldsymbol{C} - \{0\}$. Hence recalling that

$$r^2 = \zeta_1 * \bar{\zeta}_1 + \bar{\zeta}_2 * \zeta_2 = \zeta_1 \cdot \bar{\zeta}_1 + \bar{\zeta}_2 \cdot \zeta_2$$

and that $r^2 = (r^2)_0$, we define r by

$$r = \sqrt[*]{r^2}. \tag{3.14}$$

By Theorem 3.3, r is defined as an element of $C^\infty(\boldsymbol{C}^2 - \{0\})[[\mu]]$. It is evident by Lemma 3.5 that $r * r = r^2$.

As in §1, (1.6), we denote by $\sqrt[*]{\ }$ the square root of an element in the algebra. We call $f \in C^\infty(M)[[\mu]]$ a *unitary element* if $f * f^* = f^* * f = 1$. Using Theorem 3.3 and Lemma 3.7, we have the following application:

LEMMA 3.8 (Polar decomposition). *Assume that* $f = \sum \mu^k * f_k \in C^\infty(M)[[\mu]]$ *satisfies* $|f_0| > 0$. *Then,* $|f| = {}^*\!\sqrt{f * f^*}$ *and* $|f|^{-1}$ *are defined as elements of* $C^\infty(M)[[\mu]]$. *Moreover* $|f|^{-1} * f$ *is a unitary element.*

PROOF. By Theorem 3.3, $|f| = {}^*\!\sqrt{f * f^*}$ is well defined. $|f|^{-1}$ exists and $f^* * f$ is invertible by Lemma 3.1. Using Lemma 3.7, we have

$$\begin{aligned} |f|^{-1} * f * (|f|^{-1} * f)^* &= |f|^{-1} * f * f^* * |f|^{-1} \\ &= |f|^{-2} * f * f^* = 1 \\ (|f|^{-1} * f)^* * |f|^{-1} * f &= f^* * |f|^{-2} * f \\ &= f^* * (f * f^*)^{-1} * f \\ &= (f^* * f)^{-1} * f^* * f = 1. \end{aligned} \tag{3.15}$$

□

Let $\tilde{M}$ be the universal covering space of M and let $C^\infty_H(\tilde{M})[[\mu]]$ be the subspace consisting of all hermitian elements, i.e. elements such that $f^* = f$.

LEMMA 3.9. *For any unitary element* $a \in C^\infty(M)[[\mu]]$, *there is an element* $\tau \in C^\infty_H(\tilde{M})[[\mu]]$ *such that* $a = e_*^{i\tau}$.

PROOF. It is obvious that $e_*^{i\tau}$ is a unitary element. For the converse, we set the universal covering $\tilde{\boldsymbol{C}}_*$ of the domain $\boldsymbol{C}_* = \boldsymbol{C} - \{0\}$ and consider the function log as a function on $\tilde{\boldsymbol{C}}_*$. We denote by $\tilde{z}$ the lift of $z \in \boldsymbol{C}_*$ to the universal covering. The μ^0-term a_0 of $a = \sum \mu^k * a_k$ satisfies $|a_0(p)| = 1$ for all $p \in M$. Notice the contour integral around $a_0(p)$, $(1/2\pi i)\int_{c_p} \log \tilde{z}\,(z - a_0(p))_*^{-1}\,dz$ defines a smooth function on the universal covering $\tilde{M}$. Hence, we define the element $\log_* a \in C^\infty(\tilde{M})[[\mu]]$ by

$$\log_* a\,(\tilde{p}) = \frac{1}{2\pi i}\int_{c_p} \log \tilde{z}\,(z - a)_*^{-1}(\tilde{p})\,dz, \quad \tilde{p} \in \tilde{M} \tag{3.16}$$

where c_p is a small contour around $a_0(p)$.

Lemma 3.5 (ii) shows $\exp_*(\log_* a) = a$. Thus, $i\tau = \log_* a$ is a desired element of $C^\infty(\tilde{M})[[\mu]]$. □

As a byproduct, one can define "noncommutative polar coordinates" on $\boldsymbol{C}_* = \boldsymbol{C} - \{0\}$. Consider the smooth algebra $C^\infty(\boldsymbol{C}_*)[[\hbar]]$ such that $[\zeta, \bar{\zeta}] = -2\hbar$ (cf. §1). By the above argument we set $r = {}^*\sqrt{\zeta * \bar{\zeta}}$,

$$e_*^{i\tau} = r^{-1} * \zeta, \quad \text{and} \quad \rho = r^2/2 \tag{3.17}$$

where τ is an element of the universal covering space of $\boldsymbol{C}_*$.

LEMMA 3.10 (Noncommutative polar coordinates). *On* $C^\infty(\boldsymbol{C}_*)[[\hbar]]$, $[\zeta, \bar{\zeta}] = -2\hbar$ *if and only if* $[\rho, \tau] = -i\hbar$.

PROOF. If $[\rho, \tau] = -i\hbar$ holds then

$$\zeta * \bar{\zeta} = r^2, \quad \bar{\zeta} * \zeta = e^{-i\tau} * 2\rho * e^{i\tau} = r^2 + 2\hbar.$$

For the converse, taking an appropriate sector or considering the universal covering space of $\boldsymbol{C}_*$, we compute

$$[\rho, i\tau] = \left[\frac{r^2}{2}, \log_*(r^{-1} * \zeta)\right] = \zeta^{-1} * r * r^{-1} * \left[\frac{r^2}{2}, \zeta\right] = \hbar. \tag{3.18}$$ □

4. Local generator systems for $\mathscr{A}^\infty$

Recall that $\mathscr{A}^\infty \subset C^\infty(\boldsymbol{C}_*^2)[[\hbar]]$. Let $U_+ = \{\zeta_1 \neq 0\}$, $U_- = \{\zeta_2 \neq 0\}$. By the local property (cf. Proposition 2.7), the algebra extends to $C^\infty(U_\pm)[[\hbar]]$, in which one can consider subalgebras of $R(e^t)$-invariant elements. These are in fact the localizations of $\mathscr{A}^\infty$, and will be denoted by $\mathscr{A}^\infty_{U_+}$, $\mathscr{A}^\infty_{U_-}$. The algebra $\mathscr{A}^\infty$ is understood as the patch work of these localized algebras.

In $\mathscr{A}^\infty_{U_+}$ (resp. $\mathscr{A}^\infty_{U_-}$), we have an element $z = \zeta_1^{-1} * \zeta_2 = \Xi_1^{-1} * \Xi_2$ (resp. $w = \zeta_2^{-1} * \zeta_1 = \Xi_2^{-1} * \Xi_1$). It is easy to see by $\zeta_i * r^2 = (r^2 - 2\hbar) * \zeta_i$ that

$$[\mu, z] = [\mu, \bar{z}] = 0, \quad [\mu, w] = [\mu, \bar{w}] = 0, \tag{4.1}$$

but z and w are not represented by matrices. Although we lose the matrix representations, we can obtain some geometric pictures.

First of all we note the following:

LEMMA 4.1.

$$[z * \bar{z}, \bar{z} * z] = 0, \quad [w * \bar{w}, \bar{w} * w] = 0. \tag{4.2}$$

PROOF. This is easy to see from $[\zeta_i * \bar{\zeta}_i, \bar{\zeta}_i * \zeta_i] = 0$.

On $\mathscr{A}^\infty_{U_+}$ (resp. $\mathscr{A}^\infty_{U_-}$), $\{\mu, z, \bar{z}, \Xi_1, \Xi_1^*\}$ (resp. $\{\mu, w, \bar{w}, \Xi_2, \Xi_2^*\}$) generates a dense subalgebra.

The following relations are given for $\{\mu, z, \bar{z}, \Xi_1, \Xi_1^*\}$, with the corresponding relations holding for $\{\mu, w, \bar{w}, \Xi_2, \Xi_2^*\}$:

LEMMA 4.2.

$$\begin{aligned} &[\mu^{-1}, z] = 0, \qquad [z, \Xi_1] = [z, \Xi_2] = 0, \\ &[\bar{z}, \Xi_1^*] = [\bar{z}, \Xi_2^*] = 0, \\ &[z, \Xi_1^*] = -\mu * \Xi_1^{-1} * z * (1 + \mu). \end{aligned} \tag{4.3}$$

$$(4.4)\qquad \begin{aligned}&[z,\bar{z}] = \mu(1+z*\bar{z})*(1+\bar{z}*z),\\ &[z,(1+\bar{z}*z)^{-1}*\bar{z}] = \mu.\end{aligned}$$

$$(4.5)\qquad \begin{aligned}&(1+\bar{z}*z)^{-1} = \Xi_1*\Xi_1^*,\\ &\mu+\bar{z}*z*(1+\bar{z}*z)^{-1} = \Xi_2*\Xi_2^*.\end{aligned}$$

PROOF. Recalling that $z=\Xi_1^{-1}*\Xi_2$ and using (1.17) and (1.18), we have the first two equalities in (4.3). The equality

$$[\Xi_1^{-1}*\Xi_2,\Xi_1^*] = [\Xi_1^{-1},\Xi_1^*]*\Xi_2+\Xi_1^{-1}*[\Xi_2,\Xi_1^*]$$

and (1.19) and Lemma 2.8 show the last equality of (4.3). For (4.4), we have

$$(4.6)\qquad \begin{aligned}[z,\bar{z}] &= \zeta_1^{-1}*\zeta_2*\bar{\zeta}_1^{-1}*\bar{\zeta}_2-\bar{\zeta}_1^{-1}*\bar{\zeta}_2*\zeta_1^{-1}*\zeta_2\\ &= -2\hbar(\bar{\zeta}_1*\zeta_1)^{-1}*(1+(\zeta_1*\bar{\zeta}_1)^{-1}*\bar{\zeta}_2*\zeta_2)\\ &= \mu(1+z*\bar{z})*(1+\bar{z}*z),\end{aligned}$$

and

$$(4.7)\qquad \begin{aligned}[z,(1+\bar{z}*z)^{-1}*\bar{z}] &= [z,(1+\bar{z}*z)^{-1}]*\bar{z}+(1+\bar{z}*z)^{-1}*[z,\bar{z}]\\ &= -(1+\bar{z}*z)^{-1}*[z,\bar{z}]*\{z*(1+\bar{z}*z)^{-1}*\bar{z}-1\}\\ &= \mu.\end{aligned}$$

For the first equality of (4.5), using (1.18–19) and (1.22) gives

$$(4.8)\qquad \begin{aligned}(1+\bar{z}*z)*\Xi_1*\Xi_1^* &= (1+\Xi_2^**\Xi_1^{*-1}*\Xi_1^{-1}*\Xi_2)*\Xi_1*\Xi_1^*\\ &= \Xi_1*\Xi_1^*+\Xi_2^**\Xi_1^{*-1}*\Xi_2*\Xi_1^*\\ &= 1.\end{aligned}$$

The equality $z*((1+\bar{z}*z)^{-1}-\Xi_1*\Xi_1^*)*\bar{z}=0$, combined with (4.7) gives the second equality of (4.5). □

REMARK 4.3. The first equality of (4.4) can be viewed as the quantum version of the Kähler form on the standard sphere, which is given by $\{z,\bar{z}\}=(1+z\bar{z})^2$, and the second one may interpret $(1+\bar{z}*z)^{-1}*\bar{z}$ as the quantum version of the canonical conjugate variable of z. In [St], Šťovíček treat the quantum Kähler form given in the form $[z,\bar{z}]=\mu(1+e^tz*\bar{z})*(1+e^{-t}\bar{z}*z)$.

On $U_+\cap U_-$, we see that z and w are invertible and $z*w=1$. The relations between the two given systems of generators (i.e. the coordinate transformation) are given by

$$(4.9)\qquad \{\mu,w,\bar{w},\Xi_2,\Xi_2^*\} = \{\mu,z^{-1},\bar{z}^{-1},\Xi_1*z,\bar{z}*\Xi_2^*\}.$$

The relations giving constraints are

$$(4.10)\qquad (1+\bar{z}*z)^{-1}-\Xi_1*\Xi_1^*=0,\quad (1+\bar{w}*w)^{-1}-\Xi_2*\Xi_2^*=0.$$

Note that $[z,\Xi_1]=[\bar{z},\Xi_1^*]=0$, and that $z*((1+\bar{z}*z)^{-1}-\Xi_1*\Xi_1^*)*\bar{z}=0$ gives the second equality in (4.10).

By (1.17), we see that eigenvalues of $-\mathrm{ad}(\mu^{-1})$ are integers and the eigenspace of the eigenvalue m is spanned in the coefficients $C^\infty(U_+)$ (resp. $C^\infty(U_-)$) by $\Xi_1^k*\Xi_1^{*l}$ $(k-l=m)$ on U_+ (resp. $\Xi_2^k*\Xi_2^{*l}$ $(k-l=m)$ on U_-). We denote by $\mathscr{A}_m^\infty$ the eigenspace of $-\mathrm{ad}(\mu^{-1})$ with the eigenvalue $m\in\boldsymbol{Z}$.

The coordinate transformation (4.9) may be understood as follows:

(i) First, $\{\mu,z\}$ and $\{\mu,w\}$ give holomorphic local generator systems on the Riemann sphere $P^1(\boldsymbol{C})=S^2$,

(ii) (4.9) defines a holomorphic line bundle L over S^2.

Since $[z,\Xi_i]=0$, $[w,\Xi_i]=0$ $(i=0,1)$ by Lemma 4.2, one may regard Ξ_1, Ξ_2 as *holomorphic sections* of L, although these are not mappings but only elements of the algebra. Such elements will be referred to as *holomorphic q-sections.*

On U_+ (resp. U_-), we see that $\Xi_2=z*\Xi_1$ (resp. $\Xi_1=w*\Xi_2$).

Similarly,

$$(4.11)\qquad \Xi_1^k*\Xi_2^l,\quad (k+l=m)$$

forms a linear basis in the coefficients $\boldsymbol{C}[[\mu]]$ of the space $\mathscr{H}_m$ of all holomorphic q-sections of the holomorphic line bundle L^m. We see easily that $\mathscr{H}_m=\{0\}$ for $m<0$ and $\mathscr{H}_m\subset\mathscr{A}_m^\infty$ for $m\geq 0$. On U_+ (resp. U_-), we see that

$$\Xi_1^k*\Xi_2^l=z^l*\Xi_1^m,\quad (\text{resp.}=w^k*\Xi_2^m).$$

The coordinate transformation is given by

$$(4.12)\qquad \Xi_1^k*\Xi_2^l=w^k*\Xi_2^m=(w^k*\Xi_1^m)*z^m=z^l*\Xi_1^m.$$

Here we define precisely the notion of *constraint relations* (cf. (4.10)). A relation will be called a constraint relation if it gives the defining equation of S^3 in $\boldsymbol{R}^4$ under the condition $\hbar=0$ or $\mu=0$. In our system, relations expressed via commutator brackets do not give constraint relations since commutators are vanishing under $\hbar=0$ or $\mu=0$.

To obtain a local generator system without constraint relations, we use Lemma 3.10. By this lemma, we can define $\tau_\pm\in C^\infty(\tilde{W}_\pm)$ where $\tilde{W}_\pm$ is the universal covering space of $U_\pm-\{0\}$ as follows: set

$$e_*^{i\tau_+}=|\Xi_1|^{-1}*\Xi_1,\quad e_*^{i\tau_-}=|\Xi_2|^{-1}*\Xi_2,\quad (\text{cf. Lemma 3.8}).$$

Then $\{\mu,z,\bar{z},e^{i\tau_+}\}$, $\{\mu,w,\bar{w},e^{i\tau_-}\}$ are local generator systems without constraint relations. The coordinate transformation is given by

$$(4.13)\qquad \{\mu,w,\bar{w},e_*^{i\tau_-}\}=\{\mu,z^{-1},\bar{z}^{-1},e_*^{i\theta_+}*e_*^{i\tau_+}\}$$

where $e_*^{i\theta_+}$ is the unitary part of the polar decomposition of z. By using (1.10) and $z=\zeta_1^{-1}*\zeta_2$, it is not hard to see that

$$(4.14)\qquad e_*^{i\theta_+}*e_*^{i\tau_+}=e_*^{i\tau_+}*e_*^{i\theta_+}=e_*^{i\tau_-}.$$

The coordinate transformation (4.13) may be understood as that of quantum version of Hopf fibration of S^3 over the Riemann sphere $P^1(\boldsymbol{C}) = S^2$.

By (4.10) we have

$$\Xi_1 = {}^*\sqrt{1+\bar{z}*z}^{\,-1} * e^{i\tau_+}, \quad \Xi_2 = {}^*\sqrt{1+\bar{w}*w}^{\,-1} * e^{i\tau_-}. \tag{4.15}$$

By (1.17) we have

$$[\mu^{-1}, e^{i\tau_\pm}] = -e^{i\tau_\pm}.$$

It follows that the eigenspace of $-\mathrm{ad}(\mu^{-1})$ is given by $\mathscr{A}_m^\infty = C^\infty(S^2)[[\mu]] * e^{mi\tau_\pm}$.

Using $\tau_\pm = -i\log e^{i\tau_\pm}$, we see

$$[\mu^{-1}, \tau_\pm] = i, \qquad \text{hence} \quad [\tau_\pm, \mu] = \mu^2 i. \tag{4.16}$$

Functions on S^2 are elements of $\mathscr{A}^\infty$ which commutes with μ. Set

$$\mathscr{A}_0^\infty = \{f \in \mathscr{A}^\infty; [\mu, f] = 0\}. \tag{4.17}$$

$\{U_+; \mu, z\}$ and $\{U_-; \mu, w\}$ may be understood as local holomorphic coordinate systems on S^2 respectively. Since the coordinate transformation $w = 1/z$ does not involve $\bar{z}$, one may say that is *holomorphic*. This motivates the definition of a noncommutative Kähler manifolds:

Definition 4.4. A μ-regulated smooth algebra $(C^\infty(M)[[\mu]], *)$ is called a *noncommutative Kähler manifold*, if there is a simple open covering $\{U_\alpha\}_\alpha$ of M with the following properties:

(1) On each $C^\infty(U_\alpha)[[\mu]]$ there is a local generator system

$$z_\alpha^1, \dots, z_\alpha^m, \bar{z}_\alpha^1, \dots, \bar{z}_\alpha^m \quad \text{such that} \quad [z_\alpha^i, z_\alpha^j] = [\bar{z}_\alpha^i, \bar{z}_\alpha^j] = 0$$

and the matrix $([z_i, \bar{z}_j])$ is non-degenerate.

(2) On any intersection $C^\infty(U_\alpha \cap U_\beta)[[\mu]]$, the coordinate transformation $\Phi_{\alpha\beta}$ is holomorphic, that is

$$\Phi_{\alpha\beta}(z_\beta^i) = f_{\alpha\beta}(z_\alpha^1, \cdots, z_\alpha^m)$$
$$\Phi_{\alpha\beta}(\bar{z}_\beta^i) = \bar{f}_{\alpha\beta}(\bar{z}_\alpha^1, \cdots, \bar{z}_\alpha^m).$$

It is clear that S^2 given above is a noncommutative Kähler manifold.

Remark 4.5. Here we give a little long remark involving a summary of the result of our paper [**OMMY2**].

(i) Karabegov [**K**] gives a slightly different definition of noncommutative Kähler manifolds using both of the $*$-product and the usual commutative product. Definition 4.3 contains only the $*$-product. Note that the commutative structure cannot be specified from a given $*$-product.

(ii) By the localization theorem (cf. [**OMY2**]), combined with the quantum version of Darboux theorem (cf. [**O**]), we see that every deformation quantization $(C^\infty(M)[[\mu]], *)$ on a symplectic manifold M is obtained as a Weyl function algebra of a Weyl manifold W_M constructed on M. In [**OMMY2**], we gave that Weyl manifold W_M for a fixed symplectic manifold M, and hence the deformation quantization of $C^\infty(M)$ is parameterized by a de Rham cohomology class, called Poincaré-Cartan class, $\sum_{k\geq 0}\mu^{2k}c_k(W_M)\in H^2(M)[[\mu^2]]$. This corresponds to the result of Deligne [**De**], which gives Fedosov's construction of $*$-products is parameterized by the same classes. $c_0(W_M)$ is known to be the class determined by the symplectic 2-form.

$\mathscr{A}_0^\infty$ in (4.17) regarded as a deformation quantization of $C^\infty(S^2)$ is labeled by the natural volume form of 2-sphere. Namely, $\mathscr{A}_0^\infty$ gives the Poincaré-Cartan class such that $\sum_{k\geq 1}\mu^{2k}c_k(W_M)=0$.

In [**OMMY2**], we showed that on any Kähler manifold M, we can construct a noncommutative Kähler manifold. More precisely, for a Weyl manifold W_M over M, if the Poincaré-Cartan class $\sum_{k\geq 0}\mu^{2k}c_k(W_M)$ has the property that $\sum_{k\geq 1}\mu^{2k}c_k(W_M)$ vanishes in the cohomology group $H^2(M,\mathscr{O})[[\mu^2]]$ of sheaf of holomorphic functions, then $(C^\infty(M)[[\mu]],*)$ is a noncommutative Kähler manifold.

Furthermore, if the original simplectic manifold is of integral class, then we can construct a noncommutative contact algebra on which $\exp t\,\mathrm{ad}(\mu^{-1})$ gives a free S^1-action, which may be regarded as an S^1-bundle over M.

In the last part of this section, we give a local generator system which may be understood as a canonical local coordinate system on a contact manifold.

On U_+ (resp. U_-) we set

$$\xi_+=\sqrt{2}\,{}^*\!\sqrt{1+z*\bar{z}}^{\,-1}*z,\quad (\text{resp. } \xi_-=\sqrt{2}\,{}^*\!\sqrt{1+w*\bar{w}}^{\,-1}*w).$$

Using (4.4) of Lemma 4.2 and bumping lemma (Lemma 3.7), we see that

$$[\xi_+,\bar{\xi}_+]=2\mu\qquad (\text{resp. } [\xi_-,\bar{\xi}_-]=2\mu). \tag{4.18}$$

Take the polar decomposition $\xi_\pm=\rho_\pm*e_*^{i\theta_\pm}$. It is obvious that $e_*^{i\theta_\pm}$ are the unitary parts of z, w respectively. Then, we have

Lemma 4.6.

$$\left[\frac{1}{2}\rho_+^2,\theta_+\right]=i\mu. \tag{4.19}$$

Proof. By (4.18), we have $\rho_+^2-e_*^{-i\theta_+}*\rho_+^2*e_*^{i\theta_+}=2\mu$. Hence we have

$$[\rho_+^2,e_*^{i\theta_+}]=-2\mu*e_*^{i\theta_+}. \tag{4.20}$$

Taking log in (4.20) we have Lemma 4.6. □

The following is not hard to see:

Lemma 4.7. *On U_+ it holds the following relation*:

$$[\mu,\xi_+]=0,\quad \xi_+*e_*^{i\tau_+}=e_*^{i\tau_+}*\xi_+*{}^*\!\sqrt{1-\mu}^{\,-1}. \tag{4.21}$$

Similar relations hold also on U_-.

Regard $\{\mu,(1/2)\rho_+^2,\theta_+,\tau_+\}$ (resp. $\{\mu,(1/2)\rho_-^2,\theta_-,\tau_-\}$) as a local generator system.

These may be regarded as a canonical local coordinate system on a noncommutative contact algebra, because the commutation relations are given in the following Lemma. Note that $\mathrm{ad}(\tau_\pm)$ plays the role of the degree operator field (cf. [**OMY**]).

Proposition 4.8. *Following relations hold on* $U_\pm$:

$$[\mu,\rho_\pm^2]=[\mu,\theta_\pm]=0,\quad \left[\frac{1}{2}\rho_\pm^2,\theta_\pm\right]=i\mu. \tag{4.22}$$

$$[\tau_\pm,\mu]=i\mu^2,[\tau_\pm,\theta_\pm]=0,\left[\tau_\pm,\frac{1}{2}\rho_\pm^2\right]=i\mu\frac{1}{2}\rho_\pm^2. \tag{4.23}$$

$$[\tau_\pm,\xi_\pm]=\frac{i\mu}{2}*\xi_\pm,\quad [\tau_\pm,\bar{\xi}_\pm]=\frac{i\mu}{2}*\bar{\xi}_\pm. \tag{4.24}$$

Proof. The first one is trivial and the second is given by Lemma 4.6. First two equalities of (4.23) are proved by (4.21) and (4.14).

To obtain the second equality of (4.23), remark at first that (4.16) yields $e_*^{i\tau_\pm}*\mu=\mu*e_*^{i(\tau_\pm+\mu i)}$. Combining this with the identity $[\mu^{-1},e_*^{i\tau_\pm}]=-e_*^{i\tau_\pm}$, we have

$$e_*^{i(\tau_\pm+\mu i)}=e_*^{i\tau_\pm}*(1-\mu).$$

By (4.21), we have

$$e_*^{i\tau_\pm}*\rho^2=\rho^2*e_*^{i\tau_\pm}*(1-\mu)=\rho^2*e_*^{i(\tau_\pm+\mu i)}.$$

Thus for any holomorphic function f we see

$$f_*(e_*^{i\tau_\pm})*\rho^2=\rho^2*f_*(e_*^{i(\tau_\pm+\mu i)}).$$

Putting $f=\log$, we have the third one of (4.23).

To obtain the last two equalities, remark that (4.23) yields $[\tau_\pm,\rho_\pm]=i\mu/2*\rho_\pm$ by taking square root. (4.24) is obtained by a similar computation. □

The coordinate transformation is given by

$$(\rho_-,\theta_-,e^{\tau_-})=(2-\rho_+,-\theta_+,e^{i\theta_+}*e^{\tau_+}). \tag{4.25}$$

For various m we consider elements $\tilde{f}*e^{im\tau_\pm}$. We compute

$$\begin{aligned}[\xi_+,\tilde{f}*e^{im\tau_+}]&=[\xi_+,\tilde{f}]*e^{im\tau_+}+\tilde{f}*[\xi_+,e^{im\tau_+}]\\&=([\xi_+,\tilde{f}]+(1-\sqrt{1+m\mu}^{-1})\tilde{f}*\xi_+)*e^{im\tau_+}.\end{aligned} \tag{4.26}$$

Setting $\kappa=\sqrt{1+m\mu}^{-1}$ and using $\tilde{f}*\xi_+=\xi_+\cdot\tilde{f}+\mu(\partial\tilde{f})/(\partial\bar{\xi}_+)$, obtained by Moyal product formula (2.1), we have

$$[\xi_+,\tilde{f}*e^{im\tau_+}]=\left\{-\mu(1+\kappa)\frac{\partial\tilde{f}}{\partial\bar{\xi}_+}+(1-\kappa)\xi_+\cdot\tilde{f}\right\}*e^{im\tau_+}. \tag{4.27}$$

The same formulae hold for $\bar{\xi}_+$ and μ:

$$[\bar{\xi}_+,\tilde{f}*e^{im\tau_+}]=\left\{\mu(1+\kappa)\frac{\partial\tilde{f}}{\partial\xi_+}+(1-\kappa)\bar{\xi}_+\cdot\tilde{f}\right\}*e^{im\tau_+}. \tag{4.28}$$

$$[\mu,\tilde{f}*e^{im\tau_+}]=\mu(1-\kappa^2)\tilde{f}*e^{im\tau_+}. \tag{4.29}$$

A calculation using Lemma 2.8 shows that these are compatible with (4.18) and Lemma 2.8. Remark that the right hand sides are differential operators.

We can obtain the same formula for ξ_-, $\bar{\xi}_-$. Thus, $\mathrm{ad}(\xi_\pm)$, $\mathrm{ad}(\bar{\xi}_\pm)$, $\mathrm{ad}(\mu)$ can be viewed as differential operators $\hat{\xi}_\pm$, $\widehat{\bar{\xi}_\pm}$, $\hat{\mu}$ acting on $C^\infty(U_\pm)[[\mu]] * e^{im\tau_\pm}$. Here $*e^{im\tau_\pm}$ plays only a role of local basis of a line bundle. It is obvious that

$$[\hat{\xi}_\pm, \widehat{\bar{\xi}_\pm}] = 2\hat{\mu}. \tag{4.30}$$

Remark now that we can write as follows:

$$\hat{\xi}_\pm = (1-\kappa)\xi_\pm + (1+\kappa)\hat{X}, \quad \hat{X} = -\mu\frac{\partial}{\partial\bar{\xi}_\pm}$$

$$\widehat{\bar{\xi}_\pm} = (1-\kappa)\bar{\xi}_\pm + (1+\kappa)\hat{Y}, \quad \hat{Y} = \mu\frac{\partial}{\partial\xi_\pm}.$$

Since μ is a formal parameter, we can define the operator $f(\hat{\xi}, \hat{\bar{\xi}}, \hat{v})$ for any smooth function. To be precise, we have to use here the notion of Weyl continuations (cf. [**OMY1**]).

So far we have treated μ as a formal parameter. Following the idea of Guillemin [**G**], we regard m as a $\boldsymbol{Z}$-asymptotics (cf. [**GS**]). Instead of this, we fix $m\mu$ as a real number, e.g. $m\mu = \hbar$. Hence we set $\kappa^{-2} = 1+\hbar$ and $\mu = \hbar/m$.

Since m appears always with μ, i.e. $m\mu$ in the above equalities, we treat the asymptotic behavior with respect to m^{-1}.

By virtue of treating m as a moving parameter, we can now restrict the representation space, by setting $\mu = \hbar/m$

$$C^\infty(U_\pm)[[\mu]] * e^{im\tau_\pm} \quad \text{to} \quad C^\infty(U_\pm) * e^{im\tau_\pm}.$$

The latter can be identified with the space of all classical sections of classical $\boldsymbol{C}$ bundle associated to S^1-principal bundle.

Thus, we obtain an operator representation of the algebra $C^\infty(S^2)[[\mu]]$. This is indeed the van Hove representation of the quantized algebra of prequantized S^1-bundle over S^2 ([**vH**]). This procedure may give the answer to the conjecture of Guillemin [**G**], because the compactness of the underlying space is not used in the above argument.

5. Global generators with matrix representations

Set $\mathscr{A}_0^\infty = \{f \in \mathscr{A}^\infty; [\mu; f] = 0\}$. By (1.17), we see that

$$\mu, \Xi_1 * \Xi_1^*, \Xi_1 * \Xi_2^*, \Xi_2 * \Xi_1^* \ (= (\Xi_1 * \Xi_2^*)^*) \tag{5.1}$$

generates a dense subalgebra $\mathscr{A}_0$ of $\mathscr{A}_0^\infty$. Besides the van Hove representation mentioned in the last paragraph of §4, the argument in §1 shows that μ^{-1} and each of the generators (5.1) are represented by blockwise diagonal matrices:

$$diag\{B_{1,1}, B_{2,2}, \dots, B_{k,k}, \dots\}.$$

Moreover, we have the following relations:

LEMMA 5.1.

$$\left(\Xi_1 * \Xi_1^* - \frac{1}{2}\right)^2 + \Xi_1 * \Xi_2^* * (\Xi_1 * \Xi_2^*)^* = \frac{1}{4} \tag{5.2}$$

$$[\Xi_1 * \Xi_2^*, \Xi_2 * \Xi_1^*] = -2\mu * \left(\Xi_1 * \Xi_1^* - \frac{1+\mu}{2}\right) \tag{5.3}$$

$$\left[\Xi_1 * \Xi_1^* - \frac{1+\mu}{2}, \Xi_1 * \Xi_2^*\right] = -\mu * \Xi_1 * \Xi_2^* \tag{5.4}$$

$$\left[\Xi_1 * \Xi_1^* - \frac{1+\mu}{2}, \Xi_2 * \Xi_1^*\right] = \mu * \Xi_2 * \Xi_1^* \tag{5.5}$$

PROOF. By a direct calculation using (1.20), we have (5.2).
For (5.3), we compute by using (1.22) as follows:

$$\begin{aligned} \Xi_1 * \Xi_2^* * \Xi_2 * \Xi_1^* &= \Xi_1 * (1 - \Xi_1^* * \Xi_1) * \Xi_1^* \\ &= \Xi_1 * \Xi_1^* - (\Xi_1 * \Xi_1^*)^2 \\ &= 1 + \mu - \Xi_2 * \Xi_2^* - (1 + \mu - \Xi_2 * \Xi_2^*)^2 \\ &= -\mu * (1+\mu) + (1+2\mu) * \Xi_2 * \Xi_2^* - (\Xi_2 * \Xi_2^*)^2 \\ &= -\mu * (1+\mu) + 2\mu * \Xi_2 * \Xi_2^* + \Xi_2 * \Xi_1^* * \Xi_1 * \Xi_2^* \\ &= \mu * (1+\mu) - 2\mu * \Xi_1 * \Xi_1^* + \Xi_2 * \Xi_1^* * \Xi_1 * \Xi_2^* \\ &= -2\mu * \left(\Xi_1 * \Xi_1^* - \frac{1+\mu}{2}\right) + \Xi_2 * \Xi_1^* * \Xi_1 * \Xi_2^*. \end{aligned} \tag{5.6}$$

Computations similar to (5.6) give (5.4) and (5.5). □

If we set

$$H = \Xi_1 * \Xi_1^* - \frac{1+\mu}{2}, \quad Z = \Xi_1 * \Xi_2^*, \quad Z^* = \Xi_2 * \Xi_1^* \tag{5.7}$$

the above lemma shows that the algebra generated is the universal enveloping algebra of the Lie algebra $sl_\mu(2; \boldsymbol{C})$:

$$[H, Z] = -\mu * Z, \qquad [H, Z^*] = \mu * Z^*, \qquad [Z, Z^*] = -2\mu * H \tag{5.8}$$

constrained by

$$\left(H + \frac{\mu}{2}\right)^2 + Z * Z^* = \frac{1}{4}. \tag{5.9}$$

Note that $(H + \mu/2)^2 + Z * Z^*$ is in the center of the enveloping algebra.

The matrix representations for H and Z are given as follows:

$$\begin{aligned} H &= diag\{B_{1,1}, B_{2,2}, \dots, B_{k,k}, \dots\}, \\ Z &= diag\{B'_{1,1}, B'_{2,2}, \dots, B'_{k,k}, \dots\}, \end{aligned} \tag{5.10}$$

with $Z^* = {}^tZ$. Here we set

$$B_{k,k} = \frac{1}{2k} diag\{k-1, k-3, \dots, -(k-3), -(k-1)\}, \tag{5.11}$$

$$B'_{k,k} = \frac{1}{k}\begin{pmatrix} 0 & \sqrt{(k-1)1} & & & & \\ & 0 & \sqrt{(k-2)2} & & & \\ & & \ddots & \ddots & & \\ & & & 0 & \sqrt{2(k-2)} & \\ & & & & 0 & \sqrt{1(k-1)} \\ & & & & & 0 \end{pmatrix}. \tag{5.12}$$

$\mathscr{A}_0$ is the algebra generated by μ, H, Z, Z^* and $\mathscr{A}_0$ is linearly isomorphic to $\boldsymbol{C}[H, Z, Z^*, \mu]$.

Let $\pi_k(a)$ be the $B_{k,k}$-component of $a \in \mathscr{A}_0$. π_k is then an algebra homomorphism of $\mathscr{A}_0$ onto the full matrix algebra of rank k. The kernel of π_k is a maximal 2-sided ideal of $\mathscr{A}_0$.

On the other hand, $\mu * \mathscr{A}_0$ is a 2-sided ideal of $\mathscr{A}_0$ with

$$\mathscr{A}_0/\mu * \mathscr{A}_0 \cong \left\{\boldsymbol{C}[H, Z, Z^*]; H^2 + ZZ^* = \frac{1}{4}\right\}.$$

Thus for each point p of $\{\boldsymbol{R} \times \boldsymbol{C} : H^2 + ZZ^* = 1/4\}$, there is a 2-sided ideal of codimension 1 and containing μ consisting of the pullback of functions vanishing at a point p. We denote such an ideal by $\mathscr{I}_p$.

The above argument shows that the kernel of π_k does not correspond to a classical point. Moreover, it is not hard to see that $\pi_k(\mathscr{I}_p) = \pi_k(\mathscr{A}_0)$ for any p and k.

It is remarkable that we have to avoid matrix representations to obtain the classical picture. As we have seen above, the classical picture does not appear directly in these matrix representations. If we want to recover the classical picture, then we have to take asymptotic expansions by $\mu = \hbar/m$. Thus it is impossible to recover the classical picture from $\pi_k(\mathscr{A}^\infty)$ for a fixed k.

Let $K_m = \bigcap_{m \geq k} \ker \pi_k$. Then, $K_m \supset K_{m+1}$, $\bigcap K_m = \{0\}$ and $\mathscr{A}_0^\infty/K_m$ is a finite dimensional algebra.

6. Orthogonal projections and Berezin representations

We give a representation of $\mathscr{A}_0$ as an algebra of linear maps on $\mathscr{H}_m$ (see between (4.11) and (4.12)) and show that the representation in this section is the same as the representation given by Berezin [**Be**].

Let $\Gamma(L^m)$ be the eigenspace of $-\mathrm{ad}(\mu^{-1})$ with the eigenvalue m. By (1.17), we have $\mathscr{H}_m \subset \Gamma(L^m)$ and

$$\begin{cases} C^\infty(S^2)[[\mu]] * \Gamma(L^m) * C^\infty(S^2)[[\mu]] = \Gamma(L^m), \\ \Xi_i * \Gamma(L^m) \subset \Gamma(L^{m+1}). \end{cases} \tag{6.1}$$

Since each element of $\Gamma(L^m)$ is written as $f(\mu,z,\bar{z})*\Xi_1^m$ (resp. $g(\mu,w,\bar{w})*\Xi_2^m$) on U_+ (resp. U_-), an element of $\Gamma(L^m)$ may be regarded as a "smooth section" of L^m.

Note that μ^{-1} commutes with each element of $C^\infty(S^2)[[\mu]]$. If μ^{-1} is represented as a linear mapping on $\sum\oplus\Gamma(L^m)$, we may assume that μ^{-1} is $-\lambda_m I$ on each $\Gamma(L^m)$.

Note that $(\mu^{-1}+1)*\Xi_i=\Xi_i*\mu^{-1}$ (cf. (1.17)). By (6.1), we have $\lambda_m=m+c$, where c is an arbitrary constant.

In what follows we set $c=1$ to obtain the Berezin representation. So far, μ is regarded as a formal parameter. In the spirit of §4, we set here $\mu^{-1}=-(m+1)$ on each $\Gamma(L^m)$. Thus, the space $\Gamma(L^m)$ is changed to the space which does not involve μ.

Recall that the holomorphic line bundle L^m is expressed in the local generator systems as

$$\{\mu,z;\Xi_1^m\},\quad \{\mu,w;\Xi_2^m\}. \tag{6.2}$$

A linear basis of the space $\mathscr{H}_m$ of all holomorphic q-sections is given by

$$\frac{\sqrt{(m+1)!}}{\sqrt{k!l!}}\Xi_1^k*\Xi_2^l=\begin{cases}\dfrac{\sqrt{(m+1)!}}{\sqrt{k!l!}}z^l*\Xi_1^m & (\text{on } U_+)\\[2ex] \dfrac{\sqrt{(m+1)!}}{\sqrt{k!l!}}w^k*\Xi_2^m & (\text{on } U_-)\end{cases}\qquad (k+l=m).$$

These form an orthonormal basis. By (1.10), (1.16) and Lemma 3.7, we have for $k+l=m$

$$\frac{1}{\sqrt{2\hbar^m}}\frac{1}{\sqrt{k!l!}}\zeta_1^k*\zeta_2^l=\sqrt{-\mu}^{-m}\prod_{j=0}^{m-1}\sqrt{1+j\mu}*\frac{1}{\sqrt{k!l!}}\Xi_1^k*\Xi_2^l. \tag{6.3}$$

Since we set $\mu^{-1}=-(m+1)$ on $\Gamma(L^m)$, we have

$$\frac{1}{\sqrt{2\hbar^m}}\frac{1}{\sqrt{k!l!}}\zeta_1^k*\zeta_2^l=\sqrt{\frac{(m+1)!}{k!l!}}\Xi_1^k*\Xi_2^l=\sqrt{\frac{(m+1)!}{k!l!}}z^l*\Xi_1^m\qquad\text{on } U_+. \tag{6.4}$$

Now recall that the generators $\Xi_1*\Xi_1^*$, $\Xi_1*\Xi_2^*$, $\Xi_2*\Xi_1^*$ are expressed as matrices. Using (1.10) and Lemma 3.7, we note the following:

$$\begin{aligned}\Xi_1*\Xi_1^*&=r^{-1}*\zeta_1*\bar{\zeta}_1*r^{-1}=r^{-2}*\zeta_1*\bar{\zeta}_1=-\frac{\mu}{2\hbar}*\zeta_1*\bar{\zeta}_1\\ \Xi_1*\Xi_2^*&=r^{-1}*\zeta_1*\bar{\zeta}_2*r^{-1}=r^{-2}*\zeta_1*\bar{\zeta}_2=-\frac{\mu}{2\hbar}*\zeta_1*\bar{\zeta}_2.\end{aligned} \tag{6.5}$$

$$\begin{aligned}\Xi_1^**\Xi_1&=-\frac{\mu}{2\hbar}*\frac{1}{1-\mu}*\bar{\zeta}_1*\zeta_1\\ \Xi_2^**\Xi_1&=-\frac{\mu}{2\hbar}*\frac{1}{1-\mu}*\bar{\zeta}_2*\zeta_1.\end{aligned} \tag{6.6}$$

Thus the matrix representations of §1 for the above generators are written for each non-negative integer m as

$$\Xi_1 * \Xi_1^* : \sqrt{\frac{(m+1)!}{(m-l)!l!}} z^l * \Xi_1^m \to \frac{m-l}{m+1} \sqrt{\frac{(m+1)!}{(m-l)!l!}} z^l * \Xi_1^m,$$

$$(6.7) \qquad \Xi_1 * \Xi_2^* : \sqrt{\frac{(m+1)!}{k!l!}} z^l * \Xi_1^m \to \frac{\sqrt{(k+1)l}}{m+1} \sqrt{\frac{(m+1)!}{(k+1)!(l-1)!}} z^{l-1} * \Xi_1^m,$$

$$\Xi_2 * \Xi_1^* : \sqrt{\frac{(m+1)!}{k!l!}} z^l * \Xi_1^m \to \frac{\sqrt{k(l+1)}}{m+1} \sqrt{\frac{(m+1)!}{(k-1)!(l+1)!}} z^{l+1} * \Xi_1^m,$$

where we set $z^{-1} = z^{m+1} = 0$.

Here, note that $\{\Xi_1^* * \Xi_1, \Xi_2^* * \Xi_1, \Xi_1^* * \Xi_2\}$ also forms a system of generators. These have also matrix representations as follows:

$$\Xi_1^* * \Xi_1 : \sqrt{\frac{(m+1)!}{(m-l)!l!}} z^l * \Xi_1^m \to \frac{m+1-l}{m+2} \sqrt{\frac{(m+1)!}{(m-l)!l!}} z^l * \Xi_1^m,$$

$$(6.8) \qquad \Xi_2^* * \Xi_1 : \sqrt{\frac{(m+1)!}{k!l!}} z^l * \Xi_1^m \to \frac{\sqrt{(k+1)l}}{m+2} \sqrt{\frac{(m+1)!}{(k+1)!(l-1)!}} z^{l-1} * \Xi_1^m,$$

$$\Xi_1^* * \Xi_2 : \sqrt{\frac{(m+1)!}{k!l!}} z^l * \Xi_1^m \to \frac{\sqrt{k(l+1)}}{m+2} \sqrt{\frac{(m+1)!}{(k-1)!(l+1)!}} z^{l+1} * \Xi_1^m,$$

where we set $z^{-1} = z^{m+1} = 0$. To give the Berezin representation, it is convenient to use $\{\Xi_1^* * \Xi_1, \Xi_2^* * \Xi_1, \Xi_1^* * \Xi_2\}$, rather than $\{\Xi_1 * \Xi_1^*, \Xi_1 * \Xi_2^*, \Xi_2 * \Xi_1^*\}$.

We now define integral operators for representing (6.8). The following is easily obtained:

LEMMA 6.1. *Let z and v denote the complex variables on U_+. Then, for each non-negative integer m the mapping*

$$I_m(p)(z) = \frac{m+1}{\pi} \int_{C^2} p(v) \frac{(1+z\bar{v})^m}{(1+v\bar{v})^m} \frac{1}{(1+v\bar{v})^2} dv\, d\bar{v}$$

defines the identity on the space $\mathscr{P}_m$ of all polynomials of degree up to m.

Since $1/(1+v\bar{v})^2\, dv\, d\bar{v}$ is the volume form on S^2, the right hand of the equality in Lemma 6.1 can be viewed as the integral over S^2. Using Lemma 6.1, we define the projection operator P_m by

$$(6.9) \qquad (P_m f)(z) = \frac{m+1}{\pi} \int_{C^2} f(v,\bar{v}) \frac{(1+z\bar{v})^m}{(1+v\bar{v})^m} \frac{1}{(1+v\bar{v})^2} dv\, d\bar{v}$$

and we define $B(a)f$ for any $f = \sum_m f_m * \Xi_1^m$, $f_m \in \mathscr{P}_m$ by

$$B(a)f = \sum_{m \geq 0} P_m(af_m) * \Xi_1^m$$

for any $a \in C^\infty(S^2)$. It is easily seen that $B(a)$ defines a linear operator of $\sum \oplus \mathscr{P}_m * \Xi_1^m$ into itself.

A direct computation gives the following identities:

$$\Xi_1^* * \Xi_1 = B\left(\frac{1}{1+\bar{z}z}\right), \quad \Xi_2^* * \Xi_1 = B\left(\frac{\bar{z}}{1+\bar{z}z}\right), \quad \Xi_1^* * \Xi_2 = B\left(\frac{z}{1+\bar{z}z}\right). \tag{6.10}$$

This constitutes the Berezin representation [Be]. Using (6.10), we have an operator representation of the algebra $\mathscr{A}_0$. This coincides with the matrix representation given in §1.

By (6.9), $B(a)f$ can be expressed as an integral operator

$$B(a)f = \sum_{m\geq 0}\left\{\frac{m+1}{\pi}\int_{C^2} a(v,\bar{v})f_m(v)\frac{(1+z\bar{v})^m}{(1+v\bar{v})^m}\frac{1}{(1+v\bar{v})^2}dv\,d\bar{v}\right\} * \Xi_1^m.$$

We have already seen in §1, any element of the algebra $\mathscr{A}$ are represented as matrices, although these are not blockwise diagonal matrices.

Recall the topology on matrices given in §1. Since the above representation is given via the $*$-product, we see this representation gives a continuous homomorphism of $\mathscr{A}$ into the space of matrices with the weak topology. Hence, we have the following:

THEOREM 6.2. *The matrix representation* (6.8) *extends to the algebra* $\mathscr{A}$. *The algebra* $\mathscr{A}_0$ *is represented by blockwise diagonal matrices, and* μ^{-1} *is represented as a diagonal matrix. Thus, we have a matrix representation for the Lie algebra* $\mu^{-1} * \mathscr{A}$.

In particular, the Lie algebra $\mu^{-1} * \mathscr{A}_0$ is represented by blockwise diagonal matrices. Since each block is finite rank, we see that the group G generated by $\exp\mu^{-1} * \mathscr{A}_0$ is also represented by blockwise diagonal matrices. This means that $\mu^{-1} * \mathscr{A}_0$ is continuously embedded in a projective limit of finite dimensional Lie algebras.

It follows also that G has a series of finite codimensional normal subgroups N_k such that $N_k \supset N_{k+1}$, $\bigcap N_k = \{e\}$. Thus, there is an isomorphism from G into the group F obtained as the projective limit of finite dimensional Lie groups.

We finish with some remarks on the noncommutative Riemann sphere from a more general point of view. Thinking of S^2 as a compact Kähler manifold, we have an S^1-bundle $S^3 = S^1_{S^2}$, and then a holomorphic line bundle L_{S^2} associated with the Kähler polarization of S^2. Let L_{*,S^2} be $L_{S^2} - \{0\text{-section}\}$, which is in fact the space $\boldsymbol{C}^2 - \{0\}$. Denoting a point of L_{*,S^2} by $(z;s)$, we may regard the radial element r as an element of L_{*,S^2} defined by

$$r(z;s) = |s|.$$

Here $|s|$ is defined via the hermitian structures on L_{S^2}. Define a holomorphic transformation $R(e^t) : L_{*,S^2} \to L_{*,S^2}$ by

$$R(e^t)(z;s) = (z;e^t s). \tag{6.11}$$

By applying Lemma 3.10 to each fiber, the pullback of $R(e^t)$ induces an automorphism

(6.12) $$R(e^t)^* : (C^\infty(L_{*,S^2})[[\hbar]], *) \to (C^\infty(L_{*,S^2})[[\hbar]], *)$$

by setting $R(e^t)^*\hbar = e^{2t}\hbar$.

Besides representing elements of the algebra $\mathscr{A}_0$, it is possible to represent the elements of the algebra $\mathscr{A}$ which are $R(e^t)$-invariant on the space $\sum_{m\geq 0}\oplus \mathscr{H}_m$ of all holomorphic q-sections. Note again that the element μ which is not a central element is represented as an operator. $\mathrm{ad}(\mu^{-1})$ acts as a fiber preserving diffeomorphism on $L^m_{S^2}$ and each $\mathscr{H}_m$ is an eigenspace of μ^{-1} with eigenvalue $-m$. Since S^2 is compact, every $\mathscr{H}_m$ is a finite dimensional vector space. Hence the algebra $\mathscr{A}_0$ is represented by blockwise diagonal matrices. As μ, μ^{-1} are represented, one obtain the representation of $\mu^{-1} * \mathscr{A}$, the Lie algebra of all infinitesimal automorphisms of $\mathscr{A}$.

It appears that this method can be applied for any Kähler manifold of integral class whenever the associated line bundle of the S^1-bundle over M has a nontrivial holomorphic section.

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