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A special Lagrangian fibration in the Taub-NUT space

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Abstract. In this paper we construct explicitly a special Lagrangian fibration in the Taub-NUT space. The Taub-NUT space is a complex 2-fold with a Ricci-flat metric and it is well known to physicists. For this space, we find S^1-invariant special Lagrangian submanifolds by using moment map techniques and show that a family of special Lagrangian submanifolds give a fibration of the Taub-NUT space. We also study a topology of special Lagrangian fibers using explicit description of special Lagrangians.

1. Introduction.

A study of calibrated geometry is initiated by R. Harvey and H. B. Lawson [**4**]. Many results have been obtained on calibrated geometry. In particular, calibrated submanifolds (special Lagrangian, associative, coassociative, etc) were given by many researchers. Calibrated submanifolds are very important geometrical objects. For instance, these submanifolds are homologically volume minimizing.

In this paper, we construct a special Lagrangian fibration in the Taub-NUT space. The Taub-NUT space is a complex 2-fold which appears in physics ([**2**], [**3**]). Although this space is a standard complex space $\boldsymbol{C}^2$ as a complex manifold, the Kähler metric is complete non-flat Ricci-flat metric on $\boldsymbol{C}^2$. This Kähler metric is called Taub-NUT metric. The Taub-NUT metric has also the following property. At distant points from the origin, this metric is decomposed asymptotically into S^1 direction and $\boldsymbol{R}^3$ direction like standard metric of the $S^1 \times \boldsymbol{R}^3$. From this property, we construct explicitly a special Lagrangian fibration using the method of Ionel and Min-Oo ([**6**], [**7**]). They found special Lagrangian submanifolds of the deformed conifold T^*S^3 and the resolved conifold (the total space of $\mathscr{O}(-1) \oplus \mathscr{O}(-1) \to \boldsymbol{CP}^1$) using the moment map technique in ([**6**], [**7**]). We use their moment map technique to construct explicitly special Lagrangian fibration.

This paper is organized as follows. In Section 2 we introduce the Taub-NUT space and study a hyper-Kähler structure of the space. In Section 3 we construct

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S^1-invariant special Lagrangian submanifolds in the Taub-NUT space using the method of Ionel Min-Oo and show that 2-parameter family of special Lagrangians gives a special Lagrangian fibration of the Taub-NUT space.

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2. Taub-NUT space.

In this section we introduce the Taub-NUT space and describe a Kähler 2-form and a holomorphic 2-form corresponding to the hyper-Kähler structure of the Taub-NUT space. We refer the reader to the paper [**3**].

First, we consider the Hopf map. For the 3-dimensional sphere $S^3 \subset \boldsymbol{C}^2$ and the 2-dimensional sphere $S^2 \subset \boldsymbol{R}^3$, the Hopf map $\varphi : S^3 \to S^2$ is defined by

$$(2\,\mathrm{Re}(z_1\overline{z_2}),\ 2\,\mathrm{Im}(z_1\overline{z_2}),\ |z_1|^2 - |z_2|^2) \in S^2, \tag{1}$$

where, $(z_1, z_2) \in S^3$. We consider the following map $\tilde{\varphi} : \boldsymbol{C}^2\backslash\{(0,\ 0)\} \to \boldsymbol{R}^3\backslash\{(0,\ 0,\ 0)\}$ as an extension of the Hopf map (1). For $(z_1, z_2) \in \boldsymbol{C}^2\backslash\{(0,\ 0)\}$,

$$(2\,\mathrm{Re}(z_1 z_2),\ 2\,\mathrm{Im}(z_1 z_2),\ |z_1|^2 - |z_2|^2) \in \boldsymbol{R}^3\backslash\{(0,\ 0,\ 0)\}. \tag{2}$$

The punctured complex space $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$ has a principal S^1-bundle structure by the map (2). The S^1-action on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$ is the following.
For $t \in S^1$,

$$\phi_t(z_1,\ z_2) = (e^{it} z_1,\ e^{-it} z_2). \tag{3}$$

Next, we try to construct a hyper-Kähler structure on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$. We choose the local coordinate system on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$ in the following way. We put (u_1, u_2, u_3) and t as coordinates of $\boldsymbol{R}^3\backslash\{(0,\ 0,\ 0)\}$ and S^1 respectively. (i.e. $u_1 = 2\,\mathrm{Re}(z_1 z_2)$, $u_2 = 2\,\mathrm{Im}(z_1 z_2)$, $u_3 = |z_1|^2 - |z_2|^2$.) By the local trivialization of the principal bundle, we have a local coordinate system (u_1, u_2, u_3, t) on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$. Now, it is important that how to construct a hyper-Kähler structure on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$. In fact, there is a well-known method which is called the Gibbons-Hawking Ansatz for this problem ([**2**], [**3**]). Hence, we construct a hyper-Kähler structure by using the Gibbons-Hawking Ansatz.

Remark 2.1. Although the Gibbons-Hawking Ansatz is a method which constructs a hyper-Kähler structure, an obtained Ricci-flat metric is a not only the Taub-NUT metric. Hence, we first explain the Gibbons-Hawking Ansatz on the S^1-bundle $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$, and we show that the Taub-NUT metric is the special case.

Let U be an open set on $\boldsymbol{R}^3\backslash\{(0,\ 0,\ 0)\}$ and V a positive harmonic function on U which satisfies the following condition. For a Hodge star operator $*$ with respect to the standard flat metric of $\boldsymbol{R}^3$, $- * dV$ represents the first Chern class of the S^1-bundle $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$ if $\int_{S^2} - * dV = 1$. Then, the connection 1-form θ on the S^1 bundle is the $\mathfrak{u}(1) \cong i\boldsymbol{R}$-valued 1-form which satisfies

$$\frac{d\theta}{2\pi i} = *dV. \tag{4}$$

This is compatible with the requirement $d(d\theta) = 0$. We put the curvature form for the connection form θ:

$$d\theta = \tilde{\varphi}^*\alpha \quad (\text{for } \alpha : \text{2-form on } U).$$

From the above choice of V, $*dV$ satisfies (4). Then, we consider the integrable almost complex structure (i.e. complex structure) J on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$ defined by

$$J(du_1) = -du_2,\ J(du_3) = -V^{-1}\theta_0, \tag{5}$$

where, $\theta_0 = \frac{\theta}{2\pi i}$. We consider a triple of 2-forms $(\omega_1,\ \omega_2,\ \omega_3)$.

$$\omega_1 := \frac{1}{2\pi i} du_1 \wedge \theta + V du_2 \wedge du_3, \tag{6a}$$

$$\omega_2 := \frac{1}{2\pi i} du_2 \wedge \theta + V du_3 \wedge du_1, \tag{6b}$$

$$\omega_3 := \frac{1}{2\pi i} du_3 \wedge \theta + V du_1 \wedge du_2. \tag{6c}$$

The above pair of 2-forms satisfies the following relations.

$$\omega_i^2 \neq 0,\ \omega_i \wedge \omega_j = 0\ (i \neq j),\ d\omega_i = 0.$$

Therefore, the triple $(\omega_1,\ \omega_2,\ \omega_3)$ defines a hyper-Kähler structure on $\boldsymbol{C}^2\backslash\{(0,\ 0)\}$. For this hyper-Kähler structure, a Kähler 2-form ω and a holomorphic 2-form Ω corresponding to the complex structure J are given by

$$\begin{aligned}\omega :&= \omega_3\\&= \frac{1}{2\pi i} du_3 \wedge \theta + V du_1 \wedge du_2, \qquad (7)\\ \Omega :&= -\omega_1 - i\omega_2\\&= (\theta_0 - iV du_3) \wedge (du_1 + i du_2). \qquad (8)\end{aligned}$$

We obtain the following Ricci-flat Kähler metric corresponding to the above Kähler structure by the relation $g(\zeta, \xi) = \omega(\zeta, J\xi)$:

$$g = V du \cdot du + V^{-1} \theta_0^2. \qquad (9)$$

In the above discussion, we have constructed a hyper-Kähler structure on $\boldsymbol{C}^2 \backslash \{(0,\ 0)\}$. In fact, it is easy to see by direct calculation that this hyper-Kähler structure extends smoothly across the origin $\{0\}$. Hence, we have a hyper-Kähler structure on $\boldsymbol{C}^2$ and this complex structure is isomorphic to the standard one on $\boldsymbol{C}^2$ (we show this explicitly later, see (30) and (31)). This is an outline of the Gibbons-Hawking Ansatz.

From now on, we construct the Taub-NUT metric by using this Ansatz. For the purpose, we define the positive harmonic function V on open set U of $\boldsymbol{R}^3 \backslash \{(0,\ 0,\ 0)\}$ and the corresponding $\mathfrak{u}(1)$-valued connection form θ on $\boldsymbol{C}^2 \backslash \{(0,\ 0)\}$ given by:

$$V := \varepsilon + \frac{1}{4\pi \sqrt{{u_1}^2 + {u_2}^2 + {u_3}^2}} \quad \text{for } \varepsilon \geq 0, \qquad (10)$$

$$\theta := \frac{i\, \mathrm{Im}(\overline{z_1} dz_1 - \overline{z_2} dz_2)}{|z_1|^2 + |z_2|^2}. \qquad (11)$$

The above function V and the connection form θ defines a hyper-Kähler structure. Fixing a value of ε in the expression of V, we obtain a Ricci-flat metric.

We now proceed to examples. We first look at the case of $\varepsilon = 0$. The Ricci-flat Kähler metric g obtained from (6) turns out to be the standard Kähler metric of $\boldsymbol{C}^2$. Indeed the 2-forms $(\omega_1,\ \omega_2,\ \omega_3)$ corresponding to (6) are given by

$$\omega_1 := \frac{1}{\pi}(dx_2 \wedge dy_1 - dx_1 \wedge dy_2), \qquad (12a)$$

$$\omega_2 := \frac{1}{\pi}(dx_1 \wedge dx_2 - dy_1 \wedge dy_2), \qquad (12b)$$

$$\omega_3 := \frac{1}{\pi}(dx_1 \wedge dy_1 + dx_2 \wedge dy_2), \tag{12c}$$

where, $z_j = x_j + iy_j$ are the standard coordinate of $\boldsymbol{C}^2$ for $j = 1,\ 2$. Thus, we obtain the standard Kähler metric from the description of the Kähler form ω_3. From this fact, we see by direct calculation that for any positive number ε, the complex structure J and 2-forms (6.a, b, c) extend smoothly across $\{0\}$ and so the resulting hyper-Kähler structure also does across $\{0\}$. In particular, the hyper-Kähler structure corresponding to positive harmonic function (10) are deformations of the standard one on the Euclidian $\boldsymbol{C}^2$.

We consider the case of $\varepsilon = 1$ as one of the hyper-Kähler structures corresponding to $\varepsilon \neq 0$.

For $\varepsilon = 1$, we obtain a Kähler form ω and a holomorphic 2-form Ω and a Ricci-flat metric g by substituting $V = 1 + \frac{1}{4\pi\sqrt{{u_1}^2+{u_2}^2+{u_3}^2}}$ into (7), (8) and (9).

$$\begin{aligned}
\omega :&= \omega_3 \\
&= \frac{1}{2\pi i} du_3 \wedge \theta + \left(1 + \frac{1}{4\pi\sqrt{{u_1}^2+{u_2}^2+{u_3}^2}}\right) du_1 \wedge du_2, \qquad (13)\\
\Omega :&= -\omega_1 - i\omega_2 \\
&= \left\{\theta_0 - i\left(1 + \frac{1}{4\pi\sqrt{{u_1}^2+{u_2}^2+{u_3}^2}}\right) du_3\right\} \wedge (du_1 + i du_2), \qquad (14)
\end{aligned}$$

$$ds^2 = \left(1 + \frac{1}{4\pi\sqrt{{u_1}^2+{u_2}^2+{u_3}^2}}\right) du \cdot du + \left(1 + \frac{1}{4\pi\sqrt{{u_1}^2+{u_2}^2+{u_3}^2}}\right)^{-1} \theta_0^2, \tag{15}$$

where $u = (u_1, u_2, u_3)$.

The space $\boldsymbol{C}^2$ obtained using this procedure is called the Taub-NUT space and the above Ricci-flat metric on this space is called Taub-NUT metric.

3. A construction of special Lagrangian submanifolds in the Taub-NUT space.

In this section, we find special Lagrangian submanifolds L^2 in the Taub-NUT space $\boldsymbol{C}^2$: i.e. $L^2 \subset \boldsymbol{C}^2$ such that

$$\text{(i) } \omega|_L \equiv 0, \tag{16}$$

$$\text{(ii) } \operatorname{Im}\Omega|_L \equiv 0, \tag{17}$$

where ω and Ω are a Kähler 2-form and a holomorphic 2-form respectively. We rewrite ω and Ω by using coordinate (z_1, z_2). Consequently, we obtain the following description.

$$\omega = \left(\frac{i}{2\pi} + 2i|z_2|^2\right) dz_1 \wedge d\,\overline{z_1} + \left(\frac{i}{2\pi} + 2i|z_1|^2\right) dz_2 \wedge d\,\overline{z_2}$$
$$- 2iz_1\overline{z_2} d\,\overline{z_1} \wedge dz_2 + 2i\overline{z_1}z_2 dz_1 \wedge d\,\overline{z_2}, \tag{18}$$
$$\Omega = \left(\frac{1}{\pi i} - 2i|z_1|^2 - 2i|z_2|^2\right) dz_1 \wedge dz_2 + 2iz_1z_2 dz_1 \wedge d\,\overline{z_1} - 2iz_1z_2 dz_2 \wedge d\,\overline{z_2}$$
$$- 2iz_2^2 dz_1 \wedge d\,\overline{z_2} - 2iz_1^2 d\,\overline{z_1} \wedge dz_2. \tag{19}$$

The holomorphic 2-form Ω contains $d\bar{z}$. This is because the complex coordinate system (z_1, z_2) of $\boldsymbol{C}^2$ is not holomorphic with respect to the complex structure (5). On the other hand, the Calabi-Yau condition $2\omega \wedge \omega = \Omega \wedge \overline{\Omega}$ is satisfied.

Remark 3.1. We choose the following process. First we describe special Lagrangian submanifolds in the Taub-NUT space in terms of non-holomorphic coordinates (z_1, z_2). Next, we rewrite special Lagrangian submanifolds in terms of holomorphic coordinates.

We try to construct a special Lagrangian fibration in the Taub-NUT space using the method of Ionel and Min-Oo. Their method is based on the moment map technique. First, we calculate the moment map of the canonical S^1-action on the Taub-NUT space. Now, a Hamiltonian circle action $S^1 \to \boldsymbol{C}^2$ is given by:

$$\phi_t(z_1,\ z_2) = (e^{it}z_1,\ e^{-it}z_2), \quad (\text{for} \ \ t \in S^1) \tag{20}$$

from (3).

Lemma 3.2. *A moment map* $\mu : \boldsymbol{C}^2 \to \mathfrak{u}(1)^*$ *corresponding to the* S^1*-action* (20) *is given by*

$$\mu(z_1,\ z_2) = \frac{1}{2\pi}(|z_1|^2 - |z_2|^2) \in \mathfrak{u}(1)^* \cong \boldsymbol{R}. \tag{21}$$

Proof. We show that the above map μ satisfies the following conditions: (I) For $\forall X \in \boldsymbol{R}$, we consider the map $f_X : M \to \boldsymbol{R}$ defined by $f_X(p) = \mu(p)(X)$ $(p \in \boldsymbol{C}^2)$. Then, the condition $\iota_{v_X}\omega = -df_X$ is satisfied, where ω is the Kähler form (18) and v_X is the vector field given by:

$$v_X := X\left(iz_1\frac{\partial}{\partial z_1} - iz_2\frac{\partial}{\partial z_2} - i\overline{z_1}\frac{\partial}{\partial \overline{z_1}} + i\overline{z_2}\frac{\partial}{\partial \overline{z_2}}\right), \text{ for } X \in \boldsymbol{R}.$$

(II) $\mu \circ \phi_t = Ad^* \circ \mu$.

First, we prove the condition (I). From the expression of the Kähler form (18), we have the following:

$$\begin{aligned}
\iota_{v_X}\omega = X\Big\{&\left(\frac{i}{2\pi} + 2i|z_2|^2\right)iz_1 d\,\overline{z_1} + \left(\frac{i}{2\pi} + 2i|z_2|^2\right)i\overline{z_1}dz_1 \\
&- \left(\frac{i}{2\pi} + 2i|z_1|^2\right)iz_2 d\,\overline{z_2} - \left(\frac{i}{2\pi} + 2i|z_1|^2\right)i\overline{z_2}dz_2 \\
&+ 2i|z_1|^2 i\overline{z_2}dz_2 - 2i|z_2|^2 iz_1 d\,\overline{z_1} + 2i|z_1|^2 iz_2 d\,\overline{z_2} - 2i|z_2|^2 i\overline{z_1}dz_1\Big\} \\
= -&\frac{X}{2\pi}d\big(|z_1|^2 - |z_2|^2\big) \\
= -&df_X.
\end{aligned}$$

Thus, we obtain the condition (I). Also, we have the condition (II) using the S^1-invariance of the map μ and the commutativity of the S^1-action. Thus, we complete the proof. □

For the above moment map, we use the following proposition from Ionel, Min-Oo's papers: ([**6**], [**7**]).

Proposition 3.3 ([**6**], pp. 7–9). *Let M be a real $2n$-dimensional Calabi-Yau manifold and* $\mathrm{Aut}(M)$ *be the automorphism group for the Calabi-Yau structure of M and $G \subset \mathrm{Aut}(M)$ be a connected Lie subgroup with Lie algebra $\mathfrak{g}$. Moreover, $\mu : M \to \mathfrak{g}^*$ be the moment map corresponding to a automorphism given by $g \in G$. If L is a G-invariant connected special Lagrangian submanifold, then $L \subseteq \mu^{-1}(c)$* for $c \in Z(\mathfrak{g}^*)$*, where c is any fixed element of $Z(\mathfrak{g}^*)$.*

Since S^1 is connected and the S^1-action (20) preserves the Calabi-Yau structure (i.e. the Kähler form (18) and the holomorphic 2-form (19)) on the Taub-NUT space, we can apply the Proposition in the case of the space $\boldsymbol{C}^2$. From $\mathfrak{u}(1)^* \cong \boldsymbol{R}$, we understand clearly $Z(\mathfrak{g}^*) \cong \boldsymbol{R}$. Hence, S^1-invariant special Lagrangian submanifolds must satisfy

$$\mu(z_1, z_2) = c_1 \Longleftrightarrow |z_1|^2 - |z_2|^2 = c_1 \text{ for any fixed real const } c_1. \tag{22}$$

Next, we calculate (ii) $\operatorname{Im}\Omega|_L \equiv 0$ which is one of the special Lagrangian condition. We choose a frame v_1, v_2 of a special Lagrangian submanifold in the following way. Let v_1 be the infinitesimal S^1-action:

$$v_1 := iz_1\frac{\partial}{\partial z_1} - iz_2\frac{\partial}{\partial z_2} - i\overline{z_1}\frac{\partial}{\partial \overline{z_1}} + i\overline{z_2}\frac{\partial}{\partial \overline{z_2}},$$

and v_2 a tangent vector of an infinitesimal curve:

$$v_2 := \dot{z}_1\frac{\partial}{\partial z_1} + \dot{z}_2\frac{\partial}{\partial z_2} + \dot{\overline{z}}_1\frac{\partial}{\partial \overline{z_1}} + \dot{\overline{z}}_2\frac{\partial}{\partial \overline{z_2}}.$$

We substitute the frame (v_1, v_2) into holomorphic 2-form Ω.

$$\begin{aligned}\Omega(v_1, v_2) = &\left(\frac{1}{\pi i} - 2i|z_1|^2 - 2i|z_2|^2\right)dz_1 \wedge dz_2(v_1, v_2) + 2iz_1z_2dz_1 \wedge d\,\overline{z_1}(v_1, v_2) \\ &- 2iz_1z_2dz_2\wedge d\,\overline{z_2}(v_1, v_2) - 2iz_2^2dz_1\wedge d\,\overline{z_2}(v_1, v_2) - 2iz_1^2d\,\overline{z_1}\wedge dz_2(v_1, v_2) \\ = &\left(\frac{1}{\pi i} - 2i|z_1|^2 - 2i|z_2|^2\right)\cdot i(z_1z_2)^{\cdot} + 2iz_1z_2(iz_1\dot{\overline{z}}_1 + i\dot{z}_1\overline{z_1}) \\ &- 2iz_1z_2(-iz_2\dot{\overline{z}}_2 - i\dot{z}_2\overline{z_2}) - 2iz_2^2(iz_1\dot{\overline{z}}_2 - i\dot{z}_1\overline{z_2}) - 2iz_1^2(-i\overline{z_1}\dot{z}_2 + i\dot{\overline{z}}_1z_2) \\ = &\frac{1}{\pi}(z_1z_2)^{\cdot}.\end{aligned}$$

Hence we have the following condition.

$$\operatorname{Im}\Omega|_L \equiv 0 \Longleftrightarrow \operatorname{Im}(z_1z_2) = c_2 \quad \text{for any fixed real const } c_2.$$

We consider a pair of equations:

$$\text{(I) } |z_1|^2 - |z_2|^2 = c_1 \text{ (for } c_1\text{: any fixed real const)}, \tag{23}$$

$$\text{(II) } \operatorname{Im}(z_1z_2) = c_2 \text{ (for } c_2\text{: any fixed real const)}. \tag{24}$$

Both (23) and (24) are clearly S^1-invariant. Also, the equations (23), (24) satisfy the special Lagrangian condition (i) (i.e. Kähler form (13) vanish). Consequently, we obtain the following result.

Proposition 3.4. *Non-holomorphic expressions of S^1-invariant special Lagrangian submanifolds in the Taub-NUT space $\boldsymbol{C}^2$ are given by* (23) *and* (24).

We rewrite the equations giving special Lagrangian submanifolds in terms of holomorphic coordinates. We recall a few of settings of the Taub-NUT space. The S^1-fibration is given by

$$\begin{aligned} u_1 &= 2\,\mathrm{Re}(z_1 z_2) = z_1 z_2 + \overline{z_1 z_2}, \\ u_2 &= 2\,\mathrm{Im}(z_1 z_2) = -i(z_1 z_2 - \overline{z_1 z_2}), \\ u_3 &= |z_1|^2 - |z_2|^2. \end{aligned}$$

Recall that

$$\Omega = (\theta_0 - iV du_3) \wedge (du_1 + i du_2).$$

We consider the following basis of the holomorphic cotangent space of the Taub-NUT space.

$$du_1 + i du_2 = 2(z_1 dz_2 + z_2 dz_1), \tag{25}$$

$$\begin{aligned} \theta_0 - iV du_3 &= \frac{1}{2\pi} \frac{\mathrm{Im}(\overline{z_1} dz_1 - \overline{z_2} dz_2)}{|z_1|^2 + |z_2|^2} - i\left\{1 + \frac{1}{4\pi\sqrt{u_1{}^2 + u_2{}^2 + u_3{}^2}}\right\} du_3 \\ &= \frac{1}{4\pi i} \frac{\overline{z_1} dz_1 - \overline{z_2} dz_2 - z_1 d\overline{z_1} + z_2 d\overline{z_2}}{|z_1|^2 + |z_2|^2} - i du_3 \\ &\quad + \frac{1}{4\pi i(|z_1|^2 + |z_2|^2)} d(|z_1|^2 - |z_2|^2) \\ &= \frac{\overline{z_1} dz_1 - \overline{z_2} dz_2}{2\pi i(|z_1|^2 + |z_2|^2)} - i du_3. \end{aligned} \tag{26}$$

Now, we modify (26) as follows.

$$(\theta_0 - iV du_3) \times 2\pi i(|z_1|^2 + |z_2|^2) = \overline{z_1} dz_1 - \overline{z_2} dz_2 + 2\pi(|z_1|^2 + |z_2|^2) du_3. \tag{27}$$

Then we can choose another basis of the holomorphic cotangent space:

$$dz_1 + 2\pi z_1 du_3, \tag{28}$$

$$dz_2 - 2\pi z_2 du_3. \tag{29}$$

For the above basis, we try to choose a holomorphic coordinate of the Taub-NUT space. These equations do not give holomorphic coordinate.

$$d\widetilde{z_1} = dz_1 + 2\pi z_1 du_3,$$
$$d\widetilde{z_2} = dz_2 - 2\pi z_2 du_3.$$

We see that the left hand side of equations are closed 1-forms and the right hand side of equations are non-closed 1-forms. Therefore, let us calculate as follows:

$$\begin{aligned} dz_1 + 2\pi z_1 du_3 &= z_1\left(\frac{dz_1}{z_1} + 2\pi du_3\right) \\ &= d(\log z_1 + 2\pi u_3) \times z_1. \end{aligned}$$

Similarly,

$$\begin{aligned} dz_2 - 2\pi z_2 du_3 &= z_2\left(\frac{dz_2}{z_2} - 2\pi du_3\right) \\ &= d(\log z_2 - 2\pi u_3) \times z_2. \end{aligned}$$

Putting $\log \widetilde{z_1} := \log z_1 + 2\pi u_3$ and $\log \widetilde{z_2} := \log z_2 - 2\pi u_3$, we obtain a holomorphic coordinate functions:

$$\widetilde{z_1} = e^{2\pi u_3} z_1 = e^{2\pi(|z_1|^2 - |z_2|^2)} z_1, \tag{30}$$

$$\widetilde{z_2} = e^{-2\pi u_3} z_2 = e^{-2\pi(|z_1|^2 - |z_2|^2)} z_2. \tag{31}$$

Hence we have a holomorphic coordinate system $(\widetilde{z_1}, \widetilde{z_2})$ of the Taub-NUT space and the Taub-NUT space is identified with standard $\boldsymbol{C}^2$ as a complex manifold. By using this holomorphic coordinate system, we obtain the following result.

Theorem 3.5. *S^1-invariant special Lagrangian submanifolds in the Taub-NUT space $\boldsymbol{C}^2$ with coordinate $(\widetilde{z_1}, \widetilde{z_2})$ are given by equations*:

$$e^{-4\pi c_1}|\widetilde{z_1}|^2 - e^{4\pi c_1}|\widetilde{z_2}|^2 = c_1, \tag{32}$$

$$\mathrm{Im}(\widetilde{z_1}\widetilde{z_2}) = c_2, \tag{33}$$

where, c_1, c_2 are real consts.

Remark 3.6. Since the above special Lagrangian submanifolds are S^1-invariant, the above equations (32), (33) express curves obtained by projection of special Lagrangian submanifolds to base space $\boldsymbol{R}^3$.

We obtain a special Lagrangian fibration on the Taub-NUT space by the above two parameter family of special Lagrangian submanifolds.

Finally, we study the topology of special Lagrangian fibers. We show easily that generic topology of special Lagrangians is cylinder $S^1 \times \boldsymbol{R}$ by non-compactness of the Taub-NUT space and S^1-invariance of the special Lagrangians. However the generic fiber don't intersect the origin of the Taub-NUT space. A special Lagrangian fiber intersecting the origin is the unique singular fiber (S^1-invariant cone) obtained in the case of $c_1 = c_2 = 0$.

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