The skein index for link invariants

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Abstract. We introduce the skein index, which is an integer-valued index of a link invariant. It is used to compare link invariants and to find a skein relation. We give the complete list for link invariants of skein index less than or equal to two, and discuss the skein indices of operator invariants, and in particular quantum invariants and quandle coloring numbers. By determining the skein index of the JKSS invariant, we construct a new skein relation and give a formula for the invariant of a closed 2-braid.

1. Introduction.

We introduce the skein index for link invariants. In general, we may define skein indices for invariants of virtual links, spatial graphs, and so on. The skein index is an integer-valued index of a link invariant, which is used to compare link invariants and to find a skein relation.

A skein relation of a link invariant is a nontrivial linear relation among the values of the invariants of links which are identical except in the neighborhood of a point. Skein relations are helpful for evaluating link invariants and investigating their properties. In particular, Alexander [1], Jones [9] and HOMFLY [4], [18] polynomials can be characterized by their skein relations.

A link invariant defines an equivalence relation on the set of all links by its value. By determining the equivalence class, we may see how powerful the invariant is. Unfortunately, it is not easy to determine the equivalence class. The skein index of a link invariant is defined by the minimum n which satisfies that there exists a skein relation of the invariant among any $n + 1$ oriented $(2, 2)$-tangles. Hence it is an index which grasps the complexity of the invariant. We give the complete list for link invariants of skein index less than or equal to two.

Since the discovery of the Jones polynomial, many link invariants have been defined, which include so called quantum invariants. The skein index of a quantum

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invariant is less than or equal to the dimension of the space of intertwiners, which are equivariant with respect to the actions of the quantum algebra. In general, the skein index of an operator invariant is bounded by the dimension of a certain vector space. A quandle cocycle invariant [2], and then the number of quandle colorings of a link diagram, are operator invariants [7]. So we may also compare quandles by the skein index of the quandle coloring number.

By the definition, we may use the skein index to find a skein relation. We determine the skein index of the JKSS invariant, and show its applications. Virtual knot theory is an extension of classical knot theory, which was introduced by Kauffman [13] (see also [14]). The JKSS invariant is a two-variable polynomial invariant for oriented virtual links, which was defined by Sawollek [19] based on an invariant for oriented links in a thickened surface defined by Jaeger, Kauffman and Saleur [8]. It is equivalent to the 0th virtual Alexander polynomial defined by Silver and Williams [20]. An R-matrix for this invariant was given by Kauffman and Radford [15], where an R-matrix is a solution of the Yang-Baxter equation.

The rest of the paper is organized as follows. In Section 2, we introduce the skein index for link invariants. In Section 3, we give the complete list for link invariants of skein index less than or equal to two. In Section 4, we discuss the skein indices of operator invariants, and in particular quantum invariants and quandle coloring numbers. In Section 5, we determine the skein index of the JKSS invariant. In Section 6, we introduce a new skein relation of the JKSS invariant by using the skein index, and give a formula for the invariant of a virtual closed 2-braid.

2.   The skein index for link invariants.

A link invariant is a map from the set of all links to some set of mathematical objects such that its value does not change under isotopies of S^3. In this paper, we assume that the set of mathematical objects is a field of characteristic 0.

We denote by $\mathscr{T}$ the set of all oriented tangles of the (2,2)-tangle form as depicted in Figure 1. Let $T(0) \in \mathscr{T}$ be the trivial tangle. Let $T(1), T(-1) \in \mathscr{T}$ be the positive and negative crossings, respectively. For tangles T_1, T_2, we denote by $T_1 \circ T_2$ the tangle obtained by stacking T_1 on T_2 (see Figure 2). For an integer $n > 1$, set

$$T(n) := \underbrace{T(1) \circ \cdots \circ T(1)}_{n}, \quad T(-n) := \underbrace{T(-1) \circ \cdots \circ T(-1)}_{n}.$$
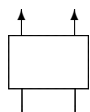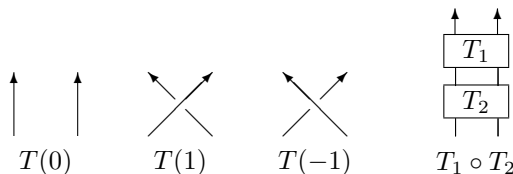
Figure 1.

$$T(0) \qquad T(1) \qquad T(-1) \qquad T_1 \circ T_2$$

Figure 2.

Let F be a field. Let $f(L)$ be a F-valued invariant of an oriented link L. For given scalars $a_1, \ldots, a_n \in F$ and tangles $T_1, \ldots, T_n \in \mathscr{T}$, the relation

$$a_1 f(T_1) + \cdots + a_n f(T_n) = 0 \tag{1}$$

means that the equality

$$a_1 f(L(T_1)) + \cdots + a_n f(L(T_n)) = 0$$

holds for any links $L(T_1), \ldots, L(T_n)$ which are identical except in the neighborhood of a point where they are the tangles $T_1, \ldots, T_n$. When the relation (1) holds for some nontrivial coefficients $(a_1, \ldots, a_n) \neq (0, \ldots, 0)$, we call it the skein relation of the invariant f among the tangles $T_1, \ldots, T_n$. Let $P(L)$ be the HOMFLY polynomial [**4**], [**18**] of an oriented link L, which is a Laurent polynomial in variables l, m assigned to the link L. The HOMFLY polynomial satisfies the relation

$$l P(T(1)) + l^{-1} P(T(-1)) + m P(T(0)) = 0,$$

which is an example of a skein relation.

The skein index $s(f)$ of a link invariant f is the minimum n which satisfies that, for any $n + 1$ tangles $T_1, \ldots, T_{n+1} \in \mathscr{T}$, there exist nontrivial coefficients $(a_1, \ldots, a_{n+1}) \neq (0, \ldots, 0)$ such that

$$a_1 f(T_1) + \cdots + a_{n+1} f(T_{n+1}) = 0$$

holds. If no such n exists, we say that the skein index of f is ∞. In the case of the HOMFLY polynomial, for any tangles T_1, T_2, T_3, there exist coefficients

$(a_1, a_2, a_3) \neq (0, 0, 0)$ such that

$$a_1 P(T_1) + a_2 P(T_2) + a_3 P(T_3) = 0,$$

which implies that $s(P) \leq 2$. Similarly, we may define skein indices for invariants of virtual links, spatial graphs, and so on.

3.   Link invariants of skein index less than or equal to two.

Let $x_1, \ldots, x_n$ be variables. Put $F_1 = \boldsymbol{C}(x_1, \ldots, x_n)$. Let F_2 be a field. A map $g : F_1 \to F_2$ is a substitution map if there exist $\alpha_1, \ldots, \alpha_n \in F_2$ such that $g(X(x_1, \ldots, x_n)) = X(\alpha_1, \ldots, \alpha_n)$ for any $X(x_1, \ldots, x_n) \in F_1$. For a F_i-valued invariant f_i $(i = 1, 2)$, f_2 is a specific evaluation of f_1 if there exists a substitution map $g : F_1 \to F_2$ such that $f_2 = g \circ f_1$. For example, the one-variable polynomial invariants t and $t(-1)^{\#L}$ are specific evaluations of the two-variable polynomial invariant $s + t(-1)^{\#L}$, where $\#L$ is the number of components of a link L. We denote by S_f the set of all specific evaluations of a link invariant f. We note that $f \in S_f$. Since $f' \in S_f$ satisfies any skein relation of f, we have $s(f') \leq s(f)$.

THEOREM 1.   *Link invariants of skein index less than or equal to two are classified as follows.*

1. *An invariant f is of skein index 0 if and only if $f = 0$.*
2. *An invariant f is of skein index 1 if and only if $f \neq 0$ is a specific evaluation of one of the invariants*

$$t, \quad t(-1)^{\#L}.$$

3. *An invariant f is of skein index 2 if and only if $f \notin S_t \bigcup S_{t(-1)^{\#L}}$ is a specific evaluation of one of the invariants*

$$st^{\#L-1}, \quad s + t(-1)^{\#L}, \quad tP(L).$$

We prepare two lemmas for a proof of Theorem 1.

LEMMA 2.   *Let $f \neq 0$ be a link invariant which satisfies the skein relation*

$$af(T(1)) + bf(T(-1)) = 0.$$

Then we have $a + b = 0$ and $f(L) = c_{\#L}$, where $c_{\#L}$ is an element of the set $\{c_1, c_2, \ldots\}$ which depends only on the number of components of a link L.

PROOF.   By the skein relation, we have

$$af\big(\,S\!\!>\!\!<\,\big) + bf\big(\,S\!\!><\!\!\,\big) = 0.$$

Since

$$f\big(\,S\!\!>\!\!<\,\big) = f\big(\,S\!\!><\!\!\,\big) \neq 0$$

for some tangle S, we have $a + b = 0$. By the skein relation $f(T(1)) = f(T(-1))$, we have $f(L) = c_{\#L}$. □

LEMMA 3.   *Let $f \neq 0$ be a link invariant which satisfies the skein relation*

$$af(T(1)) + bf(T(0)) = 0.$$

Then $f \in S_t \bigcup S_{t(-1)^{\#L}}$.

PROOF.   By the skein relation, we have

$$a^2 f(T(1)) - b^2 f(T(-1)) = -abf(T(0)) + abf(T(0)) = 0.$$

By Lemma 2, we have $a^2 - b^2 = 0$ and $f(L) = c_{\#L}$. Then we have the skein relation

$$f(T(1)) - f(T(0)) = 0 \quad \text{or} \quad f(T(1)) + f(T(0)) = 0.$$

The first skein relation implies that $c_n - c_{n+1} = 0$ for any n, and then $f \in S_t$. The second skein relation implies that $c_n + c_{n+1} = 0$ for any n, and then $f \in S_{t(-1)^{\#L}}$. □

Let $T \cup \bigcirc$ be the tangle consisting of a tangle T and an unlinked trivial knot $\bigcirc$. We denote the tangle $T \cup \underbrace{\bigcirc \cup \cdots \cup \bigcirc}_{n}$ by $T \cup \bigcirc^n$.

PROOF OF THEOREM 1.   Let f be a link invariant of skein index 0. By the skein relation

$$af(T(1)) = 0,$$

we have

$$af\big(\,T\!\!>\!\!<\,\big) = 0$$

for any tangle T, which implies that $f = 0$. Conversely, $f = 0$ is of skein index 0, since we have the skein relation

$$f(T) = 0,$$

for any tangle $T \in \mathscr{T}$.

Let f be a link invariant of skein index 1. By Lemma 3, $f \in S_t \bigcup S_{t(-1)^{\#L}}$. Conversely, the invariant $f(L) = t$ is of skein index 1, since we have the skein relation

$$f(T_1) - f(T_2) = 0,$$

for any tangles $T_1, T_2 \in \mathscr{T}$. We show that the invariant $f(L) = t(-1)^{\#L}$ is of skein index 1. Let T_1, T_2 be tangles in $\mathscr{T}$. By the skein relation $f(T(1)) = f(T(-1))$, we have

$$f(T_1) = f(T(i_1) \cup \bigcirc^{n_1}),$$
$$f(T_2) = f(T(i_2) \cup \bigcirc^{n_2}),$$

for some $i_1, i_2 \in \{0, 1\}$ and $n_1, n_2 \in \mathbf{Z}_{\geq 0}$. Then we have the skein relation

$$f(T_1) - (-1)^{n_1 - n_2 + i_1 - i_2} f(T_2) = 0.$$

Let f be a link invariant of skein index 2. Then we have the skein relations

$$a_1 f(T(1)) + a_2 f(T(-1)) + a_3 f(T(0)) = 0,$$
$$b_1 f(T(1)) + b_2 f(T(1) \cup \bigcirc) + b_3 f(T(0)) = 0.$$

By Lemma 3, if $a_1 = 0$ or $a_2 = 0$, then $s(f) < 2$. Thus $a_1, a_2 \neq 0$. When $a_3 \neq 0$, we have $f \in S_{tP(L)}$ [4], [18]. Suppose that $a_3 = 0$. By Lemma 2, we have $f(L) = c_{\#L}$. By the relation

$$b_1 f(\uparrow) + b_2 f(\uparrow \cup \bigcirc) + b_3 f(\uparrow \cup \bigcirc) = 0,$$

we have

$$b_1 = 0 \qquad\qquad \text{if } b_2 + b_3 = 0, \tag{2}$$

$$c_{n+1} = -\frac{b_1}{b_2 + b_3} c_n \quad \text{otherwise.} \tag{3}$$

The equality (3) implies that $f \in S_{st\#L-1}$. Suppose $b_2 + b_3 = 0$, and then $b_1 = 0$. Since the skein relation

$$f(T(1) \cup \bigcirc) - f(T(0)) = 0$$

is equivalent to the skein relation

$$f(T(0) \cup \bigcirc) - f(T(-1)) = 0,$$

we have the relation

$$f(\uparrow \cup \bigcirc^2) - f(\uparrow) = 0,$$

which implies that $c_{n+2} - c_n = 0$ for any n, and then $f \in S_{s+t(-1)\#L}$.

Conversely, the invariants $f(L) = st^{\#L-1}$, $f(L) = s + t(-1)^{\#L}$, and $f(L) = tP(L)$ are of skein index 2: Let T_1, T_2, T_3 be tangles in $\mathcal{T}$. If $f(L) = st^{\#L-1}$ or $s + t(-1)^{\#L}$, then we have the skein relation $f(T(1)) = f(T(-1))$. And then, we have

$$f(T_1) = f(T(i_1) \cup \bigcirc^{n_1}),$$
$$f(T_2) = f(T(i_2) \cup \bigcirc^{n_2}),$$
$$f(T_3) = f(T(i_3) \cup \bigcirc^{n_3}),$$

for some $i_1, i_2, i_3 \in \{0, 1\}$ and $n_1, n_2, n_3 \in \mathbf{Z}_{\geq 0}$. We show that $f(L) = st^{\#L-1}$ is of skein index 2. We may assume that $i_1 = i_2$ and $n_1 \geq n_2$. Then we have the skein relation

$$f(T_1) - t^{n_1-n_2} f(T_2) = 0.$$

We show that $f(L) = s + t(-1)^{\#L}$ is of skein index 2. We may assume that $i_1 + n_1 \equiv i_2 + n_2 \pmod{2}$. Then we have the skein relation

$$f(T_1) - f(T_2) = 0.$$

We show that $f(L) = tP(L)$ is of skein index 2. Since we may reduce the number of crossings and that of components by using the defining skein relation

$$lf(T(1)) + l^{-1}f(T(-1)) + mf(T(0)) = 0,$$

we have

$$f(T_1) = \alpha_1 f(T(0)) + \alpha_4 f(T(1)),$$
$$f(T_2) = \alpha_2 f(T(0)) + \alpha_5 f(T(1)),$$
$$f(T_3) = \alpha_3 f(T(0)) + \alpha_6 f(T(1)),$$

for some coefficients $\alpha_1, \ldots, \alpha_6$. Then we have a skein relation among the tangles T_1, T_2, T_3. $\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad\qquad$ □
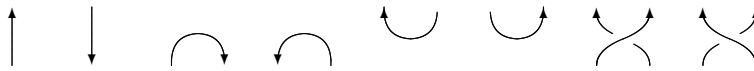

Figure 3.

4.   The skein index of an operator invariant.

We recall the definition of an operator invariant. Any oriented tangle diagram can be expressed up to isotopy as a diagram composed from the elementary oriented tangle diagrams shown in Figure 3. Furthermore any oriented tangle diagram can be expressed up to isotopy as an oriented sliced tangle diagram which is such a diagram sliced by horizontal lines such that each domain between adjacent horizontal lines has either a single crossing or a single critical point.

Let V be a finite dimensional vector space over a field F, and let V^* be the dual vector space. We prepare appropriate linear maps

$$R : V \otimes V \to V \otimes V, \tag{4}$$
$$\Omega_+ : V \otimes V^* \to F, \tag{5}$$
$$\Omega_- : V^* \otimes V \to F, \tag{6}$$
$$\mho_+ : F \to V \otimes V^*, \tag{7}$$
$$\mho_- : F \to V^* \otimes V. \tag{8}$$

For example, R is a solution of the Yang-Baxter equation:

$$(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V) = (\mathrm{id}_V \otimes R)(R \otimes \mathrm{id}_V)(\mathrm{id}_V \otimes R),$$

for the invariance under the Reidemeister move R3. We associate these linear maps to the elementary oriented tangle diagrams as described in Figure 4. Corresponding to any oriented tangle diagram D, we may then obtain a linear map

$[D]$ as the composition of tensor products of copies of the linear maps associated to the elementary oriented tangle diagrams within D. For example,

$$\left[\vcenter{\hbox{\includegraphics{tangle}}} \right] = (\mathrm{id}_V \otimes \Omega_+)(R \otimes \mathrm{id}_{V^*})(\mathrm{id}_V \otimes \mho_+).$$

$V \otimes V \qquad\qquad V \otimes V \qquad\qquad V \qquad\qquad V^*$

$\uparrow R \qquad\qquad \uparrow R^{-1} \qquad\qquad \uparrow \mathrm{id}_V \qquad\qquad \uparrow \mathrm{id}_{V^*}$

$V \otimes V \qquad\qquad V \otimes V \qquad\qquad V \qquad\qquad V^*$

$F \qquad\qquad F \qquad\qquad V \otimes V^* \qquad\qquad V^* \otimes V$

$\uparrow \Omega_+ \qquad\qquad \uparrow \Omega_- \qquad\qquad \uparrow \mho_+ \qquad\qquad \uparrow \mho_-$

$V \otimes V^* \qquad\qquad V^* \otimes V \qquad\qquad F \qquad\qquad F$
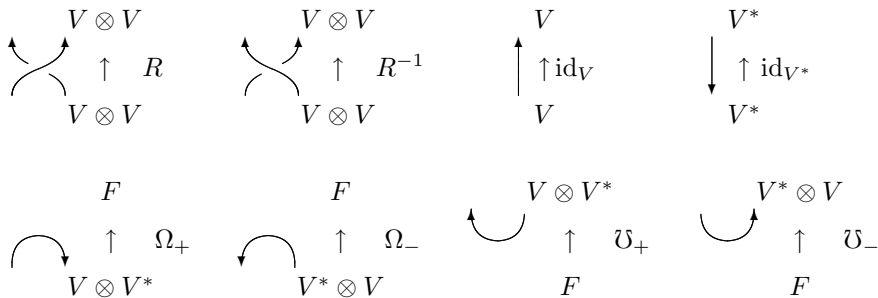
Figure 4.

For an oriented tangle T represented by D, we define $[T] := [D]$, where we note that, the linear map $[D]$ is invariant under the Reidemeister moves because of the "appropriate" linear maps. For an oriented link L, the linear map $[L] : F \to F$ is represented by a scalar. An operator invariant of L is defined by this scalar. For the details of operator invariants we refer the reader to [**17**].

Let f be an operator invariant. We denote by $[\mathscr{T}]$ the vector space spanned by $\{[D] \mid T \in \mathscr{T}\}$. Put $n := \dim[\mathscr{T}]$. Then, for any tangles $T_1, \ldots, T_{n+1} \in \mathscr{T}$, there exist nontrivial coefficients $(a_1, \ldots, a_{n+1}) \neq (0, \ldots, 0)$ such that $a_1[T_1] + \cdots + a_{n+1}[T_{n+1}] = 0$. This linear relation implies the skein relation $a_1 f(T_1) + \cdots + a_{n+1} f(T_{n+1}) = 0$. Then we have

$$s(f) \leq \dim[\mathscr{T}] \leq \dim \mathrm{End}(V \otimes V).$$

Let $B : \mathscr{T} \times \mathscr{T} \to F$ be the bilinear form defined by

$$B(X, Y) = \Omega_+ \circ (\mathrm{id}_V \otimes \Omega_+ \otimes \mathrm{id}_{V^*}) \circ ((X \circ Y) \otimes \mathrm{id}_{V^* \otimes V^*}) \circ (\mathrm{id}_V \otimes \mho_- \otimes \mathrm{id}_{V^*}) \circ \mho_-.$$

We show that, if B is nondegenerate, then

$$s(f) = \dim[\mathscr{T}].$$

Let $\{[T_1], \ldots, [T_n]\}$ be a basis of $[\mathscr{T}]$, where $T_i \in \mathscr{T}$. We suppose that $a_1 f(T_1) + \cdots + a_n f(T_n) = 0$, which implies that $a_1 B([T_1], [T_i]) + \cdots + a_n B([T_n], [T_i]) = 0$ for $i = 1, \ldots, n$. Since B is nondegenerate, we have $a_1 = \cdots = a_n = 0$. Thus the relation $a_1 f(T_1) + \cdots + a_n f(T_n) = 0$ is not a skein relation, which implies that

$$s(f) \geq \dim[\mathscr{T}].$$

We remark that quantum invariants and quandle cocycle invariants are operator invariants.

We focus on the skein index of a quantum invariant. We note that an irreducible representation λ of the quantum algebra $U_q(\mathfrak{sl}_n)$ is represented by a Young diagram:

$$\lambda = (\lambda_1, \ldots, \lambda_{n-1}), \quad (\lambda_1 \geq \cdots \geq \lambda_{n-1} \geq 0).$$

THEOREM 4.   *Let f_λ be the quantum invariant derived from the representation λ of the quantum algebra $U_q(\mathfrak{g})$: V_λ is the $U_q(\mathfrak{g})$-module.*

(a) *Let n_λ^ν be the integer determined by the irreducible decomposition*

$$V_\lambda \otimes V_\lambda \cong \bigoplus_\nu (V_\nu)^{\oplus n_\lambda^\nu}.$$

Then we have

$$s(f_\lambda) \leq \sum_\nu (n_\lambda^\nu)^2.$$

(b) *Set $\mathfrak{g} = \mathfrak{sl}_2$, $\lambda = (n)$. The invariant f_λ is the colored Jones polynomial. Then we have*

$$s(f_\lambda) \leq n + 1.$$

(c) *Set $\mathfrak{g} = \mathfrak{sl}_3$, $\lambda = (m + n, n)$. Then we have*

$$s(f_\lambda) \leq \begin{cases} d(m, n) & \text{if } m \geq n, \\ d(n, m) & \text{otherwise,} \end{cases}$$

where

$$d(m,n) = (n+1)^2(n^2+2n+3)/3 + (m-n)(n+1)(2n^2+4n+3)/3.$$

For the representation $\lambda = (1,0,\dots,0)$ of the quantum algebra $U_q(\mathfrak{sl}_n)$, we have

$$V_\lambda \otimes V_\lambda \cong V_{(2,0,\dots,0)} \oplus V_{(0,0,\dots,0)}.$$

By Theorem 4 (a), we have $s(f_\lambda) \leq 2$, where we remark that the invariant f_λ is the HOMFLY polynomial. In Section 3, we showed that the skein index of the HOMFLY polynomial is 2.

PROOF.

(a) Let $\mathrm{End}_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda) \subset \mathrm{End}(V_\lambda \otimes V_\lambda)$ be the space of intertwiners, which are equivariant with respect to the actions of the quantum algebra $U_q(\mathfrak{g})$. For the details of intertwiners we refer the reader to [**12**], [**17**]. Schur's lemma implies that

$$\dim \mathrm{End}_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda) = \sum_\nu (n_\lambda^\nu)^2.$$

Since $[D] \in \mathrm{End}_{U_q(\mathfrak{g})}(V_\lambda \otimes V_\lambda)$ for a tangle diagram D, we have

$$s(f_\lambda) \leq \sum_\nu (n_\lambda^\nu)^2.$$

(b) By the Clebsch-Gordan formula

$$V_{(n)} \otimes V_{(n)} \cong V_{(2n)} \oplus V_{(2n-2)} \oplus \cdots \oplus V_{(0)},$$

we have

$$s(f_\lambda) \leq n+1.$$

(c) We show that

$$\dim \mathrm{End}_{U_q(\mathfrak{sl}_3)}(V_{(m+n,n)} \otimes V_{(m+n,n)}) = \begin{cases} d(m,n) & \text{if } m \geq n, \\ d(n,m) & \text{otherwise.} \end{cases}$$

Since $V_{(n+m,m)}$ is dual to $V_{(m+n,n)}$, it is sufficient to show the equality for $m \geq n$. We suppose that $m \geq n$.

For an irreducible representation $\lambda = (\lambda_1, \ldots, \lambda_{n-1})$ of the quantum algebra $U_q(\mathfrak{sl}_n)$, set

$$\Lambda(\lambda) = \left\{ (\nu_1, \ldots, \nu_n) \left| \begin{array}{l} \nu_1 \geq \cdots \geq \nu_n \geq 0, \\ \nu_i \geq \lambda_i \ \text{for} \ i = 1, \ldots, n-1, \\ \nu_1 + \cdots + \nu_n = 2(\lambda_1 + \cdots + \lambda_{n-1}) \end{array} \right. \right\}.$$

For $\nu = (\nu_1, \ldots, \nu_n) \in \Lambda(\lambda)$, put $\bar\nu = (\nu_1 - \nu_n, \ldots, \nu_{n-1} - \nu_n)$. Then we have

$$V_\lambda \otimes V_\lambda \cong \bigoplus_{\nu \in \Lambda(\lambda)} (V_{\bar\nu})^{\oplus c_{\lambda\lambda}^\nu},$$

where $c_{\lambda\mu}^\nu$ is the Littlewood-Richardson number (see, for example, [3], [5], [6]).

In the case of $n = 3$, for $\lambda = (m+n, n)$ and $\nu = (m+n, n+j, i)$, we have

$$c_{\lambda\lambda}^\nu = \begin{cases} i+j-n+1 & \text{if } 0 \leq i \leq n \text{ and } n-i \leq j \leq n, \\ i+1 & \text{if } 0 \leq i \leq n \text{ and } n < j \leq m+n-i, \\ 2m+2n-i-2j+1 & \text{if } 0 \leq i \leq n \text{ and } m+n-i < j \leq m+n-i/2, \\ n-i+j+1 & \text{if } n < i \leq m+n \text{ and } i-n \leq j \leq n, \\ 2n-i+1 & \text{if } n < i \leq m+n \text{ and } n < j \leq m, \\ 2m+2n-i-2j+1 & \text{if } n < i \leq m+n \text{ and } m < j \leq m+n-i/2, \\ 0 & \text{otherwise.} \end{cases}$$

Then we have

$$\dim \mathrm{End}_{U_q(\mathfrak{sl}_3)}\big(V_{(m+n,n)} \otimes V_{(m+n,n)}\big)$$

$$= \sum_{i=1}^{n}(2m+2n-4i+4)i^2 + (m-n+1)(n+1)^2$$

$$= d(m,n). \qquad\qquad\qquad\qquad \square$$

We conjectured that the equalities in Theorem 4 (b) and (c) hold.

We focus on the skein index of a coloring number, which is a quandle cocycle invariant. A quandle [10], [11], [16] is a non-empty set X with a binary operation $(a,b) \mapsto a * b$ satisfying the following axioms:

(Q1) For any $a \in X$, $a * a = a$.

(Q2) For any $a, b \in X$, there is a unique $c \in X$ such that $a = c * b$.
(Q3) For any $a, b, c \in X$, $(a * b) * c = (a * c) * (b * c)$.

We denote by $a * \bar{b}$ the element c in the axiom (Q2).

Let X be a finite quandle. Put $\overline{X} = \{\bar{x} \mid x \in X\}$. Let V be a vector space over the field $\boldsymbol{C}$ with a basis $\{e_x\}_{x \in X}$. We denote by $\{e_{\bar{x}}\}_{x \in X}$ the dual basis of V^*:

$$e_{\bar{x}}(e_y) = \delta_y^x,$$

where δ_y^x is the Kronecker delta. For $x_1, \ldots, x_n, y_1, \ldots, y_m \in X \bigcup \overline{X}$, we define the linear map $e_{y_1 \cdots y_m}^{x_1 \cdots x_n} : \bigotimes_{k=1}^m V_{y_k} \to \bigotimes_{k=1}^n V_{x_k}$ by

$$e_{y_1 \cdots y_m}^{x_1 \cdots x_n}(e_{z_1} \otimes \cdots \otimes e_{z_m}) = \delta_{z_1}^{y_1} \cdots \delta_{z_m}^{y_m} e_{x_1} \otimes \cdots \otimes e_{x_n},$$

where $V_x = V$ and $V_{\bar{x}} = V^*$ for $x \in X$. For the linear maps (4)–(8), we set

$$R := \sum_{x,y \in X} e_{x \ \ y}^{(y*x)x},$$

$$\Omega_+ := \sum_{x \in X} e_{x\bar{x}}, \qquad \Omega_- := \sum_{x \in X} e_{\bar{x}x},$$

$$\mho_+ := \sum_{x \in X} e^{x\bar{x}}, \qquad \mho_- := \sum_{x \in X} e^{\bar{x}x}.$$

Then, the linear map $[D]$ is an invariant of an oriented tangle T represented by a diagram D. For an oriented link diagram D, the linear map $[D]$ gives the number of the colorings by the quandle X. We denote it by $\#\mathrm{Col}_X$.

Let W be the vector space spanned by

$$\left\{ e_{y_1 y_2}^{x_1 x_2} \mid x_1, x_2, y_1, y_2 \in X \text{ satisfy } (z * x_2) * x_1 = (z * y_2) * y_1 \text{ for any } z \in X \right\}.$$

We denote by $\mathrm{Aut}(X)$ the automorphism group of X:

$$\mathrm{Aut}(X) = \{\varphi : X \to X \mid \varphi\text{: bijection}, \varphi(x * y) = \varphi(x) * \varphi(y) \text{ for any } x, y \in X\}.$$

We define the action of $\mathrm{Aut}(X)$ on W by

$$\varphi \cdot e_{y_1 y_2}^{x_1 x_2} = e_{\varphi(y_1)\varphi(y_2)}^{\varphi(x_1)\varphi(x_2)},$$

for $\varphi \in \mathrm{Aut}(X)$. Set $W^{\mathrm{Aut}(X)} := \{f \in W \mid \varphi \cdot f = f \text{ for any } \varphi \in \mathrm{Aut}(X)\}$. Since $[D] \in W^{\mathrm{Aut}(X)}$ for a tangle diagram D, we have

$$s(\#\mathrm{Col}_X) \leq \dim W^{\mathrm{Aut}(X)}.$$

Let R_p be the dihedral quandle, consisting of the set $\mathbf{Z}/p\mathbf{Z}$ with the binary operation defined by $a * b = 2b - a$. We suppose that p is an odd prime. In the case of R_p, W is spanned by

$$\left\{ e_{y_1 y_2}^{x_1 x_2} \mid x_1, x_2, y_1, y_2 \in X \text{ satisfy } x_1 - x_2 = y_1 - y_2 \right\},$$

and

$$\mathrm{Aut}(R_p) = \{\varphi : \mathbf{Z}/p\mathbf{Z} \to \mathbf{Z}/p\mathbf{Z} \mid \varphi(x) = ax + b, \ p \nmid a\}.$$

Then

$$s(\#\mathrm{Col}_{R_p}) \leq \dim W^{\mathrm{Aut}(R_p)} = p + 2.$$

5.   The skein index of the JKSS invariant.

A virtual link diagram is a link diagram with virtual crossings where a virtual crossing is an encircled crossing with no over/under information. Two virtual link diagrams are said to be equivalent if they are related by a finite sequence of the generalized Reidemeister moves R1–3, V1–4 shown in Figure 5. A virtual link is an equivalence class of virtual link diagrams under these moves. A virtual tangle diagram is a tangle diagram with virtual crossings. A virtual tangle is an equivalence class of virtual tangle diagrams under the generalized Reidemeister moves.
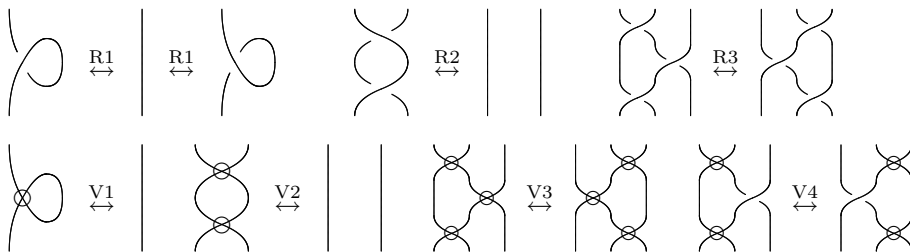

Figure 5.

We recall the definition of the JKSS invariant as an operator invariant. Let V be a vector space over the field $\boldsymbol{C}(x,y)$ with a basis $\{e_1, e_2\}$. We denote by $\{e_{-1}, e_{-2}\}$ the dual basis of V^*:

$$e_{-i}(e_j) = \delta_j^i.$$

We define the linear map $e_{j_1 \cdots j_m}^{i_1 \cdots i_n} : \bigotimes_{k=1}^m V_{j_k} \to \bigotimes_{k=1}^n V_{i_k}$ by

$$e_{j_1 \cdots j_m}^{i_1 \cdots i_n}(e_{k_1} \otimes \cdots \otimes e_{k_m}) = \delta_{k_1}^{j_1} \cdots \delta_{k_m}^{j_m} e_{i_1} \otimes \cdots \otimes e_{i_n},$$

where $V_1 = V_2 = V$ and $V_{-1} = V_{-2} = V^*$. For the linear maps (4)–(8) and $P : V \otimes V \to V \otimes V$ associated to the oriented virtual crossing shown in Figure 6, we set

$$R := x^{-1/2} e_{11}^{11} + (x^{-1/2} - x^{1/2}) e_{12}^{12} - x^{-1/2} y e_{21}^{12} - x^{1/2} y^{-1} e_{12}^{21} - x^{1/2} e_{22}^{22},$$

$$\Omega_+ := x^{-1/2} e_{1(-1)} - x^{-1/2} e_{2(-2)},$$

$$\Omega_- := e_{(-1)1} + e_{(-2)2},$$

$$\mho_+ := e^{1(-1)} + e^{2(-2)},$$

$$\mho_- := x^{1/2} e^{(-1)1} - x^{1/2} e^{(-2)2},$$

$$P := e_{11}^{11} + e_{21}^{12} + e_{12}^{21} - e_{22}^{22}.$$

Figure 6.

Then, the linear map $[D]$ of an oriented virtual tangle diagram D is invariant under the moves R1–R3, V2–V4. For the move V1, we have the relations

$$\left[\begin{array}{c}\text{\reflectbox{$\otimes$}}\end{array}\right] = x^{-1/2} \left[\begin{array}{c}\uparrow\end{array}\right], \qquad \left[\begin{array}{c}\otimes\end{array}\right] = x^{1/2} \left[\begin{array}{c}\uparrow\end{array}\right].$$

Thus the operator invariant is actually an invariant of an oriented virtual link up to multiplication by powers of $x^{\pm 1/2}$. To avoid this ambiguity, we introduce a virtual framed link/tangle. A virtual framed link/tangle is an equivalence class of virtual link/tangle diagrams under the moves R1–3, V2–4. The JKSS invariant $Z(L; x, y)$ of an oriented virtual framed link L is defined by the operator invariant.

Theorem 5.   *The skein index of the JKSS invariant is* 6.

We prepare a lemma for a proof of Theorem 5. Let T_v be the oriented virtual crossing shown in Figure 6. Set

$$T_1 := T(0), \qquad T_2 := T(1), \qquad T_3 := T_v,$$

$$T_4 := T(1) \circ T_v, \qquad T_5 := T_v \circ T(1), \qquad T_6 := T(1) \circ T_v \circ T(1).$$

Let $T^\wedge$ be the closure of the oriented virtual framed tangle T as shown in Figure 7. We denote $(T_i \circ T)^\wedge$ by $T^{\wedge i}$. Let B be the 6×6 matrix defined by

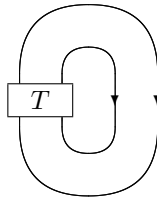$$B_{ij} = Z(T_j^{\wedge i}; x, y).$$

Figure 7.

We give an explicit form of B in the following lemma without a proof, because the data are obtained immediately through the definition of the JKSS invariant.

Lemma 6.   *The matrix B is the symmetric matrix given by*

$$B = \begin{bmatrix} 0 & 0 & 0 & b_1 & b_1 & b_2 \\ 0 & 0 & b_1 & b_2 & b_2 & b_3 \\ 0 & b_1 & 0 & 0 & 0 & b_4 \\ b_1 & b_2 & 0 & b_4 & 0 & b_5 \\ b_1 & b_2 & 0 & 0 & b_4 & b_5 \\ b_2 & b_3 & b_4 & b_5 & b_5 & b_6 \end{bmatrix},$$

where

$$b_1 = x^{-1/2}y^{-1} + x^{-1/2} + x^{-3/2}y + x^{-3/2},$$

$$b_2 = -x^{-1}y + x^{-1}y^{-1} + x^{-2}y + x^{-2} - y^{-1} - 1,$$

$$b_3 = x^{1/2}y^{-1} + x^{1/2} + x^{-1/2}y - x^{-1/2}y^{-1} - x^{-3/2}y + x^{-3/2}y^{-1}$$
$$+ x^{-5/2}y + x^{-5/2},$$

$$b_4 = -x^{-2}y^2 + x^{-2} - y^{-2} + 1,$$

$$b_5 = x^{1/2}y^{-2} - x^{1/2} - x^{-1/2}y^{-2} + x^{-1/2} + x^{-3/2}y^2 - x^{-3/2} - x^{-5/2}y^2 + x^{-5/2},$$

$$b_6 = -xy^{-2} + x - x^{-1}y^2 - x^{-1}y^{-2} + 2x^{-1} + 2x^{-2}y^2 - 2x^{-2} - x^{-3}y^2$$
$$+ x^{-3} + 2y^{-2} - 2.$$

Furthermore the matrix B is invertible, since

$$\det(B) = -x^{-11}y^{-8}(x+1)^2(y+1)^6(y-1)^2(x+y)^6(x-y)^2 \neq 0.$$

PROOF OF THEOREM 5. The linear maps $R, R^{-1}, \mathrm{id}_V, \mathrm{id}_{V^*}, n, \tilde{n}, u, \tilde{u}$, and P are of the form

$$\sum_{i_1 + \cdots + i_n = j_1 + \cdots + j_m} a^{i_1 \cdots i_n}_{j_1 \cdots j_m} e^{i_1 \cdots i_n}_{j_1 \cdots j_m}, \tag{9}$$

where $a^{i_1 \cdots i_n}_{j_1 \cdots j_m} \in \boldsymbol{C}(x,y)$. A linear map associated to an oriented virtual tangle diagram is of this form, since the linear map is obtained as the composition of tensor products of copies of the linear maps $R, R^{-1}, \mathrm{id}_V, \mathrm{id}_{V^*}, n, \tilde{n}, u, \tilde{u}$, and P. Then, for any oriented virtual (2,2)-tangle diagram D, we have

$$[D] = a_1 e^{11}_{11} + a_2 e^{12}_{12} + a_3 e^{12}_{21} + a_4 e^{21}_{12} + a_5 e^{21}_{21} + a_6 e^{22}_{22},$$

for some $a_1, \ldots, a_6 \in \boldsymbol{C}(x,y)$. Hence the skein index of the JKSS invariant is less than or equal to 6.

We suppose that $\sum_{j=1}^{6} c_j Z(T_j; x, y) = 0$. Then we have

$$0 = \sum_{j=1}^{6} c_j Z(T_j^{\wedge i}; x, y) = \sum_{j=1}^{6} B_{ij} c_j.$$

Since the matrix B is invertible, we have $c_1 = \cdots = c_6 = 0$, which implies that there is no skein relation between $T_1, \ldots, T_6$. Therefore the skein index of the JKSS invariant is 6. $\qquad\square$

6.   A formula for the JKSS invariant of a virtual closed 2-braid.

As an application of Theorem 5, we give a formula for the JKSS invariant of a virtual closed 2-braid. Set $T_7 := T_v \circ T(1) \circ T_v$.

LEMMA 7.   *We have the following skein relations*:

$$Z(T(n); x, y) = a_n Z(T(1); x, y) + a_{n-1} Z(T(0); x, y), \tag{10}$$

$$Z(T_7; x, y) = \sum_{i=1}^{6} c_i Z(T_i; x, y), \tag{11}$$

where

$$a_n = \frac{(x^{-1/2})^n - (-x^{1/2})^n}{x^{-1/2} + x^{1/2}},$$

$$c_1 = (x^{3/2}y^{-1} - x^{3/2} + x^{1/2}y - x^{1/2}y^{-1} - x^{-1/2}y + x^{-1/2})(x+1)^{-1},$$

$$c_2 = (2xy^{-1} - x + 2y - 1)(x+1)^{-1},$$

$$c_3 = (-x^{3/2}y^{-1} + x^{3/2} - x^{1/2}y - x^{1/2}y^{-1} - x^{-1/2}y + x^{-1/2})(x+1)^{-1},$$

$$c_4 = (x - 1)(x+1)^{-1},$$

$$c_5 = (x - 1)(x+1)^{-1},$$

$$c_6 = 2x^{1/2}(x+1)^{-1}.$$

PROOF.   The skein relation (10) follows from the skein relation

$$Z(T(1); x, y) - Z(T(-1); x, y) = (x^{-1/2} - x^{1/2})Z(T(0); x, y),$$

which was given by Sawollek in [**19**, Theorem 4].

Since the skein index of the JKSS invariant is 6, we have a skein relation between $T_1, \ldots, T_7$:

$$\sum_{j=1}^{6} c'_j Z(T_j; x, y) = c'_7 Z(T_7; x, y).$$

Then we have

$$\sum_{j=1}^{6} B_{ij} c'_j = \sum_{j=1}^{6} c'_j Z(T_j^{\wedge_i}; x, y) = c'_7 Z(T_7^{\wedge_i}; x, y).$$

Since the matrix B is invertible, we have

$$\begin{bmatrix} c'_1 \\ c'_2 \\ c'_3 \\ c'_4 \\ c'_5 \\ c'_6 \end{bmatrix} = c'_7 B^{-1} \begin{bmatrix} Z(T_7^{\wedge_1}; x, y) \\ Z(T_7^{\wedge_2}; x, y) \\ Z(T_7^{\wedge_3}; x, y) \\ Z(T_7^{\wedge_4}; x, y) \\ Z(T_7^{\wedge_5}; x, y) \\ Z(T_7^{\wedge_6}; x, y) \end{bmatrix} = c'_7 B^{-1} \begin{bmatrix} 0 \\ b_4 \\ b_1 \\ b_2 \\ b_2 \\ b_7 \end{bmatrix},$$

where

$$b_7 = x^{1/2} y^{-3} + x^{1/2} + x^{-5/2} y^3 + x^{-5/2}.$$

Since $(c'_1, \ldots, c'_7) \neq (0, \ldots, 0)$, we have $c'_7 \neq 0$. Then, by putting $c_j = c'_j / c'_7$, we obtain the skein relation (11). □

Any virtual closed 2-braid is given by $(T(n_1) \circ T_v \circ \cdots \circ T(n_l) \circ T_v)^{\wedge}$.

PROPOSITION 8. *The JKSS invariant of a virtual (framed) closed 2-braid is given by*

$$Z((T(n_1) \circ T_v \circ \cdots \circ T(n_l) \circ T_v)^{\wedge}; x, y) = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 & 0 \end{bmatrix} A(n_1) \cdots A(n_l) \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_1 \\ b_2 \end{bmatrix},$$

where $A(n)$ is defined by

$$\begin{bmatrix} 0 & 0 & a_{n-1} & a_n & 0 & 0 \\ 0 & 0 & a_n & a_{n+1} & 0 & 0 \\ a_{n-1} + c_1 a_n & c_2 a_n & c_3 a_n & c_4 a_n & c_5 a_n & c_6 a_n \\ a_1 c_2 a_n & d_n & a_1 c_4 a_n & e_n & a_1 c_6 a_n & f_n \\ a_n + c_1 a_{n+1} & c_2 a_{n+1} & c_3 a_{n+1} & c_4 a_{n+1} & c_5 a_{n+1} & c_6 a_{n+1} \\ a_1 c_2 a_{n+1} & d_{n+1} & a_1 c_4 a_{n+1} & e_{n+1} & a_1 c_6 a_{n+1} & f_{n+1} \end{bmatrix}$$

with

$$d_n := a_{n-1} + c_1 a_n + a_2 c_2 a_n,$$

$$e_n := c_3 a_n + a_2 c_4 a_n,$$

$$f_n := c_5 a_n + a_2 c_6 a_n.$$

PROOF.   The skein relations (10) and (11) imply that

$$
\begin{bmatrix}
Z(T_1 \circ T(n) \circ T_v; x, y) \\
Z(T_2 \circ T(n) \circ T_v; x, y) \\
Z(T_3 \circ T(n) \circ T_v; x, y) \\
Z(T_4 \circ T(n) \circ T_v; x, y) \\
Z(T_5 \circ T(n) \circ T_v; x, y) \\
Z(T_6 \circ T(n) \circ T_v; x, y)
\end{bmatrix}
= A(n)
\begin{bmatrix}
Z(T_1; x, y) \\
Z(T_2; x, y) \\
Z(T_3; x, y) \\
Z(T_4; x, y) \\
Z(T_5; x, y) \\
Z(T_6; x, y)
\end{bmatrix}.
\tag{12}
$$

Set

$$
X(i_1, \ldots, i_j) :=
\begin{bmatrix}
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_1}; x, y) \\
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_2}; x, y) \\
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_3}; x, y) \\
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_4}; x, y) \\
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_5}; x, y) \\
Z((T(i_1) \circ T_v \circ \cdots \circ T(i_j) \circ T_v)^{\wedge_6}; x, y)
\end{bmatrix}.
$$

The equality (12) implies that

$$X(i_1, \ldots, i_j) = A(i_1) X(i_2, \ldots, i_j)$$

and

$$
X(n) = A(n)
\begin{bmatrix}
Z(T_1^{\wedge}; x, y) \\
Z(T_2^{\wedge}; x, y) \\
Z(T_3^{\wedge}; x, y) \\
Z(T_4^{\wedge}; x, y) \\
Z(T_5^{\wedge}; x, y) \\
Z(T_6^{\wedge}; x, y)
\end{bmatrix}
= A(n)
\begin{bmatrix}
B_{11} \\
B_{12} \\
B_{13} \\
B_{14} \\
B_{15} \\
B_{16}
\end{bmatrix}
= A(n)
\begin{bmatrix}
0 \\
0 \\
0 \\
b_1 \\
b_1 \\
b_2
\end{bmatrix}.
$$

Then we have the formula as follows.

$$Z((T(n_1) \circ T_v \circ \cdots \circ T(n_l) \circ T_v)^\wedge; x, y)$$

$$= [\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]X(n_1, n_2, \ldots, n_l)$$

$$= [\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]A(n_1)X(n_2, \ldots, n_l)$$

$$= \cdots = [\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]A(n_1) \cdots A(n_{l-1})X(n_l)$$

$$= [\ 1\ \ 0\ \ 0\ \ 0\ \ 0\ \ 0\ ]A(n_1) \cdots A(n_l) \begin{bmatrix} 0 \\ 0 \\ 0 \\ b_1 \\ b_1 \\ b_2 \end{bmatrix}. \qquad \square$$

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