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Asymptotically holomorphic embeddings of presymplectic manifolds

By Takayuki Moriyama

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Abstract. We apply Donaldson-Auroux's asymptotically holomorphic methods to construct asymptotically holomorphic embeddings of presymplectic closed manifolds of constant rank with integral form into Grassmannians $\mathrm{Gr}(r, N)$. In particular, we obtain asymptotically holomorphic embeddings into the projective spaces $\boldsymbol{C}\mathrm{P}^{N-1}$ such that the pull-back of the Fubini-Study form is cohomologous to $k\omega/2\pi$ for large integers k. Moreover, we can construct asymptotically holomorphic immersions along the symplectic distribution of presymplectic manifolds into the projective spaces.

1. Introduction.

Donaldson provided an asymptotically holomorphic method to extend the notion of ampleness in Kähler geometry to general symplectic manifolds [**3**]. Let (M, ω) be a $2n$-dimensional symplectic manifold with integral form ω, i.e. $[\omega/2\pi] \in H^2(M, \boldsymbol{R})$ lifts to an integral cohomology class. For any fixed such lift $h \in H^2(M, \boldsymbol{Z})$, he showed that there exist symplectic submanifolds of M which realize the Poincaré dual of kh, for all k large enough. These submanifolds are constructed as the zero sets of asymptotically holomorphic sections of a complex line bundle $L^{\otimes k}$ with $c_1(L) = h$. Auroux generalized Donaldson's method to the case of one-parameter families of asymptotically holomorphic sections of vector bundles $E \otimes L^{\otimes k}$ for any Hermitian bundle E over M [**1**].

There has been much success in symplectic geometry by applying Donaldson-Auroux's asymptotically holomorphic methods. V. Muñoz, F. Presas and I. Sols proved that there exists an asymptotically holomorphic embedding of any $2n$-dimensional symplectic closed manifold (M, ω) with integral form ω into the projective space $\boldsymbol{C}\mathrm{P}^{2n+1}$ such that the pull-back of the Fubini-Study form is cohomologous to $k\omega/2\pi$ for large integers k [**8**]. They also proved a similar result in contact geometry [**9**]. These results are analogous to the Kodaira embedding theorem.

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In this paper, we introduce presymplectic manifolds as a generalization of symplectic manifolds and apply Donaldson-Auroux's asymptotically holomorphic technique to presymplectic manifolds. We show that there exist asymptotically holomorphic immersions along the symplectic distribution of $(2n+\ell)$-dimensional presymplectic closed manifolds with an integral presymplectic form into the $(2n+\ell)$-dimensional projective space $\boldsymbol{C}\mathrm{P}^{2n+\ell}$.

Definition 1. Let M be a C^∞-manifold of dimension $(2n+\ell)$ for integers $n>0$ and $\ell \geq 0$. A closed 2-form ω on M is a presymplectic form of rank $2n$ if $\omega_x^n \neq 0$ and $\omega_x^{n+1} = 0$ for all $x \in M$. We call (M,ω) a presymplectic manifold of rank $2n$.

Let (M,ω) be a presymplectic manifold of rank $2n$. Then the ℓ-dimensional distribution $F = \{v \in TM \mid i(v)\omega = 0\}$ is integrable since ω is closed. Such a distribution F is called the characteristic distribution. We choose a complementary subbundle $W \subset TM$ to F. The $2n$-dimensional distribution W is a symplectic distribution with the restricted form ω. We can always find almost complex structures on W compatible with ω. We call such a structure J an almost pre-complex structure on M with respect to the symplectic distribution W. Moreover we obtain the Riemannian metric g on M with the orthogonal decomposition $TM = W \oplus_g F$, such that g restricted to W is compatible for J and ω. The pair (J,g) is said to be a pre-compatible pair on (M,ω).

Given a pre-compatible pair (J,g), we define an operator $\bar\partial$ which acts on any map ϕ from M to a complex manifold X as the anti-holomorphic part of the derivative $d\phi$ restricted to W. To define asymptotically holomorphicity of sequence of maps we use a sequence of the rescaled metrics $g_k = kg$ for integers k.

Definition 2. Let (X, J_X, g_X) be a Kähler manifold with an integral Kähler class. A sequence of maps $\phi_k : M \to X$ is an asymptotically holomorphic sequence of maps if there exist constants C_p which are independent on k, such that

$$\left|\nabla^p \phi_k\right|_{g_k} \leq C_p,\ \left|\nabla^{p-1}\bar\partial\phi_k\right|_{g_k} \leq C_p k^{-\frac{1}{2}}$$

for all $p \geq 1$, where ∇ is the covariant derivative with respect to g_k and the Kähler metric g_X. A sequence of maps $\phi_k : M \to X$ is an asymptotically holomorphic sequence of immersions along W if ϕ_k is an asymptotically holomorphic sequence of maps and there exists a constant $\gamma > 0$ which is independent of k, satisfying that $d\phi_k$ is non-degenerate on W and

> $d\phi_k|_W : W_x \to T_{\phi(x)}X$ has a left inverse $\theta_{W,k}$ whose norm is less than γ^{-1} at every point $x \in M$.

A sequence of embeddings $\phi_k : M \to X$ is an asymptotically holomorphic sequence of embeddings if ϕ_k is an asymptotically holomorphic sequence of maps and there exists a constant $\gamma' > 0$ which is independent of k, satisfying that

> $d\phi_k : T_x M \to T_{\phi(x)} X$ has a left inverse θ_k whose norm is less than γ'^{-1} at every point $x \in M$.

We construct the following maps to the projective space of dimension $2n+\ell$:

THEOREM 1. *Let (M,ω) be a $(2n+\ell)$-dimensional closed presymplectic manifold of rank $2n$ with a pre-compatible pair (J,g) and an integral form ω. Then there exist asymptotically holomorphic sequences of immersions $\phi_k : M \to \boldsymbol{C}\mathrm{P}^{2n+\ell}$ along the symplectic distribution W with $[\phi_k^* \omega_{FS}] = [k\omega/2\pi]$ for large integers k.*

The following theorem is a generalization of Kodaira embedding theorem to presymplectic manifolds:

THEOREM 2. *Let (M,ω) be a $(2n+\ell)$-dimensional closed presymplectic manifold of rank $2n$ with a pre-compatible pair (J,g) and an integral form ω. Then there exist asymptotically holomorphic sequences of embeddings $\phi_k : M \to \boldsymbol{C}\mathrm{P}^m$ with $[\phi_k^* \omega_{FS}] = [k\omega/2\pi]$ for large integers k where $m = 2n + \max\{2\ell, 1\}$.*

Let L be a complex line bundle with $c_1(L)$ an integral lift of $[\omega/2\pi]$ and $\mathscr{U} \to \mathrm{Gr}(r,N)$ the universal bundle of rank r over the Grassmannian. Then we construct asymptotically holomorphic sequences of maps to Grassmannians $\mathrm{Gr}(r,N)$:

THEOREM 3. *Let (M,ω) be a $(2n+\ell)$-dimensional closed presymplectic manifold of rank $2n$ with a pre-compatible pair (J,g) and an integral form ω. Suppose that $E \to M$ is a Hermitian vector bundle of rank r and N is a positive integer satisfying $N \geq n+\ell+r$ and $r(N-r) \geq 2n+\ell$. Then there exist asymptotically holomorphic sequences of immersions $\phi_k : M \to \mathrm{Gr}(r,N)$ along W with $\phi_k^* \mathscr{U} = E \otimes L^{\otimes k}$ for large integers k.*

THEOREM 4. *Let (M,ω) be a $(2n+\ell)$-dimensional closed presymplectic manifold of rank $2n$ with a pre-compatible pair (J,g) and an integral form ω. Suppose that $E \to M$ is a Hermitian vector bundle of rank r and N is a positive integer satisfying $N \geq n+\ell+r$ and $r(N-r) \geq 2n+\max\{2\ell,1\}$. Then there exist asymptotically holomorphic sequences of embeddings $\phi_k : M \to \mathrm{Gr}(r,N)$ with $\phi_k^* \mathscr{U} = E \otimes L^{\otimes k}$ for large integers k.*

The above constructions are independent of the choice of pre-compatible pairs in the following sense:

Theorem 5. *Let (J_i, g_i) be a pre-compatible pair and $\phi_{i,k}$ an asymptotically holomorphic sequence of maps as in Theorem 1, Theorem 2, Theorem 3 and Theorem 4 for $i = 0, 1$. Then $\phi_{0,k}$ and $\phi_{1,k}$ are isotopic for sufficiently large integers k.*

We say that $\phi_{0,k}$ and $\phi_{1,k}$ are isotopic (cf. [**8**]) if there exists a homotopy $\{\phi_{t,k}\}_{t\in[0,1]}$ such that $\phi_{t,k}$ is an asymptotically holomorphic sequence of maps for each t.

V. Muñoz, F. Presas and I. Sols have proved above results in the case of $\ell = 0$ (the symplectic case) [**8**]. In the case of $\ell = 1$, a stronger, from the view of the dimension of projective spaces, result has been obtained by A. Ibort and D. Martínez [**6**]. However it seems to be difficult to show the isotopic uniqueness (as in Theorem 5) by their method.

We consider Theorem 2 as an analogue of Kodaira embedding theorem to presymplectic geometry. Narasimhan and Ramanan showed that any integral closed 2-form on a compact manifold is the pull-back of the Fubini-Study form by a map into the projective space [**10**]. In our result (Theorem 2) the pull-back of the Fubini-Study form by the asymptotically holomorphic embedding is not necessary the presymplectic form. However, the dimension of the projective space in our result is smaller than that of Narasimhan and Ramanan's embedding.

In section 2, we provide a definition of an asymptotically holomorphic section of the k-th tensor $L^{\otimes k}$ of the complex line bundle L in presymplectic manifolds. In the case of $\ell \geq 2$, transversality problems of asymptotically holomorphic sections in presymplectic geometry are difficult since we cannot directly apply the Donaldson-Auroux's transversality theorem to asymptotically holomorphic functions in presymplectic manifolds. To overcome this problem we coisotropically embedd a presymplectic manifold (M, ω) into the symplectic manifold (X, Ω) where X is a neighbourhood of the zero section of the dual bundle F^*. There exists a complex line bundle $\tilde{L}$ over X such that $\tilde{L}|_M = L$. Then we see that any asymptotically holomorphic section $\tilde{s}_k$ of the vector bundles $\underline{\boldsymbol{C}}^{m+1} \otimes \tilde{L}^{\otimes k}$ on (X, Ω) induces an asymptotically holomorphic section s_k of $\underline{\boldsymbol{C}}^{m+1} \otimes L^{\otimes k}$ on (M, ω) where $\underline{\boldsymbol{C}}^{m+1}$ denotes the trivial complex line bundle of rank $m+1$. We solve the transversality problems of the asymptotically holomorphic sections s_k in presymplectic geometry by applying Donaldson-Auroux's theory to asymptotically holomorphic sections $\tilde{s}_k$ on the symplectic manifold (X, Ω).

In section 3, we perturb asymptotically holomorphic sections $\tilde{s}_k$ to be projectizable and non-degenerate on X. Then the induced section s_k is also projectizable and non-degenerate on M and we obtain the asymptotically holomorphic embeddings as the projectization $\boldsymbol{P}(s_k) : M \to \boldsymbol{C}\mathrm{P}^m$. Thus we prove the asymptotically holomorphic embedding theorems (Theorem 2 and Theorem 4).

In section 4, we improve the argument of the perturbation in previous embedding theorems by using relative Darboux coordinates. Then we can construct asymptotically holomorphic projectizable sections s_k which is non-degenerate along W. It is a strongly estimated transversality theorem along W for a sequence of asymptotically holomorphic sections. Therefore we have the asymptotically holomorphic immersion $\boldsymbol{P}(s_k)$ along W (Theorem 1 and Theorem 3). Finally, we show Theorem 5 by using the construction of one-parameter families of relative asymptotically holomorphic sections.

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2. Asymptotically holomorphic theory in presymplectic geometry.

In this section, we provide definition of an asymptotically holomorphic sequence of sections over presymplectic manifolds and construct such sections.

2.1. Asymptotically holomorphic section.

Let (M, ω) be a presymplectic manifold of rank $2n$ and (J, g) a pre-compatible pair. Then there exists the decomposition $W^* \otimes \boldsymbol{C} = W^{1,0} \oplus W^{0,1}$ by the almost pre-complex structure J. We also have the decomposition

$$W^* \otimes E = W_E^{1,0} \oplus W_E^{0,1} \tag{1}$$

for any complex vector bundle E. Hence (1) gives rise to the decomposition

$$\begin{aligned} T^*M \otimes E &= (W \oplus_g F)^* \otimes E \\ &= (W^* \otimes E) \oplus_g (F^* \otimes E) \\ &= W_E^{1,0} \oplus_g W_E^{0,1} \oplus_g (F^* \otimes E). \end{aligned} \tag{2}$$

The last decomposition (2) yields projections

$$p_g : T^*M \otimes E \to W^* \otimes E \tag{3}$$

$$q_J^{1,0} : W^* \otimes E \to W_E^{1,0} \tag{4}$$

$$q_J^{0,1} : W^* \otimes E \to W_E^{0,1}. \tag{5}$$

Given a connection ∇, then we define ∂s and $\bar{\partial} s$ for any section s of E as follows:

$$\partial s = q_J^{1,0} \circ p_g(\nabla s)$$
$$\bar{\partial} s = q_J^{0,1} \circ p_g(\nabla s).$$

It denotes that ∂s and $\bar{\partial} s$ are sections of $W_E^{1,0}$ and $W_E^{0,1}$, respectively.

Definition 3. A sequence of sections s_k of Hermitian bundles E_k with a compatible connection over M is called asymptotically J-holomorphic if there exists a family of constants C_p such that

$$\left|s_k\right| \leq C_0, \quad \left|\nabla^p s_k\right|_{g_k} \leq C_p, \quad \left|\nabla^{p-1}\bar{\partial} s_k\right|_{g_k} \leq C_p k^{-\frac{1}{2}}, \quad p \geq 1$$

for all non-negative integers k large enough, where the covariant derivative ∇ is defined by Levi-Civita connection for g_k and the connection on E.

Remark 1. Let (X, J_0) be a complex manifold and ϕ a map from M to X. We consider E as the pull-back bundle $\phi^{-1}TX$. Applying (3), (4) and (5), we have operators

$$\partial\phi = q_J^{1,0} \circ p_g(d\phi)$$
$$\bar{\partial}\phi = q_J^{0,1} \circ p_g(d\phi)$$

where the derivative $d\phi \in \Gamma(M, T^*M \otimes \phi^{-1}TX)$.

We consider a sequence of sections with good bounds by the rescaled metric g_k:

Definition 4. Let E_k be Hermitian bundles with a connection. A sequence of sections s_k of E_k has Gaussian decay in C^r-norm away from the point $x \in M$ if there exists a polynomial P and a constant $\lambda > 0$ such that for all $y \in M$, $|s_k(y)|, |\nabla s_k(y)|_{g_k}, \dots, |\nabla^r s_k(y)|_{g_k}$ are bounded by $P(d_k(x,y))\exp(-\lambda d_k^2(x,y))$, where P and λ are independent on k, and d_k is the distance by g_k.

2.2. Local theory.

We denote by M_0 the product space $\boldsymbol{C}^n \times \boldsymbol{R}^\ell$ of the complex space $\boldsymbol{C}^n$ and the Euclidean space $\boldsymbol{R}^\ell$ with the coordinates $(z^1, \dots, z^n, t^1, \dots, t^\ell)$. Let ω_0 be the 2-form $\frac{i}{2}\sum_{j=1}^n dz^j \wedge d\bar{z}^j$ and g_0 the standard Riemannian metric $\sum_{j=1}^n dz^j d\bar{z}^j + \sum_{j=1}^\ell dt^j dt^j$ on M_0. The pair (M_0, ω_0) is a presymplectic manifold of rank $2n$. By Darboux's theorem for presymplectic manifolds (see Section 7. Chapter III [**7**]), a presymplectic manifold of rank $2n$ is locally isomorphic to the standard model (M_0, ω_0). We have the characteristic distribution

$$F_0 = \{v \in TM_0 \mid i(v)\omega_0 = 0\}$$

on M_0 and the orthogonal decomposition

$$TM_0 = W_0 \oplus_{g_0} F_0$$

with respect to the metric g_0 where W_0 is the orthogonal complement to F_0. The vector bundles F_0 and W_0 are spanned by the vector fields $\{\frac{\partial}{\partial t^1}, \ldots, \frac{\partial}{\partial t^\ell}\}$ and $\{\frac{\partial}{\partial z^1}, \ldots, \frac{\partial}{\partial z^n}\}$, respectively.

We denote by J_0 the almost pre-complex structure defined by the metric g_0 and the presymplectic form ω_0.

In general, for a pre-compatible pair (J, g) on M_0 there exists a subbundle $W \subset TM_0$ over M_0 which is orthogonal to F_0 with respect to the metric g. Pre-compatible pairs (J, g) and (J_0, g_0) on M_0 induce two operators $\partial_{J,g}$ and ∂_{J_0,g_0}, respectively. Then we have following relations for any section σ of a complex vector bundle ξ with a connection over M_0:

$$\begin{aligned}
\partial_{J,g}\sigma &= \partial_{J_0,g_0}\sigma + \mu_{J,g}(\nabla\sigma) \\
\bar{\partial}_{J,g}\sigma &= \bar{\partial}_{J_0,g_0}\sigma + \mu'_{J,g}(\nabla\sigma) \\
\nabla_W\sigma &= \nabla_{W_0}\sigma + \big(\mu_{J,g} + \mu'_{J,g}\big)(\nabla\sigma)
\end{aligned}$$

where

$$\begin{aligned}
\mu_{J,g} &= q_J^{1,0} \circ p_g - q_{J_0}^{1,0} \circ p_{g_0} \\
\mu'_{J,g} &= q_J^{0,1} \circ p_g - q_{J_0}^{0,1} \circ p_{g_0}
\end{aligned}$$

are sections of $\mathrm{End}(T^*M_0 \otimes \xi)$.

Now we only consider a pre-compatible pair (J, g) satisfying

$$J(0) = J_0(0), \ \ g(0) = g_0(0)$$

at the origin 0 of M_0. In other words, sections $\mu_{J,g}$ and $\mu'_{J,g}$ vanish at the origin. Then for the product space $\mathbf{B}^{2n} \times \mathbf{B}^\ell$ of the unit balls of $\boldsymbol{C}^n$ and $\boldsymbol{R}^\ell$ with center 0 there exists a constant $C > 0$ such that

$$|\mu(z,t)| < C|(z,t)|,\ |\nabla\mu(z,t)| < C$$

$$|\mu'(z,t)| < C|(z,t)|,\ |\nabla\mu'(z,t)| < C$$

for $(z,t) \in \mathbf{B}^{2n} \times \mathbf{B}^{\ell} \subset M_0$, where the connection ∇ is the Levi-Civita connection for g and the norm is defined by g.

We define $\delta_\rho : \rho^{-1}(\mathbf{B}^{2n} \times \mathbf{B}^{\ell}) \to \mathbf{B}^{2n} \times \mathbf{B}^{\ell}$ as the dilation

$$\delta_\rho(z,t) = (\rho z, \rho t)$$

for a real number $\rho > 0$. Denote $\tilde{\mu}$ and $\tilde{\mu}'$ by the pull-back of the sections

$$\tilde{\mu} = \delta^*(\mu_{J,g})$$

$$\tilde{\mu}' = \delta^*(\mu'_{J,g}).$$

Then there exists a constant $C > 0$ such that

$$|\tilde{\mu}(z,t)| < C\rho|(z,t)|, \quad |\nabla\tilde{\mu}(z,t)| < C\rho$$

$$|\tilde{\mu}'(z,t)| < C\rho|(z,t)|, \quad |\nabla\tilde{\mu}'(z,t)| < C\rho$$

$$|(\tilde{\mu}+\tilde{\mu}')(z,t)| < C\rho|(z,t)|, \quad |\nabla(\tilde{\mu}+\tilde{\mu}')(z,t)| < C\rho.$$

In the case $\rho = k^{-\frac{1}{2}}$ for an integer $k > 0$ we have

$$|\partial_{J,g}\sigma| = |\partial_{J_0,g_0}\sigma| + O\big(k^{-\frac{1}{2}}|\nabla\sigma|\big) \tag{6}$$

$$\big|\bar{\partial}_{J,g}\sigma\big| = \big|\bar{\partial}_{J_0,g_0}\sigma\big| + O\big(k^{-\frac{1}{2}}|\nabla\sigma|\big) \tag{7}$$

$$|\nabla_W\sigma| = |\nabla_{W_0}\sigma| + O\big(k^{-\frac{1}{2}}|\nabla\sigma|\big) \tag{8}$$

for any section σ of a complex vector bundle ξ with a connection over $k^{\frac{1}{2}}(\mathbf{B}^{2n} \times \mathbf{B}^{\ell})$.

Let (M,ω) be a compact presymplectic manifold of rank $2n$ of dimension $2n+\ell$ with a pre-compatible pair (J,g). We denote by $B_{g_k}(x,c)$ the ball of radius c around x in M with respect to $g_k = kg$ and by $B(0,c')$ the ball of radius c' with respect to the standard metric g_0 around the origin 0 of M_0.

As in the case with the symplectic geometry, Darboux coordinates for the presymplectic form $k\omega$ play an important role to construct asymptotically holomorphic sections on presymplectic manifolds. A universal constant is defined as a number which is independent of integers k and any point x of M.

LEMMA 1. *For $x \in M$ and an integer $k > 0$, there exist local Darboux coordinates $(z_k^1, \dots, z_k^n, t_k^1, \dots, t_k^\ell) = \Phi_k : (M, x) \to (\boldsymbol{C}^n \times \boldsymbol{R}^\ell, 0)$ for the presymplectic form $k\omega$ such that for universal constants c, c' and integers $r \geq 1$,*

1. $\left|\Phi_k(y)\right|^2 = O(d_k(x,y)^2)$ *on* $B_{g_k}(x, c)$.
2. $\left|\nabla^r \Phi_k^{-1}\right|_{g_k} = O(1)$ *on* $B(0, c')$.
3. $\left|\bar{\partial}\Phi_k^{-1}(z_k, t_k)\right|_{g_k} = O\big(k^{-\frac{1}{2}}|z_k, t_k|\big)$ *and* $\left|\nabla^{r-1}\bar{\partial}\Phi_k^{-1}\right|_{g_k} = O\big(k^{-\frac{1}{2}}\big)$ *on* $B(0, c')$,

where $\bar{\partial}$ is the operator induced by J and the standard complex structure J_0.

Moreover, given a one-parameter family of pre-compatible pairs $(J_t, g_t)_{t \in [0,1]}$ and a one-parameter family of points $(x_t)_{t \in [0,1]}$, there exists a one-parameter family of local Darboux coordinates $\{\Phi_{t,k} : (M, x_t) \to (\boldsymbol{C}^n \times \boldsymbol{R}^\ell, 0)\}_{t \in [0,1]}$ which depend continuously on t and satisfy the same properties.

PROOF. This proof is similar to that of Lemma 3 in [**2**]. By using Darboux's theorem for the presymplectic form ω (Theorem 3.4.5 in [**11**]), there exist a neighbourhood $U_x \subset M$ of x and a local diffeomorphism $\Phi : U_x \to V_0 \subset \boldsymbol{C}^n \times \boldsymbol{R}^\ell$ such that $\Phi^*\omega_0 = \omega$. The pull-back $((\Phi^{-1})^*J, (\Phi^{-1})^*g)$ is the pre-compatible pair on V_0. We can construct a linear transformation $\Psi : V_0 \to V_0$ such that $(\Phi^{-1})^*g(0) = \Psi^*g_0(0)$ and $\Psi^*\omega_0 = \omega_0$. Then Ψ satisfies $(\Phi^{-1})^*J(0) = \Psi^*J_0(0)$. Hence we have the diffeomorphism $\Psi \circ \Phi : U_x \to V_0$ such that $(\Psi \circ \Phi)^*J_0(x) = J(x)$ and $(\Psi \circ \Phi)^*\omega_0 = \omega$.

For simplicity we write Φ instead of $\Psi \circ \Phi$. Then it follows from $(\Phi^{-1})^*J(0) = J_0(0)$ $\bar{\partial}\Phi^{-1}(0) = 0$. Since M is compact, Φ can be chosen so that the derivative of Φ^{-1} is bounded by a universal constant:

$$|\nabla\Phi^{-1}|_g = O(1).$$

Define $\Phi_k = \delta_{k^{\frac{1}{2}}} \circ \Phi$ then we have $\bar{\partial}\Phi_k^{-1}(0) = 0$ and $|\nabla\Phi_k^{-1}|_{g_k} = O(1)$. In addition $|\nabla^r\Phi_k^{-1}|_{g_k} = O(1)$ and $|\nabla^r\bar{\partial}\Phi_k^{-1}|_{g_k} = O(k^{-1/2})$ for $r \leq 1$. It follows that $|\bar{\partial}\Phi_k^{-1}|_{g_k} = O(k^{-1/2}|z_k, t_k|)$.

For a one-parameter family of the pairs $(J_t, g_t)_{t \in [0,1]}$ and points $(x_t)_{t \in [0,1]}$ we obtain a one-parameter family of linear transformations $(\Psi_t)_{t \in [0,1]}$ and Darboux charts $(\Phi_t)_{t \in [0,1]}$ for ω with $\Phi_t(x_t) = 0$, such that $(\Psi_t \circ \Phi_t)^*\omega_0 = \omega$ and $(\Psi_t \circ \Phi_t)^*J_0(x) = J_t(x)$. We define $\Phi_{t,k} = \delta_{k^{\frac{1}{2}}} \circ \Psi_t \circ \Phi_t$, and it finishes the proof. □

We consider local properties of a sequence of sections s_k of Hermitian bundles E_k over M with a connection. Now we may suppose that a trivialization of E_k includes the ball $B_{g_k}(x, c)$. Since the Darboux chart Φ_k is constructed by composition Φ with a rescaled map $\delta_{k^{\frac{1}{2}}}$ of order $k^{-\frac{1}{2}}$, the equations (6), (7) and (8) in previous subsection imply the estimates

$$\left|\bar{\partial}_{J,g} s_k\right|_{g_k} \leq C\left(\left|\bar{\partial}_{J_0,g_0} s_k\right| + k^{-\frac{1}{2}}|\nabla s_k|\right) \tag{9}$$

$$\left|\partial_{J,g} s_k\right|_{g_k} \leq C\left(\left|\partial_{J_0,g_0} s_k\right| + k^{-\frac{1}{2}}|\nabla s_k|\right) \tag{10}$$

$$\left|\nabla_W s_k\right|_{g_k} \leq C\left(\left|\nabla_{W_0} s_k\right| + k^{-\frac{1}{2}}|\nabla s_k|\right) \tag{11}$$

on $B_{g_k}(x,c)$, where the sections s_k on the right hand side are identified with sections over the ball $\Phi_k(B_{g_k}(x,c))$ in $\boldsymbol{C}^n \times \boldsymbol{R}^\ell$ by Φ_k. The larger k grows, the closer the operators ∂, $\bar{\partial}$ and ∇_W become to ∂_{J_0,g_0}, $\bar{\partial}_{J_0,g_0}$ and ∇_{W_0} on the ball $B_{g_k}(x,c)$. The reason why this happens is the situation that (J,g) is sufficiently close to (J_0,g_0) for large integer k on the ball $B_{g_k}(x,c)$. This is the idea of asymptotically holomorphic theory in presymplectic geometry.

2.3. Construction of asymptotically holomorphic sections.

In this subsection, we see that an asymptotically holomorphic sequence of sections on a certain symplectic manifold induces the asymptotically holomorphic sequence of sections on (M,ω).

Proposition 1 (Gotay [**5**]). *Let (M,ω) be a closed presymplectic manifold of rank $2n$ of dimension $2n+\ell$. Then there exist a symplectic manifold (X,Ω) and a coisotropic embedding $i : M \to X$, i.e., X is a $2(n+\ell)$-dimensional manifold and $i^*\Omega = \omega$.*

The symplectic manifold X is obtained as a tubular neighbourhood of the zero section of the dual bundle of the characteristic distribution F. The symplectic form Ω is constructed with Weinstein's technique [**12**]. Then $\Omega|_M$ splits as $\omega + \sum_{j=1}^{\ell} ds^j \wedge dt^j$ on $TX|_M = W \oplus F \oplus F^*$ where $\{\frac{\partial}{\partial t^1}, \dots, \frac{\partial}{\partial t^\ell}\}$ and $\{\frac{\partial}{\partial s^1}, \dots, \frac{\partial}{\partial s^\ell}\}$ are local frames of F and F^*, respectively.

Given a pre-compatible pair (J,g) on M, then we obtain a Riemannian metric $\tilde{g}$ on X such that $i^*\tilde{g} = g$ and $TM \oplus_{\tilde{g}} F^*$ over M. In addition, we also have an almost complex structure $\tilde{J}$ on X which is compatible for Ω and $\tilde{g}$. Then the endmorphism $i^*\tilde{J}$ on W is well-defined and satisfies $i^*\tilde{J} = J$. For a one-parameter family of pre-compatible pairs $\{(J_t,g_t)\}_{t\in[0,1]}$ there exists the family $\{(\tilde{J}_t,\tilde{g}_t)\}_{t\in[0,1]}$.

From now on, we fix the symplectic manifold (X,Ω), the Riemannian metric $\tilde{g}$ and the almost complex structure $\tilde{J}$ on X for the presymplectic manifold (M,ω) with the pre-compatible pair (J,g). In the one-parameter case, $\{(\tilde{J}_t,\tilde{g}_t)\}_{t\in[0,1]}$ is fixed. We have appropriate Darboux coordinates which are compatible for the coisotropic embedding $i : M \to X$.

Lemma 2 ([**11**, Theorem 3.4.10]). *Let M be a coisotropic submanifold of codimension ℓ of a $2(n+\ell)$-dimensional symplectic manifold (X,Ω). Then for* *any $x \in M$ there exist local Darboux coordinates $(z^1, \ldots, z^n, w^1, \ldots, w^\ell) = \tilde{\Phi}$: $(X, x) \to (\boldsymbol{C}^n \times \boldsymbol{C}^\ell, 0)$ for the symplectic form Ω such that $\Phi = i^*\tilde{\Phi}$ is given by Darboux coordinates $(z^1, \ldots, z^n, t^1, \ldots, t^\ell) : (M, x) \to (\boldsymbol{C}^n \times \boldsymbol{R}^\ell, 0)$ for the presymplectic form ω with $t^j = \mathrm{Re}(w^j)$.*

Moreover we produce following relative Darboux coordinates for the symplectic form $k\Omega$ and presymplectic form $k\omega$.

Proposition 2. *For $x \in M$ and an integer $k > 0$, there exists local Darboux coordinates $(z_k^1, \ldots, z_k^n, w_k^1, \ldots, w_k^\ell) = \tilde{\Phi}_k : (X, x) \to (\boldsymbol{C}^n \times \boldsymbol{C}^\ell, 0)$ for the symplectic form $k\Omega$ such that $\Phi_k = i^*\tilde{\Phi}_k = (z_k^1, \ldots, z_k^n, t_k^1, \ldots, t_k^\ell) : (M, x) \to (\boldsymbol{C}^n \times \boldsymbol{R}^\ell, 0)$ is Darboux coordinates for the presymplectic form $k\omega$, where $t_k^j = \mathrm{Re}(w_k^j)$ for $1 \le j \le \ell$, and for universal constants c, c' and integers $r \ge 1$,*

1. $\left|\tilde{\Phi}_k(y)\right|^2 = O(d_k(x, y)^2)$ *on* $B_{\tilde{g}_k}(x, c)$.
2. $\left|\nabla^r \tilde{\Phi}_k^{-1}\right|_{\tilde{g}_k} = O(1)$ *on* $B(0, c')$.
3. $\left|\bar{\partial}_{\tilde{J}} \tilde{\Phi}_k^{-1}(z_k, w_k)\right|_{\tilde{g}_k} = O\big(k^{-\frac{1}{2}}|z_k, w_k|\big)$ *and* $\left|\nabla^{r-1}\bar{\partial}_{\tilde{J}} \tilde{\Phi}_k^{-1}\right|_{\tilde{g}_k} = O\big(k^{-\frac{1}{2}}\big)$ *on* $B(0, c')$.
4. $\left|\bar{\partial}_J \Phi_k^{-1}(z_k, t_k)\right|_{g_k} = O\big(k^{-\frac{1}{2}}|z_k, t_k|\big)$ *and* $\left|\nabla^{r-1}\bar{\partial}_J \Phi_k^{-1}\right|_{g_k} = O\big(k^{-\frac{1}{2}}\big)$ *on* $B(0, c') \cap \phi_k(M)$,

where $\bar{\partial}_{\tilde{J}}$ (resp. $\bar{\partial}_J$) is the operator induced by $\tilde{J}$ (resp. J) and the standard complex (resp. almost pre-complex) structure J_0 in $\boldsymbol{C}^n \times \boldsymbol{C}^\ell$ (resp. $\boldsymbol{C}^n \times \boldsymbol{R}^\ell$).

Moreover, given a one-parameter family of pre-compatible pairs $\{(J_t, g_t)\}_{t\in[0,1]}$ and a one-parameter family of points $(x_t)_{t\in[0,1]}$ of M, there exists a one-parameter family of Darboux coordinates $\{\tilde{\Phi}_{t,k}\}_{t\in[0,1]}$ which depend continuously on t and satisfy the same properties.

Proof. For the Darboux chart $\tilde{\Phi}$ in Lemma 2, the pull-back $((\tilde{\Phi}^{-1})^*\tilde{J}, (\tilde{\Phi}^{-1})^*\tilde{g})$ is the pair of the complex structure and the Riemannian metric on a neighbourhood $\tilde{V}_0 \subset \boldsymbol{C}^n \times \boldsymbol{C}^\ell$ of the origin 0, respectively. Then we can construct a linear transformation $\tilde{\Psi} : \tilde{V}_0 \to \tilde{V}_0$ such that $(\tilde{\Phi}^{-1})^*\tilde{g}(0) = \tilde{\Psi}^*\tilde{g}_0(0)$ and $\tilde{\Psi}^*\Omega_0 = \Omega_0$, where $\tilde{g}_0$ is the standard Riemannian metric on $\tilde{V}_0$. In addition, $\tilde{\Psi}$ can be chosen to satisfy $\tilde{\Psi}(\mathrm{Im}(w)) = \mathrm{Im}(w)$ and $\tilde{\Psi}^*\omega_0 = \omega_0$. We have $(\tilde{\Phi}^{-1})^*\tilde{J}(0) = \tilde{\Psi}^*\tilde{J}_0(0)$ and $(\tilde{\Phi}^{-1})^*J(0) = \tilde{\Psi}^*J_0(0)$. Therefore the composition $\tilde{\Psi} \circ \tilde{\Phi}$ satisfies $(\tilde{\Psi} \circ \tilde{\Phi})^*\tilde{J}_0(x) = \tilde{J}(x)$ and $(\tilde{\Psi} \circ \tilde{\Phi})^*J_0(x) = J(x)$. We define $\tilde{\Phi}_k = \delta_{k^{\frac{1}{2}}} \circ \tilde{\Psi} \circ \tilde{\Phi}$. Then it follows from the proof of Lemma 1 that $\tilde{\Phi}_k$ satisfies the conditions 1, 2, 3 and 4. This completes the proof in non-parameter case.

For one-parameter families of the pairs $(J_t, g_t)_{t\in[0,1]}$ and points $(x_t)_{t\in[0,1]}$ we only need to apply the above discussion to a one-parameter family of relative Darboux charts $(\tilde{\Phi}_t)_{t\in[0,1]}$ for Ω and ω with $\tilde{\Phi}_t(x_t) = 0$. Hence it finishes the proof. □

Since ω is integral, then so Ω is. Moreover, for any lift h of $[\omega/2\pi]$ we can select $\tilde{L} \to X$ a Hermitian line bundle with $\tilde{L}|_M = L$. A compatible connection on $\tilde{L}$ with curvature $-i\Omega$ can also be fixed.

Now, we recall the Donaldson's construction of asymptotically holomorphic sections over symplectic manifold. For $x \in M$ asymptotically holomorphic local sections $\tilde{s}^{\mathrm{ref}}_{k,x}$ are constructed from a local section

$$\tilde{\sigma}_0 = e^{-(|z_k|^2+|w_k|^2)/4}$$

of a trivial line bundle ξ over $\boldsymbol{C}^{n+\ell}$. Therefore we define a local section

$$\tilde{s}^{\mathrm{ref}}_{k,x} = \tilde{\Phi}^*_k(\tilde{\beta}_k\tilde{\sigma}_0)$$

of $\tilde{L}^{\otimes k}$ over M, where $\tilde{\beta}_k$ are appropriate cut-off functions. In the one-parameter case, we define a one-parameter family of local sections $\tilde{s}^{\mathrm{ref}}_{t,k,x_t} = \tilde{\Phi}^*_{t,k}(\tilde{\beta}_k\tilde{\sigma}_0)$ where $\{\tilde{\Phi}_{t,k}\}_{t\in[0,1]}$ is a one-parameter family of relative Darboux charts. Then $\tilde{s}^{\mathrm{ref}}_{k,x}$ have Gaussian decay in C^r-norm away from x such that $|\tilde{s}^{\mathrm{ref}}_{k,x}| > c$ on a ball $B_{\tilde{g}_k}(x,1)$ of $\tilde{g}_k$-radius 1 centered at x. In the one-parameter case, $\tilde{s}^{\mathrm{ref}}_{t,k,x_t}$ satisfy the same properties. According to Donaldson, we can choose a "lattice" of points Λ in M such that

(i) $M = \cup_{p_i\in\Lambda} B_{g_k}(p_i,1)$,
(ii) $\sum_{p_i\in\Lambda} d_k(p_i,q)^r e_k(p_i,q) \leq C$ for any $q \in M$ and $r = 0,1,2,3$,

where $e_k(\cdot,q)$ is the function on M such that $e_k(p,q) = \exp(-\frac{1}{5}d_k(p,q)^2)$ if $d_k(p,q) \leq k^{\frac{1}{4}}$ and $e_k(p,q) = 0$ if $d_k(p,q) \leq k^{\frac{1}{4}}$ for $p \in M$. Then we obtain a sequence of asymptotically holomorphic global sections

$$\tilde{s}_k = \sum_{p_i\in\Lambda} w_i \tilde{s}^{\mathrm{ref}}_{k,p_i} \tag{12}$$

of $\tilde{L}^{\otimes k}$ over X, where the each w_i is a complex number with $|w_i| < 1$. It follows from Donaldson's computations (see Section 3 in [**3**]) that $\tilde{s}_k$ is a sequence of asymptotically holomorphic sections of $\tilde{L}^{\otimes k}$. Moreover we can construct a sequence of asymptotically holomorphic sections of $\tilde{L}^{\otimes k} \otimes \tilde{E}$ over X for any Hermitian bundle $\tilde{E}$ by considering the local sections $(w_i^1\tilde{s}^{\mathrm{ref}}_{k,p_i},\ldots,w_i^r\tilde{s}^{\mathrm{ref}}_{k,p_i})$ with complex vectors $w_i = (w_i^1,\ldots,w_i^r)$ such that $\|w_i\| \leq 1$. For a one-parameter family of points $(x_t)_{t\in[0,1]}$ we obtain a one-parameter family of sections $\{\tilde{s}_{t,k}\}_{t\in[0,1]}$ by using the one-parameter family of the local sections $\{\tilde{s}^{\mathrm{ref}}_{t,k,p_i}\}_{t\in[0,1]}$ instead of $\tilde{s}^{\mathrm{ref}}_{k,p_i}$ (see Section 3 in [**1**]). Therefore we have

PROPOSITION 3. *Let $\tilde{E}$ be a Hermitian bundle with a compatible connection over X. There exists a sequence of asymptotically holomorphic (non-trivial) sections $\tilde{s}_k$ of the Hermitian vector bundles $\tilde{L}^{\otimes k}\otimes\tilde{E}$. Moreover, given a one-parameter family of pre-compatible pairs $(J_t, g_t)_{t\in[0,1]}$, there exists a one-parameter family of asymptotically holomorphic sections $\tilde{s}_{t,k}$ which depend continuously on t.*

LEMMA 3. *Let $\tilde{E}$ be a Hermitian bundle with a compatible connection over X. If a sequence of sections $\tilde{s}_k$ of $\tilde{L}^{\otimes k}\otimes\tilde{E}$ is asymptotically holomorphic, then a sequence of pull-back sections $i^*\tilde{s}_k$ of $L^{\otimes k}\otimes i^*\tilde{E}$ is also asymptotically holomorphic.*

PROOF. Let $\tilde{s}$ be a section of $\tilde{E}$ over X and s the pull-back section $i^*\tilde{s}$ over M. Then we have

$$i^*(\nabla\tilde{s}) = \nabla s + (\nabla\tilde{s})^{\perp}$$

where $(\nabla\tilde{s})^{\perp}$ is the restriction of $\nabla\tilde{s}|_M$ to $TM^{\perp}\otimes E$. It follows that $\|\nabla s\|_g \leq \|\nabla\tilde{s}\|_{\tilde{g}}$ on M. Iterating this argument to $\nabla^r\tilde{s}$, we obtain

$$\|s\|_{C^r,g} \leq \|\tilde{s}\|_{C^r,\tilde{g}}$$

on M. For the holomorphic part $\bar{\partial}\tilde{s}$, we also have

$$i^*(\bar{\partial}\tilde{s}) = \bar{\partial}s + (\bar{\partial}\tilde{s})^{\perp}.$$

Hence it follows that

$$\|\bar{\partial}s\|_{C^r,g} \leq \|\bar{\partial}\tilde{s}\|_{C^r,\tilde{g}}$$

on M, and the proof is finished. □

The following corollary follows from Proposition 3 and Lemma 3.

COROLLARY 1. *Let (M,ω) be a closed presymplectic manifold of rank $2n$ and ω integral. Given a pre-compatible pair (J,g), then there exist sequences of asymptotically holomorphic sections s_k of the Hermitian bundles $L^{\otimes k}\otimes E$ for any Hermitian bundle E with a compatible connection.*

Moreover, given a one-parameter family of pre-compatible pairs $(J_t, g_t)_{t\in[0,1]}$, there exists a one-parameter family of asymptotically holomorphic sections $s_{t,k}$

which depend continuously on t.

Asymptotically holomorphic sections $\tilde{s}_k$ constructed as in (12) which depend on choices of w_i with $|w_i| \leq 1$. Then we can take a appropriate constant w_i so that $\tilde{s}_k$ satisfies the properties relating transversality, and prove our main theorems in Section 3 and 4.

Remark 2. In the above construction, the support of each section $\tilde{s}^{\mathrm{ref}}_{k,p_i}$ contains the open ball $B_{\tilde{g_k}}(p_i, c) \subset X$ for a universal constant c. We define $B_k(M)$ as the open set $\cup_{p_i \in \Lambda} B_{\tilde{g_k}}(p_i, c)$. Then $(B_k(M), \Omega)$ is the symplectic manifold in which M is coisotropically embedded. Later, we consider to perturb the asymptotically holomorphic section $\tilde{s}_k$ on the neighbourhood $B_k(M)$ of M in X.

3. Asymptotically holomorphic embeddings.

3.1. The case of projective spaces.

Definition 5. Let Y be a C^∞-manifold and L' a Hermitian line bundle over Y. A section s of the vector bundle $\underline{\boldsymbol{C}}^{2(n+\ell)+1} \otimes L'$ is ν-projectizable on Y if $|s(y)| > \nu$ for all $y \in Y$, where ν is a positive constant and $\underline{\boldsymbol{C}}^{2(n+\ell)+1}$ is the trivial bundle over Y.

If a section s of the vector bundle $\underline{\boldsymbol{C}}^{2(n+\ell)+1} \otimes L'$ is ν-projectizable for a positive constant ν, then we obtain a map

$$\boldsymbol{P}(s) : Y \to \boldsymbol{C}\mathrm{P}^{2(n+\ell)}.$$

The next result follows from Proposition 2.15 in [**8**]:

Lemma 4. *Let* $\tilde{s}_k$ *be an asymptotically holomorphic sequence of sections of the vector bundles* $\underline{\boldsymbol{C}}^{2(n+\ell)+1} \otimes \tilde{L}^{\otimes k}$ *over* X. *Then for* $\alpha > 0$ *there exists another sequence* $\tilde{\sigma}_k$ *satisfying that*

1. $|\tilde{s}_k - \tilde{\sigma}_k|_{C^1, \tilde{g}_k} < \alpha$.
2. $\tilde{\sigma}_k$ *is* ν*-projectizable on* $B_k(M)$.
3. $| \wedge^{n+\ell} \partial \boldsymbol{P}(\tilde{\sigma}_k)|_{\tilde{g}_k} > \nu$ *on* $B_k(M)$

for a universal constant $\nu > 0$. *Moreover, for a continuous family of relative asymptotically holomorphic sequences of sections* $\tilde{s}_{t,k}$ *and any* $\alpha > 0$, *there exists a continuous family of sections* $\tilde{\sigma}_{t,k}$ *satisfying the conditions* 1, 2 *and* 3 *for each* t.

Proposition 4. *Let* (M, ω) *be a closed presymplectic manifold of rank* $2n$ *of dimension* $2n + \ell$ *with a pre-compatible pair* (J, g). *There exists an asymptoti-*

cally J-holomorphic sequence of sections σ_k of the vector bundles $\underline{\boldsymbol{C}}^{2(n+\ell)+1} \otimes L^{\otimes k}$ satisfying that

1. *σ_k is ν-projectizable on M.*
2. $| \wedge^{2n+\ell} d\boldsymbol{P}(\sigma_k)|_{g_k} > \nu$

for a universal constant $\nu > 0$.

Moreover, given a one-parameter family of pre-compatible pairs $(J_t, g_t)_{t\in[0,1]}$, there exists a continuous family of asymptotically J_t-holomorphic sections $\sigma_{t,k}$ satisfying the same properties.

Proof. Let $\tilde{\sigma}_k$ be the section for a small α in Lemma 4, then the pullback $\sigma_k = i^*\tilde{\sigma}_k$ is asymptotically J-holomorphic and ν-projectizable on M. It is sufficient to show that $| \wedge^{n+\ell} \partial\boldsymbol{P}(\tilde{\sigma}_k)|_{\tilde{g}_k} > \nu$ implies $| \wedge^{2n+\ell} d\boldsymbol{P}(\sigma_k)|_{g_k} > \frac{\nu}{2}$ for a constant $\nu > 0$.

For a point $x \in M$, we can assume that $\tilde{\sigma}_k(x) = (\tilde{\sigma}_k^0(x), \tilde{\sigma}_k^1(x), \ldots, \tilde{\sigma}_k^{2(n+\ell)}(x)) = (\tilde{\sigma}_k^0(x), 0, \ldots, 0)$ by the action of $\mathrm{U}(2(n+\ell)+1)$ on $\boldsymbol{C}^{2(n+\ell)+1}$. The ν-projectizability of the section $\tilde{\sigma}_k$ implies that $|\tilde{\sigma}_k^0(x)| \geq \nu$. Hence we can take a universal constant c such that $|\tilde{\sigma}_k^0| \geq \nu/2$ on $B_{\tilde{g}_k}(x,c)$ because of the upper bound $|\tilde{\sigma}_k|_{C^1} < C$ for a universal constant C. Let Φ_0 be a trivialization of $\boldsymbol{C}\mathrm{P}^{2(n+\ell)}$ for the affine coordinates $U_k = \{[z^0, \cdots, z^{2(n+\ell)}] | z^0 \neq 0\}$ and we define the map $f_k = \Phi_0 \circ \boldsymbol{P}(\tilde{\sigma}_k)$:

$$
\begin{aligned}
f_k : B_{\tilde{g}_k}(x,c) &\to \boldsymbol{C}^{2(n+\ell)}, \\
&\qquad y \mapsto \left(\frac{\tilde{\sigma}_k^1(y)}{\tilde{\sigma}_k^0(y)}, \cdots, \frac{\tilde{\sigma}_k^{2(n+\ell)}(y)}{\tilde{\sigma}_k^0(y)} \right).
\end{aligned}
$$

Since the map f_k satisfies $|\nabla^p f_k| = O(1)$ and $|\nabla^p \bar{\partial} f_k| = O(k^{-\frac{1}{2}})$, and Φ_0 is an isometry at $[1,0,\ldots,0]$ for the standard metric of $\boldsymbol{C}^{2(n+\ell)+1}$, the map $\boldsymbol{P}(\tilde{\sigma}_k)$ satisfies

$$
\left|\nabla^p \boldsymbol{P}(\tilde{\sigma}_k)\right|_{\tilde{g}_k} = O(1), \quad \left|\nabla^{p-1} \bar{\partial} \boldsymbol{P}(\tilde{\sigma}_k)\right|_{\tilde{g}_k} = O\big(k^{-\frac{1}{2}}\big) \tag{13}
$$

for all $p \geq 1$. This implies

$$
\left| \wedge^{2(n+\ell)} d\boldsymbol{P}(\tilde{\sigma}_k)\right|_{\tilde{g}_k} = \left| \wedge^{n+\ell} \partial\boldsymbol{P}(\tilde{\sigma}_k)\right|_{\tilde{g}_k} + O\big(k^{-\frac{1}{2}}\big) > \frac{\nu}{2}
$$

for a large integer k. Hence we have

$$\left| \wedge^{2n+\ell} d\boldsymbol{P}(\sigma_k)\right|_{g_k} > \frac{\nu}{2C}$$

on $B_{g_k}(x,c)$ for universal constants c and C, and the proof is finished. □

Proof of Theorem 2. We prove that $\phi_k = \boldsymbol{P}(\sigma_k)$ in Proposition 4 is an asymptotically holomorphic embedding of M. The property $[\phi^*\omega_{FS}] = [k\omega/2\pi]$ is obvious from the definition of the hyperplane bundle H on $\boldsymbol{C}\mathrm{P}^{2(n+\ell)}$. Next we can perturb this immersion ϕ_k to be an embedding by a small C^r-perturbation keeping the asymptotic holomorphicity since $2\dim M < \dim \boldsymbol{C}\mathrm{P}^{2(n+\ell)}$ for $\ell \neq 0$. When $\ell = 0$ we may modify Proposition 4 to obtain the section σ_k of the bundle $\underline{\boldsymbol{C}}^{2n+2} \otimes L^{\otimes k}$ and consider the map $\boldsymbol{P}(\sigma_k)$ into $\boldsymbol{C}\mathrm{P}^{2n+1}$ satisfying $2\dim M < \dim \boldsymbol{C}\mathrm{P}^{2n+1}$. Hence we have embeddings. We set $m = 2n + \max\{2\ell, 1\}$ and denote by ϕ_k such an embedding into $\boldsymbol{C}\mathrm{P}^m$ for simplicity. Moreover the equation (13) in the proof of Proposition 4 implies that

$$\left|\nabla^p \boldsymbol{P}(\sigma_k)\right|_{g_k} = O(1), \quad \left|\nabla^{p-1}\bar{\partial}\boldsymbol{P}(\sigma_k)\right|_{g_k} = O(k^{-\frac{1}{2}}).$$

Therefore ϕ_k is the asymptotically holomorphic sequence of embeddings.

To complete the proof of our main theorem we only need to estimate the left inverse of $d\phi_k$. The differential $d\phi_k$ gives the isomorphism

$$(d\phi_k)_x : T_xM \to (\phi_k)_*T_xM \subset T_{\phi(x)}\boldsymbol{C}\mathrm{P}^m.$$

We define $(\bar{\theta}_k)_x$ to be the inverse of $(d\phi_k)_x$. The map $\bar{\theta}_k$ has the estimate $|\bar{\theta}_k| > C\nu^{-1}$ for a universal constant C since the lower bound $| \wedge^{2n+\ell} d\boldsymbol{P}(\sigma_k)|_{g_k} > \nu$. Then we consider the left inverse $\theta_k = \bar{\theta}_k \circ \mathrm{pr}^{\perp}$ by composing $\bar{\theta}_k$ and the orthogonal projection $\mathrm{pr}^{\perp}$ from $T_{\phi(x)}\boldsymbol{C}\mathrm{P}^m$ to $(\phi_k)_*(T_xM)$, and have the estimate of θ_k in Definition 2. □

3.2. The case of grassmannians.

Definition 6. Let Y be a C^∞-manifold and E' a Hermitian bundle of rank r over Y. A section s of the vector bundle $\underline{\boldsymbol{C}}^N \otimes E'$ is ν-grassmannizable on Y if $| \wedge^r s(y)| > \nu$ for all $y \in Y$.

If a section s of the vector bundle $\underline{\boldsymbol{C}}^N \otimes E'$ is ν-grassmannizable for a positive constant ν, then we obtain a map

$$\mathrm{Gr}(s) : Y \to \mathrm{Gr}(r, N).$$

The following result is proved by Proposition 4.6. in [**8**]

LEMMA 5. *Let $\tilde{s}_k$ be an asymptotically holomorphic sequence of sections of the vector bundles $\underline{\boldsymbol{C}}^N \otimes \tilde{E} \otimes \tilde{L}^{\otimes k}$. Then for $\alpha > 0$ there exists another sequence $\tilde{\sigma}_k$ satisfying that*

1. $|\tilde{s}_k - \tilde{\sigma}_k|_{C^1, \tilde{g}_k} < \alpha$.
2. *$\tilde{\sigma}_k$ is ν-grassmannizable on $B_k(M)$.*
3. *$| \wedge^{n+\ell} \partial \mathrm{Gr}(\tilde{\sigma}_k)|_{\tilde{g}_k} > \nu$ on $B_k(M)$ for a positive constant ν.*

Moreover the result holds for a continuous family of relative asymptotically holomorphic sections $\tilde{\sigma}_{t,k}$.

PROPOSITION 5. *Suppose $N \geq n + \ell + r$ and $r(N - r) \geq 2n + \max\{2\ell, 1\}$. For any Hermitian bundle E of rank r over M, there exists an asymptotically J-holomorphic sequence of sections σ_k of the vector bundles $\underline{\boldsymbol{C}}^N \otimes E \otimes L^{\otimes k}$ satisfying that*

1. *σ_k is ν-grassmannizable on M.*
2. *$| \wedge^{2n+\ell} d\mathrm{Gr}(\sigma_k)|_{g_k} > \nu$ for a universal constant $\nu > 0$.*

Moreover the result holds for a continuous one-parameter family of asymptotically J_t-holomorphic sections $\sigma_{t,k}$.

PROOF. Given the section $\tilde{\sigma}_k$ in Lemma 5, then the pull-back $\sigma_k = i^* \tilde{\sigma}_k$ is asymptotically J-holomorphic and ν-grassmannizable on M. Then we can show that $\mathrm{Gr}(\sigma_k)$ satisfies $| \wedge^{2n+\ell} d\mathrm{Gr}(\sigma_k)|_{g_k} > \nu$ for a constant $\nu > 0$ by applying the same argument in the proof of Proposition 4 to this map $\mathrm{Gr}(\sigma_k)$, and this finishes the proof. $\square$

PROOF OF THEOREM 4. Let σ_k be the section in Proposition 5. We prove that $\phi_k = \mathrm{Gr}(\sigma_k)$ satisfies the condition in Theorem 4. The property $E = \phi^* \mathscr{U}$ is obvious from the definition of the universal bundle $\mathscr{U}$ on $\mathrm{Gr}(r, N)$. Moreover the condition 2 in Proposition 5 implies that the map ϕ_k is an immersion. Now we have $2 \dim M < \dim \mathrm{Gr}(r, N)$ from the assumption $r(N - r) \geq 2n + \max\{2\ell, 1\}$. Hence we can perturb this immersion ϕ to be an embedding by a small perturbation keeping the asymptotic holomorphicity. We denote this embedding by ϕ_k again. We obtain the estimate of the left inverse of ϕ_k by repeating the argument in the proof of Theorem 2, and hence this finishes the proof. $\square$

4. Asymptotically holomorphic immersions.

Let (M, ω) be a $(2n+\ell)$-dimensional closed presymplectic manifold of rank $2n$ with a pre-compatible pair (J, g) and an integral form ω. We suppose that (X, Ω) is the symplectic manifold in Proposition 1 with the almost complex structure $\tilde{J}$

and the Riemannian metric $\tilde{g}$ such that $i^*\tilde{J} = J$ and $i^*\tilde{g} = g$ for the embedding $i : M \to X$.

4.1. Local perturbation and transversality.

Our main tools to prove the main theorem are a local perturbation of asymptotically holomorphic sections and a transversality theorem on symplectic manifolds. We recall these results in this subsection.

Definition 7. A family of properties $P(\epsilon, x)_{x\in X,\epsilon>0}$ of sections of bundles over X is local and C^r-open if for a section s satisfying $P(\epsilon, x)$, any section σ such that $|s(x) - \sigma(x)|_{C^r} < \eta$ satisfies $P(\epsilon - C\eta, x)$ where C is a universal constant.

The following lemma is the result of a perturbation of asymptotically holomorphic sections in symplectic geometry.

Proposition 6 (Auroux [**2**, Proposition 3]). *Let $\tilde{s}_k$ be asymptotically holomorphic sections of $\tilde{E}_k$ and $P(\epsilon, x)_{x\in X,\epsilon>0}$ a local and C^r-open properties of sections of vector bundles $\tilde{E}_k$ over X. Assume that there exist universal constants c, c', c'' and p such that given any $x \in M$, any small $\delta > 0$, there exist asymptotically holomorphic sections $\tilde{\tau}_{k,x}$ of $\tilde{E}_k$ with the following properties for large integer k:*

1. *$|\tilde{\tau}_{k,x}|_{C^r,\tilde{g}_k} < c''\delta$.*
2. *The sections $\frac{1}{\delta}\tilde{\tau}_{k,x}$ have Gaussian decay in C^r-norm away from x.*
3. *The sections $\tilde{s}_k + \tilde{\tau}_{k,x}$ satisfy the property $P(c'\delta(\log(\delta^{-1}))^{-p}, y)$ for all $y \in B_{\tilde{g}_k}(x, c)$.*

Then for any $\alpha > 0$ there exist asymptotically holomorphic sections $\tilde{\sigma}_k$ of $\tilde{E}_k$ such that

- *$|\tilde{s}_k - \tilde{\sigma}_k|_{C^r,\tilde{g}_k} < \alpha$.*
- *The sections $\tilde{\sigma}_k$ satisfy the property $P(\epsilon, y)$ for all $y \in B_k(M)$ with a universal constant $\epsilon > 0$.*

Moreover, given a one-parameter family of sections $\tilde{s}_{t,k}$ and a one-parameter family of sections $\tilde{\tau}_{t,k,x}$ with the conditions 1, 2 *and* 3*, there exists a one-parameter family of sections $\tilde{\sigma}_{t,k}$ which depends continuously on t and satisfies the same properties.*

Definition 8. A map $f : \boldsymbol{C}^m \to \boldsymbol{C}^r$ is η-transverse to 0 at a point $x \in \boldsymbol{C}^m$ if it satisfies at least one of the following properties:

1. $|f(x)| > \eta$.
2. The derivative $df(x)$ has a right inverse θ with $|\theta| < \eta^{-1}$.

Proposition 7 (Donaldson [**4**, Theorem 12]). *There exists an integer p depending on m and r, with the following property: let $\eta = \delta(\log(\delta^{-1}))^{-p}$ for $0 < \delta < \frac{1}{2}$. Let f be a $\boldsymbol{C}^r$-valued function over the ball $B^+ = B(0, \frac{11}{10}) \subset \boldsymbol{C}^m$ satisfying the following bounds over B^+:*

$$|f| \leq 1, \quad |\bar{\partial} f| \leq \eta, \quad |\nabla \bar{\partial} f| \leq \eta.$$

Then there exists $w \in \boldsymbol{C}^r$ with $|w| < \delta$ such that $f - w$ is η-transverse to 0 over the unit ball $B \subset B^+$. Moreover, given a one-parameter family of a $\boldsymbol{C}^r$-valued functions $(f_t)_{t\in[0,1]}$ over the ball B^+ depending continuously on t with the above bounds for all t, there exists a continuous family $(w_t)_{t\in[0,1]}$ satisfying the same properties.

4.2. The case of projective spaces.

Let $\tilde{W}$ be a vector bundle which is an extension of W to X.

Proposition 8. *Let $\tilde{s}_k$ be an asymptotically holomorphic sequence of sections of the vector bundles $\underline{\boldsymbol{C}}^{2n+\ell+1} \otimes \tilde{L}^{\otimes k}$ which is ν-projectizable. Then for $\alpha > 0$ there exists another sequence $\tilde{\sigma}_k$ satisfying that*

1. $\left|\tilde{s}_k - \tilde{\sigma}_k\right|_{C^1, \tilde{g}_k} < \alpha$.
2. $\left| \wedge^{2n} d_{\tilde{W}} \boldsymbol{P}(\tilde{\sigma}_k)\right|_{\tilde{g}_k} > \nu'$ *for a constant $\nu' > 0$ on $B_k(M)$.*

Moreover, for a continuous family of relative asymptotically holomorphic sequences of sections $\tilde{s}_{t,k}$ and any $\alpha > 0$, there exists a continuous family of sections $\tilde{\sigma}_{t,k}$ satisfying the conditions 1 *and* 2.

Proof. We define the property $P^{(m)}(\epsilon, y)$ of sections as $| \wedge^{2m} d_{\tilde{W}} \boldsymbol{P}(\tilde{s}_k)(y)| > \epsilon$ for $y \in X$. We will construct a section with the property $P^{(m+1)}(\epsilon, y)$ by induction on m for $m = 0, \ldots, n-1$. If we construct a local perturbation $\tilde{\tau}_{k,x}^{(m+1)}$ for $x \in M$ satisfying the condition (1), (2) and (3) in Proposition 6, then we have the section $\tilde{\sigma}_k^{(m+1)}$ satisfying $P^{(m+1)}(\epsilon, y)$ on $B_k(M)$ for a constant ϵ. Finally we will obtain a global section $\tilde{\sigma}_k^{(n)}$ such that $| \wedge^{2n} d_{\tilde{W}} \boldsymbol{P}(\tilde{\sigma}_k^{(n)})| > \epsilon$ on $B_k(M)$. Hence it suffices to construct such a local perturbation $\tilde{\tau}_{k,x}^{(m+1)}$ on $B_{\tilde{g}_k}(x, c)$ for a universal constant c.

Before we start the induction, we see a local representation of a section $\tilde{\sigma}_k^{(m)}$ satisfying the property $P^{(m)}(\epsilon, y)$ on $B_k(M)$ for a universal constant ϵ. By trivializing $\tilde{L}^{\otimes k}$ with $\tilde{s}_{k,x}^{\mathrm{ref}}$, we can denote the section $\tilde{\sigma}_k^{(m)}$ restricted to the ball $B_{\tilde{g}_k}(x, 1)$ as follows:

$$\tilde{\sigma}_k^{(m)} = \left(s_k^0\, \tilde{s}_{k,x}^{\mathrm{ref}},\ s_k^1\, \tilde{s}_{k,x}^{\mathrm{ref}},\ \ldots,\ s_k^{2n+\ell}\, \tilde{s}_{k,x}^{\mathrm{ref}}\right)$$

over $B_{\tilde{g}_k}(x,1)$. Hence we identify $\tilde{\sigma}_k^{(m)}$ with the $\boldsymbol{C}^{2n+\ell+1}$-valued function

$$\tilde{\sigma}_k^{(m)} = \left(s_k^0, s_k^1, \ldots, s_k^{2n+\ell}\right) : B_{\tilde{g}_k}(x,c) \to \boldsymbol{C}^{2n+\ell+1}$$

on $B_{\tilde{g}_k}(x,c)$ for a universal constant c. Now we consider the composition $f_k^{(m)} = \Phi_0 \circ \boldsymbol{P}(\tilde{\sigma}_k^{(m)})$ by the trivialization Φ_0 of $\boldsymbol{C}\mathrm{P}^{2n+\ell}$ for $U_k = \{[z^0, \ldots, z^{2n+\ell}] | z^0 \neq 0\}$:

$$\begin{aligned} f_k^{(m)} : B_{\tilde{g}_k}(x,c) &\to \boldsymbol{C}^{2n+\ell}, \\ y &\mapsto \left(\frac{s_k^1(y)}{s_k^0(y)}, \ldots, \frac{s_k^{2n+\ell}(y)}{s_k^0(y)}\right). \end{aligned}$$

Then the map $f_k^{(m)}$ satisfies

$$C^{-1}\left| \wedge^{2m} d\boldsymbol{P}\left(\tilde{\sigma}_k^{(m)}\right)\right| \leq \left| \wedge^{2m} df_k^{(m)}\right| \leq C\left| \wedge^{2m} d\boldsymbol{P}\left(\tilde{\sigma}_k^{(m)}\right)\right|$$

on $B_{\tilde{g}_k}(x,c)$ for a universal constant C. Hence we perturb this map $f_k^{(m)}$ instead of $\boldsymbol{P}(\tilde{\sigma}_k^{(m)})$. Now we define asymptotic holomorphic 1-forms

$$\mu_k^j = \partial\left(\frac{z_k^j}{s_k^0}\right) \tag{14}$$

where $(z_k^1, \ldots, z_k^{n+\ell})$ denotes the Darboux coordinates with $z_k^{n+i} = w_k^i$ for $i = 1, \ldots, \ell$ in Proposition 2. The 1-forms $\{\mu_k^j\}_{j=1,\ldots,n+\ell}$ can be taken a basis of 1-forms on $B_{\tilde{g}_k}(x,c)$ which is an orthogonal basis at x. In addition, we have the lower bound $|\mu_k^j| > C\nu$ on $B_{\tilde{g}_k}(x,c)$ for a universal constant C. The form $\partial f_k^{(m)}$ is the section of $T^*X \otimes \boldsymbol{C}^{2n+\ell}$. Hence for a basis $\{e_j\}_{j=1,\ldots,2n+\ell}$ of $\boldsymbol{C}^{2n+\ell}$, $\partial f_k^{(m)}$ is written as

$$\partial f_k^{(m)} = \sum_{\substack{i=1,\ldots,n+\ell, \\ j=1,\ldots,2n+\ell}} u^{ij} \mu_k^i \otimes e_j \tag{15}$$

where $\{u^{ij}\}_{i=1,\ldots,n+\ell, j=1,\ldots,2n+\ell}$ are complex valued functions on $B_{\tilde{g}_k}(x,c)$. Under the representation (15) we identify $\partial f_k^{(m)}$ with the complex $(n+\ell, 2n+\ell)$-matrix

$$\partial f_k^{(m)} = \begin{pmatrix} u_k^{11} & \cdots & u_k^{1\ 2n+\ell} \\ \vdots & \ddots & \vdots \\ u_k^{n+\ell\ 1} & \cdots & u_k^{n+\ell\ 2n+\ell} \end{pmatrix}. \tag{16}$$

Let V be the subbundle of $T^{1,0}X|_{B_{\tilde{g}_k}(x,c)}$ which is spanned by the 1-forms $\{\mu_k^j\}_{j=1,\ldots,n}$. Then the estimate $|\partial_V f_k| > \epsilon$ implies that $|d_{\tilde{W}} f_k| > C^{-1}\epsilon$ on $B_{\tilde{g}_k}(x,c)$ for a sufficiently large integer k and a universal constant C shrinking c if necessary, since $\tilde{W}(x) = W(x)$ equals to $V(x)$ and $\partial_W f(x) = \partial_V f(x)$ at the origin x of the relative Darboux coordinates. Hence we can represent $\partial_V f_k^{(m)}$ as

$$\partial_V f_k^{(m)} = \begin{pmatrix} u_k^{11} & \cdots & u_k^{1\ 2n+\ell} \\ \vdots & \ddots & \vdots \\ u_k^{n\ 1} & \cdots & u_k^{n\ 2n+\ell} \end{pmatrix} \tag{17}$$

and assume that

$$\partial_V f_k^{(m)}(x) = \begin{pmatrix} u_k^{11}(x) & 0 & \cdots & & 0 & \cdots & 0 \\ 0 & u_k^{22}(x) & 0 & \cdots & 0 & & 0 \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & u_k^{nn}(x) & 0 & \cdots & 0 \end{pmatrix}$$

by applying a unitary transformation $\mathrm{U}(n) \subset \mathrm{U}(n+\ell)$ on the Darboux coordinates and $\mathrm{U}(2n+\ell)$ on $\boldsymbol{C}^{2n+\ell}$ fixing $(1, 0, \ldots, 0)$.

We start the inductive construction of local sections $\tilde{\tau}_{k,x}^{(m+1)}$. We apply the proof of Proposition 19 in [**8**] to $\partial_V f_k^{(m)}$ instead of ∂f_k. At first, we construct a local perturbation $\tilde{\tau}_{k,x}^{(1)}$. We define $\tilde{\sigma}_k^{(0)}$ as the section $\tilde{s}_k$ in the assumption of this proposition. Then we consider the function $f_k^{(0)} = \Phi_0 \circ \boldsymbol{P}(\tilde{\sigma}_k^{(0)})$ and represent $\partial f_k^{(0)}$ as in (16). Now we set the $\boldsymbol{C}^{2n+\ell}$-valued function

$$h_k = \left(u_k^{11}, u_k^{12}, \ldots, u_k^{1\ 2n+\ell}\right) : B_{\tilde{g}_k}(x,c) \to \boldsymbol{C}^{2n+\ell}$$

which is the first row of (17). Applying Proposition 7 to h_k, we obtain a constant $w_k = (w_k^1, \ldots, w_k^{2n+\ell}) \in \boldsymbol{C}^{2n+\ell}$ such that

$$|h_k - w_k| > \eta = \delta(\log(\delta^{-1}))^{-p}$$

and $|w_k| < \delta$. The function $h_k - w_k$ gives rise to the $\boldsymbol{C}^{2n+\ell}$-valued 1-form

$$\begin{aligned}\partial f_k^{(0)} - \sum_{j=1}^{2n+\ell} w_k^j \mu_k^1 \otimes e_j &= \partial\biggl(f_k^{(0)} - \sum_{j=1}^{2n+\ell} \frac{w_k^j z_k^1}{s_k^0} \otimes e_j \biggr) \\ &= \partial\biggl(\frac{s_k^1 - w_k^1 z_k^1}{s_k^0}, \dots, \frac{s_k^{2n+\ell} - w_k^{2n+\ell} z_k^1}{s_k^0} \biggr)\end{aligned}$$

whose norm has lower bound $C\eta$. This perturbation is induced by adding the local section

$$\tilde{\tau}_{k,x}^{(1)} = -\bigl(0, w_k^1 z_k^1 \tilde{s}_{k,x}^{\mathrm{ref}}, \dots, w_k^{2n+\ell} z_k^1 \tilde{s}_{k,x}^{\mathrm{ref}}\bigr)$$

to $\tilde{\sigma}_k^{(0)} = \tilde{s}_k$ on $B_{\tilde{g}_k}(x,c)$. Then we obtain $\tilde{\sigma}_k^{(1)}$ satisfying the property $P^{(1)}(\epsilon, y)$ on $B_k(M)$. Therefore we complete the first step in the induction.

Next we assume that there exists a section $\tilde{\sigma}_k^{(m)}$ with the property $P^{(m)}(\epsilon, y)$ on $B_k(M)$ for $1 \le m \le n-1$. Then we define a function $f_k^{(m)}$ by

$$f_k^{(m)} = \Phi_0 \circ \boldsymbol{P}\bigl(\tilde{\sigma}_k^{(m)}\bigr).$$

It follows from our assumption that $| \wedge^{2m} d_{\tilde{W}} f_k^{(m)}| > \nu$ for a universal constant ν. Then we have

$$\bigl| \wedge^m \partial_V f_k^{(m)} \bigr| > C\nu \tag{18}$$

for a universal constant C on $B_{\tilde{g}_k}(x,c)$ since $\tilde{W}(x) = W(x)$ and $W^{1.0}(x) = V(x)$. We find a local perturbation $\tilde{\tau}_{k,x}^{(m+1)}$ of this 1-forms $\partial f_k^{(m)}$ to satisfy $| \wedge^{m+1} \partial_V (f_k^{(m)} + \boldsymbol{P}(\tilde{\tau}_{k,x}^{(m+1)}))| > \eta$ for a universal constant η. The inequality (18) and the representation (17) implies that $|u_k^{11}(x) \cdots u_k^{mm}(x)| > \nu/C'$ for a universal constant C'. We may assume $| \det(u_k^{ij})_{i,j=1,\dots,m}| > \nu/2C'$ on $B_{\tilde{g}_k}(x,c)$ by taking a small constant c. Define the functions

$$\begin{aligned}\theta_k &= \det \bigl(u_k^{ij}\bigr)_{i,j=1,\dots,m} \\ M_k^p &= \det \bigl(u_k^{ij}\bigr)_{i=1,\dots,m+1,\ j=1,\dots,m,p},\end{aligned}$$

that is, θ_k and M_k^p are the determinants of the following matrices:

$$\begin{pmatrix} u_k^{11} & \cdots & u_k^{1m} & 0 \\ \vdots & \ddots & \vdots & \vdots \\ u_k^{m1} & \cdots & u_k^{mm} & 0 \\ 0 & \cdots & 0 & 1 \end{pmatrix}, \qquad \begin{pmatrix} u_k^{11} & \cdots & u_k^{1m} & u_k^{1p} \\ \vdots & \ddots & \vdots & \vdots \\ u_k^{m1} & \cdots & u_k^{mm} & u_k^{mp} \\ u_k^{m+1\ 1} & \cdots & u_k^{m+1\ m} & u_k^{m+1\ p} \end{pmatrix}$$

for integer p with $m + 1 \leq p \leq 2n + \ell$, respectively. We set $M_k = (M_k^{m+1}, \ldots, M_k^{2n+\ell})$ and consider a perturbation of M_k. With the lower bounds of θ_k we can define a $\boldsymbol{C}^{2n+\ell-m}$-valued function

$$h_k = \left(\frac{M_k^{m+1}}{\theta_k}, \ldots, \frac{M_k^{2n+\ell}}{\theta_k}\right) : B_{\tilde{g}_k}(x, c) \to \boldsymbol{C}^{2n+\ell-m}.$$

Since we have $n + \ell < 2n + \ell - m$ for $0 \leq m \leq n - 1$, we can apply Proposition 7 to h_k on any step of the induction. Hence there exists a constant $w_k = (w_k^{m+1}, \ldots, w_k^{2n+\ell}) \in \boldsymbol{C}^{2n+\ell-m}$ such that

$$|h_k - w_k| > \delta(\log(\delta^{-1}))^{-p}$$

and $|w_k| < \delta$. The above estimate is equivalent to

$$\begin{aligned} |M_k - w_k\theta_k| &= \left|\left(M_k^{m+1} - w_k^{m+1}\theta_k, \ldots, M_k^{2n+\ell} - w_k^{2n+\ell}\theta_k\right)\right| > \eta \\ &= \frac{\nu}{2C'}\delta(\log(\delta^{-1}))^{-p}. \end{aligned}$$

The each component $M_k^p - w_k^p\theta_k$ is the determinant of the matrix

$$\begin{pmatrix} u_k^{11} & \cdots & u_k^{1m} & u_k^{1p} \\ \vdots & \ddots & \vdots & \vdots \\ u_k^{m1} & \cdots & u_k^{mm} & u_k^{mp} \\ u_k^{m+1\ 1} & \cdots & u_k^{m+1\ m} & u_k^{m+1\ p} - w_k^p \end{pmatrix}.$$

Hence the perturbation $M_k - w_k\theta_k$ corresponds to the 1-form

$$\begin{aligned} \partial f_k^{(m)} - \sum_{j=m+1}^{2n+\ell} w_k^j \mu_k^{m+1} \otimes e_j &= \partial\left(f_k^{(m)} - \sum_{j=m+1}^{2n+\ell} \frac{w_k^j z_k^{m+1}}{s_k^0} \otimes e_j\right) \\ &= \partial\left(\frac{s_k^1}{s_k^0}, \ldots, \frac{s_k^m}{s_k^0}, \frac{s_k^{m+1} - w_k^{m+1} z_k^{m+1}}{s_k^0}, \ldots, \frac{s_k^{2n+\ell} - w_k^{2n+\ell} z_k^{m+1}}{s_k^0}\right). \end{aligned}$$

This form is induced by adding the local section

$$\tilde{\tau}_{k,x}^{(m+1)} = -\big(0, \ldots, 0, w_k^{m+1} z_k^{m+1} \tilde{s}_{k,x}^{\mathrm{ref}}, \ldots, w_k^{2n+\ell} z_k^{m+1} \tilde{s}_{k,x}^{\mathrm{ref}}\big)$$

to the section $\tilde{\sigma}_k^{(m)}$ on $B_{\tilde{g}_k}(x, c)$. This implies that there exists a section $\tilde{\sigma}_k^{(m+1)}$ with the property $P^{(m+1)}(\epsilon, y)$ on $B_k(M)$. Thus we inductively construct the section $\tilde{\sigma}_k^{(n)}$ and finish the proof for the non-parameter version.

In the parameter case we only apply the same argument to a one-parameter family of sections $\tilde{\sigma}_{t,k}^{(0)} = \tilde{s}_{t,k}$. □

Proof of Theorem 1. We have an asymptotically $\tilde{J}$-holomorphic and ν-projectizable sequence of sections $\tilde{s}_k$ over X ([**8**, Proposition 2.15]). Given the section $\tilde{\sigma}_k$ for the sections $\tilde{s}_k$ and a small α in Proposition 8, then the pull-back $\sigma_k = i^*\tilde{\sigma}_k$ is asymptotically J-holomorphic and ν-projectizable on M. In addition, the map $\boldsymbol{P}(\sigma_k)$ satisfies that $|\wedge^{2n} d_W \boldsymbol{P}(\sigma_k)|_{g_k} > \nu'$ on M for a constant C and $\nu' > 0$. Therefore the map $\phi_k = \boldsymbol{P}(\sigma_k) : M \to \boldsymbol{C}\mathrm{P}^{2n+\ell}$ is non-degenerate along W. We obtain the estimate of the left inverse of $d\phi_k|_W$ by repeating the argument in the proof of Theorem 2. □

4.3. The case of Grassmannians.

Proposition 9. *Let $\tilde{s}_k$ be an asymptotically holomorphic sequence of sections of the vector bundles $\underline{\boldsymbol{C}}^N \otimes \tilde{E} \otimes \tilde{L}^{\otimes k}$, which is ν-grassmannizable on $B_k(M)$. Then for $\alpha > 0$, there exists another sequence $\tilde{\sigma}_k$ satisfying that*

1. $|\tilde{s}_k - \tilde{\sigma}_k|_{C^1, \tilde{g}_k} < \alpha$.
2. $\big|\wedge^{2n} d_{\tilde{W}} \mathrm{Gr}(\tilde{\sigma}_k)\big|_{\tilde{g}_k} > \nu'$ *on $B_k(M)$ for a universal constant $\nu' > 0$.*

Moreover, for a continuous family of relative asymptotically holomorphic sequences of sections $\tilde{s}_{t,k}$ and any $\alpha > 0$, there exists a continuous family of sections $\tilde{\sigma}_{t,k}$ satisfying the conditions 1 *and* 2*.*

Proof. We repeat the inductive argument of the proof of Proposition 8. We define a property $P^{(m)}(\epsilon, y)$ of sections as $|\wedge^{2m} d_{\tilde{W}} \mathrm{Gr}(\tilde{\sigma}_k^{(m)})| > \epsilon$ for $y \in X$. Suppose that there exists a section $\sigma_k^{(m)}$ satisfying the property $P^{(m)}(\epsilon, y)$ on $B_k(M)$ for each number $0 \le m \le n-1$. Then it suffices to show there exists a local perturbation $\tilde{\tau}_{k,x}^{(m+1)}$ such that $|\wedge^{2(m+1)} d_{\tilde{W}} \mathrm{Gr}(\tilde{\sigma}_k^{(m)} + \tilde{\tau}_{k,x}^{(m+1)})| > \eta$ on $B_{\tilde{g}_k}(x, c)$.

Given a local frame $\{e_1, \ldots, e_r\}$ trivializing $\tilde{E}$ and a frame $\{v_1, \ldots, v_r\}$ of $\underline{\boldsymbol{C}}^N$, we can represent $\tilde{\sigma}_k^{(m)}$ restricted to $B_{\tilde{g}_k}(x, c)$ as

$$\tilde{\sigma}_k^{(m)} = \sum_{\substack{i=1,\dots,r,\\ j=1,\dots,N}} s_k^{ij} v_j \otimes e_i \otimes \tilde{s}_{k,x}^{\mathrm{ref}} \tag{19}$$

where $\{s_k^{ij}\}_{i=1,\dots,r,\ j=1,\dots,N}$ are complex valued functions on $B_{\tilde{g}_k}(x,c)$. Under the representation (19) we identify $\tilde{\sigma}_k^{(m)}$ with the complex (r,N)-matrix

$$\tilde{\sigma}_k^{(m)} = \begin{pmatrix} s_k^{11} & \cdots & s_k^{1N} \\ \vdots & \ddots & \vdots \\ s_k^{r1} & \cdots & s_k^{rN} \end{pmatrix} \tag{20}$$

and suppose

$$\tilde{\sigma}_k^{(m)}(x) = \begin{pmatrix} s_k^{11}(x) & 0 & \cdots & & 0 & \cdots & 0 \\ 0 & s_k^{22}(x) & 0 & \cdots & 0 & & 0 \\ \vdots & & \ddots & 0 & \vdots & & \vdots \\ 0 & \cdots & 0 & s_k^{rr}(x) & 0 & \cdots & 0 \end{pmatrix} \tag{21}$$

by unitary transformations $\mathrm{U}(r)$ and $\mathrm{U}(N)$. Then it follows that $|\det(s_k^{ij})_{i,j=1,\dots,r}| > C^{-1}\nu$ from (21) and the ν-grassmannizability of $\tilde{\sigma}_k^{(m)}$. Hence we can define the composition $f_k = \Phi_0 \circ \mathrm{Gr}(\tilde{\sigma}_k^{(m)})$ by the trivialization Φ_0 of $\mathrm{Gr}(r,N)$ for $U_0 = \{[v_1,\cdots,v_N] | [v_1,\cdots,v_N] \cap [0,\dots,0,v_{r+1},\cdots,v_N] = 0\}$:

$$f_k : B_{\tilde{g}_k}(x,c) \to \boldsymbol{C}^{r(N-r)},$$

$$y \mapsto \begin{pmatrix} s_k^{11}(y) & \cdots & s_k^{1r}(y) \\ \vdots & \ddots & \vdots \\ s_k^{r1}(y) & \cdots & s_k^{rr}(y) \end{pmatrix}^{-1} \begin{pmatrix} s_k^{1\ r+1}(y) & \cdots & s_k^{1N}(y) \\ \vdots & \ddots & \vdots \\ s_k^{r\ r+1}(y) & \cdots & s_k^{rN}(y) \end{pmatrix}. \tag{22}$$

Let A_k be the matrix valued function such that $A_k(y) = (s_k^{ij}(y))_{i,j=1,\dots,r}$ for $y \in B_{\tilde{g}_k}(x,c)$ and B a constant matrix of $\mathrm{GL}(r,\boldsymbol{C})$. Then we define asymptotic holomorphic $\boldsymbol{C}^r$-valued 1-forms

$$\mu_k^{ij} = \partial\big(A_k^{-1} B z_k^i e_j\big) \tag{23}$$

for $i = 1,\dots,n+\ell$ and $j = 1,\dots,r$, where $(z_k^1,\dots,z_k^{n+\ell})$ are the Darboux coordinates in Proposition 2. We choose the matrix $B \in \mathrm{GL}(r,\boldsymbol{C})$ such that

$\{\mu_k^{ij}\}_{i=1,\dots,n+\ell,\ j=1,\dots,r}$ is an orthogonal basis of $T^*X\otimes \boldsymbol{C}^r$ at the origin x. Hence $\{\mu_k^{ij}\}_{i=1,\dots,n+\ell,\ j=1,\dots,r}$ is a local frame of $T^*X\otimes \boldsymbol{C}^r$ on $B_{\tilde{g}_k}(x,c)$ and satisfies the estimate $|\mu_k^{ij}|>C\nu$ for a universal constant C.

We denote $f_k=(f_k^1,\dots,f_k^{N-r})$ where each f_k^t is the column $A_k^{-1}(s_k^{jt})_{j=1,\dots,r}$ of (22). Then the section ∂f_k^t of $T^*X\otimes \boldsymbol{C}^r$ is represented as

$$\partial f_k^t=\sum_{\substack{i=1,\dots,n+\ell,\\ j=1,\dots,r}} u_k^{ijt}\mu_k^{ij}$$

where $\{u_k^{ijt}\}_{i=1,\dots,n+\ell,\ j=1,\dots,r}$ are complex valued functions on $B_{\tilde{g}_k}(x,c)$. We identify ∂f_k^t with the complex $(n+\ell,r)$-matrix valued function $(u_k^{ijt})_{i=1,\dots,n+\ell,\ j=1,\dots,r}$ by the basis $\{\mu_k^{ij}\}_{i=1,\dots,n+\ell,\ j=1,\dots,r}$. Then we consider $\partial f_k=(\partial f_k^1,\dots,\partial f_k^{N-r})$ as the complex $(n+\ell,r(N-r))$-matrix valued function and write ∂f_k as follows:

$$\partial f_k=\begin{pmatrix} u_k^{11} & \cdots & u_k^{1\ r(N-r)}\\ \vdots & \ddots & \vdots\\ u_k^{n+\ell\ 1} & \cdots & u_k^{n+\ell\ r(N-r)}\end{pmatrix}$$

where we identify $\boldsymbol{C}^{r(N-r)}$ with complex $(r,N-r)$-matrices by a function

$$\begin{gathered}\{1,\dots,r(N-r)\}\to\{1,\dots,r\}\times\{1,\dots,N-r\},\\ \alpha\mapsto(j(\alpha),t(\alpha)).\end{gathered}$$

Let V be the subbundle of $T^{1,0}X\otimes\boldsymbol{C}^r|_{B_{\tilde{g}_k}(x,c)}$ which is spanned by the $\boldsymbol{C}^r$-valued 1-forms $\{\mu_k^{ij}\}_{i=1,\dots,n,\ j=1,\dots,r}$. Then the estimate $|\partial_V f_k|>\epsilon$ implies that $|d_{\tilde{W}}f_k|>C^{-1}\epsilon$ on $B_{\tilde{g}_k}(x,c)$ for a universal constant C, shrinking c if necessary. Hence we can represent $\partial_V f_k$ as

$$\partial_V f_k=\begin{pmatrix} u_k^{11} & \cdots & u_k^{1\ r(N-r)}\\ \vdots & \ddots & \vdots\\ u_k^{n\ 1} & \cdots & u_k^{n\ r(N-r)}\end{pmatrix}, \tag{24}$$

and we may further suppose that the matrix at x is upper triangle:

$$\partial_V f_k(x) = \begin{pmatrix} u_k^{11}(x) & * & \cdots & & * & \cdots & * \\ 0 & u_k^{22}(x) & * & \cdots & * & \cdots & * \\ \vdots & & \ddots & * & \vdots & & \vdots \\ 0 & \cdots & 0 & u_k^{n\ n}(x) & * & \cdots & * \end{pmatrix}$$

by the unitary transformation $\mathrm{U}(n)$ which acts on the relative Darboux coordinates. Let $\theta_k = \det(u_k^{i\alpha})_{i,\alpha=1,\ldots,m}$ and $M_k^p = \det(u_k^{i\alpha})_{i=1,\ldots,m+1,\ \alpha=1,\ldots,m,p}$ where the matrix valued functions $(u_k^{i\alpha})_{i,\alpha=1,\ldots,m}$ and $(u_k^{i\alpha})_{i=1,\ldots,m+1,\ \alpha=1,\ldots,m,p}$ for $m+1 \le p \le r(N-r)$ are the first (m,m)-minor and the $(m+1,m+1)$-minors of (24), respectively. Applying the same argument in the proof of Proposition 8 to the 1-form $\partial_V f_k$ of (24), then we obtain a constant $w_k = (w_k^{m+1},\ldots,w_k^{r(N-r)}) \in \boldsymbol{C}^{r(N-r)-m}$ such that $|w_k| < \delta$ and

$$|M_k - w_k\theta_k| = \left|\left(M_k^{m+1} - w_k^{m+1}\theta_k,\ldots,M_k^{r(N-r)} - w_k^{r(N-r)}\theta_k\right)\right| > \eta.$$

This perturbation is induced by adding the local section

$$\tilde{\tau}_{k,x}^{(m+1)} = -B\left(0,\ldots,0,\sum_{\substack{t(\alpha)=r+1,\\ \alpha>m}} w_k^\alpha z_k^{m+1} e_{j(\alpha)}\tilde{s}_{k,x}^{\mathrm{ref}},\ldots,\sum_{\substack{t(\alpha)=N,\\ \alpha>m}} w_k^\alpha z_k^{m+1} e_{j(\alpha)}\tilde{s}_{k,x}^{\mathrm{ref}}\right)$$

to $\tilde{\sigma}_k^{(m)}$ on $B_{\tilde{g}_k}(x,c)$, and it finishes the proof. □

Proof of Theorem 3. We have an asymptotically $\tilde{J}$-holomorphic and ν-grassmannizable sequence of sections $\tilde{s}_k$ over X ([**8**, Proposition 4.4]). Let σ_k be the section in Proposition 9. We prove that $\phi_k = \mathrm{Gr}(\sigma_k)$ satisfies the condition 2 in Theorem 3. The condition 2 in Proposition 9 implies that the map ϕ_k is the immersion along W. We obtain the estimate of the left inverse of $d\phi_k|_W$ by repeating the argument in the proof of Theorem 2, and hence this finishes the proof. □

4.4. Uniqueness of constructed maps.

Proof of Theorem 5. Let $i : (M,\omega) \to (X,\Omega)$ be the coisotropic embedding as in Proposition 1. We suppose that (J_0, g_0) and (J_1, g_1) are two pre-compatible pairs on (M,ω). Then it is sufficient to construct an isotopy for two sequences $\phi_{j,k}$ of asymptotically J_j-holomorphic maps for $j = 0,1$ as in Theorem 1. We use the construction of an isotopy in the proof of Theorem 2.11 in [**8**]. Let $s_{j,k}$ be an asymptotically J_j-holomorphic sequence of sections for $j = 0,1$ such that $\phi_{j,k} = \boldsymbol{P}(s_{j,k})$ as in proof of Theorem 1. Then there exists an asymptotically holomorphic sequence of sections $\tilde{s}_{j,k}$ such that $i^*\tilde{s}_{j,k} = s_{j,k}$ for each $j = 0, 1$. The section $\tilde{s}_{j,k}$ is ν-projectizable and satisfies $|\wedge^{2n} d_{\tilde{W}}\boldsymbol{P}(\tilde{s}_{j,k})|_{\tilde{g}_{j,k}} > \nu$ for a universal constant ν. Let $\{(J_t, g_t)\}_{t\in[0,1]}$ be a continuous family of pre-compatible pairs on (M, ω) such that $J_t = J_0$ for $t \in [0, \frac{1}{3}]$ and $J_t = J_1$ for $t \in [\frac{2}{3}, 1]$. Then we have the continuous family of almost complex structures $\{\tilde{J}_t\}_{t\in[0,1]}$ and Riemannian metrics $\{\tilde{g}_t\}_{t\in[0,1]}$ on (X, Ω). We define a family of asymptotically holomorphic sections over X:

$$\tilde{s}_{t,k} = \begin{cases} (1-3t)\tilde{s}_{0,k} & t \in [0, \frac{1}{3}] \\ 0 & t \in [\frac{1}{3}, \frac{2}{3}] \\ (3t-2)\tilde{s}_{1,k} & t \in [\frac{2}{3}, 1]. \end{cases}$$

Let α be a positive constant such that any perturbation of $\tilde{s}_{j,k}$ of C^1-norm less than α is still $\frac{\nu}{2}$-projectizable and $|\wedge^{2n} d_{\tilde{W}}\boldsymbol{P}(\tilde{s}_{j,k})|_{\tilde{g}_{j,k}} > \frac{\nu}{2}$ for $j = 0, 1$. Applying Proposition 8 to a one-parameter family $\tilde{s}_{t,k}$, then we have $\tilde{\sigma}_{t,k}$ which is ν'-projectizable and $|\wedge^{2n} d_{\tilde{W}}\boldsymbol{P}(\tilde{\sigma}_{t,k})|_{\tilde{g}_{t,k}} > \nu'$ for a universal constant ν'. We define a one-parameter family of asymptotically holomorphic sequences of sections

$$\tilde{\tau}_{t,k} = \begin{cases} (1-3t)\tilde{s}_{0,k} + 3t\tilde{\sigma}_{0,k} & t \in [0, \frac{1}{3}] \\ \tilde{\sigma}_{t,k} & t \in [\frac{1}{3}, \frac{2}{3}] \\ (3t-2)\tilde{s}_{1,k} + (3-3t)\tilde{\sigma}_{1,k} & t \in [\frac{2}{3}, 1]. \end{cases}$$

Then $\tilde{\tau}_{t,k}$ is ν''-projectizable and satisfies $|\wedge^{2n} d_{\tilde{W}}\boldsymbol{P}(\tilde{\tau}_{t,k})|_{\tilde{g}_{t,k}} > \nu''$ for $\nu'' = \min\{\frac{\nu}{2}, \nu'\}$. The pull-back $\tau_{t,k} = i^*\tilde{\tau}_{t,k}$ is the J_t-asymptotically holomorphic sequence of sections over M which is ν''-projectizable and satisfies $|\wedge^{2n} d_W\boldsymbol{P}(\tau_{t,k})|_{g_{t,k}} > \nu''$. Hence the map $\phi_{t,k} = \boldsymbol{P}(\tau_{t,k})$ is the J_t-asymptotically holomorphic immersion along W for each t. Thus $\phi_{0,k}$ and $\phi_{1,k}$ are isotopic. □

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Takayuki Moriyama

Department of Mathematics
Osaka University
Machikaneyama1-1, Toyonaka
Osaka 560-0043, Japan
E-mail: t-moriyama@cr.math.sci.osaka-u.ac.jp