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Higher v_n torsion in Lie groups

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Abstract. We study the Morava K-theory of the exceptional Lie groups at the prime 2, and of certain projective Lie groups at a variety of primes.

§1. Introduction.

In this paper we consider $K(n)^*(G)$, the Morava K-theory of certain Lie groups. Specifically, we compute $K(n)^*(G)$ for $G = G_2$, F_4, E_6, E_7, E_8, PE_7 and $PSp(m)$ at the prime 2, and PE_6 and $PU(m)$ for odd primes. In particular, together with [**Ho**], [**Y2**] (see also [**Hu**]) and certain elementary observations, this completes the computations of the Morava K-theory of all the connected, simple, simply connected compact Lie groups with the sole exception the groups $\mathrm{Spin}(m)$ at the prime 2.

Our principal tool in every case is the Atiyah-Hirzebruch spectral sequence

$$H^*(G; K(n)^*) \Rightarrow K(n)^*(G).$$

If this spectral sequence has $E^{**}_{2r(p^n-1)+2} = E^{**}_{\infty}$ for some positive integer r then the connective theory $k(n)^*(G)$ has at most v_n^r torsion. This observation was used in [**Ka1**] to show for odd primes that $k(n)_*(X)$ has at most v_n^1 torsion for any simply connected H-space X. Our work thus examines the v_n torsion for the above mentioned simply connected groups at the prime 2 and for certain non-simply connected groups at general primes. As Kane observes, Hodgkin shows in [**Ho**] that $k(1)^*(G)$ has higher v_1 torsion at the prime 2 for $G = E_7$ and E_8. Kane also notes in [**Ka2**] that there is v_1^2 torsion at the prime 2 for $G = PE_7$; however, our results prove that in all other 2-primary cases $k(n)^*(G)$ has at most v_n^1 torsion for $PSp(m)$, the exceptional groups and their projective counterparts, while similar results also hold at the appropriate primes for various projective unitary groups (see §6 for details).

This paper is arranged as follows. In the next section we collect a number of common observations about our computations and methods. In §3 to §6 we then proceed to compute the spectral sequences for, respectively, the exceptional groups, their projective versions, the projective symplectic groups and the projective unitary groups. Precise statements of our results can be found in the relevant sections.

It may be of use to record at this point where the previous computations of $K(n)^*(G)$ for various Lie groups G may be found. The paper of Hodgkin [**Ho**] computes the integral complex K-theory of the connected, simple, simply connected

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compact Lie groups. These are all torsion free and as algebras are exterior of rank equal to the rank of the individual groups. From this the first Morava K-theory of such groups is easily deduced: $K(1)$ is of course the mod p reduction of the Adams summand of p local complex K-theory. The complex K-theory of non-simply connected Lie groups is studied in the work of Held and Suter [**HS**]. For the higher Morava K-theories, these are computed for all the connected, simple, simply connected compact Lie groups at odd primes by Yagita in [**Y2**], where computations of their BP and $P(n)$ theories are also made. Some of the computations for the exceptionals at the prime 2 appeared in the first author's thesis [**Hu**], where the corresponding odd primary calculations are also repeated. For the non-exceptional groups, the $K(n)$-theory of $SO(2m+1)$ is extensively studied in [**Rao1**], and a few remarks on the behaviour of $K(n)^*(\mathrm{Spin}(m))$ can be found in [**Hu**]. We see in the next section that the Morava K-theory of groups G with no torsion in $H^*(G;\boldsymbol{Z})$ is essentially trivial.

§2. Methods.

As remarked in the introduction, our main tool for studying $K(n)^*(G)$ is the Atiyah-Hirzebruch spectral sequence (AHSS)

$$H^*(G;K(n)^*) \Rightarrow K(n)^*(G).$$

The following four results provide the computational control we require.

LEMMA 2.1. *Let X be a finite CW complex for which $H^*(X;\boldsymbol{Z})$ is free of p torsion. Then the* AHSS *for $K(n)^*(X)$ collapses for all $n \in \boldsymbol{N}$.* □

This result is standard and follows, for example, from a simple rank counting argument.

LEMMA 2.2. *Let X be a finite CW complex. If the* AHSS *for $K(n)^*(X)$ collapses for some $n \geq 1$, then so does that for $K(n+1)^*(X)$. Moreover,*

$$\mathrm{rank}_{K(n)^*} K(n)^*(X) \geq \mathrm{rank}_{\boldsymbol{Q}} H^*(X;\boldsymbol{Q}).$$

PROOF. Rank counting shows that $\mathrm{rank}_{K(n)^*} K(n)^*(X) \leq \mathrm{rank}_{\boldsymbol{F}_p} H^*(X;\boldsymbol{F}_p)$ with equality if and only if the AHSS for $K(n)^*(X)$ collapses. As X is finite, [**Rav**] (2.11) tells us

$$\mathrm{rank}_{K(n)^*} K(n)^*(X) \leq \mathrm{rank}_{K(n+1)^*} K(n+1)^*(X).$$

The first result now follows. The second, which provides a lower bound for all these ranks, follows for example by considering the Bockstein spectral sequence for $K(n)^*(X)$. □

It is well known that the Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$ is a spectral sequence of $K(n)^*$ algebras. As the Morava K-theories all have Künneth isomorphisms, it is easy to check that the spectral sequence for $K(n)^*(G)$, where G is an associative H-space, is a spectral sequence of $K(n)^*$ Hopf algebras, the Hopf algebra structures on E_2^{**} and E_∞^{**} being those given by or related to the Hopf algebra structures on $H^*(G; K(n)^*)$ and $K(n)^*(G)$ respectively. Our third lemma provides a useful restriction on the behaviour of differentials in our spectral sequences (cf. Lemma 6.3 in [**Br**]).

LEMMA 2.3. *If $x \in E_r^{m,0}$ and $d_r(x') = 0$ for all $x' \in E_r^{u,0}$ with $u < m$, then $d_r(x)$ is primitive.*

PROOF. Let us write ψ for the coproduct in E_r^{**} and $\bar{\psi}$ for the reduced coproduct given by

$$\psi(x) = x \otimes 1 + 1 \otimes x + \bar{\psi}(x).$$

Thus an element y is primitive if and only if $\bar{\psi}(y) = 0$. Note that $\bar{\psi}(x) = \sum_i y_i \otimes z_i$ where the degrees of y_i and z_i are both strictly positive and strictly less than the degree of x. So, in the case hypothesized,

$$\begin{aligned} \psi d_r(x) &= (d_r \otimes 1 + 1 \otimes d_r)\psi(x) \\ &= (d_r \otimes 1 + 1 \otimes d_r)(x \otimes 1 + 1 \otimes x + \sum y_i \otimes z_i) \\ &= (d_r x) \otimes 1 + 1 \otimes (d_r x) + 0. \end{aligned}$$ □

This last result will be used many times to demonstrate the triviality of a differential in the AHSS. To show that a given differential d_r is trivial we examine the dimension of each $d_r(x_i)$ as x_i runs through the generators of E_r^{**}. If there are no primitive elements in any of these dimensions then (2.3) shows that d_r must be zero.

Finally, we note a result of Yagita [**Y1**] identifying the first non-trivial differential in the Atiyah-Hirzebruch spectral sequence.

LEMMA 2.4. *The first non-trivial differential in the Atiyah-Hirzebruch spectral sequence for $K(n)^*(X)$ is $d_{2(p^n-1)+1}$ and is represented by a unit multiple of Milnor's operation Q_n.* □

Recall that the cohomology operation Q_n is defined inductively by

$$Q_0 = \beta, \quad Q_r = \mathscr{P}^{p^{r-1}} Q_{r-1} - Q_{r-1} \mathscr{P}^{p^{r-1}} \qquad \text{if } p > 2,$$
$$Q_0 = Sq^1, \quad Q_r = Sq^{2^r} Q_{r-1} + Q_{r-1} Sq^{2^r} \qquad \text{if } p = 2.$$

§3. The exceptional groups.

We begin our study with the compact, connected, simply connected, simple Lie groups. We recall that the only such groups to have 2 torsion in their integral cohomology are

$$G_2, F_4, E_6, E_7, E_8 \quad \text{and} \quad \mathrm{Spin}(m) \text{ for } m \geq 7.$$

Thus, by (2.1), the Atiyah-Hirzebruch spectral sequence for every other compact, connected, simply connected, simple Lie group collapses and $k(n)^*(G)$ for such G contains no v_n torsion. We concentrate in this section on the remaining exceptional groups.

The following proposition, all of whose contents are to be found in [**Mi**], lists the ordinary mod 2 cohomology and Steenrod algebra action on the groups concerned. We use the notation that an element x_r lies in dimension r. For a group G we denote by $\mathrm{P}(G)$ a basis for the module of primitive elements.

PROPOSITION 3.1.
(a) *$H^*(G_2;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3]/(x_3^4) \otimes \Lambda(x_5)$ and $\mathrm{P}(G_2) = \{x_3, x_5, x_3^2\}$.*
(b) *$H^*(F_4;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_{15}, x_{23})$ and $\mathrm{P}(F_4) = \{x_3, x_5, x_3^2, x_{15}, x_{23}\}$.*
(c) *$H^*(E_6;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3]/(x_3^4) \otimes \Lambda(x_5, x_9, x_{15}, x_{17}, x_{23})$ and $\mathrm{P}(E_6) = \{x_3, x_5, x_3^2, x_9, x_{17}\}$.*
(d) *$H^*(E_7;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3, x_5, x_9]/(x_3^4, x_5^4, x_9^4) \otimes \Lambda(x_{15}, x_{17}, x_{23}, x_{27})$ and $\mathrm{P}(E_7) = \{x_3, x_5, x_3^2, x_9, x_5^2, x_{17}, x_9^2\}$. Furthermore, the reduced coproduct acts on the other generators as follows:*

$$\bar{\psi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5,$$
$$\bar{\psi}(x_{23}) = x_3^2 \otimes x_{17} + x_9^2 \otimes x_5,$$
$$\bar{\psi}(x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9.$$

The Steenrod algebra acts by

$$\begin{array}{llll}
Sq^2 x_3 = x_5, & & & \\
Sq^1 x_5 = x_3^2, & Sq^4 x_5 = x_9, & & \\
Sq^1 x_9 = x_5^2, & Sq^8 x_9 = x_{17}, & & \\
Sq^1 x_{15} = x_3^2 x_5^2, & Sq^2 x_{15} = x_{17}, & Sq^8 x_{15} = x_{23}, & Sq^{12} x_{15} = x_{27}, \\
Sq^1 x_{17} = x_9^2, & & & \\
Sq^1 x_{23} = x_3^2 x_9^2, & Sq^4 x_{23} = x_{27}, & & \\
Sq^1 x_{27} = x_5^2 x_9^2 & & \text{and all other operations are zero.} &
\end{array}$$

(e) *$H^*(E_8;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, x_{23}, x_{27}, x_{29})$ and $\mathrm{P}(E_8) = \{x_3, x_5, x_3^2, x_9, x_5^2, x_3^4, x_{17}, x_9^2, x_5^4, x_3^8\}$. In this Hopf algebra the reduced coproduct acts on the other generators as*

$$\bar{\psi}(x_{15}) = x_3^2 \otimes x_9 + x_5^2 \otimes x_5 + x_3^4 \otimes x_3,$$
$$\bar{\psi}(x_{23}) = x_3^2 \otimes x_{17} + x_9^2 \otimes x_5 + x_5^4 \otimes x_3,$$
$$\bar{\psi}(x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9 + x_3^8 \otimes x_3,$$
$$\bar{\psi}(x_{29}) = x_3^4 \otimes x_{17} + x_5^4 \otimes x_9 + x_3^8 \otimes x_5.$$

The action of the Steenrod algebra is as follows:

$$\begin{array}{ll}
Sq^2 x_3 = x_5, & \\
Sq^1 x_5 = x_3^2, & Sq^4 x_5 = x_9, \\
Sq^1 x_9 = x_5^2, & Sq^8 x_9 = x_{17},
\end{array}$$

$$\begin{array}{llll} Sq^1 x_{15} = x_3^2 x_5^2, & Sq^2 x_{15} = x_{17}, & Sq^8 x_{15} = x_{23}, & Sq^{12} x_{15} = x_{27}, \\ Sq^1 x_{17} = x_9^2, & & & Sq^{14} x_{15} = x_{29}, \\ Sq^1 x_{23} = x_3^2 x_9^2, & Sq^4 x_{23} = x_{27}, & Sq^6 x_{23} = x_{29}, & \\ Sq^1 x_{27} = x_5^2 x_9^2, & Sq^2 x_{27} = x_{29}, & & \\ Sq^1 x_{29} = x_{15}^2 & & \text{and all other operations are zero.} & \end{array}$$

□

We consider now the computations of $K(n)^*(G)$ for these G at the prime 2 and for $n \geq 1$. The case of $n = 1$ is essentially that of mod 2 K-theory, whose integral version has been fully calculated by Hodgkin, [**Ho**]. The integral K-groups $K^*(G)$ are torsion free and so the $K(1)$ result can be immediately read off. In all cases $K(1)^*(G)$ is an exterior algebra over $K(1)^*$ of rank equal to that of the Lie group in question; the Atiyah-Hirzebruch spectral sequence fails to collapse for any of the exceptionals. Moreover, as noted in the introduction, for $G = E_7$ or E_8 there is more than one non-trivial differential in the spectral sequence for $K(1)^*(G)$ (indeed, both these cases have v_1^2 torsion, but no v_1^3 torsion). We turn our attention to the cases $K(n)^*(G)$ with $n \geq 2$.

PROPOSITION 3.2. *There are no non-trivial differentials in the spectral sequence for $K(2)^*(G)$ with $G = G_2$, F_4 or E_6. Hence, by* (2.2), *the sequences for $K(n)^*(G)$ collapse for these groups for all $n \geq 2$.*

PROOF. The possible non-trivial differentials in the spectral sequence for $K(n)^*(G)$ are $d_{2s(2^n-1)+1}$ with $s = 1, 2, \ldots$. Thus, for $n = 2$ the differentials we must consider are $d_7, d_{13}, d_{19}, \ldots$. Just as outlined at the end of §2, we examine for each group the ring generators, x_r say, of $E_2^{**} = H^*(G; \boldsymbol{F}_2) \otimes K(2)^*$ in order of increasing dimension and inspect whether there are any primitive elements firstly in dimensions $r + 7$, then in dimensions $r + 13$, and so on. In each case we find that the generators and primitive elements fail to occur in dimensions that would allow the possibility of a non-trivial differential and so the sequences all collapse by (2.3). □

The same method of argument readily gives the following result for E_7 and E_8.

PROPOSITION 3.3. *There are no non-trivial differentials in the spectral sequence for $K(4)^*(G)$ with $G = E_7$ or E_8. Hence, by* (2.2), *the sequences for $K(n)^*(G)$ collapse for these groups for all $n \geq 4$.* □

We consider next the cases of $K(2)^*(E_7)$ and $K(3)^*(E_7)$. Using (2.4) and the Steenrod action listed in (3.1) we compute the first potentially non-zero differentials in the corresponding Atiyah-Hirzebruch spectral sequences. We find

$$Q_2(x_i) = \begin{cases} x_5^2 & \text{if } i = 3, \\ 0 & \text{otherwise,} \end{cases}$$

and

$$Q_3(x_i) = \begin{cases} x_9^2 & \text{if } i = 3, \\ 0 & \text{otherwise.} \end{cases}$$

In the spectral sequence for $K(3)^*(E_7)$ the differentials we must consider are d_{15}, d_{29}, d_{43}, and so on. The differential d_{15} is given by the calculation of Q_3 and we can compute E_{16}^{**} as

$$E_{16}^{**} = K(3)^*[x_5]/(x_5^4) \otimes \Lambda(y_6, x_9, x_{15}, x_{17}, y_{21}, x_{23}, x_{27}),$$

where y_6 represents the old x_3^2 and y_{21} stands for the old $x_3x_9^2$. From (3.1)(d) we can compute that a basis for the module of primitive elements is given by $\{x_5, y_6, x_9, x_5^2, x_{17}, y_{21}\}$. Consequently, as in the proofs of (3.2) and (3.3), the differentials from this point on are all zero. Following the coproduct formulae through these calculations yields the following summary of our results on $K(3)^*(E_7)$.

THEOREM 3.4. *The Atiyah-Hirzebruch spectral sequence for $K(3)^*(E_7)$ at the prime 2 has only one non-trivial differential, namely d_{15}. The E_∞-term has algebra structure*

$$K(3)^*[x_5]/(x_5^4) \otimes \Lambda(y_6, x_9, x_{15}, x_{17}, y_{21}, x_{23}, x_{27})$$

in which the classes x_5, y_6, x_9, x_{17} and y_{21} are primitive and the reduced coproduct on the remaining generators acts by

$$\begin{aligned} \bar{\psi}(x_{15}) &= y_6 \otimes x_9 + x_5^2 \otimes x_5, \\ \bar{\psi}(x_{23}) &= y_6 \otimes x_{17}, \\ \bar{\psi}(x_{27}) &= x_5^2 \otimes x_{17}. \end{aligned}$$ □

We should point out that the statement about the Hopf algebra structure is a statement about the E_∞-term only, not about the Morava K-theory of E_7. In the case of $K(2)^*(G)$ at the prime 2 for $G = G_2$, E_7 and E_8 among others, **[Rao2]** and **[Y3]** show there to be non-trivial extension problems in the Hopf algebra structure; in particular, the product is non-commutative and there are v_n terms in the coproduct. These comments are of course also of relevance to (3.7)–(3.9) below.

Turning to the case of $K(2)^*(E_7)$, the differentials in question are now d_7, d_{13}, d_{19}, d_{25}, and so on. Calculating the operation Q_2 we compute the E_8-term as

$$K(2)^*[x_9]/(x_9^4) \otimes \Lambda(x_5, y_6, y_{13}, x_{15}, x_{17}, x_{23}, x_{27}), \tag{3.5}$$

where y_6 once again represents the old x_3^2 and y_{13} represents $x_3x_5^2$. The elements $\{x_5, y_6, x_9, y_{13}, x_{17}, x_9^2\}$ provide a basis for the module of primitives. At this point we find that the differential d_{13} is potentially non-zero on the element x_5, perhaps sending it to $v_n^2x_9^2$.

LEMMA 3.6. *The differential d_{13} is trivial.*

PROOF. Note that whether $d_{13}(x_5)$ is trivial or not, d_{13} must act trivially on all the generators other than x_5: if $d_{13}(x_5) = 0$ then d_{13} is zero everywhere by consideration of the primitives as before, while if $d_{13}(x_5) \neq 0$ then there can be no more differential action in the spectral sequence since

$$\mathrm{rank}_{K(2)^*} K(2)^*(E_7) \geq \mathrm{rank}_{\mathbf{Q}} H^*(E_7; \mathbf{Q}) = 2^7$$

by (2.2) and the classical computations of the rational cohomology of Lie groups. So in particular, $d_{13}(x_{23}) = 0$. Suppose $d_{13}(x_5) = v_n^2 x_9^2$ and consider the coproduct and d_{13} on x_{23}:

$$\begin{aligned} 0 = \bar{\psi} d_{13}(x_{23}) &= d_{13}\bar{\psi}(x_{23}) \\ &= d_{13}(x_3^2 \otimes x_{17} + x_9^2 \otimes x_5) \\ &= v_n^2 \cdot x_9^2 \otimes x_9^2. \end{aligned}$$

But of course $x_9^2 \otimes x_9^2$ is not zero in this algebra and so $d_{13}(x_5)$ must be zero after all. □

Repeated application of (2.3) now demonstrates that there are no more differentials in this spectral sequence and (3.4) gives the final E_∞-term for $K(2)^*(E_7)$.

THEOREM 3.7. *The Atiyah-Hirzebruch spectral sequence for $K(2)^*(E_7)$ at the prime 2 has only one non-trivial differential, namely d_7. The E_∞-term has algebra structure*

$$K(2)^*[x_9]/(x_9^4) \otimes \Lambda(x_5, y_6, y_{13}, x_{15}, x_{17}, x_{23}, x_{27})$$

in which the classes x_5, y_6, x_9, y_{13} and x_{17} are primitive and the reduced coproduct on the remaining generators acts by

$$\begin{aligned} \bar{\psi}(x_{15}) &= y_6 \otimes x_9, \\ \bar{\psi}(x_{23}) &= y_6 \otimes x_{17} + x_9^2 \otimes x_5, \\ \bar{\psi}(x_{27}) &= x_9^2 \otimes x_9. \end{aligned}$$ □

Next we turn to the case of E_8. By (3.3), we need only consider the cases $K(2)^*(E_8)$ and $K(3)^*(E_8)$; we begin with the latter. The following summarises the action of Milnor's operation Q_3 on $H^*(E_8; \boldsymbol{F}_2)$ and is calculated from the Steenrod action given in (3.1):

$$\begin{array}{lll} Q_3 x_3 = x_9^2, & Q_3 x_5 = x_5^4, & Q_3 x_9 = x_3^8, \\ Q_3 x_{15} = x_{15}^2 + x_3^{10} + x_5^6, & Q_3 x_{17} = 0, & Q_3 x_{23} = x_5^4 x_9^2, \\ Q_3 x_{27} = x_3^8 x_9^2, & & Q_3 x_{29} = x_3^8 x_5^4. \end{array}$$

We rechoose generators as follows:

$$y_{23} = x_{23} + x_5 x_9^2, \qquad y_{27} = x_{27} + x_3^9,$$
$$y_{29} = x_{29} + x_5^4 x_9.$$

Then

$$H^*(E_8; \boldsymbol{F}_2) = \boldsymbol{F}_2[x_3, x_5, x_9, x_{15}]/(x_3^{16}, x_5^8, x_9^4, x_{15}^4) \otimes \Lambda(x_{17}, y_{23}, y_{27}, y_{29}),$$

where

$$Q_3x_3 = x_9^2, \qquad Q_3x_5 = x_5^4,$$
$$Q_3x_9 = x_3^8, \qquad Q_3x_{15} = x_{15}^2 + x_3^{10} + x_5^6,$$

and Q_3 is zero on all other generators.

Consider the subalgebras

$$A = \boldsymbol{F}_2[x_3, x_5, x_9]/(x_3^{16}, x_5^8, x_9^4),$$
$$B = A \otimes \boldsymbol{F}_2[x_{15}]/(x_{15}^4)$$

and write d_A and d_B for the restriction of the differential Q_3 to A and B respectively. If we put $y_{30} = x_{15}^2 + x_5^6 + x_3^{10}$ then

$$B = A \oplus Ax_{15} \oplus Ay_{30} \oplus Ax_{15}y_{30} \quad \text{as } A \text{ modules.}$$

Obviously, $d_A(A) \subset A$, $d_B(Ax_{15} \oplus Ay_{30}) \subset Ax_{15} \oplus Ay_{30}$ and $d_B(Ax_{15}y_{30}) \subset Ax_{15}y_{30}$. Hence

$$H(B) = H(A) \oplus H(Ax_{15} \oplus Ay_{30}) \oplus H(Ax_{15}y_{30}),$$

where

$$H(A) = \boldsymbol{F}_2[w_6]/(w_6^4) \oplus \Lambda(w_{10}, w_{21}, w_{25}, w_{33}).$$

Here, w_6, w_{10}, w_{21}, w_{25} and w_{33} represent the old elements x_3^2, x_5^2, $x_3x_9^2$, x_5^5 and $x_3^8x_9$ respectively.

Also, $H(Ax_{15}y_{30}) = H(A)x_{15}y_{30}$ and $H(Ax_{15} \oplus Ay_{30}) = 0$. Thus

$$H(B) = H(A) \oplus H(A)x_{15}y_{30},$$

where it is easily seen that $(x_{15}y_{30})^2 = 0$. Therefore

$$H(B) = \boldsymbol{F}_2[w_6]/(w_6^4) \otimes \Lambda(w_{10}, w_{21}, w_{25}, w_{33}, w_{45}),$$

where $w_{45} = x_{15}y_{30} = x_{15}(x_{15}^2 + x_5^6 + x_3^{10})$.

Putting this all together, we have

$$E_{16}^{**} = K(3)^*[w_6]/(w_6^4) \otimes \Lambda(w_{10}, x_{17}, w_{21}, y_{23}, w_{25}, y_{27}, y_{29}, w_{33}, w_{45}).$$

The next differentials to consider are d_{33}, d_{47} and so on, but these are all forced to be zero for dimensional reasons. Following the coproduct formulae through these calculations yields the following summary of our results on $K(3)^*(E_8)$.

THEOREM 3.8. *The Atiyah-Hirzebruch spectral sequence for $K(3)^*(E_8)$ at the prime 2 has only one non-trivial differential, namely d_{15}. The E_∞-term has algebra structure*

$$K(3)^*[w_6]/(w_6^4) \otimes \Lambda(w_{10}, x_{17}, w_{21}, y_{23}, w_{25}, y_{27}, y_{29}, w_{33}, w_{45})$$

in which the classes w_6, w_{10}, x_{17}, w_{21}, w_{25} and w_{33} are primitive and the reduced coproduct on the remaining generators acts by

$$\bar{\psi}(y_{23}) = w_6 \otimes x_{17}, \qquad \bar{\psi}(y_{27}) = w_{10} \otimes x_{17},$$
$$\bar{\psi}(y_{29}) = w_6^2 \otimes x_{17}, \qquad \bar{\psi}(w_{45}) = w_{33} \otimes w_6^2. \qquad \square$$

Lastly, we consider the calculations for $K(2)^*(E_8)$. The Steenrod action on $H^*(E_8;\boldsymbol{F}_2)$ given in (3.1) provides the following description of the action of Q_2, effectively the first non-zero differential in the Atiyah-Hirzebruch spectral sequence:

$$\begin{aligned} &Q_2x_3 = x_5^2, && Q_2x_5 = x_3^4, && Q_2x_9 = 0, \\ &Q_2x_{15} = x_3^4x_5^2, && Q_2x_{17} = x_3^8, && Q_2x_{23} = x_{15}^2 + x_3^{10} + x_3^4x_9^2, \\ &Q_2x_{27} = x_3^8x_5^2, && && Q_2x_{29} = x_3^{12}. \end{aligned}$$

We rechoose generators as follows:

$$\begin{aligned} y_{15} &= x_{15} + x_3^5, & y_{17} &= x_{17} + x_3^4x_5, \\ y_{23} &= x_{23} + x_5x_9^2, & y_{27} &= x_{27} + x_3^9, \end{aligned}$$
$$y_{29} = x_{29} + x_3^8x_5.$$

Then

$$H^*(E_8;\boldsymbol{F}_2) = \boldsymbol{F}_2[x_3,x_5,x_9,y_{15},y_{17}]/(x_3^{16},x_5^8,x_9^4,y_{15}^4,y_{17}^2 + x_3^8x_5^2) \otimes \Lambda(y_{23},y_{27},y_{29}),$$

where

$$Q_2x_3 = x_5^2, \qquad Q_2x_5 = x_3^4, \qquad Q_2y_{23} = y_{15}^2,$$
$$Q_2x_9 = Q_2y_{15} = Q_2y_{17} = Q_2y_{27} = Q_2y_{29} = 0.$$

Consider the subalgebra

$$A = \boldsymbol{F}_2[x_3,x_5,y_{17}]/(x_3^{16},x_5^8,y_{17}^2 + x_3^8x_5^2),$$

with the differential $d_A : A \to A$ defined by

$$d_Ax_3 = x_5^2, \qquad d_Ax_5 = x_3^4, \qquad d_Ay_{17} = 0.$$

This gives $H(A) = \Lambda(w_6,y_{17},w_{33},w_{41})$, where $w_6 = x_3^2$, $w_{33} = x_3x_5^6$ and $w_{41} = x_3^{12}x_5$.

We also consider

$$B = \boldsymbol{F}_2[y_{15}]/(y_{15}^4) \otimes \Lambda(y_{23})$$

with the differential $d_B : B \to B$ defined by

$$d_By_{15} = 0, \qquad d_By_{23} = y_{15}^2.$$

Then $H(B) = \Lambda(y_{15},w_{53})$, where $w_{53} = y_{15}^2y_{23}$.

Thirdly, we consider

$$C = \boldsymbol{F}_2[x_9]/(x_9^4) \otimes \Lambda(y_{27},y_{29}).$$

This has the trivial differential $d_C = 0$ and so $H(C) = C$.

Thus the E_8-term of the Atiyah-Hirzebruch spectral sequence is given by

$$\begin{aligned} E_8^{**} &= K(2)^* \otimes H(A) \otimes H(B) \otimes H(C) \\ &= K(2)^*[x_9]/(x_9^4) \otimes \Lambda(w_6,y_{15},y_{17},y_{27},y_{29},w_{33},w_{41},w_{53}). \end{aligned}$$

The module of primitives is generated by the set $\{y_6,x_9,y_{17},x_9^2,y_{29},w_{33},w_{41},w_{53}\}$.

The next differentials to consider are of the form d_{6k+1} for $k = 2, 3, 4, \ldots$. Examination of the degrees of the generators and the primitive elements shows all these must be zero, again using (2.3). Thus the spectral sequence collapses from the E_8-term onwards. Following the coproduct formulae through the above calculations yields the following summary of our results on $K(2)^*(E_8)$.

THEOREM 3.9. *The Atiyah-Hirzebruch spectral sequence for $K(2)^*(E_8)$ at the prime 2 has only one non-trivial differential, namely d_7. The E_∞-term has the algebra structure*

$$K(2)^*[x_9]/(x_9^4) \otimes \Lambda(w_6, y_{15}, y_{17}, y_{27}, y_{29}, w_{33}, w_{41}, w_{53})$$

in which the classes w_6, x_9, y_{17}, y_{29}, w_{33}, w_{41} and w_{53} are primitive and the reduced coproduct on the remaining generators acts by

$$\bar{\psi}(y_{15}) = w_6 \otimes x_9 \quad \text{and} \quad \bar{\psi}(y_{27}) = x_9^2 \otimes x_9. \qquad \square$$

The table below summarises our results on the $K(n)$-theory of the exceptional Lie groups for the prime 2. The factors represent the heights of individual generators in the relevant E_∞-terms and the bold face figures indicate that the respective spectral sequences fail to collapse.

Table (3.10)
Ranks of $K(n)^*(G)$ over $K(n)^*$

Group	dim	p	$K(1)$	$K(2)$	$K(3)$	$K(n)$ $n \geq 4$
G_2	14	2	$\mathbf{2^2}$	$4 \cdot 2$	$4 \cdot 2$	$4 \cdot 2$
F_4	52	2	$\mathbf{2^4}$	$4 \cdot 2^3$	$4 \cdot 2^3$	$4 \cdot 2^3$
E_6	78	2	$\mathbf{2^6}$	$4 \cdot 2^5$	$4 \cdot 2^5$	$4 \cdot 2^5$
E_7	133	2	$\mathbf{2^7}$	$\mathbf{4 \cdot 2^7}$	$\mathbf{4 \cdot 2^7}$	$4^3 \cdot 2^4$
E_8	248	2	$\mathbf{2^8}$	$\mathbf{4 \cdot 2^8}$	$\mathbf{4 \cdot 2^9}$	$16 \cdot 8 \cdot 4^2 \cdot 2^4$

§4. The projective exceptional groups.

In this section we present calculations for the Morava K-theory of the projective groups $\mathrm{Ad}E_6 = PE_6 = E_6/\boldsymbol{Z}_3$ and $\mathrm{Ad}E_7 = PE_7 = E_7/\boldsymbol{Z}_2$, where $\boldsymbol{Z}_n$ stands for a cyclic group of order n. For PE_6 the only prime for which our results will differ from those for E_6 is 3, while for PE_7 the only prime to consider is 2. We note the existence of fibrations

$$E_6 \to PE_6 \to B\boldsymbol{Z}_3 \quad \text{and} \quad E_7 \to PE_7 \to \boldsymbol{RP}^\infty.$$

We shall need calculations of the ordinary cohomologies for these groups. The following proposition and (4.4) later on PE_7 provide the necessary detail; once again, the material can be found in the survey article [**Mi**].

PROPOSITION 4.1. $H^*(PE_6; \boldsymbol{F}_3) = \boldsymbol{F}_3[x_2, x_8]/(x_2^9, x_8^3) \otimes \Lambda(x_1, x_3, x_7, x_9, x_{11}, x_{15})$ *and* $\mathrm{P}(PE_6) = \{x_1, x_2, x_2^3\}$. *Furthermore, in this Hopf algebra the reduced coproduct acts on* *the other generators as follows*:

$$\bar{\psi}(x_3) = x_2 \otimes x_1,$$
$$\bar{\psi}(x_7) = x_2^3 \otimes x_1,$$
$$\bar{\psi}(x_8) = x_2^3 \otimes x_2,$$
$$\bar{\psi}(x_9) = x_2 \otimes x_7 - x_2^3 \otimes x_3 + x_8 \otimes x_1 + x_2^4 \otimes x_1,$$
$$\bar{\psi}(x_{11}) = x_2 \otimes x_9 - x_2^2 \otimes x_7 + x_8 \otimes x_3 - x_2^4 \otimes x_3 + x_8 x_2 \otimes x_1 - x_2^5 \otimes x_1,$$
$$\bar{\psi}(x_{15}) = x_2^3 \otimes x_9 + x_8 \otimes x_7 + x_2^6 \otimes x_3 + x_8 x_2^3 \otimes x_1.$$

The Steenrod operations β and $\mathscr{P}^1$ act as follows:

$$\beta x_1 = x_2, \qquad \beta x_3 = -x_2^2, \qquad \beta x_7 = x_8,$$
$$\beta x_9 = x_2 x_8, \qquad \beta x_{11} = -x_2^6 - x_2^2 x_8, \qquad \beta x_{15} = -x_8^2,$$
$$\mathscr{P}^1 x_2 = x_2^3, \qquad \mathscr{P}^1 x_3 = x_7, \qquad \mathscr{P}^1 x_8 = -x_2^6,$$
$$\mathscr{P}^1 x_{11} = x_{15},$$
$$\beta x_2 = \beta x_8 = \mathscr{P}^1 x_1 = \mathscr{P}^1 x_7 = \mathscr{P}^1 x_9 = \mathscr{P}^1 x_{15} = 0. \qquad \square$$

We now state our results on the Morava K-theory at the prime 3 of PE_6.

THEOREM 4.2. *The Atiyah-Hirzebruch spectral sequence for $K(n)^*(PE_6)$ collapses for $n \geq 2$.*

PROOF. For $n = 2$ the differentials in the AHSS are d_{16s+1}, thus we must consider d_{17}, d_{33} and so on. However, as the largest primitive is of dimension 6, none of these differentials can take primitive values and hence, by (2.3), must all be zero. The result for $K(n)^*(PE_6)$ for $n > 2$ now follows by (2.2). $\square$

The calculation for $K(1)^*(PE_6)$ can be read off from the results of Held and Suter [**HS**] on the K-theory of PE_6. However, a direct calculation is possible using the methods developed here for the other Lie groups.

THEOREM 4.3. *The Atiyah-Hirzebruch spectral sequence for $K(1)^*(PE_6)$ at the prime 3 has only one non-trivial differential, namely d_5. The E_∞-term has algebra structure*

$$K(1)^*[x_2]/(x_2^3) \otimes \Lambda(y_7, y_9, y_{11}, w_{13}, y_{15}, w_{19})$$

in which the classes x_2, y_7, w_{13} and w_{19} are primitive and the reduced coproduct on the remaining generators acts by

$$\bar{\psi}(y_9) = x_2 \otimes y_7,$$
$$\bar{\psi}(y_{11}) = x_2 \otimes y_9 - x_2^2 \otimes y_7,$$
$$\bar{\psi}(y_{15}) = -x_2 \otimes w_{13}.$$

PROOF. We compute the first possible differential, d_5, by its association with Milnor's operation $Q_1 = \mathscr{P}^1\beta - \beta\mathscr{P}^1$. This differential turns out to be non-zero.

$$Q_1x_1 = x_2^3, \qquad Q_1x_2 = 0, \qquad Q_1x_3 = x_2^4 - x_8,$$
$$Q_1x_7 = -x_2^6, \qquad Q_1x_8 = 0, \qquad Q_1x_9 = -x_2^7 + x_2^3x_8,$$
$$Q_1x_{11} = x_2^8 + x_2^4x_8 + x_8^2, \qquad Q_1x_{15} = -x_2^6x_8.$$

We make the following change of basis to allow simpler computation of the differential d_5. We replace the generators x_7, x_8, x_9, x_{11} and x_{15} by the elements

$$y_7 = x_7 + x_1x_2^3, \qquad y_8 = -x_8 + x_2^4, \qquad y_9 = x_9 + x_2^3x_3,$$
$$y_{11} = x_{11} - x_2^4x_3 + x_3x_8, \qquad y_{15} = x_{15} + x_2^3x_9,$$

for then we have

$$Q_1x_1 = x_2^3,\ Q_1x_3 = y_8, \quad \text{and } Q_1 \text{ is trivial for all other generators.}$$

Hence the E_6-term of the spectral sequence is given by the algebra

$$K(1)^*[x_2]/(x_2^3) \otimes \Lambda(y_7, y_9, y_{11}, w_{13}, y_{15}, w_{19}),$$

where w_{13} represents the old $x_1x_2^6$ and w_{19} represents the old $x_3y_8^2$.

The remaining differentials are of the form d_{4s+1} with $s \geq 2$. The element x_2 is induced from the generator of $H^2(B\boldsymbol{Z}_3; \boldsymbol{F}_3)$ and so must represent a permanent cycle. As usual, induction on the degrees of the elements shows that the images of the remaining generators must be primitive, but x_2 is the only primitive in even degrees and hence all these other differentials are zero. The sequence thus collapses from the E_6-term onwards, giving the stated result. □

We turn our attention now to the group PE_7 and examine its Morava K-theories for the prime 2. We appeal to the work of [**HS**] and [**Ka2**] for the case $n = 1$ and our calculations concentrate on the case of higher n. The following gives the mod 2 cohomology for this group. (The final claim of the theorem can be seen, for example, by considering the spectral sequence for the fibration $E_7 \to PE_7 \to \boldsymbol{R}P^\infty$.)

THEOREM 4.4. *$H^*(PE_7; \boldsymbol{F}_2) = \boldsymbol{F}_2[x_1, x_5, x_9]/(x_1^4, x_5^4, x_9^4) \otimes \Lambda(x_6, x_{15}, x_{17}, x_{23}, x_{27})$ and $\mathrm{P}(PE_7) = \{x_1, x_1^2, x_5, x_6, x_9, x_5^2, x_{17}, x_9^2\}$. In this Hopf algebra, the reduced coproduct acts on the other generators by*

$$\bar{\psi}\ (x_{15}) = x_6 \otimes x_9 + x_5^2 \otimes x_5,$$
$$\bar{\psi}\ (x_{23}) = x_6 \otimes x_{17} + x_9^2 \otimes x_5,$$
$$\bar{\psi}\ (x_{27}) = x_5^2 \otimes x_{17} + x_9^2 \otimes x_9.$$

Moreover, the quotient map $E_7 \to PE_7$ in cohomology carries each generator x_i for $i = 5, 9, 15, 17, 23, 27$ to the element of the same name as that used in §3. □

THEOREM 4.5. *The Atiyah-Hirzebruch spectral sequence for $K(n)^*(PE_7)$ collapses for all $n \geq 2$.*

PROOF. It suffices to show that the Atiyah-Hirzebruch spectral sequence collapses for $n=2$. The differentials here are d_7, d_{13}, d_{19} and so on. The first of these, d_7, must be zero on each generator, as can be seen from the usual analysis of primitives. For the second differential, d_{13}, examination of the primitives fails to rule out a possible action sending x_5 to some non-trivial multiple of x_9^2. However, this is ruled out by the naturality of the Atiyah-Hirzebruch spectral sequence with respect to the map $E_7 \to PE_7$; explicitly, by the result (3.6). Once we see that $d_{13}(x_5)=0$ then the module of primitives shows the whole of d_{13} to be trivial. Finally, d_{19} and all higher differentials are zero for dimensional reasons: the highest dimensional primitive is in degree 18. □

§5. The projective symplectic groups.

The projective symplectic groups $PSp(m)$ are the quotients of the symplectic groups $Sp(m)$ by their centres $\boldsymbol{Z}_2$. Since projective symplectic groups have only 2 torsion, the prime in this section will always be 2. The main results are Theorems 5.3 and 5.4, which describe the Morava K-theory of the groups $PSp(m)$ additively. The latter parts of the section are concerned with the $P(n)^*$-module structure of $P(n)^*(PSp(m))$ where we obtain only partial answers.

The mod 2 cohomology of $PSp(m)$ was computed, as a Hopf algebra, by Baum and Browder [**BB**]; the following summary of their results is quoted from [**Mi**].

THEOREM 5.1. *Let $m=qm'$ where $q=2^r$ is the highest power of two dividing m. Then*

$$H^*(PSp(m);\boldsymbol{F}_2) \cong \boldsymbol{F}_2[v]/v^{4q} \otimes \Lambda(b_3, b_7, \ldots, \hat{b}_{4q-1}, \ldots, b_{4m-1}),$$

where v is in degree one, b_r is of degree r and "^" indicates that an element is missing. The action of the mod 2 *Steenrod algebra is given by*

$$Sq^{4j} b_{4k+3} = \binom{k}{j} b_{4k+4j+3};$$

$Sq^j b_{4k+3} = 0$ if $j \not\equiv 0 (\operatorname{mod} 4)$ unless $r \geq 1$, $j=1$ and $4k+3 = 2q-1 : Sq^1 b_{2q-1} = v^{2q}$. The reduced diagonal is given by

$$\begin{aligned} \bar{\psi}\,(b_{4k+3}) &= \sum_{i=1}^{k-1} \binom{k}{i} b_{4i+3} \otimes v^{4k-4i} \quad (k \geq 2), \\ \bar{\psi}\,(b_7) &= b_3 \otimes v^4, \\ \bar{\psi}\,(b_3) &= 0. \end{aligned}$$

□

For simplicity we shall distinguish two cases: (1) $n > r$ and (2) $n \leq r$. The first case is essentially trivial in the sense that there are no differentials in the AHSS

$$E_2^{**} = H^*(PSp(m);\boldsymbol{F}_2) \otimes K(n)^* \Rightarrow K(n)^*(PSp(m)). \tag{5.2}$$

THEOREM 5.3. *The spectral sequence for $K(n)^*(PSp(m))$ collapses on E_2^{**} whenever $n > r$. In other words, there is an isomorphism of $K(n)^*$-modules*

$$K(n)^*(PSp(m)) \cong K(n)^* \otimes H^*(PSp(m);\boldsymbol{F}_2).$$

PROOF. Since $Q_n b_i = 0$ by (5.1) and $Q_n v = v^{2^{n+1}} = 0$, we have $E_{2^{n+1}}^{**} = E_2^{**}$. Next we show that v is a permanent cycle. As v is primitive any potential image of v under some differential d_r has to be primitive too. Since there are no even degree primitives in the target area of a potential differential, $d_r(v) = 0$ for all r.

By the same argument, $d_r(b) = 0$ for any primitive b, which are those in degrees $4k+3$ with k a power of two, except for b_7. In particular, b_3 is a permanent cycle. This suffices to start the following induction on k. Suppose $d_r(b_{4\ell+3}) = 0$ for all ℓ less than k. Then $d_r(b_{4k+3})$ is primitive by (2.3) and hence zero for the lack of primitives in the appropriate (even) degrees. □

The other case is not much more complicated either.

THEOREM 5.4. *If $r \geq n$ then, as $K(n)^*$-modules,*

$$K(n)^*(PSp(m)) \cong K(n)^* \otimes \boldsymbol{F}_2[u]/(u^{2^n}) \otimes \Lambda(w) \otimes \Lambda(b_3, b_7, \ldots, \hat{b}_{4q-1}, \ldots, b_{4m-1}),$$

where u is in degree two, and w in degree $2^{r+2} - 2^{n+1} + 1$.

A comment on degrees of elements may be in order here. When writing "x has degree k" we actually mean it to have degree k modulo the degree of v_n, since the natural grading on $K(n)^*$ is the cyclical grading modulo $|v_n| = -2(2^n - 1)$. Of course the degrees as we state them serve also to record the skeletal filtration numbers of representatives of these elements.

PROOF. The difference to the previous calculation is that we now have a non-trivial differential, namely

$$d_{2^{n+1}-1}(v) = v_n Q_n v = v_n v^{2^{n+1}}.$$

The theorem asserts that this is the only non-trivial differential. Since Q_n vanishes on the b's, the $E_{2^{n+1}}$-term is as in the statement of the theorem, where we wrote u for v^2 and w for $v^{2^{r+2}-2^{n+1}+1}$. To see that v^2 is a permanent cycle, consider the fibre bundle

$$Sp(m) \to PSp(m) \xrightarrow{i} B\boldsymbol{Z}_2,$$

where i is the map which picks up the class v in cohomology (see [**BB**]). Follow up the map i by the obvious map to $\boldsymbol{CP}^\infty$ classifying the canonical line bundle over $B\boldsymbol{Z}_2$. Then the two-dimensional generator of the cohomology of $\boldsymbol{CP}^\infty$ maps to v^2, and comparison to the Atiyah-Hirzebruch spectral sequence for $\boldsymbol{CP}^\infty$ gives the result. Thus all the even powers of v are permanent. The result now follows by an induction identical to that used in the proof of (5.3). □

We conclude the section with a few remarks on $P(n)^*(PSp(m))$, where we can offer only partial results. In the $P(n)^*$-AHSS differentials also have odd horizontal degree, so, by the same argument as above, it collapses on E_2^{**} as long as $n > r$. To determine the structure as $P(n)^*$-module, the following lemma, due to Yagita, is useful.

LEMMA 5.5 ([**Y2**], Lemma 2.1). *Let X be a finite complex. Let $x_i \in H^*(X; \boldsymbol{F}_p)$ be permanent cycles in the* AHSS *for both $P(n)^*$-theory and $K(n)^*$-theory. Then in the E_∞-term for the $P(n)^*$-theory spectral sequence the $P(n)^*$-module generated by the x_i is $P(n)^*$-free.*

We immediately deduce

THEOREM 5.6. *For $n > r$, there is a $P(n)^*$-module isomorphism*

$$P(n)^*(PSp(m)) \cong P(n)^* \otimes H^*(PSp(m); \boldsymbol{F}_2). \qquad \square$$

For $1 \leq n \leq r$ there is a $P(n)^*$-analogue to (5.4) too, in the sense that the AHSS collapses after the first differential $d_{2^{n+1}-1}$. This is proved in exactly the same way as before. The $P(n)^*$-module structure however becomes more complicated. We have only been able to determine it in the first non-trivial case, which is that of $n = r$.

PROPOSITION 5.7. *For $m = 2^r m'$ with m' odd, there is a $P(r)^*$-module isomorphism*

$$P(r)^*(PSp(m)) \cong \begin{pmatrix} P(r)^* \otimes \boldsymbol{F}_2[u]/(u^{2^r}) \otimes \Lambda(w) \\ \oplus \\ P(r+1)^*\{x, xu, \ldots, xu^{2^r-1}\} \end{pmatrix} \otimes \Lambda(b_3, \ldots, \hat{b}_{4q-1}, \ldots, b_{4m-1})$$

where $\deg(u) = 2$, $\deg(w) = 2^{r+1} + 1$ *and* $\deg(x) = 2^{r+1}$.

PROOF. The argument is completely analogous to the one used repeatedly in **[Y2]**, e.g. Lemma 5.4. The first non-zero differential is $d_{2^{r+1}-1}$, and $E_{2^{r+1}}^{**}$ is isomorphic to the right hand side in the statement of the proposition. Here the classes u and w correspond to v^2 and $v^{2^{r+1}+1}$, respectively, as in (5.4). To prove that the spectral sequence collapses on $E_{2^{r+1}}^{**}$ we use downward induction on the horizontal degree. Assume that every element in $E_{2^{r+1}}^{s,*}$ persists to the E_∞-term for $s > t$, and let $y \in E_{2^{r+1}}^{t,0}$. Then $d_m(y) = 0$ for any $m > 2^{r+1}$ by the inductive hypothesis. Since the Atiyah-Hirzebruch spectral sequence for $P(r)^*$-theory is confined to the fourth quadrant, this means that y is in E_∞^{**}. Now y will be of the form $y_1 + y_2$, where y_1 is in the first summand of the $E_{2^{r+1}}$-term and y_2 in the second. If y_1 is non-zero, then y is also a non-trivial permanent cycle in the $K(r)^*$-AHSS and thus the $P(r)^*$-module it generates is free by (5.5). If $y_1 = 0$, consider the map $i_\infty : E_\infty^{**}(P(r)) \to E_\infty^{**}(P(r+1))$ induced by $i : P(r) \to P(r+1)$. The $P(r+1)^*$-module generated by $i_\infty(y)$ is free, hence the $P(r+1)^*$-module generated by y is free, too. This completes the inductive step, and we conclude $E_{2^{r+1}}^{**} = E_\infty^{**}$. It remains to show that the isomorphism is as $P(r)^*$-modules. For that it suffices to construct an element $x \in P(r)^*(PSp(m))$ with $v_r x = 0$. Consider the exact triangle

$$\begin{tikzcd} P(r)^*(PSp(m)) \arrow[rr, "v_r"] & & P(r)^*(PSp(m)) \arrow[dl, "i"] \\ & P(r+1)^*(PSp(m)) \arrow[ul, "\delta"] & \end{tikzcd}$$

The element $i\delta v = Q_r v$ is non-zero, hence $\delta v \neq 0$; let $x = \delta v$ and by exactness, $v_r x = 0$. $\square$

§6. The projective unitary groups and other quotients of $SU(m)$.

In this section we want to study the Morava K-theory and $P(n)$-theory of central quotients of $SU(m)$. Recall that the centre of $SU(m)$ consists of elements of the form $\lambda \cdot \mathrm{Id}$ with $\lambda^m = 1$, and the groups in question are quotients of $SU(m)$ by (central) subgroups Z_ℓ where ℓ divides m. The method shall be the same as in the previous sections, namely a (brute force) calculation of the Atiyah-Hirzebruch spectral sequence. (6.1) below, due to Baum and Browder ([**BB**]), gives the cohomology of these groups; we quote once again from the survey [**Mi**].

THEOREM 6.1. *Let Z_ℓ be a subgroup of the centre Z_m of $SU(m)$, and let p be a prime dividing ℓ. Let $m = p^r m'$, $\ell = p^s \ell'$, where m' and ℓ' are prime to p, and set $G = SU(m)/Z_\ell$. If $p > 2$, or $p = 2$ and $s > 1$, there exist generators $z_i \in H^{2i-1}(G; F_p)$, $1 \le i \le m$, $i \ne p^r$, and $y \in H^2(G; F_p)$ such that*

(a) $H^*(G; F_p) \cong F_p[y]/(y^{p^r}) \otimes \Lambda(z_1, z_2, \ldots, \hat{z}_{p^r}, \ldots, z_m)$ *as algebras;*

(b) $\bar{\psi}(z_i) = \delta_{rs} z_1 \otimes y^{i-1} + \sum_{j=2}^{i-1} \binom{i-1}{j-1} z_j \otimes y^{i-j}$ *for $i > 1$, where $\delta_{rs} = 1$ if $r = s$ and zero else, $\bar{\psi}(z_1) = 0$, $\bar{\psi}(y) = 0$;*

(c) $\mathscr{P}^k z_i = \binom{i-1}{k} z_{i+k(p-1)}$, $\beta z_{jp^{r-1}} = j^{-1} y^{p^{r-1}}$, $1 \le j \le p-1$, *and in addition, $\beta z_1 = y$ if $s = 1$.*[(1)]

If $p = 2$ and $s = 1$ the above has to be modified as follows: in (a), *one has $y = z_1^2$, and in addition*

(d) $Sq^{2k+1} z_i = 0$ *except for $Sq^1 z_1 = z_1^2$ and $Sq^1 z_{2^{r-1}} = z_1^{2^r}$ if $r > 1$.* □

Let G be as in the statement of the theorem. A basis for the module of primitive elements in the mod p cohomology of G is given by

(i) $z_1, y, y^p, \ldots, y^{p^{r-1}}$ if $r = s$;

(ii) $z_1, z_{p+1}, z_{p^2+1}, \ldots, z_{p^{r-1}+1}$; $y, y^p, \ldots, y^{p^{r-1}}$ if $r \ne s$.

Furthermore, using parts (c) and (d) of the same theorem, one readily verifies

(iii) $Q_k y = 0$, $Q_k z_i = 0$ as long as $k > 0$, $i > 1$;

(iv) $Q_k z_1 = y^{p^k}$ if $s = 1$, and zero otherwise for p odd;

(v) $Q_k z_1 = z_1^{2^{k+1}}$ if $s = 1$, and zero otherwise for $p = 2$.

As in the previous section, degree arguments imply:

LEMMA 6.2. *Suppose G is as above and $n \ge r > 0$. Then we have the following isomorphisms of $K(n)^*$, respectively $P(n)^*$, modules:*

(a) $K(n)^*(G) \cong K(n)^* \otimes H^*(G; F_p)$;

(b) $P(n)^*(G) \cong P(n)^* \otimes H^*(G; F_p)$.

PROOF. We begin with the $K(n)$ result. Under the stated assumptions the first differential is trivial. Any subsequent differential has length at least $4(p^n - 1) + 1$, and thus vanishes on the primitives z_1 and y for dimensional reasons. Now use the inductive argument of (5.3) to see that the spectral sequence collapses. This shows (a); the same argument works for the $P(n)$ case and the isomorphism as $P(n)^*$-modules follows from (5.5). □

From now on we shall assume that p is odd, the case $p = 2$ being similar. To simplify further and avoid cumbersome notation, we consider only projective unitary groups in detail, although the arguments work in greater generality (see our remark 6.4). So let $m = \ell$ (p odd), with $m = p^r m'$ where p does not divide m', and $r > 0$.

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(1) This last operation seems to be missing in the original paper [**BB**] as well as in [**Mi**].

Notice that the assumption on n in (6.2) may then be relaxed to $n \geq \min\{1, r-1\}$, but for smaller n dimensional arguments alone no longer suffice to compute differentials in the AHSS, or rather show their vanishing; one has to use the fact that this is a spectral sequence of Hopf algebras more efficiently. The following theorem is stated in terms of $P(n)$, which of course implies the analogous statement for $K(n)$.

THEOREM 6.3. *The $P(n)^*$-AHSS for $P(n)^*(PU(m))$ collapses if $n > 0$, and there is an isomorphism of $P(n)^*$-modules*

$$P(n)^*(PU(m)) \cong P(n)^* \otimes H^*(PU(m); \boldsymbol{F}_p).$$

PROOF. We know this already when n is at least $r-1$; we also know that the first differential $v_n \otimes Q_n$ vanishes for $n > 0$. The only primitives in the cohomology of $PU(m)$ being z_1 and y^{p^k}, $0 \leq k < r$, one sees immediately that y is a permanent cycle. We claim that the z's are permanent cycles, too. Suppose not. Let d be the first non-trivial differential and z_j the lowest degree generator on which d does not vanish. Then $d(z_j)$ is primitive, by (2.3), thus $d(z_j) = \alpha y^{p^k}$ for some $k \geq 2$ and some $\alpha \in P(n)^*$, which we shall suppress from notation. Let j_0 be the smallest positive integer such that $z_j \otimes y^{j_0}$ is a summand in $\bar{\psi}\,(z_{j+j_0})$. For $j = 1$ we have $j_0 = 1$; if $j > 1$, j_0 is the smallest positive integer such that $\binom{j+j_0-1}{j-1} \not\equiv 0 \bmod p$. This is easily computed as follows: let $j - 1 = \sum_0^\ell a_\nu p^\nu$ be the p-adic expansion of $j-1$; pick the first coefficient, a_{ν_0} say, which is smaller than $p-1$ and set $j_0 = p^{\nu_0}$. Now ν_0 has to be smaller than k (since otherwise $j = p^k$ and $\deg d = 1$), hence $j_0 \leq p^{k-1}$. The next even degree primitive after y^{p^k}, if one exists, being $y^{p^{k+1}}$ in degree $2p^{k+1}$, one inductively concludes that for $\ell < j_0$, $d(z_{j+\ell})$ has to be primitive and hence zero. Furthermore,

$$\bar{\psi} d(z_{j+j_0}) = d\bar{\psi}(z_{j+j_0}) = d(\lambda z_j \otimes y^{j_0} + \text{other terms}) = \lambda y^{p^k} \otimes y^{j_0}$$

where λ is some non-zero constant, the "other terms" giving zero since they involve only lower degree generators. But there are no classes $x \in H^*(G; \boldsymbol{F}_p)$ with $\bar{\psi}(x) = y^{p^k} \otimes y^a$ for any $a < p^k$, by inspection, and we arrive at a contradiction. The isomorphism $P(n)^*(PU(m)) \cong P(n)^* \otimes H^*(PU(m); \boldsymbol{F}_p)$ as modules follows again from (5.5). $\square$

REMARK 6.4. The same proof goes through for $G = SU(m)/\boldsymbol{Z}_\ell$ if m and ℓ have the same p-exponent, i.e., if $r = s$ in the notation of (6.1).

References

[BB] P. F. Baum and W. Browder, The cohomology of quotients of classical groups, Topology, **3** (1965), 305–336.

[Br] W. Browder, On differential Hopf algebras, Trans. Amer. Math. Soc., **107** (1963), 153–176.

[HS] R. P. Held and U. Suter, On the K-theory of compact Lie groups with finite fundamental group, Quart. J. Maths., **24** (1973), 343–356.

[Ho] L. Hodgkin, On the K-theory of Lie groups, Topology, **6** (1967), 1–36.

[Hu] J. R. Hunton, On Morava's extraordinary K-theories, Ph.D. thesis, Cambridge, 1989.

[Ka1] R. Kane, BP torsion in finite H-spaces, Trans. Amer. Math. Soc., **264** (1981), 473–497.

[Ka2] R. Kane, Implications in Morava K-theory, Memoires Amer. Math. Soc., **340** (1986).

[Mi] M. Mimura, Homotopy theory of Lie groups, in: Handbook of Algebraic Topology, ed. by I. M. James, Elsevier, (1995), 951–991.

[Rao1] V. K. Rao, On the Morava K-theories of $SO(2n+1)$, Proc. Amer. Math. Soc., **108** (1990), 1031–1038.

[Rao2] V. K. Rao, Spin(n) is not homotopy nilpotent for $n \geq 7$, Topology, **32** (1993), 239–249.

[Rav] D. C. Ravenel, Localization with respect to certain periodic cohomology theories, Amer. J. Math., **106** (1984), 351–414.

[Y1] N. Yagita, On the Steenrod algebra of Morava K-theory, J. London Math. Soc., **22** (1980), 423–438.

[Y2] N. Yagita, On mod odd prime Brown Peterson cohomology groups of exceptional Lie groups, J. Math. Soc. Japan, **34** (1982), 293–305.

[Y3] N. Yagita, Homotopy nilpotency for simply connected Lie groups, Bull. London Math. Soc., **25** (1993), 481–486.

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