J. Math. Soc. Japan
Vol. 60, No. 3 (2008) pp. 819–863
doi: 10.2969/jmsj/06030819

The Log-effect for p-evolution type models

By Massimo Cicognani, Fumihiko Hirosawa and Michael Reissig

(Received Apr. 6, 2006)
(Revised Sep. 24, 2007)

Abstract. The goal of the paper is to study the Log-effect for special p-evolution type models. The loss of regularity is related in an optimal way due to some unboundedness conditions for the derivatives of coefficients up to the second order with respect to t. Some counter-examples show that these conditions are sharp. We present the state of art of methods to construct such counter-examples.

1. Introduction.

Recently the study of p-evolution operators with non-regular coefficients was discussed by several authors. This study was motivated by astonishing progress was obtained in the last years for strictly hyperbolic operators, in the language of p-evolution operators these operators are called 1-evolution operators, with non-regular coefficients. This progress to prove H^∞ well-posedness of the Cauchy problem bases heavily on the regularity assumptions the authors taking into account. There is a C^1 approach [**4**] which assumes only some conditions to the first derivatives of coefficients of the principal part with respect to t and which yields H^∞ well-posedness with a *finite loss of regularity*. The non-regular behavior is in fact some non-Lipschitz behavior which allows that the first derivatives in t become singular in a suitable way if t, let us say, tends to 0. Here and in the following $t = 0$ is the hyperplane where the Lipschitz behavior of coefficients is violated. Such a singular behavior is described by so-called *local conditions*. If one additionally assumes a *global condition* to the coefficients uniformly on the interval of definition $[0, T]$, e.g. some Hölder or some Log^m-Lipschitz behavior, then it might be that the global condition allows to weaken the local one [**12**]. But in this paper we shall state only local conditions to the coefficients. The question for sharpness of the obtained results was answered by introducing several counter-examples. If one studies these counter-examples in the case of local conditions in C^1 approach, that is, one takes into account the first order derivative of the coefficient only, then

2000 *Mathematics Subject Classification.* Primary 35L05; Secondary 35L15, 35Q40.

Key Words and Phrases. Schrödinger equation, p-evolution model, non-regular coefficients, loss of regularity, well-posedness.

one can find a gap (see [**3**]) between necessary and sufficient conditions which is described by a $\log 1/t$ term.

This gap was the starting point to create a C^2 approach which in the language of local conditions needs as well as conditions for the first and for the second derivatives of coefficients of the principal part with respect to t cf. with [**5**], [**8**]. This higher regularity allows to close the gap which tells us that the local condition for the first derivative is weaker than in the C^1 approach [**5**], [**8**]. Nevertheless we need to control the second derivative. Moreover, several notions of loss of regularity, *no loss, arbitrary small loss, finite loss and infinite loss* were introduced in [**8**], [**10**]. Today we have a complete optimal hierarchy of local conditions which is connected with the hierarchy of loss of regularity. This relation is called *Log-effect* because the hierarchy strongly depends on powers of $\log 1/t$ (see e.g. [**7**] or Remark 2.1).

In several papers [**1**], [**3**] the C^1 approach was generalized to classes of operators of p-evolution type. Global or local conditions or the coupling between both was considered.

The goal of the present paper is to study the *Log-effect* for classes of operators of p-evolution type.

In Section 2 we discuss classes of models of p-evolution type with time-dependent coefficients. Results for H^∞ well-posedness of the Cauchy problem is proved.

Section 3 is devoted to counter-examples which show the optimality of our assumptions in the main result. We introduce the state of art of methods. On the one hand Floquet' theory is introduced as an effective tool. On the other hand the interplay of Lyapunov and energy functional of solutions is used to understand interactions of oscillations. Finally, an instability argument coupled with an old approach from [**6**] yields the information that the loss of regularity really occurs.

Some concluding remarks will be given in Section 4.

2. *P*-evolution type models with time dependent coefficients.

2.1. Main result.

From the results for the Log-effect of [**5**] and [**8**] it seems to be reasonable that one effective class of Cauchy problems which allows to prove the Log-effect is the class of operators with the principal part in the *sense of Petrowsky*

$$D_t^2 - \sum_{|\alpha|=2p} a_\alpha(t) D_x^\alpha. \tag{2.1}$$

Therefore most of the considerations of this paper are devoted to the forward Cauchy problem

$$\begin{cases} D_t^2 u - \sum_{|\alpha|=0}^{2p} a_\alpha(t) D_x^\alpha u - \sum_{|\alpha|=0}^{p-1} b_\alpha(t) D_x^\alpha D_t u = 0, \\ u(0,x) = u_0(x),\ u_t(0,x) = u_1(x). \end{cases} \tag{2.2}$$

In the same way we can study the backward Cauchy problem. Although the principal part of the operator in (2.2) in the sense of Petrowsky is given in (2.1) we will call

$$D_t^2 - \sum_{|\alpha|=p+1}^{2p} a_\alpha(t) D_x^\alpha - \sum_{|\alpha|=1}^{p-1} b_\alpha(t) D_x^\alpha D_t \tag{2.3}$$

the *principal part* of the p-evolution operator. The terms

$$\sum_{|\alpha|=0}^{p} a_\alpha(t) D_x^\alpha + b_0(t) D_t \tag{2.4}$$

form the *part of lower order* of the p-evolution operator. Both parts are treated in a different way.

Now we are in position to formulate the main result of this paper.

THEOREM 2.1. *Let us consider the Cauchy problem* (2.2) *under the following assumptions*:

- (*for the principal part in the sense of Petrowsky*)

There exists a positive constant C_0 such that

$$C_0 |\xi|^{2p} \le \sum_{|\alpha|=2p} a_\alpha(t) \xi^\alpha \le C_0^{-1} |\xi|^{2p}.$$

Moreover, with $\gamma \in [0,1]$ it holds

$$|D_t^l a_\alpha| \lesssim \left(\frac{1}{t} \left(\log \frac{1}{t} \right)^\gamma \right)^l, \quad \textit{for } l = 1, 2, \ \textit{and } |\alpha| = 2p.$$

- (*for the remaining coefficients of the principal part* (2.3))

We assume that a_α are real and that

$$|D_t^l a_\alpha| \lesssim \left(\frac{1}{t} \left(\log \frac{1}{t} \right)^\gamma \right)^{\sigma_{l\alpha}} \quad \textit{for } l = 0, 1, 2, \ \textit{and } |\alpha| = p+1, \ldots, 2p-1,$$

where

$$0 \le \sigma_{0\alpha} \le \frac{2p-|\alpha|}{p},\ \sigma_{1\alpha} = \frac{(5p-2|\alpha|)(1-\sigma_{0\alpha})+|\alpha|-p}{2(|\alpha|-p)},$$
$$\sigma_{2\alpha} = \frac{(3p-|\alpha|)(1-\sigma_{0\alpha})+|\alpha|-p}{|\alpha|-p}.$$

Moreover, we assume that b_α are real and that

$$|D_t^l b_\alpha| \lesssim \left(\frac{1}{t}\left(\log\frac{1}{t}\right)^\gamma\right)^{\theta_{l\alpha}} \quad \textit{for } l=0,1,2, \textit{ and } |\alpha|=1,\dots,p-1,$$

where

$$0 \le \theta_{0\alpha} \le \frac{p-|\alpha|}{p},\ \theta_{1\alpha} = \frac{(p-|\alpha|)(1-\theta_{0\alpha})+|\alpha|}{|\alpha|},$$
$$\theta_{2\alpha} = \frac{(2p-|\alpha|)(1-\theta_{0\alpha})+|\alpha|}{|\alpha|}.$$

• (*for the lower order part* (2.4))
We assume the integrability condition $a_\alpha, b_0 \in L^1(0,T)$ for $|\alpha| = 0,\dots,p$. Then the Cauchy problem is H^∞ well-posed with loss of regularity $\exp(C_2(\log\langle D_x\rangle)^\gamma)$, *that is,*

$$\|(\langle D_x\rangle^p u, D_t u)(t,\cdot)\|_{H^s} \le C_1 \|\exp(C_2(\log\langle D_x\rangle)^\gamma)(\langle D_x\rangle^p u_0, u_1)\|_{H^s},$$

where C_1 and C_2 are suitable positive constants.

Remark 2.1. The statement of this theorem describes the Log-effect. Log-effect means that the power γ of the Log-term $\log 1/t$ has a strong influence on the loss of regularity. Namely, if $\gamma = 0$, then we have no loss. If $\gamma \in (0,1)$, then $\exp(C(\log\langle D_x\rangle)^\gamma)$ corresponds to an arbitrary small loss $\langle D_x\rangle^\varepsilon$, ε is positive and arbitrary small. If $\gamma = 1$, then we have a finite loss $\langle D_x\rangle^C$. Finally, we want to mention again that, if we would include the sum $\sum_{|\alpha|=p} b_\alpha(t) D_x^\alpha D_t u$ in (2.2), then due to the counter-example from [**9**] $\gamma = 0$ in the above assumptions can already imply a finite loss. This result corresponds to expected results from C^1 theory, consequently, the Log-effect cannot be shown (cf. with Theorem 3.1).

Remark 2.2. To fix the general model with time dependent coefficients (2.2) we recall one counter-example from [**9**] which tells us that for the strictly hyperbolic Cauchy problem

$$D_t^2 u + b(t) D_{xt}^2 u - a(t) D_x^2 u = 0, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{2.5}$$

we cannot expect the Log-effect. The main reason is that the interaction of oscillations of coefficients $a(t)$ and $b(t)$ does in general not allow to observe this effect. Consequently, the Log-effect can be proved only under special assumptions to the coefficients $a(t)$ and $b(t)$ of (2.5), see e.g. the papers for the case $b(t) \equiv 0$.

In Section 3.1 this counter-example will be generalized to the p-evolution type model

$$D_t^2 u + b(t) D_x^p D_t u - a(t) D_x^{2p} u = 0, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x) \tag{2.6}$$

(see Theorem 3.1).

2.2. Proof.

The proof is divided into several steps. After partial Fourier transformation with respect to x we get from (2.2)

$$D_t^2 v - \sum_{|\alpha|=0}^{2p} a_\alpha(t)\xi^\alpha v - \sum_{|\alpha|=0}^{p-1} b_\alpha(t)\xi^\alpha D_t v = 0, \tag{2.7}$$

where v denotes the partial Fourier transform of u.

2.2.1. Regularization and energy.

With a smooth function $\chi = \chi(s)$, $\chi(s) = 1$ for $|s| \leq 1$, $\chi(s) = 0$ for $|s| \geq 2$, $\chi(s) \in [0,1]$ we define the regularization

$$a_\alpha^{\varepsilon_\alpha}(t) := \chi\Big(\frac{t}{\varepsilon_\alpha}\Big) a_\alpha(\varepsilon_\alpha) + \Big(1 - \chi\Big(\frac{t}{\varepsilon_\alpha}\Big)\Big) a_\alpha(t) \text{ for } |\alpha| = p+1, \dots, 2p,$$

$$b_\alpha^{\eta_\alpha}(t) := \chi\Big(\frac{t}{\eta_\alpha}\Big) b_\alpha(\eta_\alpha) + \Big(1 - \chi\Big(\frac{t}{\eta_\alpha}\Big)\Big) b_\alpha(t) \text{ for } |\alpha| = 1, \dots, p-1,$$

where ε_α and η_α are positive constants. We introduce the following notations:

$$\lambda_1(t,\xi) := \Big(\sum_{|\alpha|=p+1}^{2p} a_\alpha(t)\xi^\alpha \Big)^{\frac{1}{2}}, \quad \lambda_1^\varepsilon(t,\xi) := \Big(\sum_{|\alpha|=p+1}^{2p} a_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha \Big)^{\frac{1}{2}},$$

$$\mu_1(t,\xi) := \frac{1}{2}\sum_{|\alpha|=1}^{p-1} b_\alpha(t)\xi^\alpha, \quad \mu_1^\eta(t,\xi) := \frac{1}{2}\sum_{|\alpha|=1}^{p-1} b_\alpha^{\eta_\alpha}(t)\xi^\alpha,$$

$$\nu_1^{\varepsilon,\eta}(t,\xi) := \big(\lambda_1^\varepsilon(t,\xi)^2 + \mu_1^\eta(t,\xi)^2\big)^{\frac{1}{2}}, \quad h_a(t,\xi) := \frac{\sum_{|\alpha|=0}^{p} a_\alpha(t)\xi^\alpha}{\nu_1^{\varepsilon,\eta}(t,\xi)}.$$

Then the equation (2.7) is rewritten as follows:

$$\big(D_t^2 - \lambda_1^2 - \nu_1^{\varepsilon,\eta} h_a - 2\mu_1 D_t - b_0 D_t\big)v(t,\xi) = 0.$$

Let us carry out the complex dissipation transformation

$$w(t,\xi) := \exp\bigg(-i\int_0^t \mu_1^\eta(s,\xi)\,ds\bigg)v(t,\xi).$$

Then we have

$$\begin{aligned}
&\exp\bigg(-i\int_0^t \mu_1^\eta(s,\xi)\,ds\bigg)(D_t^2 - \lambda_1^2 - \nu_1^{\varepsilon,\eta}h_a - 2\mu_1 D_t - b_0 D_t)\exp\bigg(i\int_0^t \mu_1^\eta(s,\xi)\,ds\bigg)\\
&\quad = D_t^2 + 2\mu_1^\eta D_t + D_t\mu_1^\eta + (\mu_1^\eta)^2 - \lambda_1^2 - \nu_1^{\varepsilon,\eta}h_a - 2\mu_1 D_t - 2\mu_1\mu_1^\eta - b_0 D_t - b_0\mu_1^\eta\\
&\quad = D_t^2 - (\lambda_1^\varepsilon)^2 - \big(\lambda_1^2 - (\lambda_1^\varepsilon)^2\big) - (\mu_1^\eta)^2 - 2\big(\mu_1\mu_1^\eta - (\mu_1^\eta)^2\big) - 2(\mu_1 - \mu_1^\eta)D_t\\
&\qquad - b_0 D_t + D_t\mu_1^\eta - \nu_1^{\varepsilon,\eta}h_a - b_0\mu_1^\eta\\
&\quad = D_t^2 - (\nu_1^{\varepsilon,\eta})^2 - \big(\lambda_1^2 - (\lambda_1^\varepsilon)^2\big) - 2\mu_1^\eta(\mu_1 - \mu_1^\eta) - 2(\mu_1 - \mu_1^\eta)D_t - b_0 D_t\\
&\qquad + D_t\mu_1^\eta - \nu_1^{\varepsilon,\eta}h_a - b_0\mu_1^\eta\\
&\quad = D_t^2 - (\nu_1^{\varepsilon,\eta})^2 - \nu_1^{\varepsilon,\eta}g_a - 2\mu_1^\eta g_b - 2g_b D_t - b_0 D_t + D_t\mu_1^\eta - \nu_1^{\varepsilon,\eta}h_a - b_0\mu_1^\eta\\
&\quad = D_t^2 - (\nu_1^{\varepsilon,\eta})^2 - \nu_1^{\varepsilon,\eta}r_{11} - r_{12}D_t,
\end{aligned}$$

where

$$g_a = g_{a,\varepsilon,\eta} := \frac{\lambda_1^2 - (\lambda_1^\varepsilon)^2}{\nu_1^{\varepsilon,\eta}}, \quad g_b = g_{b,\varepsilon,\eta} := \mu_1 - \mu_1^\eta$$

and

$$r_{11} := g_a + \frac{2\mu_1^\eta g_b}{\nu_1^{\varepsilon,\eta}} - \frac{D_t \mu_1^\eta}{\nu_1^{\varepsilon,\eta}} + h_a + \frac{b_0 \mu_1^\eta}{\nu_1^{\varepsilon,\eta}}, \quad r_{12} := 2g_b + b_0.$$

Thus our starting equation (2.7) is reduced to

$$\big(D_t^2 - (\nu_1^{\varepsilon,\eta})^2 - \nu_1^{\varepsilon,\eta} r_{11} - r_{12} D_t\big) w(t,\xi) = 0. \tag{2.8}$$

We define the energy of w by

$$W(t,\xi) = (w_1, w_2)^T := \big(\nu_1^{\varepsilon,\eta} w, D_t w\big)^T (t,\xi).$$

Our first strategy is that for large frequencies the dominant term in $\nu_1^{\varepsilon,\eta}(t,\xi)$ is $\sum_{|\alpha|=2p} a_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha$. For this reason we have to pose some *dominance conditions* to the terms of lower order of $\nu_1^{\varepsilon,\eta}(t,\xi)$.

Lemma 2.1. *Let us suppose the dominance conditions*

$$\begin{cases} \beta_\alpha \le \dfrac{2p - |\alpha|}{\sigma_{0\alpha}} & \text{for } |\alpha| = p+1, \dots, 2p-1, \\ \delta_\alpha \le \dfrac{p - |\alpha|}{\theta_{0\alpha}} & \text{for } |\alpha| = 1, \dots, p-1, \end{cases} \tag{2.9}$$

where β_α and δ_α are defined by the balance between regularization parameters $\varepsilon_\alpha, \eta_\alpha$ and frequency variable ξ

$$\frac{1}{\varepsilon_\alpha}\left(\log\frac{1}{\varepsilon_\alpha}\right)^\gamma = \tilde{c}|\xi|^{\beta_\alpha}, \quad \frac{1}{\eta_\alpha}\left(\log\frac{1}{\eta_\alpha}\right)^\gamma = \tilde{c}|\xi|^{\delta_\alpha}, \tag{2.10}$$

with a small constant $\tilde{c}$. Then $\nu_1^{\varepsilon,\eta}(t,\xi) \ge C|\xi|^p$ for large frequencies with a suitable positive constant C.

Proof. The statement follows immediately from the estimates

$$|a_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha| \le C\left(\frac{1}{\varepsilon_\alpha}\left(\log\frac{1}{\varepsilon_\alpha}\right)^\gamma\right)^{\sigma_{0\alpha}} |\xi|^{|\alpha|} \le C\tilde{c}|\xi|^{\beta_\alpha\sigma_{0\alpha}}|\xi|^{|\alpha|} \le C\tilde{c}|\xi|^{2p},$$

$$|b_\alpha^{\eta_\alpha}(t)\xi^\alpha|^2 \le C\left(\frac{1}{\eta_\alpha}\left(\log\frac{1}{\eta_\alpha}\right)^\gamma\right)^{2\theta_{0\alpha}} |\xi|^{2|\alpha|} \le C\tilde{c}|\xi|^{2\delta_\alpha\theta_{0\alpha}}|\xi|^{2|\alpha|} \le C\tilde{c}|\xi|^{2p}$$

together with the assumptions from Theorem 2.1 for a_α with $|\alpha| = 2p$. □

2.2.2. Two steps of diagonalization procedure.

In opposite to the C^1 approach we will carry out two steps of diagonalization procedure. Using the definition of W we obtain from (2.8) and from the definition of the energy

$$D_t w_1 = \nu_1^{\varepsilon,\eta} w_2 + \frac{D_t \nu_1^{\varepsilon,\eta}}{\nu_1^{\varepsilon,\eta}} w_1, \quad D_t w_2 = (\nu_1^{\varepsilon,\eta} + r_{11}) w_1 + r_{12} w_2.$$

Thus from (2.8) we arrive at

$$D_t W - \nu_1^{\varepsilon,\eta} \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} W - \frac{D_t \nu_1^{\varepsilon,\eta}}{\nu_1^{\varepsilon,\eta}} \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} W - \begin{pmatrix} 0 & 0 \\ r_{11} & r_{12} \end{pmatrix} W = 0. \tag{2.11}$$

To carry out the first step of diagonalization procedure we set

$$W =: M_1 W_1 \quad \text{with} \quad M_1 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}.$$

Applying this transformation to system (2.11) it yields

$$\begin{cases} D_t W_1 - \nu_1^{\varepsilon,\eta} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_1 - \dfrac{D_t \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}} \begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} W_1 \\ \qquad - \dfrac{1}{2} \begin{pmatrix} r_{11} + r_{12} & r_{11} - r_{12} \\ -r_{11} - r_{12} & -r_{11} + r_{12} \end{pmatrix} W_1 = 0. \end{cases} \tag{2.12}$$

To carry out the second step of diagonalization procedure we take into consideration only the structure of the antidiagonal elements of the second matrix from (2.12). Here we will later use the special structure of $\frac{D_t \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}$. For this reason we define

$$q = q(t,\xi) := \frac{D_t \nu_1^{\varepsilon,\eta}}{4(\nu_1^{\varepsilon,\eta})^2} = \frac{D_t (\nu_1^{\varepsilon,\eta})^2}{8(\nu_1^{\varepsilon,\eta})^3} \tag{2.13}$$

and apply the transformation

$$W_1 =: M_2 W_2 \quad \text{with} \quad M_2 := \begin{pmatrix} 1 & -q \\ q & 1 \end{pmatrix}.$$

Setting this transformation into the system (2.12) gives

$$\begin{cases} D_tW_2 - \nu_1^{\varepsilon,\eta}\begin{pmatrix} 1-\frac{2q^2}{1+q^2} & -2q+\frac{2q^3}{1+q^2} \\ -2q+\frac{2q^3}{1+q^2} & -1+\frac{2q^2}{1+q^2}\end{pmatrix}W_2 \\ \quad -\dfrac{D_t\nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\begin{pmatrix} 1+\frac{2q}{1+q^2} & 1-\frac{2q^2}{1+q^2} \\ 1-\frac{2q^2}{1+q^2} & 1-\frac{2q}{1+q^2}\end{pmatrix}W_2 + M_2^{-1}(D_tM_2)W_2 \\ \quad -\dfrac{1}{2}M_2^{-1}\begin{pmatrix} r_{11}+r_{12} & r_{11}-r_{12} \\ -r_{11}-r_{12} & -r_{11}+r_{12}\end{pmatrix}M_2W_2 = 0. \end{cases} \tag{2.14}$$

But here we supposed the invertibility of M_2. It follows from the smallness of q in the definition of M_2. This smallness condition can be realized by an additional balance between the regularization parameters ε_α, η_α and the frequency variable ξ.

Lemma 2.2. *Let us suppose the conditions*

$$\begin{cases} \beta_\alpha\sigma_{1\alpha} \le 3p - |\alpha| & \text{for } |\alpha| = p+1,\dots,2p, \\ \delta_{\alpha_1}\theta_{0\alpha_1} + \delta_{\alpha_2}\theta_{1\alpha_2} \le 3p - |\alpha_1| - |\alpha_2| & \text{for } |\alpha_1|,|\alpha_2| = 1,\dots,p-1, \end{cases} \tag{2.15}$$

where all β_α, δ_{α_1}, δ_{α_2} are defined by (2.10) *with a small constant $\tilde{c}$. Then the matrix M_2 is invertible for large frequencies.*

Proof. The statement follows immediately if we can show that the above assumption guarantees the smallness of $|q|$ from (2.13). Together with the assumptions of Theorem 2.1 this smallness follows from

$$\begin{cases} \left(\dfrac{1}{\varepsilon_\alpha}\left(\log\dfrac{1}{\varepsilon_\alpha}\right)^\gamma\right)^{\sigma_{1\alpha}}|\xi|^{|\alpha|} \le \tilde{c}|\xi|^{3p} & \text{for } |\alpha| = p+1,\dots,2p, \\ \left(\dfrac{1}{\eta_{\alpha_1}}\left(\log\dfrac{1}{\eta_{\alpha_1}}\right)^\gamma\right)^{\theta_{0\alpha_1}}|\xi|^{|\alpha_1|}\left(\dfrac{1}{\eta_{\alpha_2}}\left(\log\dfrac{1}{\eta_{\alpha_2}}\right)^\gamma\right)^{\theta_{1\alpha_2}}|\xi|^{|\alpha_2|} \le \tilde{c}|\xi|^{3p} & \\ & \text{for } |\alpha_1|,|\alpha_2| = 1,\dots,p-1. \end{cases}$$

The conditions (2.10) and the above assumptions ensure these relations for large frequencies. □

2.2.3. Discussion of system (2.14).

Calculating in system (2.14) the first three matrices and leaving the other ones unchanged we arrive at the new system

$$D_t W_2 - \nu_1^{\varepsilon,\eta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_2 - \frac{D_t \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} W_2 - B_0 W_2 = 0, \tag{2.16}$$

where the matrix B_0 has the following structure:

$$B_0 := \begin{cases} \nu_1^{\varepsilon,\eta}\begin{pmatrix} -\frac{2q^2}{1+q^2} & \frac{2q^3}{1+q^2} \\ \frac{2q^3}{1+q^2} & \frac{2q^2}{1+q^2} \end{pmatrix} + \frac{D_t \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\begin{pmatrix} \frac{2q}{1+q^2} & -\frac{2q^2}{1+q^2} \\ -\frac{2q^2}{1+q^2} & -\frac{2q}{1+q^2} \end{pmatrix} \\ \quad -M_2^{-1} D_t q \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix} + \frac{1}{2} M_2^{-1}\begin{pmatrix} r_{11}+r_{12} & r_{11}-r_{12} \\ -r_{11}-r_{12} & -r_{11}+r_{12} \end{pmatrix} M_2. \end{cases}$$

We should mention that the above calculations show that the matrix

$$\frac{D_t \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$$

disappears after the second step of diagonalization. This is the main use of this step. From system (2.16) we conclude that the transformation

$$W_2 =: M_3(t,\xi) W_3 \quad \text{with} \quad M_3(t,\xi) := \exp\left(\int_0^t \frac{\partial_s \nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}} ds \right)$$

gives the system

$$\partial_t W_3 - i\nu_1^{\varepsilon,\eta}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} W_3 - B_1 W_3 = 0, \tag{2.17}$$

where B_1 has the same structure as B_0. Summarizing we have proved the following lemma:

LEMMA 2.3. *After two steps of diagonalization and suitable transformations the system* (2.11) *is transferred to system* (2.17). *All transformations are organized in such a way that no additional loss of regularity is produced.*

2.2.4. Estimate of $\int_0^T |B_1(t,\xi)|dt$.

To verify the statements of Theorem 2.1 we have to analyze the system (2.17). The diagonal matrix brings no additional loss of regularity because of the fact that $\nu_1^{\varepsilon,\eta}$ is real. The loss of regularity will be determined from the term

$$\exp\left(\int_0^t |B_1(s,\xi)|ds\right) \tag{2.18}$$

after applying Gronwall's inequality to estimate the usual energy of W_3. Thus it remains to estimate $\int_0^T |B_1(t,\xi)|dt$. The analysis of the structure of B_1 from the previous step shows that therefore we have to estimate $\int_0^T$ over

$$|\nu_1^{\varepsilon,\eta}||q|^2, \quad \left|\frac{D_t\nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\right||q|, \quad |D_tq|, \quad |r_{12}|, \quad |r_{11}|.$$

Here we are able to omit some terms by using the smallness of q (invertibility of M_2, cf. Lemma 2.2).

Estimate of $\int_0^T |\nu_1^{\varepsilon,\eta}||q|^2dt$:
Here we use

$$\begin{aligned}
|\nu_1^{\varepsilon,\eta}||q|^2 &\lesssim \frac{|D_t(\nu_1^{\varepsilon,\eta})^2|^2}{|\xi|^{5p}} \lesssim |\xi|^{-5p}\big|D_t\lambda_1^{\varepsilon}(t,\xi)^2 + D_t\mu_1^{\eta}(t,\xi)^2\big|^2 \\
&\lesssim |\xi|^{-5p}\bigg(\sum_{|\alpha|=p+1}^{2p}\big|D_ta_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\big| + 2\big|\mu_1^\eta(t,\xi)D_t\mu_1^\eta(t,\xi)\big|\bigg)^2 \\
&\lesssim |\xi|^{-5p}\bigg(\sum_{|\alpha|=p+1}^{2p}\big|D_ta_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\big|^2 + |\xi|^{2p}\big|D_t\mu_1^\eta(t,\xi)\big|^2\bigg) \\
&\lesssim |\xi|^{-5p}\bigg(\sum_{|\alpha|=p+1}^{2p}\big|D_ta_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\big|^2 + |\xi|^{2p}\sum_{|\alpha|=1}^{p-1}\big|D_tb_\alpha^{\eta_\alpha}(t)\xi^\alpha\big|^2\bigg).
\end{aligned}$$

From the supposed unboundedness of $D_ta_\alpha(t)$ (with $\sigma_{1\alpha}\geq 1$ for $|\alpha|=p+1,\ldots,2p$) and of $D_tb_\alpha(t)$ (with $\theta_{1\alpha}>1$ for $|\alpha|=1,\ldots,p-1$) we conclude

$$\int_{\varepsilon_\alpha}^T \big|D_ta_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\big|^2dt|\xi|^{-5p} \lesssim \left(\frac{1}{\varepsilon_\alpha}\left(\log\frac{1}{\varepsilon_\alpha}\right)^\gamma\right)^{2\sigma_{1\alpha}}\varepsilon_\alpha|\xi|^{2|\alpha|-5p},$$

$$\int_{\eta_\alpha}^{T} \left|D_t b_\alpha^{\eta_\alpha}(t)\xi^\alpha\right|^2 dt|\xi|^{-3p} \lesssim \left(\frac{1}{\eta_\alpha}\left(\log\frac{1}{\eta_\alpha}\right)^\gamma\right)^{2\theta_{1\alpha}} \eta_\alpha|\xi|^{2|\alpha|-3p}.$$

If we introduce the balance between ε_α, η_α and ξ as in (2.10), then the conditions

$$\begin{cases} 2\sigma_{1\alpha}\beta_\alpha - \beta_\alpha + 2|\alpha| - 5p \le 0,\ \sigma_{1\alpha} \ge 1,\ |\alpha| = p+1, \dots, 2p, \\ 2\theta_{1\alpha}\delta_\alpha - \delta_\alpha + 2|\alpha| - 3p \le 0,\ \theta_{1\alpha} > 1,\ |\alpha| = 1, \dots, p-1, \quad \text{imply} \\ \displaystyle\int_0^T |\lambda_1^\varepsilon||q|^2 dt \lesssim (\log|\xi|)^\gamma \quad \text{for large frequencies.} \end{cases} \tag{2.19}$$

In the same way we can estimate $\int_0^T \left|\frac{D_t\nu_1^{\varepsilon,\eta}}{2\nu_1^{\varepsilon,\eta}}\right||q|dt$.

ESTIMATE OF $\int_0^T |D_t q|dt$:

If we differentiate q, then the only interesting new integral is that one over $\left|\frac{D_t^2(\nu_1^{\varepsilon,\eta})^2}{8(\nu_1^{\varepsilon,\eta})^3}\right|$. The other integral which appears can be estimated as the previous integral. Here we use

$$\left|\frac{D_t^2(\nu_1^{\varepsilon,\eta})^2}{8(\nu_1^{\varepsilon,\eta})^3}\right| \lesssim |\xi|^{-3p}\left(\sum_{|\alpha|=p+1}^{2p} \left|D_t^2 a_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\right|\right) + |\xi|^{-2p}\left(\sum_{|\alpha|=1}^{p-1} \left|D_t^2 b_\alpha^{\eta_\alpha}(t)\xi^\alpha\right|\right) + |\xi|^{-3p}\left(\sum_{|\alpha|=1}^{p-1} \left|D_t b_\alpha^{\eta_\alpha}(t)\xi^\alpha\right|^2\right).$$

From the supposed unboundedness of $D_t^2 a_\alpha(t)$ (with $\sigma_{2\alpha} > 1$ for $|\alpha| = p+1, \dots, 2p$), of $D_t^2 b_\alpha(t)$ (with $\theta_{2\alpha} > 1$ for $|\alpha| = 1, \dots, p-1$) and of $D_t b_\alpha(t)$ (with $\theta_{1\alpha} > 1$ for $|\alpha| = 1, \dots, p-1$) we conclude

$$\int_{\varepsilon_\alpha}^{T} \left|D_t^2 a_\alpha^{\varepsilon_\alpha}(t)\xi^\alpha\right| dt|\xi|^{-3p} \lesssim \left(\frac{1}{\varepsilon_\alpha}\left(\log\frac{1}{\varepsilon_\alpha}\right)^\gamma\right)^{\sigma_{2\alpha}} \varepsilon_\alpha|\xi|^{|\alpha|-3p},$$

$$\int_{\eta_\alpha}^{T} \left|D_t^2 b_\alpha^{\eta_\alpha}(t)\xi^\alpha\right| dt|\xi|^{-2p} \lesssim \left(\frac{1}{\eta_\alpha}\left(\log\frac{1}{\eta_\alpha}\right)^\gamma\right)^{\theta_{2\alpha}} \eta_\alpha|\xi|^{|\alpha|-2p},$$

$$\int_{\eta_\alpha}^{T} \left|D_t b_\alpha^{\eta_\alpha}(t)\xi^\alpha\right|^2 dt|\xi|^{-3p} \lesssim \left(\frac{1}{\eta_\alpha}\left(\log\frac{1}{\eta_\alpha}\right)^\gamma\right)^{2\theta_{1\alpha}} \eta_\alpha|\xi|^{2|\alpha|-3p}.$$

If we introduce the balance between ε_α, η_α and ξ as in (2.10), then the conditions

$$\begin{cases} \sigma_{2\alpha}\beta_\alpha - \beta_\alpha + |\alpha| - 3p \le 0,\ \sigma_{2\alpha} > 1,\ |\alpha| = p+1, \dots, 2p, \\ \theta_{2\alpha}\delta_\alpha - \delta_\alpha + |\alpha| - 2p \le 0,\ \theta_{2\alpha} > 1,\ |\alpha| = 1, \dots, p-1, \\ 2\theta_{1\alpha}\delta_\alpha - \delta_\alpha + 2|\alpha| - 3p \le 0,\ \theta_{1\alpha} > 1,\ |\alpha| = 1, \dots, p-1, \quad \text{imply} \\ \displaystyle\int_0^T |D_t q| dt \lesssim (\log|\xi|)^\gamma \quad \text{for large frequencies.} \end{cases} \tag{2.20}$$

ESTIMATE OF $\int_0^T |r_{12}|dt$:

The desired estimate for this integral follows from the estimates for $\int_0^T |b_0(t)|dt$ and for $\int_0^T |g_b(t,\xi)|dt$. To estimate the first integral we use the $L^1(0,T)$ property. To estimate the other integral we proceed as follows:

$$\int_0^T |g_b(t,\xi)|dt = \int_0^T \Big| \sum_{|\alpha|=1}^{p-1} (b_\alpha(t) - b_\alpha^{\eta_\alpha}(t))\xi^\alpha \Big| dt \lesssim \sum_{|\alpha|=1}^{p-1} \int_0^{2\eta_\alpha} \big|(b_\alpha(t) - b_\alpha^{\eta_\alpha}(t))\xi^\alpha\big| dt.$$

If we suppose some unbounded behavior for $b_\alpha(t)$ as it is described in the assumptions of the theorem, then for $\theta_{0\alpha} < 1$ it holds

$$\begin{aligned} \int_0^{2\eta_\alpha} \big|(b_\alpha(t) - b_\alpha^{\eta_\alpha}(t))\xi^\alpha\big| dt &\lesssim |\xi|^{|\alpha|} \int_0^{2\eta_\alpha} \Big(\frac{1}{t}\Big(\log\frac{1}{t}\Big)^\gamma\Big)^{\theta_{0\alpha}} dt \\ &\lesssim |\xi|^{|\alpha|} \Big(\frac{1}{\eta_\alpha}\Big(\log\frac{1}{\eta_\alpha}\Big)^\gamma\Big)^{\theta_{0\alpha}} \eta_\alpha. \end{aligned}$$

Using the definition (2.10) for δ_α implies that $\delta_\alpha \ge \frac{|\alpha|}{1-\theta_{0\alpha}}$ gives the estimate $\int_0^T |g_b(t,\xi)|dt \lesssim (\log|\xi|)^\gamma$. Summarizing we have shown that the conditions

$$\begin{cases} b_0 \in L^1(0,T), \quad \delta_\alpha \ge \dfrac{|\alpha|}{1-\theta_{0\alpha}} \text{ for } \theta_{0\alpha} < 1,\ |\alpha| = 1, \dots, p-1, \\ \quad \text{imply} \quad \displaystyle\int_0^T |r_{12}| dt \lesssim (\log|\xi|)^\gamma \quad \text{for large frequencies.} \end{cases} \tag{2.21}$$

ESTIMATE OF $\int_0^T |r_{11}|dt$:

To estimate this integral we firstly devote to $\int_0^T \big|b_0 \frac{\mu_1^\eta}{\nu_1^{\varepsilon,\eta}}\big|dt$. The property of b_0 to belong to $L^1(0,T)$ implies $b_0 \frac{\mu_1^\eta}{\nu_1^{\varepsilon,\eta}} \in L^1(0,T)$.

We can estimate $\int_0^T |g_b \frac{\mu_1^\eta}{\lambda_1^\varepsilon}|dt$ as it was done above to estimate $\int_0^T |r_{12}|dt$.

To estimate

$$\int_0^T |g_a(t,\xi)|dt = \int_0^T \Big| \sum_{|\alpha|=p+1}^{2p} (a_\alpha(t) - a_\alpha^{\varepsilon_\alpha}(t))|\xi|^{|\alpha|-p} \Big| dt$$
$$\lesssim \sum_{|\alpha|=p+1}^{2p} \int_0^{2\varepsilon_\alpha} \big|(a_\alpha(t) - a_\alpha^{\varepsilon_\alpha}(t))|\xi|^{|\alpha|-p}\big| dt$$

we suppose some possible unbounded behavior for $a_\alpha(t)$ as it is described in the assumptions of the theorem by $\sigma_{0\alpha} \geq 0$. Then for $\sigma_{0\alpha} < 1$ and for $|\alpha| = p+1, \ldots, 2p$, it holds

$$\int_0^{2\varepsilon_\alpha} \big|(a_\alpha(t) - a_\alpha^{\varepsilon_\alpha}(t))|\xi|^{|\alpha|-p}\big| dt \lesssim |\xi|^{|\alpha|-p} \int_0^{2\varepsilon_\alpha} \Big(\frac{1}{t}\Big(\log\frac{1}{t}\Big)^\gamma\Big)^{\sigma_{0\alpha}} dt$$
$$\lesssim |\xi|^{|\alpha|-p} \Big(\frac{1}{\varepsilon_\alpha}\Big(\log\frac{1}{\varepsilon_\alpha}\Big)^\gamma\Big)^{\sigma_{0\alpha}} \varepsilon_\alpha.$$

Using the definition (2.10) for β_α implies that $\beta_\alpha \geq \frac{|\alpha|-p}{1-\sigma_{0\alpha}}$ gives the estimate $\int_0^T |g_{\varepsilon,a}(t,\xi)|dt \lesssim (\log|\xi|)^\gamma$. To estimate $\int_0^T |h_a(t,\xi)|dt$ we use the supposed $L^1(0,T)$ property for $a_\alpha(t)$, $|\alpha| = 0,\ldots,p$. Finally, it remains to estimate $\int_0^T |\frac{D_t\mu_1^\eta}{\nu_1^{\varepsilon,\eta}}|dt$. It holds

$$\int_0^T \Big|\frac{D_t\mu_1^\eta}{\nu_1^{\varepsilon,\eta}}\Big| dt \lesssim |\xi|^{-p} \int_{\eta_\alpha}^T \big|D_t b_\alpha^{\eta_\alpha}(t)\xi^\alpha\big| dt.$$

From the supposed unboundedness of $D_t b_\alpha(t)$ (with $\theta_{1\alpha} > 1$ for $|\alpha| = 1,\ldots,p-1$) we conclude

$$\int_{\eta_\alpha}^T \big|D_t b_\alpha^{\eta_\alpha}(t)\xi^\alpha\big| dt |\xi|^{-p} \lesssim \Big(\frac{1}{\eta_\alpha}\Big(\log\frac{1}{\eta_\alpha}\Big)^\gamma\Big)^{\theta_{1\alpha}} \eta_\alpha |\xi|^{|\alpha|-p}.$$

Summarizing we have shown that the conditions

$$\begin{cases} \theta_{1\alpha}\delta_\alpha - \delta_\alpha + |\alpha| - p \leq 0, \ \theta_{1\alpha} > 1, \ \text{for } |\alpha| = 1,\ldots,p-1, \\ a_\alpha \in L^1(0,T), \quad \text{for } |\alpha| = 0,\ldots,p, \\ \beta_\alpha \geq \dfrac{|\alpha|-p}{1-\sigma_{0\alpha}} \ \text{for } \sigma_{0\alpha} < 1, \ |\alpha| = p+1,\ldots,2p, \\ \text{imply} \quad \displaystyle\int_0^T |r_{11}|dt \lesssim (\log|\xi|)^\gamma \quad \text{for large frequencies.} \end{cases} \tag{2.22}$$

2.2.5. Verification.

To verify the statement of the theorem we have to take into consideration the conditions from (2.9), (2.15), (2.19), (2.20), (2.21), (2.22). From (2.9) and (2.21) we conclude $\frac{|\alpha|}{1-\theta_{0\alpha}} \leq \frac{p-|\alpha|}{\theta_{0\alpha}}$ which is equivalent to $\theta_{0\alpha} \leq \frac{p-|\alpha|}{p} < 1$ for $|\alpha| = 1, \ldots, p-1$. The constants $\delta_\alpha = \frac{|\alpha|}{1-\theta_{0\alpha}}$. From (2.9) and (2.22) we conclude $\frac{|\alpha|-p}{1-\sigma_{0\alpha}} \leq \frac{2p-|\alpha|}{\sigma_{0\alpha}}$ which is equivalent to $\sigma_{0\alpha} \leq \frac{2p-|\alpha|}{p} < 1$ for $|\alpha| = p+1, \ldots, 2p-1$. The constants $\beta_\alpha = \frac{|\alpha|-p}{1-\sigma_{0\alpha}}$. The assumptions for a_α, $|\alpha| = 2p$, imply $\sigma_{0\alpha} = 0$, hence $\beta_\alpha = p$ for $|\alpha| = 2p$. Using $\beta_\alpha \leq p$ we see that the restriction for $\sigma_{1\alpha}$ coming from (2.19) is more restrictive than that one coming from (2.15). Thus the optimal $\sigma_{1\alpha}$ is

$$\sigma_{1\alpha} = \frac{5p-2|\alpha|+\beta_\alpha}{2\beta_\alpha} = \frac{(5p-2|\alpha|)(1-\sigma_{0\alpha})+|\alpha|-p}{2(|\alpha|-p)} > 1$$

$$\text{for } |\alpha| = p+1, \ldots, 2p-1, \quad \sigma_{1\alpha} = 1 \text{ for } |\alpha| = 2p.$$

From (2.20) we get immediately

$$\sigma_{2\alpha} = \frac{3p-|\alpha|+\beta_\alpha}{\beta_\alpha} = \frac{(3p-|\alpha|)(1-\sigma_{0\alpha})+|\alpha|-p}{|\alpha|-p} > 1 \text{ for } |\alpha| = p+1, \ldots, 2p.$$

Using $\delta_\alpha \leq p$ we see that the restriction for $\theta_{1\alpha}$ coming from (2.22) is more restrictive than those ones coming from (2.20), (2.19) or from (2.15) together with (2.9). Thus the optimal $\theta_{1\alpha}$ is

$$\theta_{1\alpha} = \frac{p-|\alpha|+\delta_\alpha}{\delta_\alpha} = \frac{(p-|\alpha|)(1-\theta_{0\alpha})+|\alpha|}{|\alpha|} > 1 \text{ for } |\alpha| = 1, \ldots, p-1.$$

At the end we conclude from (2.20)

$$\theta_{2\alpha} = \frac{2p-|\alpha|+\delta_\alpha}{\delta_\alpha} = \frac{(2p-|\alpha|)(1-\theta_{0\alpha})+|\alpha|}{|\alpha|} > 1 \text{ for } |\alpha| = 1, \ldots, p-1.$$

Thus all assumptions we made during the proof are satisfied. This completes the proof of the theorem. □

2.3. Examples.

2.3.1. Bounded coefficients in principal part.

Let us suppose that $\sigma_{0\alpha} = 0$ for $|\alpha| = p+1, \ldots, 2p-1$ and $\theta_{0\alpha} = 0$ for $|\alpha| = 1, \ldots, p-1$. We will call this case *regular case*. Then $\sigma_{1\alpha} = \frac{4p-|\alpha|}{2(|\alpha|-p)} < \frac{2p}{|\alpha|-p} = \sigma_{2\alpha}$

for $|\alpha| = p+1, \dots, 2p-1$ and $\theta_{1\alpha} = \frac{p}{|\alpha|} < \frac{2p}{|\alpha|} = \theta_{2\alpha}$ for $|\alpha| = 1, \dots, p-1$. If we set $|\alpha| = 2p$, then we have $\sigma_{1\alpha} = 1$, $\sigma_{2\alpha} = 2$ for $|\alpha| = 2p$. These are the typical orders for the leading coefficients a_α, $|\alpha| = 2p$, to observe the Log-effect. If we compare our assumptions for the first derivatives with respect to t with those from [**1**] to carry out the C^1 approach, then our orders $\sigma_{1\alpha}$ are larger. The orders $\theta_{1\alpha}$ coincide with those from [**1**]. But we have to control derivatives of second order, too.

2.3.2. Optimal unbounded coefficients in principal part.

Now let us allow the optimal unbounded behavior for coefficients a_α, $|\alpha| = p+1, \dots, 2p-1$ and b_α, $|\alpha| = 1, \dots, p-1$, that is, $\sigma_{0\alpha} = \frac{2p-|\alpha|}{p}$ for $|\alpha| = p+1, \dots, 2p-1$ and $\theta_{0\alpha} = \frac{p-|\alpha|}{p}$ for $|\alpha| = 1, \dots, p-1$. We call this case *critical singular case*. Then $\sigma_{1\alpha} = \frac{3p-|\alpha|}{p} < \frac{4p-|\alpha|}{p} = \sigma_{2\alpha}$ for $|\alpha| = p+1, \dots, 2p-1$ and $\theta_{1\alpha} = \frac{2p-|\alpha|}{p} < \frac{3p-|\alpha|}{p} = \theta_{2\alpha}$ for $|\alpha| = 1, \dots, p-1$.

Remark 2.3. Not only from the examples but also from Theorem 2.1 we see the following two tendencies:

- As more singular the allowed behavior of the above coefficients a_α and b_α is, as more restrictive are the conditions for the derivatives of first and second order.
- The allowed behavior of the derivatives of second order is more singular than that one for the derivatives of first order.

3. Counter-examples.

3.1. Possible interactions of oscillations.

The possible interaction of oscillations of coefficients of the principal part in the sense of Petrowsky of a p-evolution operator is described in the following result (cf. Remarks 2.1 and 2.2).

Theorem 3.1. *Let us consider the Cauchy problem*

$$D_t^2 u + b(t) D_x^p D_t u - a(t) D_x^{2p} u = 0, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{3.1}$$

where $D_t^2 + b(t) D_x^p D_t - a(t) D_x^{2p}$ is a p-evolution operator. There exist coefficients a and b from $C^2(0,T]$ which satisfy the conditions

- $a_0 \le a(t) \le a_1,\ b_0 \le b(t) \le b_1,$
- $|a'(t)|^2 + |a''(t)| \lesssim \big(\frac{1}{t}\big(\log\frac{1}{t}\big)^\gamma\big)^2,$
- $|b'(t)|^2 + |b''(t)| \lesssim \big(\frac{1}{t}\big(\log\frac{1}{t}\big)^\gamma\big)^2,$

with positive constants a_0, a_1, b_0, b_1 and with an arbitrary small positive γ such that the Cauchy problem (3.1) *is not H^∞ well-posed.*

Proof. The proof follows exactly the lines of the proof of Theorem 1 from [**9**]. □

3.2. Counter-examples for conditions of principal part.

3.2.1. Optimality of conditions for the principal part in the sense of Petrowsky.

In this section we want to show the sharpness of the assumptions to the coefficients of the principal part from Theorem 2.1. That the assumptions for $a_\alpha(t)$, $|\alpha| = 2p$, are sharp shows the generalization of the results from [**8**] to the operator $D_t^2 - a_{2p}(t)D_x^{2p}$.

Theorem 3.2. *Let us consider the Cauchy problem*

$$D_t^2 u - \omega\left(\left(\log\frac{1}{t}\right)^{\gamma+1}\right)D_x^{2p}u = 0, \quad u(0,x) = u_0(x),\ u_t(0,x) = u_1(x), \tag{3.2}$$

where $\omega(\tau) \in C^\infty[0,\infty)$ is a positive and 1-periodic function. Then the Cauchy problem (3.2) *is not H^∞ well-posed with the loss of regularity* $\exp(C(\log\langle D_x\rangle)^{\gamma_0})$, *$\gamma_0 < \gamma$, that is, the inequality*

$$\|(\langle D_x\rangle^p u, D_t u)(t,\cdot)\|_{H^s} \le C_1\|\exp(C_2(\log\langle D_x\rangle)^{\gamma_0})(\langle D_x\rangle^p u_0, u_1)\|_{H^s}$$

does not hold, where C_1 and C_2 are arbitrary positive constants and $0 < \gamma_0 < \gamma$.

Proof. The proof is similar to the proof of Theorem 3.3. For this reason we omit it. □

3.2.2. Optimality of conditions for the coefficients a_k of the remaining principal part.

First of all we remark that instead of the forward Cauchy problem we are able to study the backward Cauchy problem under the assumptions of Theorem 2.1. This leads to the more general inequality

$$\|(\langle D_x\rangle^p u, D_t u)(t,\cdot)\|_{H^s} \le C_1\|\exp(C_2(\log\langle D_x\rangle)^{\gamma})(\langle D_x\rangle^p u, D_t u)(r,\cdot)\|_{H^s},$$

where C_1 and C_2 are suitable positive constants independent of $t, r \in [0,T]$.

In the next theorem we will give a counter-example and will formulate a corresponding result to the previous theorem. The proof bases on the application of Floquet' theory (see [**13**] and [**15**]), which is an effective tool to prove counter-examples.

Theorem 3.3. *Let us consider the Cauchy problem*

$$D_t^2 u - D_x^{2p} u - a_k(t) D_x^k u = 0, \quad u(r,x) = u_0(x),\ u_t(r,x) = u_1(x),\ r \in [0,T], \tag{3.3}$$

where $k = p+1, \ldots, 2p-1$. *Then, there exists a positive, non-constant and* 1*-periodic function* $\omega = \omega(\cdot) \in C^\infty[0,\infty)$ *such that the Cauchy problem* (3.3) *with* $a_k(t) = (\frac{1}{t}(\log\frac{1}{t})^\gamma)^{\frac{2p-k}{p}} \omega((\log\frac{1}{t})^{\gamma+1})$ *is not* H^∞ *well-posed with the loss of regularity* $\exp(C(\log\langle D_x\rangle)^{\gamma_0})$, $\gamma_0 < \gamma$, *that is, the inequality*

$$\|(\langle D_x\rangle^p u, D_t u)(t,\cdot)\|_{H^s} \le C_1 \| \exp(C_2(\log\langle D_x\rangle)^{\gamma_0})(\langle D_x\rangle^p u, D_t u)(r,\cdot)\|_{H^s} \tag{3.4}$$

does not hold, where C_1 *and* C_2 *are arbitrary positive constants which are independent of* $r, t \in [0,T]$.

Proof. Let us assume that the above inequality (3.4) is true uniformly for $r, t \in [0,T]$. Transfer into the phase space this means ($L^{2,s} := \mathscr{F}(H^s)$)

$$\|(\langle\xi\rangle^p v, D_t v)(t,\cdot)\|_{L^{2,s}} \le C_1 \| \exp(C_2(\log\langle\xi\rangle)^{\gamma_0})(\langle\xi\rangle^p v, D_t v)(r,\cdot)\|_{L^{2,s}} \tag{3.5}$$

for the partial Fourier transform v of the solution u. We understand from the $L^{2,s}$-$L^{2,s}$ estimate (3.5) that the fundamental solution $E = E(t,r,\xi)$ satisfying

$$(\langle\xi\rangle^p v, D_t v)^T(t,\xi) = E(t,r,\xi)(\langle\xi\rangle^p v, D_t v)^T(r,\xi) \tag{3.6}$$

can be estimated by

$$|E(t,r,\xi)| \le C_1 \exp(C_2(\log\langle\xi\rangle)^{\gamma_0}).$$

Consequently, the proof of the statement of the theorem follows immediately from the following proposition. □

Proposition 3.1. *Let us consider the Cauchy problem*

$$D_t^2 v - \xi^{2p} v - a_k(t)\xi^k v = 0, \quad v(r,\xi) = v_0(\xi),\ v_t(r,\xi) = v_1(\xi),\ r \in [0,T]. \tag{3.7}$$

Let $E = E(t,r,\xi)$ *be the fundamental solution from* (3.6). *Then, there exist for any given* ξ *time levels* t_ξ *and* $\tilde{t}_\xi$ *satisfying* $0 < \tilde{t}_\xi < t_\xi$ *and* $\lim_{|\xi|\to\infty} \tilde{t}_\xi = 0$ *such* *that the following inequality holds for the fundamental solution E:*

$$\left|E(t_\xi, \tilde{t}_\xi, \xi)\right| \geq \exp(C(\log\langle\xi\rangle)^\gamma), \tag{3.8}$$

where C is a positive constant which is independent of ξ.

PROOF. For a smooth, 1-periodic and non-constant function $\tilde{\omega}(s)$ satisfying

$$\tilde{\omega}_0 \leq \tilde{\omega}(s) \leq \tilde{\omega}_1 \tag{3.9}$$

with some positive constants $\tilde{\omega}_0$ and $\tilde{\omega}_1$ we consider the following Cauchy problem:

$$\begin{cases} \dfrac{d}{d\tau}X(s_0-\tau; s_0) = \begin{pmatrix} 0 & 1 \\ -\lambda_0\tilde{\omega}(s_0-\tau) & 0 \end{pmatrix} X(s_0-\tau; s_0), \\ X(s_0; s_0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{3.10}$$

where λ_0 is chosen as a positive constant taking into account the following property which results from Floquet' theory:

LEMMA 3.1 (Floquet' theory). *Let $\tilde{\omega}(s)$ be a smooth,* 1*-periodic and non-constant function, and let $X(\tau; s_0)$ be the solution to the first order system* (3.10)*. Then there exists a positive real number λ_0 such that $X(s_0+1; s_0)$ has the eigenvalues μ and μ^{-1} satisfying $|\mu| > 1$.*

(For the proof see [**13**] or [**15**], for instance.) □

REMARK 3.1. Here the constant λ_0 belongs to an interval, which is called *an interval of instability.* Generally, for any given large positive number L, there exists a real constant $\lambda_0 > L$ such that λ_0 belongs to an interval of instability. On the other hand, λ_0 can be chosen from intervals, which are called the *intervals of stability*; thus we have $|\mu| = 1$. Therefore, Lemma 3.1 can be represented as Lemma 3.4 below in the case that λ_0 belongs to an interval of stability.

Let us introduce the notations

$$\kappa(t) := \frac{1}{t}\left(\log\frac{1}{t}\right)^\gamma, \quad s(t) := \left(\log\frac{1}{t}\right)^{\gamma+1} \quad \text{and} \quad \omega(s) := \lambda_0(\gamma+1)^2\tilde{\omega}(s) - 1.$$

Then we define the coefficient $a_k(t)$ of (3.7) as follows:

$$a_k(t) := \kappa(t)^{\frac{2p-k}{p}}\omega(s(t)).$$

Here we note that $\omega(s)$ satisfies

$$\omega_0 \le \omega(s) \le \omega_1 \tag{3.11}$$

for some positive constants ω_0 and ω_1 since $L \ge 1/(\tilde{\omega}_0(\gamma+1)^2)$ by (3.9).

Let us define $\nu = \nu(s)$ by

$$\nu(s) := -s'(t(s)) = (\gamma+1)s^{\frac{\gamma}{\gamma+1}}\exp(s^{\frac{1}{\gamma+1}}) = (\gamma+1)\kappa(t(s)),$$

where $t(s) = \exp(-s^{1/(\gamma+1)})$. Then we introduce the following notations:

$$\lambda_1(s,\xi) := (\gamma+1)^{-\frac{2p-k}{p}}\nu(s)^{-\frac{k}{p}}\xi^k,$$
$$\lambda_2(s) := \frac{1}{4}\nu'(s)^2\nu(s)^{-2} - \frac{1}{2}\nu''(s)\nu(s)^{-1},$$
$$\zeta(s,\xi) := \frac{\lambda_2(s) + \nu(s)^{-2}\xi^{2p}}{\lambda_1(s,\xi)}, \quad p(s,\tilde{s},\xi) := \omega(\tilde{s}) + \zeta(s,\xi)$$

and

$$\lambda(s,\tilde{s},\xi) := \lambda_1(s,\xi)p(s,\tilde{s},\xi).$$

Denoting $w = w(s,\xi) := \nu(s)^{\frac{1}{2}}v(t(s),\xi)$, the equation of (3.7) for $t \in (0,T]$ is reduced to the following equation:

$$D_s^2 w - \lambda(s,s,\xi)w = 0 \tag{3.12}$$

for $s \in [s(T),\infty)$. Here we remark that the first variable, and the second variable of $\lambda(s,s,\cdot)$ describe the decreasing, and oscillating behavior of the coefficient respectively. Moreover, we can suppose that the parameter ξ is positive and large without loss of generality.

For a given $\xi \in \boldsymbol{R}$, we define s_ξ implicitly by

$$\nu(s_\xi) = (\gamma+1)\xi^p, \tag{3.13}$$

it follows that

$$\lambda_1(s_\xi,\xi)=(\gamma+1)^{-2} \text{ and } \zeta(s_\xi,\xi)=1+(\gamma+1)^2\lambda_2(s_\xi). \tag{3.14}$$

Therefore, noting $\lim_{s\to\infty}\lambda_2(s)=0$ we conclude $\zeta(s,\xi)>0$ if s is large and

$$\lambda(s_\xi,s,\xi)=(\gamma+1)^{-2}(\omega(s)+1)+\lambda_2(s_\xi)\to\lambda_0\tilde{\omega}(s) \text{ as } s_\xi\to\infty. \tag{3.15}$$

Then we have the following lemma:

LEMMA 3.2. *Let δ be a positive small constant. Then the following estimates are established*:

$$\sup_{\tau\in(0,1)}\left\{\left|\lambda(s_\xi,s_\xi-\tau,\xi)-\lambda(s_\xi-\tau,s_\xi-\tau,\xi)\right|\right\}\le Cs_\xi^{-\frac{\gamma}{\gamma+1}}$$

and

$$\sup_{\tau\in(0,1)}\left\{\left|\lambda(s_\xi-j+1-\tau,s_\xi-j+1-\tau,\xi)-\lambda(s_\xi-j-\tau,s_\xi-j-\tau,\xi)\right|\right\}$$
$$\le Cs_\xi^{-\frac{\gamma}{\gamma+1}}$$

for any $j+1\le\delta s_\xi^{\gamma/(\gamma+1)}$.

PROOF. Let α be a non-zero real number and d be a positive constant satisfying $0<d\le\delta s^{\gamma/(\gamma+1)}$ for large s. Then we have

$$\left|\exp\left(\alpha s^{\frac{1}{\gamma+1}}\right)-\exp\left(\alpha(s-d)^{\frac{1}{\gamma+1}}\right)\right|\begin{cases}\ge C^{-1}ds^{-\frac{\gamma}{\gamma+1}}\exp\left(\alpha^{\frac{\gamma}{\gamma+1}}\right)\\ \le Cds^{-\frac{\gamma}{\gamma+1}}\exp\left(\alpha^{\frac{\gamma}{\gamma+1}}\right)\end{cases}$$

and

$$\exp\left(s^{\frac{1}{\gamma+1}}-(s-d)^{\frac{1}{\gamma+1}}\right)\begin{cases}\ge 1+C^{-1}\delta\\ \le 1+C\delta.\end{cases}$$

Consequently, we have

$$|\lambda_1(s,\xi)-\lambda_1(s-d,\xi)|\le Cds^{-\frac{\gamma}{\gamma+1}}\lambda_1(s,\xi),$$
$$|\lambda_2(s)-\lambda_2(s-d)|\le Cds^{-\frac{\gamma}{\gamma+1}}\lambda_2(s)$$

and

$$1 - C\delta \le \frac{\lambda_1(s-d,\xi)}{\lambda_1(s,\xi)} \le 1 + C\delta.$$

It follows with the above inequalities that

$$\begin{aligned}
&|\zeta(s,\xi) - \zeta(s-d,\xi)| \\
&\quad \le Cds^{-\frac{\gamma}{\gamma+1}} \left(\frac{\lambda_2(s) + \nu(s)^{-2}\xi^{2p}}{\lambda_1(s,\xi)} + \frac{\lambda_2(s-d) + \nu(s-d)^{-2}\xi^{2p}}{\lambda_1(s-d,\xi)} \right) \\
&\quad \le Cds^{-\frac{\gamma}{\gamma+1}} \zeta(s,\xi).
\end{aligned}$$

In particular, by (3.14) we have

$$|\zeta(s_\xi,\xi) - \zeta(s_\xi - d,\xi)| \le Cds_\xi^{-\frac{\gamma}{\gamma+1}}.$$

Therefore, for $\tau \in (0,1)$ it holds

$$\begin{aligned}
&\big|\lambda(s_\xi, s_\xi - \tau, \xi) - \lambda(s_\xi - \tau, s_\xi - \tau, \xi)\big| \\
&\quad \le \big|\omega(s_\xi - \tau) + \zeta(s_\xi - \tau, \xi)\big| \big|\lambda_1(s_\xi,\xi) - \lambda_1(s_\xi - \tau, \xi)\big| \\
&\qquad + \lambda_1(s_\xi,\xi)\big|\zeta(s_\xi - \tau,\xi) - \zeta(s_\xi,\xi)\big| \\
&\quad \le Cs_\xi^{\frac{\gamma}{\gamma+1}} \big(\omega_1 + \lambda_1(s_\xi,\xi) + \zeta(s_\xi,\xi)\big) \le Cs_\xi^{\frac{\gamma}{\gamma+1}}.
\end{aligned}$$

By using the 1-periodicity of ω we conclude

$$\begin{aligned}
&\big|\lambda(s_\xi - j + 1 - \tau, s_\xi - j + 1 - \tau, \xi) - \lambda(s_\xi - j - \tau, s_\xi - j - \tau, \xi)\big| \\
&\quad \le C(s_\xi - j + 1 - \tau)^{-\frac{\gamma}{\gamma+1}} \lambda_1(s_\xi - j + 1 - \tau, \xi) \\
&\qquad \times \big(\omega(s_\xi - j + 1 - \tau) + \zeta(s_\xi - j + 1 - \tau, \xi)\big) \\
&\qquad + \lambda_1(s_\xi - j - \tau, \xi)\big|\zeta(s_\xi - j + 1 - \tau, \xi) - \zeta(s_\xi - j - \tau, \xi)\big| \\
&\quad \le Cs_\xi^{-\frac{\gamma}{\gamma+1}} \lambda_1(s_\xi,\xi)\big(1 + js_\xi^{-\frac{\gamma}{\gamma+1}}\big) \le Cs_\xi^{-\frac{\gamma}{\gamma+1}}.
\end{aligned}$$

Thus the lemma is proved. □

Let λ_0 and $\mu(= \mu(0))$ be determined from Lemma 3.1. Then, by the continuous dependence of eigenvalues on the coefficients there exists $\varepsilon_0 > 0$ depending only on $\tilde{\omega}(s)$ such that an eigenvalue $\mu(\varepsilon)$ of $X(s_0+1, s_0; \varepsilon)$, which is a solution of

$$\begin{cases} \dfrac{d}{d\tau} X(s_0-\tau; s_0, \varepsilon) = \begin{pmatrix} 0 & 1 \\ -\lambda_0 \tilde{\omega}(s_0-\tau)+\varepsilon & 0 \end{pmatrix} X(s_0-\tau; s_0, \varepsilon), \\ X(s_0; s_0, \varepsilon) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}, \end{cases} \tag{3.16}$$

satisfies $\min_{|\varepsilon|\le\varepsilon_0}\{|\mu(\varepsilon)|\} > 1$. Indeed, taking into account of (3.15), and substituting $s_0 = s_\xi$ and $\varepsilon = \lambda_2(s_\xi)$ in (3.16) we shall apply Floquet' theory to the following Cauchy problem:

$$\begin{cases} \dfrac{d}{d\tau} X(s_\xi-\tau; s_\xi) = \begin{pmatrix} 0 & 1 \\ -\lambda(s_\xi, s_\xi-\tau, \xi) & 0 \end{pmatrix} X(s_\xi-\tau; s_\xi), \\ X(s_\xi; s_\xi) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}. \end{cases} \tag{3.17}$$

Let us introduce the positive integer n by

$$n = n(s_\xi) := \left[\delta s_\xi^{\frac{\gamma}{\gamma+1}}\right] \tag{3.18}$$

for a positive small constant δ, where $[\cdot]$ denotes the Gauss symbol. Moreover, we consider the following Cauchy problems for first order systems:

$$\begin{cases} \dfrac{d}{d\tau} X_j(\tau; 0) = \begin{pmatrix} 0 & 1 \\ -\lambda(s_\xi-j+1-\tau, s_\xi-j+1-\tau, \xi) & 0 \end{pmatrix} X_j(\tau; 0), \\ X_j(0; 0) = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \text{ for } j = 1, \ldots, n. \end{cases} \tag{3.19}$$

Here we denote the solutions of (3.17) and (3.19) at $\tau = 1$ by

$$X(s_\xi-1; s_\xi) := \begin{pmatrix} x_{11} & x_{12} \\ x_{21} & x_{22} \end{pmatrix} \quad \text{and} \quad X_j(1; 0) := \begin{pmatrix} x_{11}(j) & x_{12}(j) \\ x_{21}(j) & x_{22}(j) \end{pmatrix}, \tag{3.20}$$

and we denote the eigenvalues of $X(s_\xi-1; s_\xi)$, and $X_j(1; 0)$ by $\mu(\lambda_2(s_\xi))^{\pm 1} = \mu^{\pm 1}$, and $\mu_j^{\pm 1}$ respectively. It holds

$$x_{11}+x_{22}=\mu+\mu^{-1}, \quad |x_{11}-\mu|+|x_{22}-\mu|\ge|\mu-\mu^{-1}|.$$

Therefore, due to Lemma 3.1

$$\max\{|x_{11}-\mu|,|x_{22}-\mu|\}\ge\frac{1}{2}|\mu-\mu^{-1}|>0.$$

Let us assume

$$\max\{|x_{11}-\mu|,|x_{22}-\mu|\}=|x_{11}-\mu|,$$

the other case can be treated in the same way, we have

$$|x_{22}-\mu^{-1}|\ge\frac{1}{2}|\mu-\mu^{-1}|.$$

Then the following lemma is valid:

Lemma 3.3. *There exist positive real numbers δ and C such that for any $j=1,\dots,n-1$ the following estimates hold*:

$$\max_{1\le j\le n-1}\{|x_{lm}(j)-x_{lm}(j+1)|+|\mu_j-\mu_{j+1}|\}\le Cs_\xi^{-\frac{\gamma}{\gamma+1}} \tag{3.21}$$

for any $l,m=1,2$ and

$$\min_{1\le j\le n}\{|\mu_j|\}>1. \tag{3.22}$$

Proof. By Lemma 3.2 we immediately obtain the estimate (3.21) and

$$\sup_{\tau\in(0,1)}\{|X_j(\tau;0)|\}\le C.$$

Let us introduce $Z_j(\tau):=X(s_\xi-\tau;s_\xi)-X_j(\tau;0)$. Then $Z_j(\tau)$ is a solution to

$$\begin{cases}\dfrac{d}{d\tau}Z_j(\tau)=\begin{pmatrix}0 & 1\\ -\lambda(s_\xi,s_\xi-\tau,\xi) & 0\end{pmatrix}Z_j(\tau)\\[2ex] \quad+\begin{pmatrix}0 & 0\\ \lambda(s_\xi-j+1-\tau,s_\xi-j+1-\tau,\xi)-\lambda(s_\xi,s_\xi-\tau,\xi) & 0\end{pmatrix}X_j(\tau;0)\end{cases} \tag{3.23}$$

for $j = 1, \dots, n$. A Gronwall type argument and Lemma 3.2 imply

$$\sup_{\tau\in(0,1)} \{|Z_j(\tau)|\} \le Cjs_\xi^{-\frac{\gamma}{\gamma+1}} \le Cns_\xi^{-\frac{\gamma}{\gamma+1}}.$$

Therefore, we have from the definition of x_{lm} and $x_{lm}(j)$ that

$$|\mu - \mu_j| \le C(|x_{11} - x_{11}(j)| + |x_{22} - x_{22}(j)|) \le Cns_\xi^{-\frac{\gamma}{\gamma+1}} \le C\delta,$$

and, recalling $|\mu| > 1$, the choice $\delta = (|\mu| - 1)/(2C)$ brings the estimate $|\mu - \mu_j| \le (|\mu| - 1)/2$, it follows that $|\mu_j| \ge \min\{|\mu|, (|\mu| + 1)/2\} > 1$. Thus the proof is concluded. □

Denoting

$$B_j := \begin{pmatrix} \frac{x_{12}(j)}{\mu_j - x_{11}(j)} & 1 \\ 1 & \frac{x_{21}(j)}{\mu_j^{-1} - x_{22}(j)} \end{pmatrix} \quad \text{and} \quad G_j = B_j^{-1} B_{j+1} - I,$$

we have the following representation:

$$\begin{aligned} \begin{pmatrix} w(s_\xi, \xi) \\ w_s(s_\xi, \xi) \end{pmatrix} &= X_1(1,0) \cdots X_n(1,0) \begin{pmatrix} w(s_\xi - n, \xi) \\ w_s(s_\xi - n, \xi) \end{pmatrix} \\ &= B_1 Y_n B_n^{-1} \begin{pmatrix} w(s_\xi - n, \xi) \\ w_s(s_\xi - n, \xi) \end{pmatrix}, \end{aligned}$$

where

$$Y_n := \begin{pmatrix} \mu_1 & 0 \\ 0 & \mu_1^{-1} \end{pmatrix} (I + G_1) \cdots (I + G_{n-1}) \begin{pmatrix} \mu_n & 0 \\ 0 & \mu_n^{-1} \end{pmatrix}. \tag{3.24}$$

Then Lemma 3.3 ensures the boundedness of each of the elements of B_j, B_j^{-1} and of $B_{j-1} - B_j$, and the estimate $|G_j| \le Cs_\xi^{-\gamma/(\gamma+1)}$ holds for any $j \le n$.

To estimate the entries of Y_n we write

$$Y_n = \begin{pmatrix} \prod_{k=1}^n \mu_k & 0 \\ 0 & \prod_{k=1}^n \mu_k^{-1} \end{pmatrix} + M_1 + \cdots + M_{n-1}, \tag{3.25}$$

where M_l is the matrix which is the sum of all the products of matrices from the right-hand side of (3.25) containing exactly l of the G_k matrices; observe

$$|M_l| \le \Big(\prod_{k=1}^{n} |\mu_k|\Big)\Big(\sum_{1\le i_1<\cdots<i_l\le n-1}\prod_{j=1}^{l}|G_{i_j}|\Big).$$

Therefore,

$$\begin{aligned}
\sum_{l=1}^{n-1}|M_l| &\le \Big(\prod_{k=1}^{n}|\mu_k|\Big)\sum_{l=1}^{n-1}\binom{n-1}{l}\big(Cs_\xi^{-\frac{\gamma}{\gamma+1}}\big)^l\\
&\le \Big(\prod_{k=1}^{n}|\mu_k|\Big)\frac{1}{2}Cns_\xi^{-\frac{\gamma}{\gamma+1}}\big(1+Cs_\xi^{-\frac{\gamma}{\gamma+1}}\big)^n \le \Big(\prod_{k=1}^{n}|\mu_k|\Big)\frac{1}{2}C\delta\Big(1+\frac{C\delta}{n}\Big)^n\\
&\le \Big(\prod_{k=1}^{n}|\mu_k|\Big)\frac{1}{2}C\delta e^{C\delta}.
\end{aligned}$$

Thus we obtain

$$|(Y_n)_{11}| \ge \Big(\prod_{k=1}^{n}|\mu_k|\Big)(1-C\delta) \ge \frac{1}{2}\Big(\prod_{k=1}^{n}|\mu_k|\Big)$$

taking account of the smallness of δ. On the other hand, $|(Y_n)_{kl}|$, $(k,l)\ne(1,1)$ is very small—it is less than $\varepsilon\prod_{k=1}^{n}|\mu_k|$, where we can take ε as small as we like. Introducing the notation $R_n := B_1Y_n$ we conclude from the above estimates the following ones:

$$|(R_n)_{11}| \ge C\prod_{k=1}^{n}|\mu_k|, \quad |(R_n)_{12}| \le \varepsilon\prod_{k=1}^{n}|\mu_k|, \tag{3.26}$$

where C is a positive constant and ε is positive, but as small as we wish. Taking account of

$$\begin{aligned}
w(s_\xi,\xi) &= (R_n)_{11}\big(\tilde b_{11}w(s_\xi-n,\xi)+\tilde b_{12}w_s(s_\xi-n,\xi)\big)\\
&\qquad + (R_n)_{12}\big(\tilde b_{21}w(s_\xi-n,\xi)+\tilde b_{22}w_s(s_\xi-n,\xi)\big),\\
w(s,\xi) &= \nu(s)^{\frac{1}{2}}v(t(s),\xi),
\end{aligned}$$

here $\tilde b_{kl}$ denote the entries of B_n^{-1}, the special choice of data

$$w_s(s_\xi - n, \xi) = \frac{\nu'(s)}{\nu(s)} w(s_\xi - n, \xi) \tag{3.27}$$

on $s = s_\xi - n$ implies $v_t(\tilde{t}_\xi, \xi) = 0$, where we choose $t_\xi = t(s_\xi)$ and $\tilde{t}_\xi = t(s_\xi - n)$. Moreover, noting the estimates

$$C^{-1} s_\xi \le \big(\log\langle\xi\rangle\big)^{\gamma+1} \le C s_\xi, \tag{3.28}$$

we get from (3.26) and (3.27) together with the smallness of $\frac{\nu'(s)}{\nu(s)}$ for $s \to \infty$ the estimate

$$|w(s_\xi, \xi)| \ge C \exp\big(C(\log\langle\xi\rangle)^\gamma\big) |w(s_\xi - n, \xi)|. \tag{3.29}$$

Transforming this inequality back and using the proof to Lemma 3.2 gives

$$\begin{aligned} |v(t_\xi, \xi)| &\ge C \frac{\nu(s_\xi - n)^{1/2}}{\nu(s_\xi)^{1/2}} \exp\big(C(\log\langle\xi\rangle)^\gamma\big) |v(\tilde{t}_\xi, \xi)| \\ &\ge C_0 \exp\big(C(\log\langle\xi\rangle)^\gamma\big) |v(\tilde{t}_\xi, \xi)|. \end{aligned}$$

The last inequality and $v_t(\tilde{t}_\xi, \xi) = 0$ imply

$$\langle\xi\rangle^p |v(t_\xi, \xi)| \ge C_0 \exp\big(C(\log\langle\xi\rangle)^\gamma\big) |(\langle\xi\rangle^p v(\tilde{t}_\xi, \xi), v_t(\tilde{t}_\xi, \xi))|.$$

Thus we have (3.8) and the proof of Proposition 3.1 is concluded. □

3.2.3. Optimality of conditions for the coefficients b_k of the remaining principal part.

Finally, let us check the assumptions for b_α, $|\alpha| = 1, \dots, p-1$. It turns out that a new strategy to prove counter-examples is used. The strategy bases on the *interplay between the Lyapunov and energy functional* as it was used to prove necessity of Levi conditions (see e.g. [**11**] or [**14**]). By this interplay we are able to understand interactions of oscillations in coefficients. The proof of the following theorem is an essential refinement of the proof of the main result from [**9**]. We will show that microlocal (this means, in some part of the extended phase space) the coefficient $a_{2k}(t)$ has a *stabilizing influence*, but $b_k(t)$ has a *non-stabilizing influence* there.

Let us consider the following Cauchy problem:

$$\big(D_t^2 - \xi^{2p} - a_{2k}(t)\xi^{2k} + b_k(t)\xi^k D_t\big)v = 0, \quad (v(t_\xi,\xi), v_t(t_\xi,\xi)) = (v_0(\xi), v_1(\xi)), \tag{3.30}$$

where the equation is reduced by inverse partial Fourier transformation with respect to ξ to

$$\big(D_t^2 - D_x^{2p} - a_{2k}(t)D_x^{2k} + b_k(t)D_x^k D_t\big)u = 0. \tag{3.31}$$

Let $\omega(s) \in C^\infty(\boldsymbol{R})$ be a positive 1-periodic function satisfying

$$\omega(s) = \begin{cases} 2 & \text{on} \quad [0,1/4], \\ \text{monotone decreasing} & \text{on} \quad (1/4,1/2), \\ 1 & \text{on} \quad [1/2,3/4], \\ \text{monotone increasing} & \text{on} \quad (3/4,1). \end{cases}$$

Let us define $a_{2k}(t)$ by (for the definition of $\kappa(t)$ and $s(t)$ see the previous section)

$$a_{2k}(t) := \kappa(t)^{\frac{2(p-k)}{p}}\omega\big(s(t)\big).$$

Then we verify that $a_{2k}(t)$ satisfies the conditions of Theorem 2.1, precisely, σ_{l2k} is given by $\sigma_{l2k} = l + 2(p-k)/p$. Therefore if $b_k(t)$ satisfies

$$\big|D_t^l b_k(t)\big| \le C_l \kappa(t)^{\theta_{lk}} \quad \text{with} \quad \theta_{lk} = l + \frac{p-k}{p} \quad \text{for} \quad l = 0,1,2, \tag{3.32}$$

then one can conclude that all solutions of (3.30) satisfy the following estimate:

$$\langle\xi\rangle^{2p}|v(t,\xi)|^2 + |D_t v(t,\xi)|^2 \le \exp\big(C_0(\log\langle\xi\rangle)^\gamma\big)\big(\langle\xi\rangle^{2p}|v(t_\xi,\xi)|^2 + |D_t v(t_\xi,\xi)|^2\big) \tag{3.33}$$

for any t_ξ, t satisfying $0 \le t_\xi < t$ uniformly with respect to ξ.

Let us prove now the optimality of the conditions to $b_k(t)$ from Theorem 2.1 in the following sense:

Theorem 3.4. *Let v be a solution of* (3.30) *and γ be a positive real number. If $b_k(t) \equiv 0$, then there exist initial data (v_0, v_1), which are prescribed on $t = t_\xi$ and there exist $\tilde{t}_\xi$ such that $0 < t_\xi < \tilde{t}_\xi$ and*

$$\langle\xi\rangle^{2p}|v(\tilde t_\xi,\xi)|^2+|D_tv(\tilde t_\xi,\xi)|^2\le C_1\big(\langle\xi\rangle^{2p}|v(t_\xi,\xi)|^2+|D_tv(t_\xi,\xi)|^2\big), \tag{3.34}$$

where C_1 is independent of t_ξ and $\tilde t_\xi$. On the other hand, there exist initial data (v_0,v_1), which are prescribed on $t=t_\xi$ and there exist $\tilde t_\xi$ and a coefficient $b_k(t)\in C^3(0,T]$ satisfying $|D_t^lb_k(t)|\le C_l\kappa(t)^{l+(p-k)/p}$ for $l=0,1,2,3$ such that the following estimate holds:

$$\langle\xi\rangle^{2p}|v(\tilde t_\xi,\xi)|^2+|D_tv(\tilde t_\xi,\xi)|^2\ge \exp\big(C_2(\log\langle\xi\rangle)^\gamma\big)\big(\langle\xi\rangle^{2p}|v(t_\xi,\xi)|^2+|D_tv(t_\xi,\xi)|^2\big) \tag{3.35}$$

for any ξ, where C_2 is independent of t_ξ and $\tilde t_\xi$.

Proof. The estimate (3.34) is proved by the same arguments as in the proof of Theorem 3.3 under a suitable choice of λ_0 in an interval of stability. Let us introduce the following lemma, which is another description of Lemma 3.1.

Lemma 3.4. *Let $\tilde\omega(s)$ and $X(s;s_0)$ be defined as in Lemma 3.1. Then there exist different positive real numbers $\lambda_{0\pm}$ such that for any $\lambda_0\in(\lambda_{0-},\lambda_{0+})$, $X(s_0+1;s_0)$ has the eigenvalues $\mu=\mu(\lambda_0)$ and $\mu^{-1}=\mu(\lambda_0)^{-1}$ satisfying $|\mu|=|\mu^{-1}|=1$.*

Let λ_0 be a constant from Lemma 3.4, and (s_ξ,ξ) is determined by (3.13). Then the eigenvalues $\mu=\mu(\lambda_2(s_\xi))^{\pm1}$ of $X(s_\xi+1;s_\xi)$ to the solution of (3.17) with $\tau=1$ satisfies $|\mu|=1$ for large s_ξ by the continuous dependence of eigenvalues on the coefficients. Here we note that Lemma 3.2 is valid if λ_0 belongs to intervals of stability. Therefore, the eigenvalues $\mu_j^{\pm1}$ of $X_j(1;0)$ to the solution of (3.19) with $\tau=1$ satisfy

$$\max_{1\le j\le n}\big\{|\mu_j^{\pm1}|\big\}\le 1+Cs_\xi^{-\frac{\gamma}{\gamma+1}},$$

and $|G_j|\le Cs_\xi^{-\gamma/(\gamma+1)}$ for any $j=1,\dots,n$, where $n=[\delta s_\xi^{\gamma/(\gamma+1)}]$. Consequently, we have

$$|(Y_n)_{lm}|\le\big(1+Cs_\xi^{-\frac{\gamma}{\gamma+1}}\big)^n\le\left(1+\frac{C\delta}{n}\right)^n\le e^{C\delta},$$

which implies the estimate (3.34) for $t_\xi=t(s_\xi)$ and $\tilde t_\xi=t(s_\xi-n)$.

Let us now prove the estimate (3.35). By putting $w=w(t,\xi):=e^{-i\xi^k\int_t^T b_k(s)\,ds}v(t,\xi)$ the equation from (3.30) is rewritten as

$$\big(D_t^2-c^2\xi^{2k}-(D_tb_k)\xi^k\big)w=0, \tag{3.36}$$

where

$$c = c(t,\xi) := \sqrt{\xi^{2(p-k)} + a_{2k}(t) + b_k(t)^2}.$$

Let us define for $l = 0,1,2,3$ the sequences $\{t_{j,l}\}_{j=1}^{\infty}$ by $t_{j,l} := \exp(-(j - l/4)^{1/(\gamma+1)})$, in particular, we denote $t_{j,0} = t_j$. Then we define $b_k(t)$ satisfying (3.32) and

$$b_k(t) = \begin{cases} \text{monotone increasing} & \text{on} \quad (t_j, t_{j,1}), \\ \sqrt{2}\kappa(t_j)^{\frac{p-k}{p}} & \text{on} \quad [t_{j,1}, t_{j,2}], \\ \text{monotone decreasing} & \text{on} \quad (t_{j,2}, t_{j,3}), \\ \kappa(t_{j-1})^{\frac{p-k}{p}} & \text{on} \quad [t_{j,3}, t_{j-1}] \end{cases}$$

for $j = 1,2,\ldots$. Here we remark that $s(t_{j,l}) = j - l/4$, it follows that

$$\omega(s(t)) = \begin{cases} 2 & \text{on} \quad [t_j, t_{j,1}], \\ \text{monotone decreasing} & \text{on} \quad (t_{j,1}, t_{j,2}), \\ 1 & \text{on} \quad [t_{j,2}, t_{j,3}], \\ \text{monotone increasing} & \text{on} \quad (t_{j,3}, t_{j-1}). \end{cases}$$

Let us define the zone Z_H by $Z_H := \{(t,\xi) \in (0,T] \times \boldsymbol{R}\,;\, t \ge t_\xi,\ |\xi| \gg 1\}$, where $t_\xi = t(s_\xi)$ and s_ξ is defined by (3.13) with the constant λ_0 which allows to derive the estimate (3.34). By denoting $K := \lambda_0^{p/k}(\gamma+1)^{2p/k}$ and choosing λ_0 large, Z_H is represented as follows:

$$Z_H = \{(t,\xi) \in (0,T] \times \boldsymbol{R}\,;\, K\kappa(t) \le |\xi|^p,\ |\xi| \gg 1\}. \tag{3.37}$$

We introduce the following symbol classes in Z_H:

$$\mathscr{S}_j^h(m) = \big\{ f(t,\xi) \in C^m(Z_H)\,;\, |\partial_t^l f(t,\xi)| \le C_j |\xi|^{hp} \kappa(t)^{j+l},\ l = 0,\ldots,m \big\}.$$

Then we immediately have the following properties:

- If $f \in \mathscr{S}_j^h(m)$, then $\partial_t^l f \in \mathscr{S}_{j+l}^h(m-l)$.
- If $f_1 \in \mathscr{S}_{j_1}^{h_1}(m_1)$, $f_2 \in \mathscr{S}_{j_2}^{h_2}(m_2)$, then $f_1 f_2 \in \mathscr{S}_{j_1+j_2}^{h_1+h_2}(\min\{m_1,m_2\})$.
- We have the inclusion $\mathscr{S}_j^h(m) \subset \mathscr{S}_{j-l}^{h+l}(m)$ for any $l > 0$.
- It holds $a_{2k}(t), b_k(t) \in \mathscr{S}_{(p-k)/p}^0(3)$, $c(t,\xi) \in \mathscr{S}_0^{(p-k)/p}(3)$ and $c(t,\xi)^{-1} \in$

$\mathscr{S}_0^{-(p-k)/p}(3)$.

Denoting $W_0 = W_0(t,\xi) := {}^t(D_t w(t,\xi), c(t,\xi)\xi^k w(t,\xi))$, (3.36) is represented by the following system:

$$\left(D_t - \begin{pmatrix} 0 & c\xi^k + \frac{D_t b_k}{c} \\ c\xi^k & \frac{D_t c}{c} \end{pmatrix}\right) W_0 = 0. \tag{3.38}$$

Moreover, by using the matrix M_1:

$$M_1 := \begin{pmatrix} 1 & 1 \\ 1 & -1 \end{pmatrix}$$

in the first step of diagonalization procedure, (3.38) is rewritten as follows:

$$(D_t - \Lambda_1 - B)W_1(t,\xi) = 0, \tag{3.39}$$

where $W_1 := M_1 W_0$,

$$\Lambda_1 = \Lambda_1(t,\xi) := \begin{pmatrix} c\xi^k + \frac{D_t c}{2c} & 0 \\ 0 & -c\xi^k + \frac{D_t c}{2c} \end{pmatrix}$$

and

$$B = B(t,\xi) := \frac{1}{2c}\begin{pmatrix} D_t b_k & -D_t(b_k + c) \\ D_t(b_k - c) & -D_t b_k \end{pmatrix} \in \mathscr{S}_{\frac{2p-k}{p}}^{-\frac{p-k}{p}}(2).$$

The second step of diagonalization procedure is carried out by using the following matrix $M_2 = M_2(t,\xi)$:

$$M_2 := \begin{pmatrix} 1 & -\frac{B_{12}}{2c\xi^k} \\ \frac{B_{21}}{2c\xi^k} & 1 \end{pmatrix},$$

where B_{lm} denotes the (l,m)'th element of B. Here the property $M_2 - I \in \mathscr{S}_{(2p-k)/p}^{-(2p-k)/p}(2)$ guarantees the invertibility of M_2 for a large K, and a large λ_0 respectively. Thus, (3.39) is rewritten as follows:

$$(D_t - \Lambda_2 - B_2)W_2 = 0, \tag{3.40}$$

where $W_2 := M_2^{-1} W_1$,

$$\Lambda_2 := \begin{pmatrix} c\xi^k + \frac{D_t c}{2c} + \frac{D_t b_k}{2c} & 0 \\ 0 & -c\xi^k + \frac{D_t c}{2c} - \frac{D_t b_k}{2c} \end{pmatrix},$$

and

$$B_2 := M_2^{-1} \left((I - M_2) \begin{pmatrix} B_{11} & 0 \\ 0 & B_{22} \end{pmatrix} + \begin{pmatrix} \frac{B_{12}B_{21}}{2c\xi^k} & -\frac{B_{11}B_{12}}{2c\xi^k} \\ \frac{B_{21}B_{22}}{2c\xi^k} & -\frac{B_{12}B_{21}}{2c\xi^k} \end{pmatrix} - D_t M_2 \right).$$

Our proof requires the third step of diagonalization procedure by the matrix $M_3 = M_3(t,\xi)$:

$$M_3 := \begin{pmatrix} 1 & -\frac{\{B_2\}_{12}}{2c\xi^k + \frac{D_t b_k}{c}} \\ \frac{\{B_2\}_{21}}{2c\xi^k + \frac{D_t b_k}{c}} & 1 \end{pmatrix}.$$

Then the properties $1/(2c\xi^k + D_t b_k / c) \in \mathscr{S}_0^{-1}(2)$ and $B_2 \in \mathscr{S}_{(3p-k)/p}^{-(2p-k)/p}(1)$ guarantee the invertibility of M_3 for large K, and a large λ_0 respectively; thus (3.40) is rewritten as follows:

$$(D_t - \Lambda_2 - B_3)W_3 = 0, \tag{3.41}$$

where $W_3 := M_3^{-1} W_2$ and $B_3 \in \mathscr{S}_{(4p-2k)/p}^{-(3p-2k)/p}(0)$. Here we note the following properties:

$$M_3^{-1} B_2 M_3 + M_3^{-1}[\Lambda_2, M_3] - \begin{pmatrix} \{B_2\}_{11} & 0 \\ 0 & \{B_2\}_{22} \end{pmatrix} \in \mathscr{S}_{\frac{6p-2k}{p}}^{-\frac{5p-2k}{p}}(1),$$

$$\{B_2\}_{11},\ \{B_2\}_{22} \in \mathscr{S}_{\frac{4p-2k}{p}}^{-\frac{3p-2k}{p}}(2), \quad M_3^{-1}(D_t M_3) \in \mathscr{S}_{\frac{4p-k}{p}}^{-\frac{3p-k}{p}}(0).$$

The main contribution of the third step of diagonalization procedure is that the symbol class of remainder part of the equation is improved (with respect to the ξ variable) from $\mathscr{S}_{(3p-k)/p}^{-(2p-k)/p}(1)$ to $\mathscr{S}_{(4p-2k)/p}^{-(3p-2k)/p}(0)$.

For a positive constant T we define the matrix $\Xi = \Xi(t, T, \xi)$ by

$$\Xi(t,T,\xi) := \sqrt{\frac{c(t,\xi)}{c(T,\xi)}}\begin{pmatrix} \exp\Big(i\xi^k \int_t^T c(s,\xi)\,ds\Big) & 0 \\ 0 & \exp\Big(-i\xi^k \int_t^T c(s,\xi)\,ds\Big)\end{pmatrix},$$

$W := \Xi^{-1}W_3$, and $Q := i\Xi^{-1}B_3\Xi$. Then (3.41) is rewritten as follows:

$$\left(\partial_t - \frac{b_k'(t)}{2c(t,\xi)}\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} - Q(t,\xi)\right)W(t,\xi) = 0, \tag{3.42}$$

where $Q \in \mathscr{S}^{-(3p-2k)/p}_{(4p-2k)/p}(0)$. Let $\theta = \theta(t,\xi) \in L_{1,loc}(Z_H)$, and define $\Theta = \Theta(t,\xi)$ and $Y = Y(t,\xi)$ by

$$\Theta(t,\xi) := \exp\left(\int_t^T \theta(s,\xi)ds\right) \quad \text{and} \quad Y := \begin{pmatrix} \Theta^{-1} & 0 \\ 0 & \Theta\end{pmatrix}W.$$

Then the equation (3.42) is rewritten as follows:

$$\left(\partial_t - \left(\frac{b_k'}{2c}+\theta\right)\begin{pmatrix} 1 & 0 \\ 0 & -1\end{pmatrix} - \begin{pmatrix} \{Q\}_{11} & \Theta^{-2}\{Q\}_{12} \\ \Theta^2\{Q\}_{21} & \{Q\}_{22}\end{pmatrix}\right)Y = 0. \tag{3.43}$$

Let us define the Lyapunov functional $S = S(t,\xi)$ and the energy functional $E = E(t,\xi)$ with $Y = {}^t(y_1,y_2)$ by

$$S := -|y_1|^2 + |y_2|^2 \;\text{ and }\; E := |y_1|^2 + |y_2|^2. \tag{3.44}$$

Then there exists a positive constant q_0 such that

$$\begin{aligned}\partial_t S &= -\left(\frac{b_k'}{c}+2\theta\right)E - 2\Re\{Q\}_{11}|y_1|^2 + 2\Re\{Q\}_{22}|y_2|^2 \\ &\quad - 2\Re(\Theta^{-2}\{Q\}_{12}y_1\overline{y}_2) + 2\Re(\Theta^2\{Q\}_{21}\overline{y_1}y_2) \\ &\geq \left(-\frac{b_k'(t)}{c(t,\xi)} - 2\theta(t,\xi) - q_0|\xi|^{-3p+2k}\kappa(t)^{\frac{4p-2k}{p}}\big(\Theta(t,\xi)^2+\Theta(t,\xi)^{-2}\big)\right)E.\end{aligned}$$

Let us denote

$$q(t,\xi) := q_0|\xi|^{-3p+2k}\kappa(t)^{\frac{4p-2k}{p}}$$

and

$$\psi = \psi(t,\xi) := -\frac{b_k'(t)}{c(t,\xi)} - 2\theta(t,\xi) - q(t,\xi)\big(\Theta(t,\xi)^2 + \Theta(t,\xi)^{-2}\big).$$

If $\psi \geq 0$ on $[t_\xi, \tilde{t}_\xi]$ for any fixed ξ, then we have $\partial_t S \geq \psi S$. It follows from a Gronwall type argument that

$$S(t_\xi,\xi)\exp\left(\int_{t_\xi}^{\tilde{t}_\xi} \psi(s,\xi)\,ds\right) \leq S(\tilde{t}_\xi,\xi) \leq E(\tilde{t}_\xi,\xi), \tag{3.45}$$

which is an estimate of regularity loss of order $\exp(\int_{t_\xi}^{\tilde{t}_\xi} \psi(s,\xi)\,ds)$ since $S(t_\xi,\xi) > 0$ if we can additionally guarantee $E(t_\xi,\xi) \leq CS(t_\xi,\xi)$. Indeed, the behavior of this integral is dominated by the integral over $-b_k'/c$ by grace of our special choice of a_{2k} and b_k. Moreover, one can determine the function $\theta(t,\xi)$ such that $\psi \geq 0$. Actually, these properties are contained in the following proposition:

Proposition 3.2. *For any given ξ with large $|\xi|$ we define $N \in \boldsymbol{N}$ by $N := [s_\xi]$. Then, there exist positive constants $\delta > 0$, K in* (3.37), *and $\theta(t,\xi) \in L_{1,loc}(Z_H)$ providing the following properties*:

$$\sup_{(t,\xi)\in Z_H} \left\{ \left| \int_t^T \theta(s,\xi)\,ds \right| \right\} \leq \theta_0 \tag{3.46}$$

and

$$\int_{t_N}^{t_{N-n}} \psi(s,\xi)\,ds \geq \rho(\log\langle\xi\rangle)^\gamma \tag{3.47}$$

for some positive constants ρ and θ_0, where n is defined by (3.18).

If the above statements are valid, then by taking $t_\xi = t_N$ and $\tilde{t}_\xi = t_{N-n}$ we have

$$S(t_\xi,\xi)\exp\big(\rho(\log\langle\xi\rangle)^\gamma\big) \leq S(\tilde{t}_\xi,\xi) \leq E(\tilde{t}_\xi,\xi).$$

Noting the equivalences

$$|y_1(t,\xi)| \simeq \big|D_t v(t,\xi) + \xi^k(b_k(t) + c(t,\xi))v(t,\xi)\big|,$$
$$|y_2(t,\xi)| \simeq \big|D_t v(t,\xi) + \xi^k(b_k(t) - c(t,\xi))v(t,\xi)\big|$$

and

$$E(t,\xi) \simeq |W_0(t,\xi)|^2 \simeq |\xi|^{2p}|v(t,\xi)|^2 + |v_t(t,\xi)|^2$$

in Z_H we see that a suitable choice of the initial data gives

$$|W_0(t_\xi,\xi)| \simeq E(t_\xi,\xi) \le CS(t_\xi,\xi).$$

This yields the estimate (3.35).

We shall introduce the following lemmas for the preparation of the proof of Proposition 3.2.

LEMMA 3.5. *Let us denote* $\nu_j := (\kappa(t_j)|\xi|^{-p})^{(p-k)/p}$. *There exists a positive constant* ρ_0 *independent of* K *such that*

$$\int_{t_j}^{t_{j-1}} \frac{b_k'(s)}{c(s,\xi)}\,ds \le -\rho_0\mu_j^3$$

for any $j = 2,\dots,N$.

PROOF. Let us denote $\nu_{j,l} = (\kappa(t_{j,l})|\xi|^{-p})^{(p-k)/p}$ for $l = 1,2,3$. Noting $b_k' \ge 0$ on $(t_j,t_{j,1})$, $b_k' \le 0$ on $(t_{j,2},t_{j,3})$ and $b_k' \equiv 0$ on $(t_{j,1},t_{j,2}) \cup (t_{j,3},t_{j-1})$ we have

$$\begin{aligned}
\int_{t_j}^{t_{j,1}} \frac{b_k'(s)}{c(s,\xi)}\,ds &= \int_{t_j}^{t_{j,1}} \frac{b_k'(s)ds}{\sqrt{\xi^{2(p-k)} + 2\kappa(s)^{\frac{2(p-k)}{p}} + b_k(s)^2}} \\
&\le \int_{t_j}^{t_{j,1}} \frac{b_k'(s)ds}{\sqrt{\xi^{2(p-k)} + 2\kappa(t_{j,1})^{\frac{2(p-k)}{p}} + b_k(s)^2}} \\
&= \log\left(\frac{b_k(t_{j,1}) + \sqrt{\xi^{2(p-k)} + 2\kappa(t_{j,1})^{\frac{2(p-k)}{p}} + b_k(t_{j,1})^2}}{b_k(t_j) + \sqrt{\xi^{2(p-k)} + 2\kappa(t_{j,1})^{\frac{2(p-k)}{p}} + b_k(t_j)^2}}\right) \\
&= \log\left(\frac{\sqrt{2}\kappa(t_j)^{\frac{p-k}{p}} + \sqrt{\xi^{2(p-k)} + 2\kappa(t_{j,1})^{\frac{2(p-k)}{p}} + 2\kappa(t_j)^{\frac{2(p-k)}{p}}}}{\kappa(t_j)^{\frac{p-k}{p}} + \sqrt{\xi^{2(p-k)} + 2\kappa(t_{j,1})^{\frac{2(p-k)}{p}} + \kappa(t_j)^{\frac{2(p-k)}{p}}}}\right) \\
&= \log\left(\frac{\sqrt{2}\nu_j + \sqrt{1 + 2\nu_{j,1}^2 + 2\nu_j^2}}{\nu_j + \sqrt{1 + 2\nu_{j,1}^2 + \nu_j^2}}\right).
\end{aligned}$$

In the same way we get

$$\int_{t_{j,2}}^{t_{j,3}} \frac{b_k'(s)}{c(s,\xi)}\,ds \le \log\left(\frac{\nu_{j-1}+\sqrt{1+\nu_{j,2}^2+\nu_{j-1}^2}}{\sqrt{2}\nu_j+\sqrt{1+\nu_{j,2}^2+2\nu_j^2}}\right).$$

Here we note that

$$\nu_j \le \nu_N = \left(\frac{\kappa(t_N)}{K\kappa(t_\xi)}\right)^{\frac{p-k}{p}} \le K^{-\frac{p-k}{p}}$$

for any $j \le N$; thus one can suppose that ν_j is small for large K. Therefore, by straightforward calculations we get

$$\begin{aligned}&\left(\sqrt{2}\nu_j+\sqrt{1+2\nu_{j,1}^2+2\nu_j^2}\right)\left(\nu_{j-1}+\sqrt{1+\nu_{j,2}^2+\nu_{j-1}^2}\right)\\ &\quad-\left(\nu_j+\sqrt{1+2\nu_{j,1}^2+\nu_j^2}\right)\left(\sqrt{2}\nu_j+\sqrt{1+\nu_{j,2}^2+2\nu_j}\right)\\ &\quad\le -\frac{\sqrt{2}-1}{2}\nu_{j-1}^2\nu_j+\mathscr{O}(\nu_j^4) \le -\frac{\sqrt{2}-1}{2^{2-\frac{k}{p}}}\nu_j^3+\mathscr{O}(\nu_j^4),\end{aligned}$$

where we used the inequalities

$$\left(\frac{\nu_j}{\nu_{j-1}}\right)^{\frac{p-k}{p}} = \left(\frac{j}{j-1}\right)^{\frac{1}{\gamma+1}} \exp\left(j^{\frac{1}{\gamma+1}}-(j-1)^{\frac{1}{\gamma+1}}\right) \le 2$$

for any large j. Therefore, we obtain

$$\int_{t_j}^{t_{j-1}} \frac{b_k'(s)}{c(s)}\,ds \le \log\left(1-\frac{\sqrt{2}-1}{2^{2-\frac{k}{p}}}\nu_j^3+\mathscr{O}(\nu_j^4)\right) \le -\rho_0\nu_j^3$$

for a positive constant ρ_0. □

LEMMA 3.6. *With a positive constant C_0 we define in Z_H the functions $\eta_j=\eta_j(\xi)$ and $\theta(t,\xi)$ as follows*:

$$\eta_j(\xi) := \frac{\displaystyle\int_{t_j}^{t_{j-1}} \left(\left[\tfrac{b'_k(s)}{c(s,\xi)} \right]_+ + C_0 q(s,\xi) \right) ds}{\displaystyle -\int_{t_j}^{t_{j-1}} \left[\tfrac{b'_k(s)}{c(s,\xi)} \right]_- ds},$$

where $[f]_+ = f$ $(f > 0)$, $= 0$ $(f \le 0)$ *and* $[f]_- = f - [f]_+$. *Then* $\psi(t,\xi) \ge 0$ *in* Z_H *and the estimate* (3.46) *is valid under the following choice of* θ:

$$\theta(t,\xi) := -\frac{1}{2}\left(\eta_j(\xi) \left[\frac{b'_k(t)}{c(t,\xi)} \right]_- + \left[\frac{b'_k(t)}{c(t,\xi)} \right]_+ + C_0 q(t,\xi) \right)$$

for $t \in [t_j, t_{j-1}]$.

Proof. Recalling the proof of Lemma 3.5 we have

$$\eta_j = 1 - \frac{\displaystyle -\int_{t_j}^{t_{j-1}} \tfrac{b'_k(s)}{c(s,\xi)}\, ds - \int_{t_j}^{t_{j-1}} C_0 q(s,\xi)\, ds}{\displaystyle -\int_{t_{j,2}}^{t_{j,3}} \tfrac{b'_k(s)}{c(s,\xi)}\, ds} \le 1 - \left(\rho_0 - q_0 K^{-\frac{k}{p}} \right) \nu_j^2 + \mathscr{O}\left(\nu_j^3 \right),$$

where we used the estimates

$$\int_{t_j}^{t_{j-1}} q(s,\xi)\, ds = q_0 |\xi|^{-3p+2k} \int_{t_j}^{t_{j-1}} \kappa(s)^{\frac{4p-2k}{p}}\, ds \le q_0 \left(\frac{\kappa(t_j)}{|\xi|^p} \right)^{\frac{3p-2k}{p}} \le q_0 K^{-\frac{k}{p}} \nu_j^3.$$

Therefore we have $\eta_j \le 1$ by choosing K sufficiently large. On the other hand, for $t \in [t_j, t_{j-1}]$ we have

$$\left| \int_t^T \theta(s,\xi)\, ds \right| = \left| \int_t^{t_{j-1}} \theta(s,\xi)\, ds \right| \le 1,$$

it follows that $\Theta^2 + \Theta^{-2} \le 2e^2$. Consequently, by choosing $C_0 = 2e^2$ we obtain

$$\begin{aligned} \psi(t,\xi) &\ge -\frac{b'_k(t)}{c(t,\xi)} + \eta_j(\xi) \left[\frac{b'_k(t)}{c(t,\xi)} \right]_- + \left[\frac{b'_k(t)}{c(t,\xi)} \right]_+ + (C_0 - 2e^2) q(t,\xi) \\ &= -\big(1 - \eta_j(\xi)\big) \left[\frac{b'_k(t)}{c(t,\xi)} \right]_- \ge 0. \end{aligned}$$

□

Proof of Proposition 3.2. By the previous arguments it holds

$$\int_{t_N}^{t_{N-n}} \psi(s,\xi)\,ds \ge -\int_{t_N}^{t_{N-n}} \left(\frac{b_k'(s)}{c(s,\xi)} + 2\theta(s,\xi) + C_0 q(s,\xi)\right) ds$$
$$\ge \left(\rho_0 - C_0 q_0 K^{-\frac{k}{p}}\right) \sum_{j=N-n}^{N} \nu_j^3 \ge \rho_1 \sum_{j=N-n}^{N} \nu_j^3$$

for a positive constant ρ_1 by choosing K satisfying $K > (C_0 q_0/\rho_0)^{p/k}$. Then, denoting $r = 3(p-k)/p$ we conclude

$$\sum_{j=N-n}^{N} \nu_j^3 \ge \left(\frac{1}{K\kappa(t_\xi)}\right)^r \int_{N-n-1}^{N} s^{\frac{r\gamma}{\gamma+1}} \exp\left(r s^{\frac{1}{\gamma+1}}\right) ds$$
$$\ge \frac{\gamma+1}{2r}\left(\frac{1}{K\kappa(t_\xi)}\right)^r \left[s^{\frac{r\gamma}{\gamma+1}+\frac{\gamma}{\gamma+1}} \exp\left(r s^{\frac{1}{\gamma+1}}\right)\right]_{N-n-1}^{N}$$
$$\ge \frac{\gamma+1}{4r}\left(\frac{1}{K\kappa(t_\xi)}\right)^r N^{\frac{r\gamma}{\gamma+1}+\frac{\gamma}{\gamma+1}} \left[\exp\left(r s^{\frac{1}{\gamma+1}}\right)\right]_{N-n-1}^{N}$$
$$\ge \frac{\gamma+1}{8r}\left(\frac{1}{K\kappa(t_\xi)}\right)^r N^{\frac{r\gamma}{\gamma+1}+\frac{\gamma}{\gamma+1}} (n+1) N^{-\frac{\gamma}{\gamma+1}} \exp\left(r N^{\frac{1}{\gamma+1}}\right)$$
$$\ge C\left(\frac{\kappa(t_N)}{K\kappa(t_\xi)}\right)^r N^{\frac{\gamma}{\gamma+1}} \ge C\left(\frac{\kappa(t_N)}{K\kappa(t_{N+1})}\right)^r N^{\frac{\gamma}{\gamma+1}}$$
$$\ge C N^{\frac{\gamma}{\gamma+1}} \ge C(\log\langle\xi\rangle)^\gamma.$$

This proves the statement of Proposition 3.2. □

3.3. Does the loss of regularity really appear?

Up to now we have shown for several examples that the Cauchy problem cannot be H^∞ well-posed with the loss of regularity $\exp(C(\log\langle D_x\rangle)^{\gamma_0})$, $\gamma_0 < \gamma$. But our strategies *Floquet' theory* or *interaction of Lyapunov and energy functional* do not answer the question if the γ loss does really appear. Therefore we recall a method to prove counter-examples which originated from the paper [**6**]. Using an instability argument this method was developed in [**2**] to show for hyperbolic Cauchy problems that a loss of regularity really appears. In the following we generalize these ideas to apply them to p-evolution type models.

Let us consider the Cauchy problem

$$D_t^2 u - D_x^{2p} u - a_k(t) D_x^k u = 0, \quad u(0,x) = u_0(x), \quad u_t(0,x) = u_1(x), \tag{3.48}$$

$k \in \{p+1, \dots, 2p-1\}$, with 2π-periodic data u_0, u_1.

All the derivatives $D_t^l a_k$ of the coefficient a_k, $l \in \boldsymbol{N}$, will be estimated by means of the exponents

$$\delta_{lk} = \frac{((l+1)p - k)(1 - \delta_{0k}) + k - p}{k - p}, \quad 0 \le \delta_{0k} \le \frac{2p-k}{p}. \tag{3.49}$$

Comparing with the exponents σ_{lk}, $l = 0, 1, 2$, in Theorem 2.1, we have that δ_{0k} and σ_{0k} can be chosen in the same interval $[0, \frac{2p-k}{p}]$. If we make such a choice $\delta_{0k} = \sigma_{0k}$, then we have

$$\delta_{1k} \le \sigma_{1k}, \ \delta_{2k} = \sigma_{2k}.$$

The inequality for $l = 1$ becomes an equality if and only if $\sigma_{0k} = \frac{2p-k}{p}$, so we have

$$\delta_{lk} = \sigma_{lk}, \ l = 0, 1, 2,$$

in the critical singular case (see Section 2.3.2).

For a 2π-periodic solution $u = u(t,x)$ in the x variable, we introduce the energies

$$\dot{E}_s(u(t)) := \|u(t)\|^2_{\dot{H}^s} + \|u_t(t)\|^2_{\dot{H}^{s-p}}, \ s \in \boldsymbol{R}, \tag{3.50}$$

where $\dot{H}^s$ denotes the homogeneous Sobolev space of exponent s on the torus $\boldsymbol{T} = \boldsymbol{R}/2\pi\boldsymbol{Z}$.

Theorem 3.5. *Let $\omega = \omega(t) : (0,T] \to (0,+\infty)$ be a given continuous decreasing function with $\lim_{t\to 0+} \omega(t) = +\infty$. There exists a sequence $\{a_{km} = a_{km}(t)\}$, $m \in \boldsymbol{N}$, of real coefficients from $C^\infty(0,T]$ such that*

$$\sup_m |a_{km}(t)| \lesssim \left(\frac{1}{t} \left(\log \frac{1}{t} \right)^\gamma \right)^{\delta_{0k}}, \tag{3.51}$$

$$\sup_m \left| D_t^l a_{km}(t) \right| \lesssim \omega(t) \left(\frac{1}{t} \left(\log \frac{1}{t} \right)^\gamma \right)^{\delta_{lk}} \quad \textit{for all } l \ge 1, \tag{3.52}$$

where the exponents δ_{lk} are given by (3.49), and a sequence $\{u_m = u_m(t,x)\}$ of solutions from $C^\infty([0,T] \times \boldsymbol{T})$ of

$$D_t^2 u_m - D_x^{2p} u_m - a_{km}(t) D_x^k u_m = 0$$

such that

$$\sup_m \dot{E}_p(u_m(0)) \leq 2, \tag{3.53}$$

but

$$\sup_m \dot{E}_p\big(\exp\big(-C(\log\langle D_x\rangle)^\gamma\big)u_m(t)\big) = +\infty \tag{3.54}$$

for all $t \in (0, T]$ *and all* $C > 0$, *where the energies* $\dot{E}_s$ *are given by* (3.50).

Proof. The proof is inspired by the constructions of counter-examples in [**6**], [**2**], [**3**], [**5**]. For $\varepsilon > 0$, let us define

$$\alpha_\varepsilon(t) := 1 - 4\varepsilon\varphi(t)\sin(2t) - 2\varepsilon\varphi'(t)\sin^2 t - 4\varepsilon^2(t)\varphi^2(t)\sin^4 t, \tag{3.55}$$

$$w_\varepsilon(t) := \sin t \exp\left(2\varepsilon \int_0^t \varphi(\tau)\sin^2\tau \, d\tau\right), \tag{3.56}$$

where φ is a real non-negative 2π-periodic C^∞ function on $\boldsymbol{R}$ which is identically 0 in a neighborhood of $t = 0$ and such that

$$\int_0^{2\pi} \varphi(t)\sin^2 t \, dt = \pi.$$

The functions α_ε and w_ε belong to $C^\infty(\boldsymbol{R})$ and satisfy

$$w_\varepsilon''(t) + \alpha_\varepsilon(t) w_\varepsilon(t) = 0, \quad w_\varepsilon(0) = 0, \ w_\varepsilon'(0) = 1. \tag{3.57}$$

We consider then three monotone sequences $\{\varepsilon_m\}$, $\{\nu_m\}$, $\{\varrho_m\}$ of positive real numbers such that

$$\varepsilon_m \to 0, \quad \nu_m \to +\infty, \quad \varrho_m \to 0, \tag{3.58}$$

$$\nu_m, \left(\frac{8\pi\nu_m}{\varrho_m}\right)^{\frac{1}{p}} \in \boldsymbol{N} \ \text{ for all } m \in \boldsymbol{N}, \tag{3.59}$$

and, after setting

$$t_m := \frac{3\varrho_m}{4},\ I_m := \left[t_m - \frac{\varrho_m}{4}, t_m + \frac{\varrho_m}{4}\right],$$

we define

$$a_{0m}(t) := \begin{cases} \alpha_{\varepsilon_m}\left(\dfrac{8\pi\nu_m(t-t_m)}{\varrho_m}\right) & \text{for } t \in I_m, \\ 1 & \text{for } t \in \boldsymbol{R} \setminus I_m. \end{cases} \tag{3.60}$$

Since a_{0m} is identically equal to 1 in a neighborhood of the boundary of I_m, we have $a_{0m} \in C^\infty(\boldsymbol{R})$. Moreover, from $\varepsilon_m \to 0$ we have

$$\frac{1}{2} \le a_{0m}(t) \le \frac{3}{2} \tag{3.61}$$

for all m sufficiently large.

The sequence of coefficients $\{a_{km}\}$ is then defined by

$$a_{km}(t) := (a_{0m}(t) - 1)h_m^{2p-k}, \tag{3.62}$$

where the h_m's are the positive integers given by

$$h_m := \left(\frac{8\pi\nu_m}{\varrho_m}\right)^{\frac{1}{p}}. \tag{3.63}$$

From (3.55), (3.60), (3.62) and (3.63), we make the conditions (3.51) and (3.52) to be satisfied by fixing the parameters in such a way that

$$\frac{\nu_m}{\varrho_m} \sim \log(\omega(\varrho_m))\left(\frac{1}{\varrho_m}\left(\log\frac{1}{\varrho_m}\right)^\gamma\right)^{\frac{(1-\delta_{0k})p}{k-p}}, \tag{3.64}$$

$$\varepsilon_m \sim (\log(\omega(\varrho_m)))^{-\frac{2p-k}{p}}\left(\frac{1}{\varrho_m}\left(\log\frac{1}{\varrho_m}\right)^\gamma\right)^{\frac{k-p-(1-\delta_{0k})p}{k-p}}. \tag{3.65}$$

Then, we look for a sequence $\{u_m\}$ of solutions of the type

$$u_m(t,x) = v_m(t)e^{ih_m x}, \tag{3.66}$$

where h_m is still the integer defined by (3.63). We have

$$D_t^2 u_m - D_x^{2p} u_m - a_{km}(t) D_x^k u_m = 0$$

if and only if

$$v_m''(t) + h_m^{2p} v(t) + h_m^k a_{km}(t) v(t) = 0,$$

that is, from (3.62), if and only if

$$v_m''(t) + h_m^{2p} a_{0m}(t) v(t) = 0. \tag{3.67}$$

If we take the solution v_m of (3.67) with initial values

$$v_m(t_m) = 0,\ v_m'(t_m) = 1,$$

then, from (3.57), (3.60) and (3.63), we have

$$v_m(t) = \frac{\varrho_m}{8\pi\nu_m} w_{\varepsilon_m}\left(\frac{8\pi\nu_m(t - t_m)}{\varrho_m}\right) \quad \text{for } t \in I_m. \tag{3.68}$$

In particular, by (3.56), we have

$$v_m\left(t_m - \frac{\varrho_m}{4}\right) = 0,\ v_m'\left(t_m - \frac{\varrho_m}{4}\right) = e^{-2\pi\varepsilon_m\nu_m}, \tag{3.69}$$

$$v_m\left(t_m + \frac{\varrho_m}{4}\right) = 0,\ v_m'\left(t_m + \frac{\varrho_m}{4}\right) = e^{2\pi\varepsilon_m\nu_m}. \tag{3.70}$$

Now, let us define

$$E_m(t) := |v_m'(t)|^2 + h_m^{2p} a_{0m}(t) |v_m(t)|^2. \tag{3.71}$$

From (3.67) and (3.60), we have

$$E_m'(t) = 0 \quad \text{for } t \in \boldsymbol{R} \setminus I_m,$$

so

$$E_m(t) = e^{-2\pi\varepsilon_m\nu_m}, \quad \text{for } t \le \frac{\varrho_m}{2}, \tag{3.72}$$

$$E_m(t) = e^{2\pi\varepsilon_m\nu_m}, \quad \text{for } t \ge \varrho_m, \tag{3.73}$$

by taking also (3.69) and (3.70) into account.

We get the first desired inequality (3.53) for u_m from (3.66), (3.72) and (3.61).

By (3.64) and (3.65), we have

$$\varepsilon_m \nu_m \sim \big(\log(\omega(\varrho_m))\big)^{\frac{k-p}{p}} \left(\log \frac{1}{\varrho_m}\right)^\gamma, \tag{3.74}$$

so, from (3.63) and (3.64), we obtain

$$\lim_{m\to\infty} \varepsilon_m \nu_m - C(\log h_m)^\gamma = +\infty \tag{3.75}$$

for any $C > 0$.

We get also the last desired estimate (3.54) for u_m from (3.75) by taking (3.66), (3.73) and (3.61) into account. □

Remark 3.2. If we take instead of the function $\omega = \omega(t)$ a positive constant function $\omega(\varrho) = C_0$ in Theorem 3.5, then the coefficients a_{km} satisfy all the assumptions of Theorem 2.1. All the choices of the parameters ϱ_m, ε_m, ν_m in the proof are still possible provided that $\delta_{0k} < \frac{2p-k}{p}$ (now we need this condition in (3.64) and (3.65) in order to have still $\nu_m \to +\infty$, $\varepsilon_m \to 0$). So, now (3.73) and (3.74) with $\omega(\varrho) = C_0$ ($C_0 > 1$) show that in this case the Cauchy problem (3.48) is H^∞ well-posed, where the loss of derivatives $\exp(C(\log\langle D_x\rangle)^\gamma)$ really occurs. Thus this loss cannot be avoided in Theorem 2.1, at least in the non-critical case $\delta_{0k} < \frac{2p-k}{p}$. This proves the optimality of the classification of loss of regularity for p-evolution operators (for $p = 1$ see [**2**], [**7**]).

4. Concluding remarks.

Remark 4.1. An interesting question is that for the proof of optimality of conditions in the non-critical singular case or in the bounded case. Here the question if the loss of regularity really occurs is of special interest.

Remark 4.2. We formulated our main result for C^2 coefficients a_α, $|\alpha| = p+1, \dots, 2p$ and for C^2 coefficients b_α, $|\alpha| = 1, \dots, p-1$. Special regularization methods introduced in the papers [**10**] and [**12**] might be open an opportunity to weaken these regularity conditions in t to a "bit more regular ones" than C^1.

Acknowledgements. Most of research was done during the stay of the second and third author at the Mathematical Research Institute Oberwolfach in September 2005. Both authors thank the staff of Institute for the wonderful hospitality. The authors thank the referee for valuable suggestions.

References

[1] A. Ascanelli and M. Cicognani, Schrödinger equations of higher order, Math. Nachr., **280** (2007), 717–727.

[2] M. Cicognani and F. Colombini, Modulus of continuity of the coefficients and loss of derivatives in the strictly hyperbolic Cauchy problem, J. Differential Equations, **221** (2006), 143–157.

[3] M. Cicognani and F. Colombini, Sharp regularity of the coefficients in the Cauchy problem for a class of evolution equations, Diff. Int. Equ., **16** (2003), 1321–1344.

[4] F. Colombini, D. Del Santo and T. Kinoshita, Well-posedness of the Cauchy problem for a hyperbolic equation with non-Lipschitz coefficients, Ann. Scuola Norm. Sup. Pisa, **1** (2002), 327–358.

[5] F. Colombini, D. Del Santo and M. Reissig, On the optimal regularity of coefficients in hyperbolic Cauchy problems, Bull. Sci. Math., **127** (2003), 328–347.

[6] F. Colombini and S. Spagnolo, Some examples of hyperbolic equations without local solvability, Ann. Scient. Éc. Norm. Sup., 4^e Série, **22** (1989), 657–698.

[7] F. Hirosawa, Loss of regularity for second order hyperbolic equations with singular coefficients, Osaka J. Math., **42** (2005), 767–790.

[8] F. Hirosawa, Loss of regularity for the solutions to hyperbolic equations with non-regular coefficients – an application to Kirchhoff equation, Math. Methods Appl. Sci., **26** (2003), 783–799.

[9] F. Hirosawa and M. Reissig, About the optimality of oscillations in non-Lipschitz coefficients for strictly hyperbolic equations, Ann. Scuola Norm. Sup. Pisa Cl. Sci., **3** (2004), 589–608.

[10] F. Hirosawa and M. Reissig, Well-posedness in Sobolev spaces for second order strictly hyperbolic equations with non-differentiable oscillating coefficients, Ann. Global Anal. Geom., **25** (2004), 99–119.

[11] V. Ya. Ivrii and V. M. Petkov, Necessary conditions for the Cauchy problem for non-strictly hyperbolic equations to be well-posed (in Russian), Uspheki Mat. Nauk, **29** (1974), 3–70. English translation: Russian Math. Surveys, **29** (1974), 1–70.

[12] T. Kinoshita and M. Reissig, About the loss of derivatives for strictly hyperbolic equations with non-Lipschitz coefficients, Adv. Differential Equations, **10** (2005), 191–222.

[13] W. Magnus and S. Winkler, Hill's equation, Interscience Tracts in Pure and Applied Mathematics, **20**, Interscience, New York-London-Sydney, 1966.

[14] K. Yagdjian, The Cauchy problem for hyperbolic operators, Multiple characteristics, micro-local approach, Akademie-Verlag, Berlin, 1997.

[15] K. Yagdjian, Global existence in the Cauchy problem for nonlinear wave equations with variable speed of propagation, (eds. M. Reissig and B.-W. Schulze), New trends in the theory of hyperbolic equations, Operator Theory, Advances and Applications, **159**, Birkhäuser, 2005.

Massimo Cicognani
Dipartimento di Matematica
Universita di Bologna
40127 Bologna, Italy

Fumihiko Hirosawa
Department of Mathematical Sciences
Faculty of Science
Yamaguchi University
Yamaguchi 753-8512, Japan

Michael Reissig
Fakultät für Mathematik und Informatik
TU Bergakademie Freiberg
09596 Freiberg, Germany