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Invariants for representations of Weyl groups and two-sided cells

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Abstract. The notion of two-sided cell, which was originally introduced by A. Joseph and reformulated by D. Kazhdan and G. Lusztig, has played an important role in the representation theory. Results concerning them have been obtained by very deep and sometimes ad hoc arguments. Here we introduce certain polynomial invariants for irreducible representations of Weyl groups. Our invariants are easily calculated, and the calculational results show that they almost determine the two-sided cells. Moreover, the factorization pattern of our polynomial invariants seems to be controlled by the natural parameter set $\mathscr{M}(\mathscr{G})$ of each two-sided cell.

0. Introduction.

0.1. Motivation.

Let W be a Weyl group and S the totality of simple reflections. For $w \in W$, let $l(w)$ be the minimum of the length n of various expressions $w = s_1 s_2 \cdots s_n$ $(s_i \in S)$. In [**Gy1**], a 'q-analogue' of the following matrix valued function was studied;

$$\sum_{w \in W} \rho(w) t^{l(w)}, \tag{0.1}$$

where $\rho : W \to GL_N(\boldsymbol{C})$ is an irreducible representation. In particular, it was shown that these functions have similar properties as the congruence zeta functions of algebraic varieties.

As a modification of these functions, N. Iwahori proposed to replace the length function $l(\cdot)$ in (0.1) with another length function defined as follows. Let S' be the totality of (not necessarily simple) reflections in W. Define $l'(w)$ to be the minimum of the length m of the various expressions $w = r_1 r_2 \cdots r_m$ $(r_i \in S')$. Such length function $l'(\cdot)$ is studied by R. Carter [**C1**] in his study of conjugacy classes of Weyl groups. Since l' is a class function, (0.1) becomes the scalar function

$$c(\chi; t) := \chi(e)^{-1} \sum_{w \in W} \chi(w) t^{l'(w)} \tag{0.2}$$

after such modification, where $\chi = \operatorname{trace} \rho$.

Concerning the function (0.2), we can make the following observation by a case study. Let $\chi, \chi' \in W^\vee$, where $W^\vee$ is the set of irreducible characters of W. Then the

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following equivalence is almost true:

$$c(\chi; t) = c(\chi'; t) \Leftrightarrow \chi \underset{LR}{\sim} (\chi')^{\iota} \quad \text{for some } \iota \in \mathrm{Aut}(W, S). \tag{0.3}$$

Here $\chi \underset{LR}{\sim} \chi'$ means that χ and χ' belong to the same family (= two-sided cell) in the sense of G. Lusztig [**L1, 4.2**] (cf. (b) below). More precisely, (0.3) is *true* if W is of type A_l, B_l $(= C_l)$ or G_2. In the other cases, some deviation occurs.

Let us explain why we are interested in (0.3). The partition of W as a set into left (resp. right, two-sided) cells was first introduced by A. Joseph [**J**] and then reformulated by D. Kazhdan and G. Lusztig [**KL**]. Concerning cells, we know the following, summing up works of Joseph, Kazhdan, Lusztig, Brylinski, Kashiwara, Beilinson, Bernstein, Barbasch, Vogan, ...

(a) Let $\mathfrak{g}$ be a complex semisimple Lie algebra, and W its Weyl group. The primitive ideals of $U(\mathfrak{g})$ with trivial central character are divided into several families parametrized by the set of two-sided cells of W. To each family $\mathscr{F}_{\mathfrak{g}}$ there associates a special representation of W [**L3**] (= the Goldie rank representation in the sense of Joseph). The set of primitive ideals in a fixed family $\mathscr{F}_{\mathfrak{g}}$ is in one-to-one correspondence with the set of left cells in the associated two-sided cell, and its cardinality is equal to the dimension of the corresponding special representation.

(b) Let G be a connected reductive group over a finite field $\boldsymbol{F}_q$ satisfying a certain mild assumption. We assume that its Weyl group is the same W as in (a). A crucial step of the classification of the irreducible representations of $G(\boldsymbol{F}_q)$ is the classification of the unipotent representations [**L1**]. The set of unipotent representations $G(\boldsymbol{F}_q)^{\vee}_{uni}$ is divided into several families; $G(\boldsymbol{F}_q)^{\vee}_{uni} = \coprod \mathscr{F}_G$. The set of these families is in one-to-one correspondence with the set of two-sided cells. For each family $\mathscr{F}_G$, there associates a finite group $\mathscr{G}$ $(\simeq \mathfrak{S}_2^k$ $(k \geq 0)$, $\mathfrak{S}_3, \mathfrak{S}_4$ or $\mathfrak{S}_5$ if W is irreducible). The set of unipotent representations in a fixed family $\mathscr{F}_G$ is in one-to-one correspondence with $\mathscr{M}(\mathscr{G}) := \{(x, \rho) \mid x \in \mathscr{G}, \rho \in Z_{\mathscr{G}}(x)^{\vee}\}/\mathscr{G}$-conjugacy.

(c) Two-sided cells determine two-sided ideals of the group ring $\boldsymbol{C}W$, and give a partition of $W^{\vee}$. The latter partition gives the equivalence relation $\underset{LR}{\sim}$ appeared in (0.3), and is explicitly determined (cf. [**L1**, §4]). There is a natural inclusion $W^{\vee} \to G(\boldsymbol{F}_q)^{\vee}_{uni}$. Each $\mathscr{F}_W \in W^{\vee}/\underset{LR}{\sim}$ is contained in the (unique) family $\mathscr{F}_G$ as in (b), and the correspondence $\mathscr{F}_W \mapsto \mathscr{F}_G$ gives a bijection $W^{\vee}/\underset{LR}{\sim} \to \{\mathscr{F}_G\}$. (For $\chi \in W^{\vee}$, let $\mathscr{F}_W(\chi) \in W^{\vee}/\underset{LR}{\sim}$ be the class of χ, $\mathscr{F}_G(\chi)$ the corresponding family of $G(\boldsymbol{F}_q)^{\vee}_{uni}$, and $\mathscr{G}(\chi)$ the associated finite group.)

In spite of these beautiful and important roles played by the left/two-sided cells, the proof is often very deep and sometimes quite ad hoc. Thus it is surprising that such a simple formula like (0.3) characterizes the two-sided cells even if some deviation occurs.

0.2. Modification of $c(\chi; t)$.

Motivated by the reason explained at the end of §0.1, we want to reduce the deviation tainting (0.3). For this purpose, we modify the invariant $c(\chi; t)$. We start with introducing certain invariants $\tilde{\tau}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ and $\tau^*(\chi; t) \in \boldsymbol{Z}[t]$. The latter is essentially the same as $c(\chi; t)$, and can be obtained from the former by a certain specialization. By a case study, we observe that $\tilde{\tau}(\chi; q, y)$ has a lot of factors of the form $1 + yq^c$ ($c \in \boldsymbol{Z}$), and that these factors are often the same for χ's in a fixed family $\mathscr{F}_W \in W^\vee / \underset{LR}{\sim}$. Thus it is natural to consider the largest divisor of the form $\prod_{i=1}^{\kappa}(1 + yq^{c_i})$ of $\tilde{\tau}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$. For $\chi \in W^\vee$, let

$$\tilde{c}'(\chi; q, y) = \prod_{i=1}^{\kappa'(\chi)} (1 + yq^{c_i(\chi)}) \tag{0.4}$$

be such a largest divisor. As an invariant for $\mathscr{F}_W \in W^\vee / \underset{LR}{\sim}$, we consider

$$\tilde{c}(\chi; q, y) = \mathrm{GCD}\{\tilde{c}'(\chi'; q, y) | \chi' \underset{LR}{\sim} \chi\} = \prod_{i=1}^{\kappa(\chi)} (1 + yq^{c_i(\chi)}). \tag{0.5}$$

There are a few χ's such that $\tilde{c}(\chi; q, y) \neq \tilde{c}'(\chi; q, y)$. Calculational results indicate that this discrepancy is controlled by $\mathscr{M}(\mathscr{G})$ in the cases of A_l, D_4 and E_l (cf. Theorem 4.1). To control the same phenomenon in the cases of B_l, F_4 and G_2, we need some modular representation theory (cf. §4.2). Define $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ and $\tau^*_{prim}(\chi; t) \in \boldsymbol{Z}[t]$ so that

$$\tilde{\tau}(\chi; q, y) = (1 - q)^{-\kappa(\chi)} \prod_{i=1}^{\kappa(\chi)} (1 + yq^{c_i(\chi)}) \cdot \tilde{\tau}_{prim}(\chi; q, y), \tag{0.6}$$

$$\tau^*(\chi; t) = \prod_{i=1}^{\kappa(\chi)} (t + c_i(\chi)) \cdot \tau^*_{prim}(\chi; t). \tag{0.7}$$

(As we shall see in Proposition 1.3, $\tau^*_{prim}(\chi; t)$ is a polynomial.) Our first main result is the following theorem which says that the invariant $\tilde{c}(\chi; q, y)$ separates the families.

THEOREM 0.1. *Let W be an irreducible Weyl group not of type D_l ($l \geq 5$). For $\chi, \chi' \in W^\vee$,*

$$\tilde{c}(\chi; q, y) = \tilde{c}(\chi'; q, y) \Leftrightarrow \chi \underset{LR}{\sim} (\chi')^\iota \quad \text{for some } \iota \in \mathrm{Aut}(W, S).$$

0.3. Primitive factors of the invariants.

Let K be a field. For a polynomial $p(t) \in K[t]$, let $p(t) = \prod_{i=1}^{n} p_i(t)$ be the irreducible factorization in $K[t]$. Then $\pi_K(p(t)) := \{\deg p_i\}_{1 \leq i \leq n}$ is a partition of $\deg p$, which we shall call *the factorization pattern of $p(t)$ over K.*

Let W be an irreducible Weyl group of type A_l, D_l, or E_l. Then up to a little deviation, the factorization patterns $\pi_{\boldsymbol{Q}(q)}(\tilde{\tau}_{prim}(\chi; q, y))$ and $\pi_{\boldsymbol{Q}}(\tau^*_{prim}(\chi; t))$ depend only on the element of $\mathscr{M}(\mathscr{G})$ associated to $\chi \in W^\vee$, if $\mathscr{G} = \mathscr{G}(\chi)$ is $\mathfrak{S}_m$ ($m = 1, 2, 3, 5$). (Cf. §0.1.) More precisely, we have the following.

THEOREM 0.2. *Let W_i ($i = 1, 2$) be two irreducible Weyl groups of type A_l, D_l, or E_l, $\chi_i \in W_i^\vee$, $\mathscr{G}_i := \mathscr{G}(\chi_i) \in \{\mathfrak{S}_m \,|\, m = 1, 2, 3, 5\}$, and $(x_i, \rho_i) \in \mathscr{M}(\mathscr{G}_i)$ the element associated to χ_i.*

(1) *The two factorization patterns $\pi_{\boldsymbol{Q}(q)}(\tilde{\tau}_{prim}(\chi_i; q, y))$ for $i = 1, 2$ are the same, whenever $\mathscr{G}_1 = \mathscr{G}_2$ and $(x_1, \rho_1) = (x_2, \rho_2)$.*

(2) *If* (W_i, χ_i) $(i = 1, 2)$ *are not among the following list of* (W, χ), *the same result holds for* $\pi_{\boldsymbol{Q}}(\tau^*_{prim}(\chi; t))$.

(a) $W = D_l$ $(l \geq 5)$

(b) $W = W(E_6)$, $\chi \in \{30_p, 30'_p\}$,

(c) $W = W(E_8)$, $\chi \in \{210_x, 210'_x\}$,

(d) $W = W(E_7)$, $\chi \in \{512'_a, 512_a\}$,

(e) $W = W(E_8)$, $\chi \in \{4096_z, 4096_x, 4096'_z, 4096'_x\}$.

Concerning the irreducible representations of Weyl groups, we follow the notation in [**L1**, §4].

In the above list, (2d) and (2e) seem to be real exceptions, but (2a)–(2c) do not. In fact, in the case of (2a)–(2c), $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is an irreducible polynomial of degree 2 as is predicted by the general rule (cf. Theorem 4.1 (1b)). On the other hand, the extrapolation of the factorization patterns of $\tau^*_{prim}(\chi; t)$'s except for χ's in (2a)–(2c) might predict that they would be an irreducible polynomial of degree 2. But incidentally it decomposes into two linear factors, and thus we are obliged to exclude these χ's (cf. Theorem 4.2 (1b)).

In §4.1, we shall generalize the above result on the factorization pattern so that any irreducible Weyl group not of type D_l $(l \geq 5)$ will be included. Further, in §4.2, we reformulate the result in terms of the modular representation theory of the (generic) Iwahori-Hecke algebras.

0.4. Product formula for $\tau^*(\chi; t)$ and $\tilde{\tau}(\chi; q, y)$.

As we have seen, our invariants do not completely decompose into linear factors in general. However, $\tau^*(\chi_1; q) + \tau^*(\chi_2; q)$ decomposes into linear factors for arbitrary $\chi_1 \neq \chi_2$ in a family consisting of 3-elements. This seems to be a part of more general phenomenon. We have obtained some results in this direction. See Proposition 5.1 and (5.8) in Theorem 5.2. As for $\tilde{\tau}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$, see the formula (5.9).

0.5. Convention and Notation.

We denote the rational integer ring by $\boldsymbol{Z}$, the rational number field by $\boldsymbol{Q}$, the real number field by $\boldsymbol{R}$, and the complex number field by $\boldsymbol{C}$. For a set X, $\# X$ denotes its cardinality, and $\mathfrak{S}(X)$ the symmetric group on X. Put $\mathfrak{S}_n = \mathfrak{S}(1, 2, \ldots, n) := \mathfrak{S}(\{1, 2, \ldots, n\})$. The identity element of a finite group is denoted by e, *not* by 1. For two complex valued functions f_1 and f_2 on a finite group Γ, put $\langle f_1, f_2 \rangle_\Gamma := (\# \Gamma)^{-1} \sum_{g \in \Gamma} f_1(g) f_2(g^{-1})$. The set of isomorphism classes of irreducible (complex linear) representations of Γ is denoted by $\Gamma^\vee$. For a character χ or Γ, we denote by $\dim \chi$ the value $\chi(e)$ at the identity element, which is equal to the dimension of the corresponding representation of Γ. Concerning the irreducible representations of Weyl groups, we follow the notation in [**L1**, §4].

1. Definition of invariants.

Let V be a complex vector space of dimension l on which the Weyl group W acts faithfully as a reflection group. For $w \in W$, let $V(w)$ denote the space of w-fixed vectors in V. We define $\tilde{\tau}(q,y)$, $\tau(t)$ and $\sigma(q)$ as follows;

$$\tilde{\tau}(q,y)(w) := \frac{\det(1+yw|V)}{\det(1-qw|V)}, \tag{1.1}$$

$$\tau(t)(w) := t^{\dim V(w)}, \tag{1.2}$$

$$\sigma(q)(w) := \frac{\mathrm{trace}(w|V)}{\det(1-qw|V)}, \tag{1.3}$$

where t, q and y are indeterminates. Note that $\sigma(q)(w)$ is the coefficient of the linear term of $\tilde{\tau}(q,y)(w)$ with respect to the variable y. For an irreducible character χ of W, put

$$\tilde{\tau}(\chi;q,y) := \langle \chi, \tilde{\tau}(q,y)\rangle_W, \tag{1.4}$$

$$\tau(\chi;t) := \langle \chi, \tau(t)\rangle_W, \tag{1.5}$$

$$\sigma(\chi;q) := \langle \chi, \sigma(q)\rangle_W, \tag{1.6}$$

$$\tau^*(\chi;t) := (\# W/\dim\chi)\tau(\chi;t), \tag{1.7}$$

$$\sigma^*(\chi;q) := \sigma(\chi;q)/\tilde{\tau}(\chi;q,0), \tag{1.8}$$

where $\dim\chi = \chi(e)$ is the dimension of the representation corresponding to χ. Let $l'(w)$ $(w \in W)$ be the minimum of the length m of the various expressions $w = r_1 r_2 \cdots r_m$, where r_i's are arbitrary reflections in W. Then $l'(w) = l - \dim V(w)$ [**C1**, §2], and the function $c(\chi;t)$ in Introduction is given by

$$c(\chi;t) = t^l \tau^*(\chi;t^{-1}). \tag{1.9}$$

Therefore, we shall study the functions (1.4)–(1.8) instead of $c(\chi;t)$.

Let $\{m_1,\dots,m_l\}$ be the exponents of W (See [**B**, Chapter 5, §5, Exercise 3]). Here we understand the exponents so that $\{m_i+1\}$ are the degrees of the basic W-invariant polynomials on V. In particular, if we allow W to have non-zero fixed vectors in V, some of the exponents may be zero.

Proposition 1.1. (1) *We have*

$$\prod_{i=1}^{l}(1-q^{m_i+1})\tilde{\tau}(\chi;q,y) \in \boldsymbol{Z}[q,y], \quad \prod_{i=1}^{l}(1-q^{m_i+1})\sigma(\chi;q) \in \boldsymbol{Z}[q].$$

(2) *We have an expansion*

$$\frac{\tilde{\tau}(\chi;q,y)}{\tilde{\tau}(\chi;q,0)} = 1 + \sigma^*(\chi;q)y + \sum_{i=2}^{l}\sigma_i^*(\chi;q)y^i \tag{1.10}$$

with some $\sigma_i^*(\chi;q) \in \boldsymbol{Q}(q)$.

(3) *We obtain* $\tau^*(\chi;t)$ *from* $\tilde{\tau}(\chi;q,y)$ *by the following specialization*:

$$\tau^*(\chi;t) = \frac{\# W}{\dim\chi}\lim_{q\to 1}\tilde{\tau}(\chi;q,-1+t(1-q)) \in \boldsymbol{Z}[t]. \tag{1.11}$$

This proposition is an immediate consequence of the following lemma.

LEMMA 1.2. (1) *The functions* τ *and* σ *are obtained from* $\tilde{\tau}(q,y)$ *by the following specialization*:

$$\lim_{q\to 1}\tilde{\tau}(q,-1+t(1-q))(w)=\tau(t)(w),\quad \frac{\partial}{\partial y}\tilde{\tau}(q,y)(w)\bigg|_{y=0}=\sigma(q)(w). \tag{1.12}$$

(2) *Let* $S^n(V)$ (*resp.* $\bigwedge^m(V)$) *denote the* n-*th* (*resp.* m-*th*) *homogeneous part of the symmetric algebra* (*resp. the exterior algebra*) *of* V. *We denote the space of* W-*harmonic polynomials of degree* n *by* $H^n(V)\subset S^n(V)$ (cf. [St2]). *Then we get*

$$\tilde{\tau}(\chi;q,y)=\frac{1}{\prod_{i=1}^{l}(1-q^{m_i+1})}\sum_{n,m\geq 0}[\chi:H^n(V)\otimes \textstyle\bigwedge^m(V)]q^n y^m, \tag{1.13}$$

$$\sigma(\chi;q)=\frac{1}{\prod_{i=1}^{l}(1-q^{m_i+1})}\sum_{n\geq 0}[\chi:H^n(V)\otimes V]q^n, \tag{1.14}$$

$$\sigma^*(\chi;q)=\frac{\sum_{n\geq 0}[\chi:H^n(V)\otimes V]q^n}{\sum_{n\geq 0}[\chi:H^n(V)]q^n}. \tag{1.15}$$

Here, we denote by $[U_1:U_2]=\mathrm{Hom}_W(U_1,U_2)$ *the intertwining number of* W-*modules* U_i $(i=1,2)$.

PROOF OF LEMMA. (1) is an easy consequence of the definition.
(2) By definition we can show that

$$\tilde{\tau}(q,y)(w)=\sum_{n,m\geq 0}\mathrm{trace}(w|S^n(V)\otimes \textstyle\bigwedge^m(V))q^n y^m, \tag{1.16}$$

$$\sigma(q)(w)=\sum_{n\geq 0}\mathrm{trace}(w|S^n(V)\otimes V)q^n. \tag{1.17}$$

Note that the symmetric algebra is decomposed as $S(V)=H(V)\otimes I(V)$, where $I(V)$ is the subalgebra of W-invariants in $S(V)$. Since

$$\sum_{n\geq 0}I^n(V)q^n=\frac{1}{\prod_{i=1}^{l}(1-q^{m_i+1})}\quad (I^n(V):=I(V)\cap S^n(V)),$$

we get the formulas (1.13)–(1.15) by the orthogonality relation of irreducible characters. □

The polynomial $\tilde{\tau}(\chi;q,y)$ contains a lot of factors of the form $1+yq^c$ $(c\in\boldsymbol{Z})$ (see §3 for examples). So we consider the largest divisor $\tilde{c}'(\chi;q,y)$ of $\tilde{\tau}(\chi;q,y)\in\boldsymbol{Q}(q)[y]$ which is of the form $\prod_{i=1}^{\kappa}(1+yq^{c_i})$ $(c_i\in\boldsymbol{Z})$. Put

$$\tilde{c}(\chi;q,y)=\mathrm{GCD}\{\tilde{c}'(\chi';q,y)\,|\,\chi'\underset{LR}{\sim}\chi\}=\prod_{i=1}^{\kappa(\chi)}(1+yq^{c_i(\chi)}), \tag{1.18}$$

$$\tilde{\tau}_{prim}(\chi;q,y):=\tilde{\tau}(\chi;q,y)\frac{(1-q)^{\kappa(\chi)}}{\prod_{i=1}^{\kappa(\chi)}(1+yq^{c_i(\chi)})}, \tag{1.19}$$

$$\tau^*_{prim}(\chi;t) := \frac{\tau^*(\chi;t)}{\prod_{i=1}^{\kappa(\chi)}(t+c_i(\chi))}, \tag{1.20}$$

$$\sigma^*_{prim}(\chi;q) := \sigma^*(\chi;q) - \sum_{i=1}^{\kappa(\chi)} q^{c_i(\chi)}. \tag{1.21}$$

Proposition 1.3. (1) $\tau^*_{prim}(\chi;t) \in \boldsymbol{Z}[t]$.
(2) *The following formula holds with some* $\sigma^*_{i,prim}(\chi;q) \in \boldsymbol{Q}(q)$;

$$\frac{\tilde{\tau}_{prim}(\chi;q,y)}{\tilde{\tau}_{prim}(\chi;q,0)} = 1 + \sigma^*_{prim}(\chi;q)y + \sum_{i=2}^{l-\kappa(\chi)} \sigma^*_{i,prim}(\chi;q)y^i. \tag{1.22}$$

Proof. Abbreviating $\kappa = \kappa(\chi)$ and $c_i = c_i(\chi)$, we get

$$\begin{aligned}\boldsymbol{Q}(q)[t] \ni &\frac{\# W}{\dim\chi}\,\tilde{\tau}_{prim}(\chi;q,-1+t(1-q))\\ &= \frac{\# W}{\dim\chi}\,\tilde{\tau}(\chi;q,-1+t(1-q)) \Big/ \prod_{i=1}^{\kappa}\left(tq^{c_i} + \frac{1-q^{c_i}}{1-q}\right)\\ &\to \tau^*(\chi;t) \Big/ \prod_{i=1}^{\kappa}(t+c_i) \quad (q\to 1)\end{aligned} \tag{1.23}$$

by (1.11). Therefore $\tau^*_{prim}(\chi;t)$ is a polynomial in t. Since $\tau^*(\chi;t) \in \boldsymbol{Z}[t]$, we get (1). (2) immediately follows from the definition. $\square$

Proposition 1.4. *Let* sgn *be the sign representation of* W. *Then we have*

$$\tilde{\tau}(\chi;q,y) = (-q)^{-l}\tilde{\tau}(\chi\otimes\mathrm{sgn};q^{-1},y) = y^l\tilde{\tau}(\chi\otimes\mathrm{sgn};q,y^{-1}), \tag{1.24}$$

$$\tau^*(\chi;t) = (-1)^l\tau^*(\chi\otimes\mathrm{sgn};-t), \tag{1.25}$$

$$\sigma^*(\chi;q) = (-q)^{-l}\sigma^*(\chi\otimes\mathrm{sgn};q^{-1}), \tag{1.26}$$

$$\tilde{c}'(\chi;q,y) = \tilde{c}'(\chi\otimes\mathrm{sgn};q^{-1},y), \quad \tilde{c}(\chi;q,y) = \tilde{c}(\chi\otimes\mathrm{sgn};q^{-1},y), \tag{1.27}$$

$$\kappa(\chi) = \kappa(\chi\otimes\mathrm{sgn}), \quad \{c_i(\chi)\}_i = \{-c_i(\chi\otimes\mathrm{sgn})\}_i, \tag{1.28}$$

$$\begin{aligned}\tilde{\tau}_{prim}(\chi;q,y) &= (-q)^{\kappa(\chi)-l}\tilde{\tau}_{prim}(\chi\otimes\mathrm{sgn};q^{-1},y)\\ &= y^{l-\kappa(\chi)}q^{-\sum_i c_i(\chi)}\tilde{\tau}_{prim}(\chi\otimes\mathrm{sgn};q,y^{-1}),\end{aligned} \tag{1.29}$$

$$\tau^*_{prim}(\chi;t) = (-1)^{l-\kappa(\chi)}\tau^*_{prim}(\chi\otimes\mathrm{sgn};-t). \tag{1.30}$$

Proof. Note that w and w^{-1} are conjugate in W. Then we can show that

$$\tilde{\tau}(q,y)(w) = (-q)^{-l}\,\mathrm{sgn}(w)\cdot\tilde{\tau}(q^{-1},y)(w) = y^l\,\mathrm{sgn}(w)\cdot\tilde{\tau}(q,y^{-1})(w), \tag{1.31}$$

$$\tau(t)(w) = (-1)^l \operatorname{sgn}(w) \cdot \tau(-t)(w), \tag{1.32}$$

$$\sigma(q)(w) = (-q)^{-l} \operatorname{sgn}(w) \cdot \sigma(q^{-1})(w). \tag{1.33}$$

The proposition immediately follows from the above three formulas. □

2. Some specialization.

2.1. Specialization.

Let $\{m_1, \dots, m_l\}$ be the exponents of W. Put $f_q(w) := \prod_{i=1}^{l}(1 - q^{m_i+1})/\det(1 - qw|V)$. Then

$$f(\chi; q) := \langle \chi, f_q \rangle_W = \prod_{i=1}^{l} (1 - q^{m_i+1}) \tilde{\tau}(\chi; q, 0) \tag{2.1}$$

is called the *fake degree* of χ, whose explicit form is given in [**St1**] (for type A_l), [**L2**] (for types $B_l(= C_l)$ and D_l), and [**BL**] (for exceptional types).

Proposition 2.1. *If χ is the trivial character* 1, *then χ forms an equivalence class with respect to $\underset{LR}{\sim}$ by itself, and we have*

$$\tilde{\tau}(1; q, y) = \prod_{i=1}^{l} \frac{1 + yq^{m_i}}{1 - q^{m_i+1}}, \quad \tau^*(1; t) = \prod_{i=1}^{l} (t + m_i), \quad \sigma^*(1; q) = \prod_{i=1}^{l} q^{m_i}, \tag{2.2}$$

$$\tilde{c}(1; q, y) = \tilde{c}'(1; q, y) = \prod_{i=1}^{l} (1 + yq^{m_i}). \tag{2.3}$$

Proof. For $\tilde{\tau}(1; q, y)$, see [**B**, Chapter 5, §5, Exercise 3]. The other formulas are easy consequences of it after appropriate specialization. □

Lemma 2.2. *The following special values of $\tilde{\tau}$, τ^* and σ^* are obtained:*

$$\tilde{\tau}(\chi; q, -q) = \begin{cases} 1 & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1, \end{cases} \tag{2.4}$$

$$\tilde{\tau}(\chi; q, -q^{-1}) = \begin{cases} (-q)^{-l} & \text{if } \chi = \operatorname{sgn}, \\ 0 & \text{if } \chi \neq \operatorname{sgn}, \end{cases} \tag{2.5}$$

$$\tau^*(\chi; 1) = \begin{cases} \# W & \text{if } \chi = 1, \\ 0 & \text{if } \chi \neq 1, \end{cases} \tag{2.6}$$

$$\tau^*(\chi; -1) = \begin{cases} (-1)^l \cdot (\# W) & \text{if } \chi = \operatorname{sgn}, \\ 0 & \text{if } \chi \neq \operatorname{sgn}, \end{cases} \tag{2.7}$$

$$\sigma^*(\chi; 1) = l \quad (= \dim V). \tag{2.8}$$

$$\lim_{q \to 1} \prod_{i=1}^{l} (1 - q^{m_i+1}) \cdot \tilde{\tau}(\chi; q, y)/\dim \chi = (1 + y)^l \tag{2.9}$$

Proof. By the definition of $\tilde{\tau}$ and τ^* (2.4) and (2.6) are obvious. Thus (1.24) (resp. (1.25)) yields (2.5) (resp. (2.7)). To prove (2.8), let us put $q = 1$ in (1.15). Then the numerator becomes

$$\sum_{n\geq 0}[\chi : H^n(V) \otimes V] = [\chi : \boldsymbol{C}W \otimes V] = \dim V \cdot [\chi : \boldsymbol{C}W] = \dim V \cdot \dim \chi,$$

where $\boldsymbol{C}W$ denotes the left regular representation of W. Similarly the denominator becomes $\dim \chi$ and we get the formula. For the last formula, we calculate as follows:

$$\begin{aligned}\tilde{\tau}(\chi; q, y) &= \frac{1}{\# W} \sum_{w \in W} \chi(w) \frac{\sum_m \operatorname{trace}(w|\bigwedge^m V) y^m}{\det(1 - qw)} \\ &= \sum_m \langle \chi \otimes \bigwedge^m V, f_q \rangle_W y^m \Big/ \prod_{i=1}^{l} (1 - q^{m_i+1}),\end{aligned}$$

hence we get

$$\lim_{q\to 1} \prod_{i=1}^{l} (1 - q^{m_i+1}) \tilde{\tau}(\chi; q, y) = \dim \chi \sum_m \binom{l}{m} y^m = (\dim \chi)(1 + y)^l, \tag{2.10}$$

using $f(\chi; 1) = \dim \chi$. □

2.2. Coefficients of $\tilde{\tau}(\chi; q, y)$.

Consider $\tilde{\tau}(\chi; q, y)$ as a polynomial in y with coefficients in $\boldsymbol{Q}(q)$. By (2.1) and (1.24),

$$\begin{aligned}\tilde{\tau}(\chi; q, y) &= \frac{1}{\prod_{i=1}^{l}(1 - q^{m_i+1})}(f(\chi; q) + \cdots + f(\chi \otimes \operatorname{sgn}; q) y^l) \\ &= \frac{(-q)^{-l}}{\prod_{i=1}^{l}(1 - q^{-m_i-1})}(f(\chi \otimes \operatorname{sgn}; q^{-1}) + \cdots + f(\chi; q^{-1}) y^l).\end{aligned} \tag{2.11}$$

Here only the lowest and the highest terms are written. In the sequel, we use the similar abbreviation. In order to rewrite (2.11), we review some results of Lusztig. Assume that W is an irreducible Weyl group, and let

$$W^\vee_{ex.} = \begin{cases} \{\chi \in W^\vee \mid \dim \chi = 512\}, & \text{if } W = W(E_7), \\ \{\chi \in W^\vee \mid \dim \chi = 4096\}, & \text{if } W = W(E_8), \\ \varnothing, & \text{otherwise.} \end{cases} \tag{2.12}$$

Then $W^\vee_{ex.}$ is the set of exceptional representations [**L3**]. (Note that this notation is not compatible with that of [**L1**].) Let $d(\chi; q)$ be the *generic degree* of χ (cf. [**L1**, p. 61, $\uparrow \ell\ell$. 5-1], where it is called the 'formal dimension'). Express $f(\chi; q)$ and $d(\chi; q)$ as

$$f(\chi; q) = \alpha'(\chi) q^{a'(\chi)} + \cdots + \beta'(\chi) q^{b'(\chi)}, \tag{2.13}$$

$$d(\chi; q) = \alpha(\chi) q^{a(\chi)} + \cdots + \beta(\chi) q^{b(\chi)}, \tag{2.14}$$

where $\alpha'(\chi) \neq 0$, $\beta'(\chi) \neq 0$, $\alpha(\chi) \neq 0$, $\beta(\chi) \neq 0$, $a'(\chi) \leq b'(\chi)$, $a(\chi) \leq b(\chi)$, and the omitted terms are of the degrees between them. Then it is known that

$$\chi_1 \underset{LR}{\sim} \chi_2 \Rightarrow a(\chi_1) = a(\chi_2) \quad [\mathbf{L1}, 5.27] \tag{2.15}$$

for any $\chi_1, \chi_2 \in W^\vee$. Hence we may write $a(\mathscr{F}) := a(\chi)$ for $\chi \in \mathscr{F} \in W^\vee / \underset{LR}{\sim}$. Further, if $\chi \notin W^\vee_{ex.}$, it is known that

$$q^{-a'(\chi \otimes \mathrm{sgn})} f(\chi \otimes \mathrm{sgn}; q) = q^{-a'(\chi)} f(\chi; q) \quad [\mathbf{BL}], \tag{2.16}$$

$$-a'(\chi) + a'(\chi \otimes \mathrm{sgn}) = -a(\chi) + a(\chi \otimes \mathrm{sgn}) \quad [\mathbf{L3}, (2.2) \text{ and } (2.3)], \tag{2.17}$$

$$a'(\chi \otimes \mathrm{sgn}) + b'(\chi) = a(\chi \otimes \mathrm{sgn}) + b(\chi) = \nu \quad [\mathbf{L3}, (2.3)], \tag{2.18}$$

where ν is the number of reflections in W. Put $A = A(\mathscr{F}) := -a(\mathscr{F}) + a(\mathscr{F} \otimes \mathrm{sgn})$. By (2.11), (2.16), (1.10) and (1.24), we get

$$\tilde{\tau}(\chi; q, y) = \frac{f(\chi; q)}{\prod_{i=1}^{l}(1 - q^{m_i+1})}(1 + \sigma^*(\chi; q)y + \cdots + \sigma^*(\chi \otimes \mathrm{sgn}; q)q^A y^{l-1} + q^A y^l) \tag{2.19}$$

for $\chi \notin W^\vee_{ex.}$.

2.3. Coefficients of $\tau^*(\chi; t)$.

If we put

$$\tau^*(\chi; t) = t^l + \tau_1 t^{l-1} + \cdots + \tau_l \quad (\tau_i = \tau_i(\chi) \in \boldsymbol{Z}), \tag{2.20}$$

then as is easily seen

$$\tau_1(\chi) = \frac{1}{\dim \chi} \sum_s \chi(s), \tag{2.21}$$

where s runs over the totality of the reflections in W.

Lemma 2.3. *For any $\chi \in W^\vee$, the following two formulas hold;*

$$\tau_1(\chi) = -a(\mathscr{F}) + a(\mathscr{F} \otimes \mathrm{sgn}) = A(\mathscr{F}), \tag{2.22}$$

$$\left.\frac{d}{dq}\sigma^*(\chi; q)\right|_{q=1} = -a(\mathscr{F}) + a(\mathscr{F} \otimes \mathrm{sgn}). \tag{2.23}$$

Proof. For $\chi \notin W^\vee_{ex.}$, we have $\nu - \tau_1(\chi) = a'(\chi) + b'(\chi)$ by [**BL**, Proposition B]. Hence by (2.17) and (2.18), we get (2.22) if $\chi \in \mathscr{F} \in W^\vee / \underset{LR}{\sim}$ and $\chi \notin W^\vee_{ex.}$. Since we can directly verify the same equality also for $\chi \in W^\vee_{ex.}$, (2.22) holds for any $\chi \in W^\vee$. Now consider the limit of

$$(1 - q)^{-l}(1 + \sigma^*(\chi; q)y + \cdots + \sigma^*(\chi \otimes \mathrm{sgn}; q)q^A y^{l-1} + q^A y^l)\Big|_{y=-1+t(1-q)} \tag{2.24}$$

when $q \to 1$, assuming $\chi \notin W^\vee_{ex.}$. By (1.11), this limit is $\tau^*(\chi; t)$. On the other hand, the coefficient of t^{l-1} in (2.24) is

$$\frac{\sigma^*(\chi \otimes \mathrm{sgn}; q) - l}{1 - q} q^A.$$

Hence

$$\left.\frac{d}{dq}\sigma^*(\chi;q)\right|_{q=1} = \lim_{q\to 1}\frac{\sigma^*(\chi;q)-l}{q-1} \quad \text{by (2.8)}$$

$$= -\tau_1(\chi\otimes \mathrm{sgn}) = -a(\mathscr{F}) + a(\mathscr{F}\otimes \mathrm{sgn}) \quad \text{by (2.22)} \qquad (2.25)$$

if $\chi\notin W^\vee_{ex.}$. Moreover a direct calculation shows that (2.23) holds even if $\chi\in W^\vee_{ex.}$. □

3. Explicit form of invariants.

In this section, we calculate our functions $\tilde{\tau}$, τ^*, σ^*, $\tilde{c}'$ and $\tilde{c}$ explicitly for each type of the Weyl groups, and see whether the invariant $\tilde{c}(\chi;q,y)$ separates the two-sided cells. Summing up, we get Theorem 0.1.

3.1. Type A_l.

We realize the reflection group $W = W(A_l)$ $(= \mathfrak{S}_{l+1})$ as the group of permutation matrices of size $l+1$. (By an obvious modification of the argument below, we can calculate the invariants $\tilde{\tau}$ etc. for the *irreducible* reflection group W of type A_l.) Then for $w\in W$ of cycle type $\rho = (\rho_1 \geq \cdots \geq \rho_{l+1-l'(w)} > 0)$ [**M**, p. 60],

$$\tilde{\tau}(q,y)(w) = \prod_i \frac{1-(-y)^{\rho_i}}{1-q^{\rho_i}}. \qquad (3.1)$$

Let α be a partition of $l+1$ and χ^α the corresponding irreducible representation of $W(A_l) = \mathfrak{S}_{l+1}$. We identify the partition α with the corresponding Young diagram, which we understand as a subset of $\boldsymbol{Z}^2$ as in [**M**]. For the sake of convenience, we recollect some of the notations in [M]: for α, $|\alpha| := \sum_i \alpha_i = l+1$, $n(\alpha) := \sum_{i\geq 1}(i-1)\alpha_i$ and $\alpha^* = (\alpha_1^*\geq\alpha_2^*\geq\cdots)$ the dual partition of α. For $x=(i,j)\in\alpha$, $c(x) := j-i$ (the *content*), and $h(x) := (\alpha_i - i) + (\alpha_j^* - j) + 1$ (the *hook length*).

Proposition 3.1. (1) *For the Weyl group $W(A_l)$ of type A_l, we get*

$$\tilde{\tau}(\chi^\alpha;q,y) = q^{n(\alpha)}\prod_{x\in\alpha}\frac{1+yq^{c(x)}}{1-q^{h(x)}}, \qquad (3.2)$$

$$\tau^*(\chi^\alpha;t) = \prod_{x\in\alpha}(t+c(x)), \qquad (3.3)$$

$$\sigma^*(\chi^\alpha;q) = \sum_{x\in\alpha} q^{c(x)}, \qquad (3.4)$$

$$\tilde{c}(\chi^\alpha;q,y) = \tilde{c}'(\chi^\alpha;q,y) = \prod_{x\in\alpha}(1+yq^{c(x)}). \qquad (3.5)$$

(2) *The invariant $\tilde{c}(\chi^\alpha;q,y)$ separates the two-sided cells in $W(A_l)$.*

Proof. (1) follows from [**M**, §I.2, Ex. 5 and §I.3, Ex. 3] (with $a=1$, $b=-y$, $q=q$, and $\tilde{\tau}$ being p_ρ). As for (2), see the proof of Proposition 3.3 below. □

3.2. Type $B_l(=C_l)$.

Let $W = W_l = W(B_l)$ be the centralizer of $(1,\bar{1})(2,\bar{2})\cdots(l,\bar{l})$ in $\mathfrak{S}(1,2,\ldots,l,\bar{l},\ldots,\bar{2},\bar{1})$, $\theta : W \to W/\langle(1,\bar{1}),(2,\bar{2}),\ldots,(l,\bar{l})\rangle = \mathfrak{S}_l$ the projection, $s_1 := (1,\bar{1})$, $s_i := (i-1,i)(\overline{i-1},\bar{i})$ $(1 < i \le l)$, and $S := \{s_1,\ldots,s_l\}$. (Here $\bar{i}$ should be understood as one symbol, or as $-i$.) Then (W,S) is a Coxeter system of type B_l. We realize W as a reflection group acting on $V = \bigoplus_{i=1}^{l} \boldsymbol{C}e_i$ by

$$s_1 e_j = \begin{cases} -e_1, & \text{for } j = 1 \\ e_j, & \text{for } j \neq 1, \end{cases} \quad \text{and} \quad s_i e_j = \begin{cases} e_i, & \text{for } j = i-1 \\ e_{i-1}, & \text{for } j = i \\ e_j, & \text{for } j \neq i-1, i, \end{cases} \tag{3.6}$$

for $1 < i \le l$. Define $\varepsilon \in \mathrm{Hom}(W, \boldsymbol{C}^\times)$ by $\varepsilon(s_1) = -1$ and $\varepsilon(s_i) = 1$ $(i \neq 1)$. Let $l = l' + l''$ $(l', l'' \ge 0)$, and (α,β) be an ordered pair of partitions of l' and l''. Let W' (resp. W'') be the pointwise stabilizer of $\{l''+1,\ldots,l\}$ (resp. $\{1,\ldots,l''\}$) in W. Then W' and W'' are naturally isomorphic to $W_{l'}$ and $W_{l''}$, respectively. Define $\theta' : W' \to \mathfrak{S}_{l'}$ and $\theta'' : W'' \to \mathfrak{S}_{l''}$ in the same way as θ. Put $\varepsilon'' := \varepsilon|_{W''}$ and define $\chi^{\alpha,\beta}$ as the induced character

$$\chi^{\alpha,\beta} := \mathrm{ind}_{W'\times W''}^{W}((\chi^\alpha \circ \theta') \otimes (\chi^\beta \circ \theta'' \otimes \varepsilon'')). \tag{3.7}$$

Then $\chi^{\alpha,\beta}$ are irreducible characters of W, and all the irreducible characters of W can be obtained uniquely in this way.

Express the partitions α and β as $\alpha = (\alpha_1 \ge \alpha_2 \ge \cdots \ge \alpha_{m+1} \ge 0)$ and $\beta = (\beta_1 \ge \beta_2 \ge \cdots \ge \beta_m \ge 0)$ for sufficiently large m, by adding some parts which are equal to zero. Let $\lambda_i := \alpha_i + m + 1 - i$, $\mu_i := \beta_i + m - i$, and $\{\nu_1,\ldots,\nu_{2m+1}\} = \{\lambda_1,\ldots,\lambda_{m+1},\mu_1,\ldots,\mu_m\}$ (including multiplicity).

Lemma 3.2 ([**L1**, (4.5.6)]). *Let* $\chi = \chi^{\alpha,\beta}$ *and* $\{\nu_1,\ldots,\nu_{2m+1}\}$ *as above. If* $\{\nu'_1,\ldots,\nu'_{2m+1}\}$ *is associated to* $\chi' = \chi^{\alpha',\beta'}$ *by the same manner, then*

$$\chi \underset{LR}{\sim} \chi' \Leftrightarrow \{\nu_1,\ldots,\nu_{2m+1}\} = \{\nu'_1,\ldots,\nu'_{2m+1}\} \quad \textit{up to permutation.} \tag{3.8}$$

Proposition 3.3. (1) *For the Weyl group* $W(B_l)$ *of type* B_l, *we get*

$$\tilde{\tau}(\chi^{\alpha,\beta}; q, y) = q^{2n(\alpha)+2n(\beta)+|\beta|} \prod_{x'\in\alpha} \frac{1+yq^{2c(x')+1}}{1-q^{2h(x')}} \prod_{x''\in\beta} \frac{1+yq^{2c(x'')-1}}{1-q^{2h(x'')}}, \tag{3.9}$$

$$\tau^*(\chi^{\alpha,\beta}; t) = \prod_{x'\in\alpha}(t + 2c(x') + 1) \cdot \prod_{x''\in\beta}(t + 2c(x'') - 1), \tag{3.10}$$

$$\sigma^*(\chi^{\alpha,\beta}; q) = \sum_{x'\in\alpha} q^{2c(x')+1} + \sum_{x''\in\beta} q^{2c(x'')-1}, \tag{3.11}$$

$$\tilde{c}'(\chi^{\alpha,\beta}; q, y) = \prod_{x'\in\alpha}(1 + yq^{2c(x')+1}) \cdot \prod_{x''\in\beta}(1 + yq^{2c(x'')-1}). \tag{3.12}$$

(2) *The invariant* $\tilde{c}(\chi^{\alpha,\beta}; q, y)$ *coincides with* $\tilde{c}'(\chi^{\alpha,\beta}; q, y)$, *and it separates the two-sided cells in* $W(B_l)$.

Proof. Put $V' := \bigoplus_{i>l''} \boldsymbol{C}e_i$, $V'' := \bigoplus_{i\le l''} \boldsymbol{C}e_i$, and define $\tilde{\tau}'$ and $\tilde{\tau}''$ in the same way as $\tilde{\tau}$ for $W(A_l)$ in the former subsection, replacing V with V' and V'' respectively. Then

$$
\begin{aligned}
\tilde{\tau}(\chi^{\alpha,\beta}; q, y) &= \langle \chi^{\alpha,\beta}, \tilde{\tau}(q, y)\rangle_W \\
&= \langle \chi^{\alpha} \circ \theta', \tilde{\tau}'(q, y)\rangle_{W'} \times \langle (\chi^{\beta} \circ \theta'') \otimes \varepsilon'', \tilde{\tau}''(q, y)\rangle_{W''} \\
&= \langle \chi^{\alpha}, \tilde{\tau}'(q^2, qy)\rangle_{\mathfrak{S}_{l'}} \times \langle \chi^{\beta}, q^{l''}\tilde{\tau}''(q^2, q^{-1}y)\rangle_{\mathfrak{S}_{l''}} \\
&= q^{2n(\alpha)+2n(\beta)+|\beta|} \prod_{x'\in\alpha} \frac{1+yq^{2c(x')+1}}{1-q^{2h(x')}} \cdot \prod_{x''\in\beta} \frac{1+yq^{2c(x'')-1}}{1-q^{2h(x'')}},
\end{aligned}
\tag{3.13}
$$

which proves (3.9). The rest of the formulas in (1) easily follows from (3.9).

(2) Let $\{\nu_1, \dots, \nu_{2m+1}\} = \{\lambda_1, \dots, \lambda_{m+1}, \mu_1, \dots, \mu_m\}$ be as above. Then as is easily seen from Example 3.4 below,

$$
\prod_{i=1}^{m} (1 + yq^{-2i+1})^{2i-1} \cdot \tilde{c}'\left(\begin{pmatrix} \lambda_1 \cdots \lambda_{m+1} \\ \mu_1 \cdots \mu_m \end{pmatrix}; q, y \right) = \prod_{i=1}^{2m+1} \prod_{k=1}^{\nu_i} (1 + yq^{2k-2m-1}). \tag{3.14}
$$

(Here $\begin{pmatrix} \lambda_1 \cdots \lambda_{m+1} \\ \mu_1 \cdots \mu_m \end{pmatrix}$ is the symbol of $\chi^{\alpha,\beta}$. We often identify these two objects.) Hence for $\chi, \chi' \in W(B_l)^{\vee}$, we have

$$
\tilde{c}'(\chi; q, y) = \tilde{c}'(\chi'; q, y) \Leftrightarrow \chi \underset{LR}{\sim} \chi',
$$

by Proposition 3.2. Thus we get $\tilde{c}(\chi; q, y) = \tilde{c}'(\chi; q, y)$ and clearly it separates the cells. (Note that in the B_2 -case, $\mathrm{Aut}(W, S)$ acts on $W^{\vee}/\underset{LR}{\sim}$ trivially.) □

Example 3.4. If $\alpha = (5, 4, 4, 2)$ and $\beta = (3, 2)$, we can take, for example, $m = 3$ and express $\beta = (3, 2, 0)$. For the diagram α (resp. β), we put the content $2c(x) + 1$ $(x \in \alpha)$ (resp. $2c(x) - 1$ $(x \in \beta)$) in each box. Then we get the following tableaux.

$$
\alpha: \begin{array}{|c|c|c|c|c|} \hline 1 & 3 & 5 & 7 & 9 \\ \hline -1 & 1 & 3 & 5 & \\ \cline{1-4} -3 & -1 & 1 & 3 & \\ \cline{1-4} -5 & -3 & & & \\ \cline{1-2} \end{array}
\qquad
\beta: \begin{array}{|c|c|c|} \hline -1 & 1 & 3 \\ \hline -3 & -1 & \\ \cline{1-2} & & \end{array}
$$

Now we add $m + 1 - i = 4 - i$ (resp. $m - i = 3 - i$) nodes to the left of the i-th row for α (resp. β).

$$
\begin{array}{ccc|c|c|c|c|c|} \cline{4-8} -5 & -3 & -1 & 1 & 3 & 5 & 7 & 9 \\ \cline{4-8} & -5 & -3 & -1 & 1 & 3 & 5 & \\ \cline{4-7} & & -5 & -3 & -1 & 1 & 3 & \\ \cline{4-7} & & & -5 & -3 & & & \\ \cline{4-5} \end{array}
\qquad
\begin{array}{cc|c|c|c|} \cline{3-5} -5 & -3 & -1 & 1 & 3 \\ \cline{3-5} & -5 & -3 & -1 & \\ \cline{3-4} & & & & \end{array}
$$

Then λ_i (resp. μ_i) is the number of nodes in the i-th row of the left (resp. right) tableau above. Note that we have added $2i-1$ nodes labeled by $-2i+1$ (cf. (3.14)).

3.3. Type D_l.

We keep the notations in the preceding subsection. Let $W = \tilde{W}_l := \ker \varepsilon$ $(\subset W(B_l))$. Then $\tilde{W}_l$ is the reflection group of type D_l. Let (α, β) be an unordered pair of partitions. If $\alpha \neq \beta$, put $\tilde{\chi}^{\alpha,\beta} := \chi^{\alpha,\beta}|_{\tilde{W}_l}$. If $\alpha = \beta =: \alpha$, $\chi^{\alpha,\alpha}|_{\tilde{W}_l}$ is a sum of two irreducible characters $\chi_I^\alpha \neq \chi_{II}^\alpha$. Note that $\tilde{\chi}^{\alpha,\beta} = \tilde{\chi}^{\beta,\alpha}$ because $\chi^{\alpha,\beta} \otimes \varepsilon = \chi^{\beta,\alpha}$ holds. Then $\{\tilde{\chi}^{\alpha,\beta}, \chi_I^\alpha, \chi_{II}^\alpha\}$ are irreducible characters of $\tilde{W}_l$, and all the irreducible characters of $\tilde{W}_l$ can be obtained uniquely in this way.

For α and β above, let $\lambda_i := \alpha_{m+1-i} + i - 1$, $\mu_i := \beta_{m+1-i} + i - 1$ for $1 \leq i \leq m$, and $|\lambda| := \sum_i \lambda_i$ etc. Here we assume that $\alpha_{m+1} = \beta_{m+1} = 0$. Then $\Lambda = \begin{pmatrix} \lambda_1 \cdots \lambda_m \\ \mu_1 \cdots \mu_m \end{pmatrix}$ is the symbol associated to the irreducible character χ of $W(D_l)$ ([**L1**]). We identify an irreducible character with the associated symbol.

Lemma 3.5 ([**L1**, (4.6.10)]). *The equivalence relation $\underset{LR}{\sim}$ for $W(D_l)^\vee$ can be described as follows. Each irreducible character χ_I^α (resp. χ_{II}^α) forms an equivalence class by itself. If $\chi = \tilde{\chi}^{\alpha,\beta} = \begin{pmatrix} \lambda_1 \ldots \lambda_m \\ \mu_1 \ldots \mu_m \end{pmatrix}$ with $m \gg 0$, let $\{\nu_1, \ldots, \nu_{2m}\} = \{\lambda_1, \ldots, \lambda_m, \mu_1, \ldots, \mu_m\}$ (including multiplicity). If $\{\nu'_1, \ldots, \nu'_{2m}\}$ is associated to $\chi' = \tilde{\chi}^{\alpha',\beta'} = \begin{pmatrix} \lambda'_1 \ldots \lambda'_m \\ \mu'_1 \ldots \mu'_m \end{pmatrix}$ similarly, then*

$$\chi \underset{LR}{\sim} \chi' \Leftrightarrow \{\nu_1, \ldots, \nu_{2m}\} = \{\nu'_1, \ldots, \nu'_{2m}\} \quad \textit{up to permutation.}$$

Proposition 3.6. (1) *For the representation $\tilde{\chi}^{\alpha,\beta}$ $(\alpha \neq \beta)$ we get*

$$\tilde{\tau}(\tilde{\chi}^{\alpha,\beta}; q, y) = \frac{q^{2n(\alpha)+2n(\beta)}}{2\prod_{x \in \alpha \sqcup \beta}(1 - q^{2h(x)})} \Bigg\{ q^{|\beta|} \prod_{x' \in \alpha}(1 + yq^{2c(x')+1}) \prod_{x'' \in \beta}(1 + yq^{2c(x'')-1})$$
$$+ q^{|\alpha|} \prod_{x' \in \alpha}(1 + yq^{2c(x')-1}) \prod_{x'' \in \beta}(1 + yq^{2c(x'')+1}) \Bigg\}, \tag{3.15}$$

$$\tau^*(\tilde{\chi}^{\alpha,\beta}; t) = \frac{1}{2} \Bigg\{ \prod_{x' \in \alpha}(t + 2c(x') + 1) \prod_{x'' \in \beta}(t + 2c(x'') - 1)$$
$$+ \prod_{x' \in \alpha}(t + 2c(x') - 1) \prod_{x'' \in \beta}(t + 2c(x'') + 1) \Bigg\}, \tag{3.16}$$

$$\sigma^*(\tilde{\chi}^{\alpha,\beta}; q) = \frac{1}{2(q^{|\alpha|} + q^{|\beta|})} \cdot \Bigg\{ q^{|\beta|} \left(\sum_{x' \in \alpha} q^{2c(x')+1} + \sum_{x'' \in \beta} q^{2c(x'')-1} \right)$$
$$+ q^{|\alpha|} \left(\sum_{x' \in \alpha} q^{2c(x')-1} + \sum_{x'' \in \beta} q^{2c(x'')+1} \right) \Bigg\}. \tag{3.17}$$

(2) *For the representations* χ_I^α *and* χ_{II}^α, *we get*

$$\begin{aligned}\tilde{\tau}(\chi_I^\alpha; q, y) &= \tilde{\tau}(\chi_{II}^\alpha; q, y) \\ &= q^{4n(\alpha)+|\alpha|} \prod_{x \in \alpha} \frac{(1+yq^{2c(x)+1})(1+yq^{2c(x)-1})}{(1-q^{2h(x)})^2},\end{aligned} \tag{3.18}$$

$$\tau^*(\chi_I^\alpha; t) = \tau^*(\chi_{II}^\alpha; t) = \prod_{x \in \alpha}(t+2c(x)+1)(t+2c(x)-1), \tag{3.19}$$

$$\sigma^*(\chi_I^\alpha; q) = \sigma^*(\chi_{II}^\alpha; q) = \sum_{x \in \alpha}(q^{2c(x)+1}+q^{2c(x)-1}). \tag{3.20}$$

Proof. (1) Since $\chi^{\alpha,\beta} \otimes \varepsilon = \chi^{\beta,\alpha}$, we have

$$2\tilde{\tau}(\tilde{\chi}^{\alpha,\beta}; q, y) = \tilde{\tau}(\chi^{\alpha,\beta}; q, y) + \tilde{\tau}(\chi^{\beta,\alpha}; q, y),$$

which proves (3.15). The rest of the formulas in (1) easily follows from (3.15).

(2) Since the inner automorphism by s_1 in $W(B_l)$ induces an outer automorphism ι of $W(D_l)$ which interchanges χ_I^α and χ_{II}^α, we have

$$\tilde{\tau}(\chi^{\alpha,\alpha}; q, y) = 2\tilde{\tau}(\chi_I^\alpha; q, y) = 2\tilde{\tau}(\chi_{II}^\alpha; q, y).$$

From this we get (3.18), hence the rest of the formulas. $\square$

Next we calculate $\tilde{c}(\chi; q, y)$ for type D_l.

First, we consider the representations χ_I^α and χ_{II}^α. Since each χ_I^α (resp. χ_{II}^α) forms an equivalence class with respect to $\underset{LR}{\sim}$ by itself by Lemma 3.5, we have $\tilde{c}(\chi; q, y) = \tilde{c}'(\chi; q, y)$ for $\chi = \chi_I^\alpha$ or χ_{II}^α, and

$$\tilde{c}(\chi_I^\alpha; q, y) = \tilde{c}(\chi_{II}^\alpha; q, y) = \prod_{x \in \alpha}(1+yq^{2c(x)+1})(1+yq^{2c(x)-1}). \tag{3.21}$$

Secondly, we consider $\tilde{c}$ for $\tilde{\chi}^{\alpha,\beta}$ ($\alpha \neq \beta$). Put $[i] := 1+yq^i$, $[j,k] := \prod_{\substack{j \le i \le k \\ i \text{ is odd}}}[i]$ if $\{i \mid j \le i \le k, i \text{ is odd}\} \neq \emptyset$, and $[j,k] := 1$ if otherwise. Then the inside of $\{\ \}$ of (3.15) multiplied by $\prod_{i=1}^m[-2i+1]^{2i-1}$ ($m \gg 0$) is equal to

$$\begin{aligned} &q^{-\binom{m}{2}} \prod_{i=1}^m [-2m+1, 2\lambda_i-2m-1] \cdot [-2m+1, 2\mu_i-2m-1] \\ &\qquad \times \left\{ q^{|\mu|} \prod_{i=1}^m [2\lambda_i-2m+1] + q^{|\lambda|} \prod_{i=1}^m [2\mu_i-2m+1] \right\},\end{aligned} \tag{3.22}$$

where $\lambda_i := \alpha_{m+1-i}+i-1$, $\mu_i := \beta_{m+1-i}+i-1$ for $1 \le i \le m$ as before. Let $\Lambda = \begin{pmatrix} \lambda_1 \cdots \lambda_m \\ \mu_1 \cdots \mu_m \end{pmatrix} = \begin{pmatrix} A \sqcup C \\ B \sqcup C \end{pmatrix}$ with $A \cap B = \emptyset$. From the assumption $\alpha \neq \beta$, A and B are non-empty. Then (3.22) is equal to

$$\prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]$$

$$\times q^{-\binom{m}{2}} \left\{ q^{|\mu|} \prod_{a \in A} [2a-2m+1] + q^{|\lambda|} \prod_{b \in B} [2b-2m+1] \right\}, \tag{3.23}$$

where $C+1 := \{c+1 \mid c \in C\}$.

Lemma 3.7. *Put* $\chi = \tilde{\chi}^{\alpha,\beta}$ $(\alpha \neq \beta)$.

(1) *If there exists* $d \in \boldsymbol{Z}$ *such that*

$$A = \{a_1, \dots, a_r\}, \quad B = \{d-a_1, \dots, d-a_r\}, \quad \textit{and } r \textit{ is odd}, \tag{3.24}$$

then $\tilde{c}'(\chi; q, y)$ *is given by*

$$\prod_{i=1}^{m} [-2i+1]^{2i-1} \cdot \tilde{c}'(\chi; q, y) = [d-2m+1] \cdot \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]. \tag{3.25}$$

(2) *If the condition* (3.24) *does not hold, then* $\tilde{c}'(\chi; q, y)$ *is given by*

$$\prod_{i=1}^{m} [-2i+1]^{2i-1} \cdot \tilde{c}'(\chi; q, y) = \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]. \tag{3.26}$$

Proof. We can see that $\{\,\}$ in (3.23) vanishes for $y = -q^{-d+2m-1}$ $(d \in \boldsymbol{Z})$, if and only if the condition (3.24) holds. Let us prove it. The condition which d should satisfy is

$$q^{\sum_i b_i} \prod_{i=1}^{r} (1 - q^{2a_i - d}) + q^{\sum_i a_i} \prod_{i=1}^{r} (1 - q^{2b_i - d}) = 0,$$

which implies $\{2a_i - d\}_i = \{\varepsilon_i(2b_i - d)\}_i$ with some $\varepsilon_i = \pm 1$. Since $A \cap B = \emptyset$, all the ε_i's should be -1, hence we may assume that $2a_i - d = d - 2b_i$, i.e., $b_i = d - a_i$. Thus we can easily see that (3.24) is a necessary and sufficient condition.

Next, let us show that $y = -q^{-d+2m-1}$ is a simple root. By a change of variable $y = xq^{-d+2m-1}$, it suffices to show that $x = -1$ is a simple root of

$$\varphi(x) := \prod_{i=1}^{r} (q^{a_i} + xq^{b_i}) + \prod_{i=1}^{r} (q^{b_i} + xq^{a_i}),$$

assuming (3.24). Note that

$$\varphi'(-1) = \left(\sum_{j=1}^{r} \frac{q^{a_j} + q^{b_j}}{q^{a_j} - q^{b_j}} \right) \cdot \prod_i (q^{a_i} - q^{b_i}). \tag{3.27}$$

Since $A \cap B = \emptyset$, the first factor of (3.27) tends to $\sum_{j=1}^{r} (\pm 1)$ as $q \to 0$. Since r is odd, this limit is non-zero, and hence the first factor is non-zero. Since $A \cap B = \emptyset$, the second factor is also non-zero. □

Next proposition immediately follows from the above lemma.

PROPOSITION 3.8. *Take* $\chi = \tilde{\chi}^{\alpha,\beta} \in W(D_l)^\vee$ $(\alpha \neq \beta)$, *and let* $\lambda_i := \alpha_{m+1-i} + i - 1$, $\mu_i := \beta_{m+1-i} + i - 1$ *for* $1 \leq i \leq m$. *Here we assume* $\alpha_{m+1} = \beta_{m+1} = 0$. *Put* $C = \{\lambda_i\}_i \cap \{\mu_i\}_i$, $A = \{\lambda_i\}_i - C$, *and* $B = \{\mu_i\}_i - C$.

(1) *If* $\# A > 1$ (*hence* $\# B = \# A > 1$ *also*), *then* $\tilde{c}(\chi; q, y)$ *is given by*

$$\prod_{i=1}^{m} [-2i+1]^{2i-1} \cdot \tilde{c}(\chi; q, y) = \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1], \tag{3.28}$$

where $C + 1 = \{c + 1 \mid c \in C\}$.

(2) *If* $A = \{a\}$ *and* $B = \{b\}$, *then* $\tilde{c}(\chi; q, y)$ *is given by*

$$\prod_{i=1}^{m} [-2i+1]^{2i-1} \cdot \tilde{c}(\chi; q, y) = [a+b-2m+1] \cdot \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]. \tag{3.29}$$

As the following example shows, the invariant $\tilde{c}(\chi; q, y)$ does not separate the two-sided cells in $W(D_l)^\vee$.

EXAMPLE 3.9. Let $\Lambda = \begin{pmatrix} 0 & 1 & 3 \\ 1 & 4 & 6 \end{pmatrix}$ and $\Lambda' = \begin{pmatrix} 0 & 1 & 3 \\ 2 & 3 & 6 \end{pmatrix}$, and identify them with the corresponding irreducible characters of $W(D_9)$. Then

$$\{c_i(\Lambda)\}_i = \{c_i(\Lambda')\}_i = \{-3, -1, -1, 1, 1, 1, 3, 5\}.$$

Thus $\Lambda \underset{LR}{\not\sim} \Lambda'$ but $\tilde{c}(\Lambda; q, y) = \tilde{c}(\Lambda'; q, y)$.

For a later use, we reformulate (3.16) as follows.

LEMMA 3.10. *Take* $\chi \in W(D_l)^\vee$ *corresponding to* $\Lambda = \begin{pmatrix} \lambda_1 \cdots \lambda_m \\ \mu_1 \cdots \mu_m \end{pmatrix}$ *and let the sets* A, B *and* C *be as in Proposition* 3.8. *Then we have*

$$\begin{aligned} \prod_{i=1}^{m} (t-2i+1)^{2i-1} \times 2\tau^*(\Lambda; t) = &\prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]^* \\ &\times \left\{ \prod_{a \in A} (t+2a-2m+1) + \prod_{b \in B} (t+2b-2m+1) \right\}, \end{aligned} \tag{3.30}$$

where $[j,k]^* = \prod_{\substack{j \leq i \leq k \\ i \text{ is odd}}} (t+i)$.

3.4. Type G_2.

If $\chi = 1$ (resp. $\chi = \mathrm{sgn}$), then $\tilde{c}'(1; q, y) = (1 + yq)(1 + yq^5)$ (resp. $\tilde{c}'(\mathrm{sgn}; q, y) = (1 + yq^{-1})(1 + yq^{-5})$) by (2.3) and (1.27). The trivial (resp. sign) representation forms a family by itself. The remaining four representations form one family. In this case, $\tilde{c}'(\chi; q, y) = (1 + yq^{-1})(1 + yq)$ by (2.4) and (2.5). Hence always $\tilde{\tau}_{prim}(\chi; q, y) = 1$ and $\tilde{c}(\chi; q, y) = \tilde{c}'(\chi; q, y)$. See Table 2.

3.5. Types F_4 and E_l $(l = 6, 7, 8)$.

We give $c_i(\chi)$ $(1 \leq i \leq \kappa(\chi))$ and $\tau^*_{prim}(\chi; t)$ in Tables 3–6. Note that the set $\{c_i(\chi)\ (1 \leq i \leq \kappa(\chi))\}$ completely determines the invariant $\tilde{c}(\chi; q, y)$, so the tables describe both $\tilde{c}(\chi; q, y)$ and $\tau^*(\chi; t)$. The explicit form of $\tau_{prim}(\chi; q, y)$ is too complicated to include here.

Besides, we shall give the explicit form of $\tilde{c}(\chi; q, y) = \prod_{i=1}^{\kappa(\chi)} (1 + q^{c_i(\chi)} y)$, $\tilde{\tau}(\chi; q, y)$ and $\tau^*(\chi; t)$ for $\chi \in W_{ex.}^\vee$ in this subsection.

(1) If $W = W(E_7)$ and $\dim \chi = 512$, then $\kappa = \kappa(\chi) = 2$ and $\{c_i\}_{1 \le i \le \kappa} = \{c_i(\chi)\}_{1 \le i \le \kappa} = \{-1, 1\}$. Put $\{c_i\}_{\kappa < i \le l} = \{-4, -2, 0, 2, 4\}$ $(l = 7)$. Then $\tau^*(\chi; t) = \prod_{i=1}^{l} (t + c_i)$,

$$\tilde{\tau}(\chi; q, y) = \frac{f(\chi \otimes \mathrm{sgn}; q)}{\prod_{i=1}^{l} (1 - q^{m_i+1})} \prod_{i=1}^{l} (1 + q^{c_i} y) + \varepsilon q^5 \frac{q^2 \phi_5 \phi_{10} y^4 + \phi_5 \phi_8 \phi_{10} y^2 + q^6}{\phi_2^7 \phi_6^3 \phi_{10} \phi_{14} \phi_{18}} \prod_{i=1}^{\kappa} (1 + q^{c_i} y),$$

where $\phi_i = \phi_i(q)$ is the i-th cyclotomic polynomial (e.g., $\phi_3 = q^2 + q + 1$), and $\varepsilon = -1$ (resp. $+1$) if $\chi = 512_a'$ (resp. 512_a).

(2) If $W = W(E_8)$ and $\chi \in \{4096_z, 4096_x\}$, then $\kappa = \kappa(\chi) = 3$, and $\{c_i\}_{1 \le i \le \kappa} = \{c_i(\chi)\}_{1 \le i \le \kappa} = \{-1, 1, 7\}$. Put $\{c_i\}_{\kappa < i \le l} = \{-4, -2, 2, 4, 8\}$ $(l = 8)$. Then $\tau^*(\chi; t) = \prod_{i=1}^{l} (t + c_i)$, and

$$\begin{aligned} \tilde{\tau}(\chi; q, y) = {} & q^{-15} \frac{f(\chi \otimes \mathrm{sgn}; q)}{\prod_{i=1}^{l} (1 - q^{m_i+1})} \prod_{i=1}^{l} (1 + q^{c_i} y) \\ & + \varepsilon q^5 \frac{(q^{12} + q^{10} + q^6 + q^2 + 1) q^6 y^4 + (q^{14} + q^{12} - q^{10} + q^8 + 2q^6 + 1) \phi_8 y^2 + q^6}{\phi_1 \phi_2^8 \phi_6^4 \phi_{10}^2 \phi_{14} \phi_{18} \phi_{30}} \\ & \times \prod_{i=1}^{\kappa} (1 + q^{c_i} y), \end{aligned}$$

where $\varepsilon = -1$ (resp. $+1$) if $\chi = 4096_z$ (resp. 4096_x).

(3) If $W = W(E_8)$ and $\chi \in \{4096_z', 4096_x'\}$, then $\kappa(\chi)$, $c_i(\chi)$ $(1 \le i \le \kappa(\chi))$, $\tau^*(\chi; t)$ and $\tilde{\tau}(\chi; q, y)$ can be explicitly determined by (2) and the duality (see Proposition 1.4).

3.6. The main theorem.

We sum up the results in the former subsections into the following theorem.

Theorem 3.11. *Let W be an irreducible Weyl group not of type D_l $(l \ge 5)$. For $\chi, \chi' \in W^\vee$, we have*

$$\tilde{c}(\chi; q, y) = \tilde{c}(\chi'; q, y) \Leftrightarrow \chi \underset{LR}{\sim} (\chi')^\iota \quad \textit{for some } \iota \in \mathrm{Aut}(W, S).$$

4. Primitive factors and the relation with the modular representation theory.

We extend the notion of exceptional representations in the case of type G_2; namely, we define $W_{ex.m}^\vee \subset W^\vee$ by

$$W_{ex.m}^\vee = \begin{cases} \{\chi \in W^\vee \mid \dim \chi = 2\}, & \text{if } W = W(G_2), \\ W_{ex.}^\vee, & \text{otherwise.} \end{cases}$$

We call them exceptional representations in a modified sense. For the motivation of this modification, see §4.2 below.

4.1. Factorization patterns of $\tilde{\tau}$.

If W is of type A_l, D_l or E_l, the factorization patterns of $\tilde{\tau}_{prim}(\chi; q, y)$ and $\tau^*_{prim}(\chi; t)$ seem to be controlled by $\mathscr{M}(\mathscr{G}(\chi))$ (cf. §0.1 (b) and §0.3). Theorem 0.2 in the introduction can be refined, and generalized as follows.

Theorem 4.1. (1) *Consider the Weyl group of type A_l, D_l or E_l, excluding χ's in $W^\vee_{ex.m}$.*

(a) *If $\mathscr{G} = \mathfrak{S}_1$, then $\tilde{\tau}_{prim}(\chi; q, y) = 1$.*

(b) *If $\mathscr{G} = \mathfrak{S}_2$, then $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is an irreducible polynomial of degree 2.*

(c) *If $\mathscr{G} = \mathfrak{S}_3$ (so that $W = W(E_l)$ $(l = 6, 7, 8)$), then the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is as follows:*

$\mathscr{M}(\mathfrak{S}_3)$	$(1,1)$	$(g_2,1)$	$(1,r)$	$(g_3,1)$	$(1,\varepsilon)$
$\pi_{\boldsymbol{Q}(q)}(\tilde{\tau}_{prim}(\chi; q, y))$	$\{21^2\}$	$\{4\}$	$\{4\}$	$\{4\}$	$\{21^2\}$

(See [**L1**, 4.3] *for the notation $(g_2, 1)$ etc. See* §0.3 *for the definition of factorization pattern.*)

(d) *If $\mathscr{G} = \mathfrak{S}_5$ (so that $W = W(E_8)$), the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is as follows:*

$\mathscr{M}(\mathfrak{S}_5)$	$(1,1)$	$(g_3,1)$	$(g_2',1)$	$(1,\nu)$	$(1,\lambda')$	$(g_5,1)$	(g_3,ε)	$(1,\nu')$
$\pi_{\boldsymbol{Q}(q)}(\tilde{\tau}_{prim}(\chi; q, y))$	$\{6\}$	$\{1^24\}$	$\{1^24\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{6\}$	$\{1^24\}$

(g_2',ε'')	$(1,\lambda^2)$	(g_2',ε')	$(1,\lambda^3)$	$(g_2,1)$	$(g_4,1)$	$(g_6,1)$	(g_2,r)	(g_2,ε)
$\{6\}$	$\{1^24\}$	$\{6\}$	$\{1^24\}$	$\{6\}$	$\{1^22^2\}$	$\{6\}$	$\{1^42\}$	$\{1^24\}$

(2) *Consider the case of B_l $(= C_l)$, F_4 or G_2, excluding χ's in $W^\vee_{ex.m}$. Then every $\tilde{\tau}_{prim}(\chi; q, y)$ decomposes into linear factors in $\boldsymbol{Z}[t]$ except for $\{12_1, 4_1, 6_1, 6_2, 16_1\}$ in the F_4-case. For these excepted χ's, the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y)$ is $\{2\}$.*

(3) *For $\chi \in W^\vee_{ex.m}$, $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is irreducible.*

Theorem 4.2. *The factorization pattern of $\tau^*(\chi; t) \in \boldsymbol{Z}[t]$ is always finer than that of $\tilde{\tau}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$.*

(1) *Consider the Weyl group of type A_l, D_l or E_l, excluding χ's in $W^\vee_{ex.m}$.*

(a) *If $\mathscr{G} = \mathfrak{S}_1$, then $\tau^*_{prim}(\chi; t) = 1$.*

(b) *If $\mathscr{G} = \mathfrak{S}_2$ and $W \neq W(D_l)$ $(l \geq 5)$, then $\tau^*_{prim}(\chi; t) \in \boldsymbol{Z}[t]$ is an irreducible polynomial of degree 2 except the following 4 cases:*

$$W = W(E_6) \quad \text{and} \quad \chi \in \{30_p, 30_p'\}, \qquad W = W(E_8) \quad \text{and} \quad \chi \in \{210_x, 210_x'\}.$$

In these cases, the factorization pattern of $\tau^*_{prim}(\chi; t)$ is $\{1^2\}$.

(c) *If $\mathscr{G} = \mathfrak{S}_3$ (so that $W = W(E_l)$ $(l = 6, 7, 8)$), the factorization pattern of $\tau^*_{prim}(\chi; t) \in \boldsymbol{Z}[t]$ is as follows:*

$\mathscr{M}(\mathfrak{S}_3)$	$(1,1)$	$(g_2,1)$	$(1,r)$	$(g_3,1)$	$(1,\varepsilon)$
$\pi_{\boldsymbol{Q}}(\tau^*_{prim}(\chi; t))$	$\{21^2\}$	$\{4\}$	$\{2^2\}$	$\{2^2\}$	$\{21^2\}$

(d) *If* $\mathscr{G} = \mathfrak{S}_5$ (*so that* $W = W(E_8)$), *the factorization pattern of* $\tau^*_{prim}(\chi; t) \in \boldsymbol{Z}[t]$ *is as follows*:

$\mathscr{M}(\mathfrak{S}_5)$	$(1,1)$	$(g_3,1)$	$(g_2',1)$	$(1,\nu)$	$(1,\lambda')$	$(g_5,1)$	(g_3,ε)	$(1,\nu')$
$\pi_{\boldsymbol{Q}}(\tau^*_{prim}(\chi;t))$	$\{2^3\}$	$\{1^24\}$	$\{1^22^2\}$	$\{1^22^2\}$	$\{1^22^2\}$	$\{24\}$	$\{1^42\}$	$\{1^22^2\}$

(g_2',ε'')	$(1,\lambda^2)$	(g_2',ε')	$(1,\lambda^3)$	$(g_2,1)$	$(g_4,1)$	$(g_6,1)$	(g_2,r)	(g_2,ε)
$\{2^3\}$	$\{1^22^2\}$	$\{24\}$	$\{1^22^2\}$	$\{24\}$	$\{1^22^2\}$	$\{6\}$	$\{1^42\}$	$\{1^22^2\}$

(2) *Consider the case of* B_l $(= C_l)$, F_4 *or* G_2, *excluding* χ*'s in* $W^\vee_{ex.m}$. *Then the factorization pattern of* $\tau^*_{prim}(\chi; t)$ *is the same as that of* $\tilde{\tau}_{prim}(\chi; q, y)$.

(3) *For* $\chi \in W^\vee_{ex.m}$, $\tau^*_{prim}(\chi; t)$ *decomposes into linear factors in* $\boldsymbol{Z}[t]$.

REMARK. For type D_l, it is known that $\mathscr{G} = \mathfrak{S}_2^m$ $(m \geq 0)$. We do not know anything about the case for type D_l with $\mathscr{G} = \mathfrak{S}_2^m$ $(m \geq 2)$ yet.

4.2. Relation with the modular representation theory of (generic) Iwahori-Hecke algebra.

Here we try to find a unified picture for Theorem 4.1 from the viewpoint of the modular representation theory.

Let (W, S) be a Coxeter system, $\{q_s\}_{s \in S}$ a family of indeterminates such that $q_s = q_{s'}$ if and only if s and s' are conjugate in W and such that different q_s's are algebraically independent. Put $R = \boldsymbol{Z}[q_s^{1/2}, q_s^{-1/2}; s \in S]$ and $K := \boldsymbol{Q}(q_s^{1/2}; s \in S)$. Define an associative algebra structure in $H_0(W) = \bigoplus_{s \in W} RT_w$ so that

$$T_w T_{w'} = T_{ww'} \quad \text{if } l(w) + l(w') = l(ww'), \text{ and}$$

$$(T_s + 1)(T_s - q_s) = 0 \quad \text{for } s \in S.$$

Put $H(W) := H_0(W) \otimes_R K$. For a ring A, let $A^\vee$ denote the set of simple A-modules (up to isomorphism). Then naturally we can identify $H(W)^\vee = (\boldsymbol{C}W)^\vee = W^\vee$. For $\chi \in W^\vee$, denote the corresponding simple $H(W)$-module by $\tilde{\chi}$. For a fixed prime number p, we define the equivalence relation $\underset{p}{\sim}$ in $W^\vee$ as follows: Let us consider 'the reduction modulo pR' of elements of $H(W)^\vee$. For $\chi, \chi' \in W^\vee$, define $\chi \underset{p}{\sim} \chi'$ if $\tilde{\chi}$ and $\tilde{\chi}'$ belong to the same block (in the usual sense of the modular representation theory). Define $\chi \underset{*}{\sim} \chi'$ if there exists $\chi_0, \chi_1, \ldots, \chi_n \in W^\vee$ and prime numbers $p_1, \ldots, p_n$ such that

$$\chi = \chi_0 \underset{p_1}{\sim} \chi_1 \underset{p_2}{\sim} \cdots \underset{p_n}{\sim} \chi_n = \chi'.$$

This equivalence relation can be explicitly described as follows.

(1) If W is of type A_l, D_l or E_l $(l = 6, 7)$, then $\chi \underset{*}{\sim} \chi' \Leftrightarrow \chi \underset{LR}{\sim} \chi'$ by [**Gy2**]. It is plausible that the same holds also for E_8 (see [**Gy2**]).

(2) If W is of type B_l $(= C_l)$, then $\chi \underset{*}{\sim} \chi' \Leftrightarrow \chi = \chi'$ by [**Gy3**] and [**H**].

(3) If W is of type G_2, then the equivalence classes are $\{1\}, \{\mathrm{sgn}\}, \{\varepsilon_1\}, \{\varepsilon_2\}, \{V, V'\}$ in the notation of [**L1**, 4.8].

(4) If W is of type F_4, then $\{12_1, 4_1, 6_1, 6_2, 16_1\}$ forms an equivalence class with respect to $\underset{*}{\sim}$, and each of the remaining χ's forms an equivalence class by itself.

(Proof. For a prime number p, let R_p be the localization of R by pR, and $H_p(W) := H_0(W) \otimes_R R_p$. First consider the case $p = 2$. By [**Gy3**, 2.17] and [**C2**, p. 451], $\chi \notin \{12_1, 16_1, 4_1\} = \{\phi_{12,4}, \phi_{16,5}, \phi_{4,8}\}$ can be obtained from a principal indecomposable module by a scalar extension. In particular, each such χ forms a block by itself. Then by the argument of [**Ge**, p. 2965, $\uparrow \ell\ell$. 14-7] and by the information given in [**L1**, (4.10.4)], we can see that $12_1 + 16_1$ and $12_1 + 4_1$ come from some direct summand, say U_1 and U_2, of the left regular module $H_p(W)$. By [**Gy3**, (2.16.1)] and by [**C2**, p. 451], U_i's are indecomposable. Hence if $p = 2$, $\{12_1, 16_1, 4_1\}$ forms a single block, and its Brauer tree is 4_1–16_1–12_1. If $p = 3$, the same argument yields that $\{12_1, 6_1, 6_2\} = \{\phi_{12,4}, \phi'_{6,6}, \phi''_{6,6}\}$ forms a single block, its Brauer tree is 6_1–12_1–6_2, and each of the remaining χ's forms a block by itself.)

Using or assuming (1)–(4) above, we can restate Theorem 4.1 as follows.

Theorem 4.3. *Consider an irreducible Weyl group W.*

(1) *If $\varphi(\chi) := \#\{\chi' \in W^\vee \mid \chi' \underset{*}{\sim} \chi\} = 1$, then $\tilde{\tau}_{prim}(\chi; q, y) = 1$.*

(2) *If $\varphi(\chi) = 3$ (i.e., $W = W(D_l)$ or $W(E_l)$, and $\mathscr{G} = \mathfrak{S}_2$, but $\chi \notin W^\vee_{ex.}$), then $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is an irreducible polynomial of degree 2.*

(3) *If $\varphi(\chi) = 5$ and $\mathscr{G} = \mathfrak{S}_3$ (i.e., $W = W(E_l)$ and $\mathscr{G} = \mathfrak{S}_3$), then the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y)$ is given by* (1c) *of Theorem* 4.1.

(4) *If $\varphi(\chi) = 17$ (i.e., $W = W(E_8)$ and $\mathscr{G} = \mathfrak{S}_5$), then the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y)$ is given by* (1d) *of Theorem* 4.1.

(5) *If $\varphi(\chi) = 5$ and $\mathscr{G} = \mathfrak{S}_4$ (i.e., $W = W(F_4)$ and $\chi \in \{12_1, 4_1, 6_1, 6_2, 16_1\}$), then the factorization pattern of $\tilde{\tau}_{prim}(\chi; q, y)$ is* $\{2\}$.

(6) *If $\varphi(\chi) = 2$ (i.e., $\chi \in W^\vee_{ex.m}$), then $\tilde{\tau}_{prim}(\chi; q, y) \in \boldsymbol{Q}(q)[y]$ is irreducible.*

Remark 1. Theorem 4.2 can be also reformulated in a similar way.

Remark 2. The block structure of $H_0(W) \otimes_R R_p$ can be determined if we know

(1) *the center of $H_0(W) \otimes_R R_p$*, and

(2) *the central characters of all the irreducible representations of $H_0(W) \otimes_R K_p$*, where K_p is the fractional field of R_p.

If W is of type A_l or B_l $(= C_l)$,

(1′) *the center of $H_0(W) \otimes_R K_p$*,

and (2), are determined in [**AK**]. (See from p. 239, ℓ. 23 to the end.) In this connection, it would be worth noting that $\sigma^*(\chi; q)$ (with a slight modification in the B_l-case) appears in [**AK**, Theorem 3.20].

5. Product formula.

Let W be of type A_l, D_l or E_l, $\chi \in W^\vee$, $\mathscr{F}$ the family containing χ, and $\mathscr{G}$ the group associated to $\mathscr{F}$ (cf. §0.1 (b)). If $\chi \notin W^\vee_{ex.m}$ and $\mathscr{G} \neq 1$, then $\tau^*(\chi; t)$ does not decompose into linear factors in $\boldsymbol{Z}[t]$ (with a small number of exceptions). However, if $\mathscr{G} = \mathfrak{S}_2^r$ $(r \geq 0)$, then certain linear combinations of $\tau^*(\chi; t)$ $(\chi \in \mathscr{F})$ completely decompose into linear factors. In §5.1, we give such a product formula. Moreover, if $\mathscr{G} = \mathfrak{S}_2$, we can refine this observation. See §5.2, and also §0.4.

5.1. D_l**-case.**

Let $\chi = \tilde{\chi}^{\alpha,\beta}$, λ_i, μ_i, Λ, A, B and C be as in §3.3. In the sequel, we identify χ with Λ.

Take any bijection $\varphi : A \to B$. Let Φ be the totality of permutations $\sigma \in \mathfrak{S}(A \sqcup B \sqcup C)$ such that $\sigma(\{a, \varphi(a)\}) = \{a, \varphi(a)\}$ $(a \in A)$, and $\sigma(c) = c$ $(c \in C)$. Fix $a_0 \in A$ and put $\Phi_0 := \{\sigma \in \Phi \mid \sigma(a_0) = a_0\}$. Put

$$\sigma(\Lambda) = \begin{pmatrix} \sigma(\lambda_1), \dots, \sigma(\lambda_m) \\ \sigma(\mu_1), \dots, \sigma(\mu_m) \end{pmatrix}.$$

Note that $\Lambda \underset{LR}{\sim} \sigma(\Lambda)$ for any $\sigma \in \Phi$.

Proposition 5.1.

$$\prod_{i=1}^{m} (t - 2i + 1)^{2i-1} \times \sum_{\sigma \in \Phi_0} \tau^*(\sigma(\Lambda); t) = 2^{|A|-1} \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]^* \cdot \prod_{a \in A} (t + a + \varphi(a) - 2m + 1), \tag{5.1}$$

where $[j,k]^* = \prod_{\substack{j \le i \le k \\ i:\,\mathrm{odd}}} (t + i)$.

Proof. From (3.30), we get

$$2 \prod_{i=1}^{m} (t - 2i + 1)^{2i-1} \cdot \sum_{\sigma \in \Phi_0} \tau^*(\sigma(\Lambda); t) = \prod_{x \in A \sqcup B \sqcup C \sqcup (C+1)} [-2m+1, 2x-2m-1]^* \times \sum_{\sigma \in \Phi} \prod_{a \in \sigma(A)} (t + 2a - 2m + 1). \tag{5.2}$$

The second factor in the right hand side is equal to

$$\prod_{a \in A} \{(t + 2a - 2m + 1) + (t + 2\varphi(a) - 2m + 1)\} = 2^{|A|} \prod_{a \in A} (t + a + \varphi(a) - 2m + 1). \tag{5.3}$$

□

5.2. Families consisting of 3 elements.

Let W be of type D_l or E_l, and $\mathscr{F}$ a family consisting of three elements χ_1, χ_2, χ_3. We assume that χ_1, χ_2, χ_3 correspond to $(1,1)$, $(g_2, 1)$, $(1, \varepsilon)$ in this order, and regard the suffix i of χ_i as an element of $\boldsymbol{Z}/3\boldsymbol{Z}$. (For example $\chi_{3+1} = \chi_1$.) Put

$$e = e(\mathscr{F}) = e(\chi) := -a(\chi) + a(\chi \otimes \mathrm{sgn}) - \sum_{i=1}^{\kappa(\chi)} c_i(\chi) \quad \text{for } \chi \in \mathscr{F}. \tag{5.4}$$

By Theorem 4.1 (1b), (1.10), (1.19) and (2.19), we get

$$\frac{\tilde{\tau}_{prim}(\chi_i; q, y)}{\tilde{\tau}_{prim}(\chi_i; q, 0)} = 1 + \sigma^*_{prim}(\chi_i; q) y + q^{e(\mathscr{F})} y^2. \tag{5.5}$$

THEOREM 5.2. *To each family* $\mathscr{F} \in W^\vee / \underset{LR}{\sim}$ *consisting of* 3 *elements, there associate integers* $d_i = d_i(\mathscr{F})$ *and* $d_i' = d_i'(\mathscr{F})$ $(i \in \boldsymbol{Z}/3\boldsymbol{Z})$ *such that*

$$d_1 + d_1' = d_2 + d_2' = d_3 + d_3' \tag{5.6}$$

and

$$\sigma^*_{prim}(\chi_i; q) = (q^{d_i} + q^{d_i'})^{-1}(q^{d_{i+1}} + q^{d_{i+1}'})(q^{d_{i+2}} + q^{d_{i+2}'}), \tag{5.7}$$

$$\frac{1}{2}(\tau^*_{prim}(\chi_i; t) + \tau^*_{prim}(\chi_{i+1}; t)) = (t + d_{i+2})(t + d_{i+2}'), \tag{5.8}$$

$$\begin{aligned}\frac{\prod_{i=1}^{l}(1-q^{m_i+1})}{f(\chi_i;q)}\,&\tilde{\tau}(\chi_i; q, y)\\ &= \frac{q^{d_i}}{q^{d_i}+q^{d_i'}}(1+yq^{-d_i+d_{i+1}+d_{i+2}'})(1+yq^{-d_i+d_{i+1}'+d_{i+2}})\prod_{1\le j\le \kappa(\chi_i)}(1+yq^{c(\chi_i)})\\ &\quad+\frac{q^{d_i'}}{q^{d_i}+q^{d_i'}}(1+yq^{-d_i'+d_{i+1}+d_{i+2}})(1+yq^{-d_i'+d_{i+1}'+d_{i+2}'})\prod_{1\le j\le \kappa(\chi_i)}(1+yq^{c(\chi_i)}).\end{aligned} \tag{5.9}$$

PROOF. First, let us consider the D_l-case. Let notation be as in §3.3, and assume that $\# A = \# B = 2$. For $\Lambda := \begin{pmatrix} A \sqcup C \\ B \sqcup C \end{pmatrix}$, put

$$P(\Lambda; q) := q^{|A|-2m+1} + q^{|B|-2m+1}. \tag{5.10}$$

Now assume that $A = \{a_1, a_2\}$, $B = \{b_1, b_2\}$ and $a_1 < b_1 < a_2 < b_2$. Put $A' := \{a_1, b_1\}$, $B' := \{a_2, b_2\}$, $A'' := \{a_1, b_2\}$, $B'' := \{a_2, b_1\}$, $\Lambda_1 := \Lambda$, $\Lambda_2 := \begin{pmatrix} A' \sqcup C \\ B' \sqcup C \end{pmatrix}$ and $\Lambda_3 := \begin{pmatrix} A'' \sqcup C \\ B'' \sqcup C \end{pmatrix}$. Then Λ_1, Λ_2, Λ_3 form a family, the associated elements in $\mathscr{M}(\mathfrak{S}_2)$ are $(1,1)$, $(g_2, 1)$ and $(1, \varepsilon)$ in this order [**L1**, 4.6].

Since (3.25) (resp. (3.17)) gives an explicit form of $\tilde{c}(\chi; q, y)$ (resp. $\sigma^*(\Lambda_i; q)$), we can calculate $\sigma^*_{prim}(\Lambda_i; q)$, and we get

$$\sigma^*_{prim}(\Lambda_i; q) = P(\Lambda_i; q)^{-1} P(\Lambda_{i+1}; q) P(\Lambda_{i+2}; q). \tag{5.11}$$

Then we get (5.6) and (5.7) in the D_l-case. In the E_l-case, d_i and d_i' can be obtained by a direct calculation. See Table 7 for their explicit values.

Now (5.9) follows from (2.1) and (5.5)–(5.7). Specializing (5.9) as in (1.23), we get (5.8). □

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Tables

For each irreducible character χ of $W = W(D_4)$, $W(G_2)$, $W(F_4)$ and $W(E_l)$ $(l = 6, 7, 8)$, we shall write down $c_i(\chi)$ $(1 \le i \le \kappa(\chi))$ and $\tau^*_{prim}(\chi; t)$ in Tables 1–6. In Table 7, we shall list the integers d_i and d'_i $(1 \le i \le 3)$ appeared in Theorem 5.2. Let us give some explanation on these tables.

On Tables 1–6. We shall give all the families belonging to $W^\vee / \underset{LR}{\sim}$ and describe $c_i(\chi)$ $(1 \le i \le \kappa(\chi))$ and $\tau^*_{prim}(\chi; t)$ in these tables. Note that

$$\tau^*(\chi; t) = \prod_{i=1}^{\kappa(\chi)} (t + c_i(\chi)) \cdot \tau^*_{prim}(\chi; t).$$

So the tables describe $\tau^*(\chi; t)$ completely.

For simplicity, the families which consists of only one element (i.e., 1-element families) are collected together. For these families, the associated group is $\mathscr{G} = \mathfrak{S}_1 = \{e\}$ and $\mathscr{M}(\mathfrak{S}_1) = \{(1,1)\}$. We put the parameters of each irreducible representation χ in the first and the second columns. The first one is used in [**C2**] and the second is used in [**L1**, §4].

For the families which have more than 2 irreducible representations, we also list the associated parameters $(1,1)$, $(g_2,1)$, etc. $\in \mathscr{M}(\mathscr{G})$ in the third column. We call the family which have n elements n-element family.

For 1-element families, since $\tau^*_{prim}(\chi;t) = 1$ always, we only list $\{c_i(\chi) \mid 1 \le i \le l = \kappa(\chi)\}$ for each representation χ.

For n-element families ($n \ge 2$), in the first row, we list $\{c_i(\chi) \mid 1 \le i \le \kappa(\chi)\}$ with the associated group $\mathscr{G} = \mathfrak{S}_2, \mathfrak{S}_3, \ldots$. The following rows contain two kinds of parameters of χ, by Carter [**C2**] and Lusztig [**L1**, §4], in the first and the second columns respectively; in the third column the associated parameter belonging to $\mathscr{M}(\mathscr{G})$ is given. In the last column, $\tau^*_{prim}(\chi;t)$ is displayed.

On Table 7. Here we list d_i and d'_i $(1 \le i \le 3)$ of Theorem 5.2. In the first and the second columns, we list the special representations (i.e., $\chi \in W^\vee$ corresponding to $(1,1) \in \mathscr{M}(\mathfrak{S}_2)$) of the families $\mathscr{F} \in W^\vee / \underset{LR}{\sim}$ consisting of three elements. The third column contains d_1, d'_1, d_2, d'_2, d_3, d'_3 in this order.

Table 1. Type D_4.

1-element families ($\mathfrak{S}_1 = \{e\}$).

$(4,-)$	$\binom{4}{0}$	1	3	3	5
$(1^4,-)$	$\binom{1\,2\,3\,4}{0\,1\,2\,3}$	−5	−3	−3	−1
$(3,1)$	$\binom{3}{1}$	−1	1	3	3
$(1^3,1)$	$\binom{1\,2\,3}{0\,1\,3}$	−3	−3	−1	1
$(31,-)$	$\binom{1\,4}{0\,1}$	−1	1	1	3
$(2,2)_I$	$\binom{2}{2}_I$	−1	1	1	3
$(2,2)_{II}$	$\binom{2}{2}_{II}$	−1	1	1	3
$(21^2,-)$	$\binom{1\,2\,4}{0\,1\,2}$	−3	−1	−1	1
$(1^2,1^2)_I$	$\binom{1\,2}{1\,2}_I$	−3	−1	−1	1
$(1^2,1^2)_{II}$	$\binom{1\,2}{1\,2}_{II}$	−3	−1	−1	1

3-element family ($\mathfrak{S}_2$) : −1 1

$(21,1)$	$\binom{1\,3}{0\,2}$	$(1,1)$	t^2-3
$(2^2,-)$	$\binom{2\,3}{0\,1}$	$(g_2,1)$	t^2+3
$(2,1^2)$	$\binom{0\,3}{1\,2}$	$(1,\varepsilon)$	t^2-5

Table 2. Type G_2.

1-element families ($\mathfrak{S}_1 = \{e\}$).

$[1,0]$	trivial	1	5
$[1,6]$	sign	-5	-1

4-element family ($\mathfrak{S}_3$): -1 1

$[1,3]'$	ε_1	$(1,r)$	1
$[1,3]''$	ε_2	$(g_3,1)$	1
$[2,1]$	V	$(1,1)$	1
$[2,2]$	V'	$(g_2,1)$	1

Table 3. Type F_4.

1-element families ($\mathfrak{S}_1 = \{e\}$).

$[1,0]$	1_1	1	5	7	11
$[1,24]$	1_4	-11	-7	-5	-1
$[9,2]$	9_1	-1	1	3	5
$[9,10]$	9_4	-5	-3	-1	1
$[8,3]''$	8_1	-1	1	1	5
$[8,9]'$	8_2	-5	-1	-1	1
$[8,3]'$	8_3	-1	1	1	5
$[8,9]''$	8_4	-5	-1	-1	1

3-element family ($\mathfrak{S}_2$): -1 1 5 7

$[4,1]$	4_2	$(1,1)$	1
$[2,4]''$	2_1	$(g_2,1)$	1
$[2,4]'$	2_3	$(1,\varepsilon)$	1

3-element family ($\mathfrak{S}_2$): -7 -5 -1 1

$[4,13]$	4_5	$(1,1)$	1
$[2,16]''$	2_4	$(g_2,1)$	1
$[2,16]'$	2_2	$(1,\varepsilon)$	1

11-element family ($\mathfrak{S}_4$): -1 1

$[12,4]$	12_1	$(1,1)$	t^2-5
$[9,6]''$	9_2	$(g_2',1)$	$(t+3)(t-3)$
$[9,6]'$	9_3	$(1,\lambda^1)$	$(t+3)(t-3)$
$[1,12]''$	1_2	(g_2',ε')	$(t-5)(t+5)$
$[1,12]'$	1_3	$(1,\lambda^2)$	$(t-5)(t+5)$
$[4,8]$	4_1	(g_2',ε'')	t^2-13
$[4,7]''$	4_3	$(g_4,1)$	$(t-1)(t+1)$
$[4,7]'$	4_4	(g_2,ε'')	$(t-1)(t+1)$
$[6,6]'$	6_1	$(g_3,1)$	t^2+7
$[6,6]''$	6_2	$(1,\sigma)$	t^2-17
$[16,5]$	16_1	$(g_2,1)$	t^2-7

Table 4. Type E_6.

1-element families ($\mathfrak{S}_1 = \{e\}$).

$[1,0]$	1_p	1	4	5	7	8	11
$[1,36]$	$1_p'$	-11	-8	-7	-5	-4	-1
$[6,1]$	6_p	-1	1	4	5	7	8
$[6,25]$	$6_p'$	-8	-7	-5	-4	-1	1
$[20,2]$	20_p	-1	1	2	4	5	7
$[20,20]$	$20_p'$	-7	-5	-4	-2	-1	1
$[64,4]$	64_p	-2	-1	1	2	4	5
$[64,13]$	$64_p'$	-5	-4	-2	-1	1	2
$[60,5]$	60_p	-1	-1	1	1	2	4
$[60,11]$	$60_p'$	-4	-2	-1	-1	1	1
$[81,6]$	81_p	-3	-1	0	1	3	4
$[81,10]$	$81_p'$	-4	-3	-1	0	1	3
$[24,6]$	24_p	-2	-1	1	1	2	5
$[24,12]$	$24_p'$	-5	-2	-1	-1	1	2

3-element family ($\mathfrak{S}_2$): $-1 \quad 1 \quad 4 \quad 5$

$[30,3]$	30_p	$(1,1)$	$(t+4)(t-1)$
$[15,4]$	15_q	$(g_2,1)$	t^2+3t+8
$[15,5]$	15_p	$(1,\varepsilon)$	$t^2+3t-16$

3-element family ($\mathfrak{S}_2$): $-5 \quad -4 \quad -1 \quad 1$

$[30,15]$	$30_p'$	$(1,1)$	$(t+1)(t-4)$
$[15,16]$	$15_q'$	$(g_2,1)$	t^2-3t+8
$[15,17]$	$15_p'$	$(1,\varepsilon)$	$t^2-3t-16$

5-element family ($\mathfrak{S}_3$): $-1 \quad 1$

$[80,7]$	80_s	$(1,1)$	$(t-2)(t+2)(t^2-7)$
$[60,8]$	60_s	$(g_2,1)$	t^4-5t^2-32
$[90,8]$	90_s	$(1,r)$	$(t^2+t-8)(t^2-t-8)$
$[10,9]$	10_s	$(g_3,1)$	$(t^2-3t+8)(t^2+3t+8)$
$[20,10]$	20_s	$(1,\varepsilon)$	$(t-4)(t+4)(t^2-13)$

Table 5. Type E_7.

1-element families ($\mathfrak{S}_1 = \{e\}$).

$[1,0]$	1_a	1	5	7	9	11	13	17
$[1,63]$	$1'_a$	-17	-13	-11	-9	-7	-5	-1
$[7,1]$	$7'_a$	-1	1	5	7	9	11	13
$[7,46]$	7_a	-13	-11	-9	-7	-5	-1	1
$[27,2]$	27_a	-1	1	3	5	7	9	11
$[27,37]$	$27'_a$	-11	-9	-7	-5	-3	-1	1
$[21,3]$	$21'_b$	-1	1	3	5	7	7	11
$[21,36]$	21_b	-11	-7	-7	-5	-3	-1	1
$[189,5]$	$189'_b$	-1	-1	1	3	3	5	7
$[189,22]$	189_b	-7	-5	-3	-3	-1	1	1
$[210,6]$	210_a	-3	-1	1	1	5	5	7
$[210,21]$	$210'_a$	-7	-5	-5	-1	-1	1	3
$[105,6]$	105_b	-1	1	1	1	3	5	5
$[105,21]$	$105'_b$	-5	-5	-3	-1	-1	-1	1
$[168,6]$	168_a	-1	-1	1	1	3	5	7
$[168,21]$	$168'_a$	-7	-5	-3	-1	-1	1	1
$[189,7]$	$189'_c$	-3	-1	1	1	3	5	7
$[189,20]$	189_c	-7	-5	-3	-1	-1	1	3
$[378,9]$	$378'_a$	-3	-1	-1	1	1	3	5
$[378,14]$	378_a	-5	-3	-1	-1	1	1	3
$[210,10]$	210_b	-1	-1	-1	1	1	1	3
$[210,13]$	$210'_b$	-3	-1	-1	-1	1	1	1
$[105,12]$	105_c	-5	-1	-1	1	1	3	5
$[105,15]$	$105'_c$	-5	-3	-1	-1	1	1	5

2-element family ($\mathfrak{S}_2$): -1 1

$[512,11]$	$512'_a$	$(1,1)$	$(t-2)(t-4)(t+4)(t+2)t$
$[512,12]$	512_a	$(1,\varepsilon)$	$(t-2)(t-4)(t+4)(t+2)t$

3-element family ($\mathfrak{S}_2$): -1 1 5 7 9

$[56,3]$	$56'_a$	$(1,1)$	t^2+6t-1
$[35,4]$	35_b	$(g_2,1)$	$t^2+6t+17$
$[21,6]$	21_a	$(1,\varepsilon)$	$t^2+6t-31$

3-element family ($\mathfrak{S}_2$): -9 -7 -5 -1 1

$[56,30]$	56_a	$(1,1)$	t^2-6t-1
$[35,31]$	$35'_b$	$(g_2,1)$	$t^2-6t+17$
$[21,33]$	$21'_a$	$(1,\varepsilon)$	$t^2-6t-31$

Table 5. (continued)

3-element family ($\mathfrak{S}_2$): $-1 \quad 1 \quad 3 \quad 5 \quad 7$

$[120, 4]$	120_a	$(1, 1)$	$t^2 + 6t - 13$
$[15, 7]$	$15'_a$	$(g_2, 1)$	$t^2 + 6t + 29$
$[105, 5]$	$105'_a$	$(1, \varepsilon)$	$t^2 + 6t - 19$

3-element family ($\mathfrak{S}_2$): $-7 \quad -5 \quad -3 \quad -1 \quad 1$

$[120, 25]$	$120'_a$	$(1, 1)$	$t^2 - 6t - 13$
$[15, 28]$	15_a	$(g_2, 1)$	$t^2 - 6t + 29$
$[105, 26]$	105_a	$(1, \varepsilon)$	
$t^2 - 6t + 19$			

3-element family ($\mathfrak{S}_2$): $-3 \quad -1 \quad 1 \quad 3 \quad 5$

$[405, 8]$	405_a	$(1, 1)$	$t^2 + 2t - 7$
$[216, 9]$	$216'_a$	$(g_2, 1)$	$t^2 + 2t + 7$
$[189, 10]$	189_a	$(1, \varepsilon)$	$t^2 + 2t - 23$

3-element family ($\mathfrak{S}_2$): $-5 \quad -3 \quad -1 \quad 1 \quad 3$

$[405, 15]$	$405'_a$	$(1, 1)$	$t^2 - 2t - 7$
$[216, 16]$	216_a	$(g_2, 1)$	$t^2 - 2t + 7$
$[189, 17]$	$189'_a$	$(1, \varepsilon)$	$t^2 - 2t - 23$

3-element family ($\mathfrak{S}_2$): $-3 \quad -1 \quad 1 \quad 1 \quad 5$

$[420, 10]$	420_a	$(1, 1)$	$t^2 - 13$
$[84, 12]$	84_a	$(g_2, 1)$	$t^2 + 11$
$[336, 11]$	$336'_a$	$(1, \varepsilon)$	$t^2 - 19$

3-element family ($\mathfrak{S}_2$): $-5 \quad -1 \quad -1 \quad 1 \quad 3$

$[420, 13]$	$420'_a$	$(1, 1)$	$t^2 - 13$
$[84, 15]$	$84'_a$	$(g_2, 1)$	$t^2 + 11$
$[336, 14]$	336_a	$(1, \varepsilon)$	$t^2 - 19$

5-element family ($\mathfrak{S}_3$): $-1 \quad 1 \quad 5$

$[315, 7]$	$315'_a$	$(1, 1)$	$(t + 3)(t^2 + 2t - 11)(t - 1)$
$[280, 8]$	280_b	$(g_2, 1)$	$t^4 + 4t^3 - 4t^2 - 16t - 81$
$[280, 9]$	$280'_a$	$(1, r)$	$(t^2 - 13)(t^2 + 4t - 9)$
$[70, 9]$	$70'_a$	$(g_3, 1)$	$(t^2 - 2t + 9)(t^2 + 6t + 17)$
$[35, 13]$	$35'_a$	$(1, \varepsilon)$	$(t + 7)(t - 5)(t^2 + 2t - 27)$

5-element family ($\mathfrak{S}_3$): $-5 \quad -1 \quad 1$

$[315, 16]$	315_a	$(1, 1)$	$(t - 3)(t^2 - 2t - 11)(t + 1)$
$[280, 17]$	$280'_b$	$(g_2, 1)$	$t^4 - 4t^3 - 4t^2 + 16t - 81$
$[280, 18]$	280_a	$(1, r)$	$(t^2 - 4t - 9)(t^2 - 13)$
$[70, 18]$	70_a	$(g_3, 1)$	$(t^2 - 6t + 17)(t^2 + 2t + 9)$
$[35, 22]$	35_a	$(1, \varepsilon)$	$(t + 5)(t - 7)(t^2 - 2t - 27)$

Table 6. Type E_8.

1-element families ($\mathfrak{S}_1 = \{e\}$).

[1, 0]	1_x	1	7	11	13	17	19	23	29
[1, 120]	$1'_x$	−29	−23	−19	−17	−13	−11	−7	−1
[8, 1]	8_z	−1	1	7	11	13	17	19	23
[8, 91]	$8'_z$	−23	−19	−17	−13	−11	−7	−1	1
[35, 2]	35_x	−1	1	5	7	11	13	17	19
[35, 74]	$35'_x$	−19	−17	−13	−11	−7	−5	−1	1
[560, 5]	560_z	−1	−1	1	5	7	7	11	13
[560, 47]	$560'_z$	−13	−11	−7	−7	−5	−1	1	1
[567, 6]	567_x	−3	−1	1	3	7	9	11	13
[567, 46]	$567'_x$	−13	−11	−9	−7	−3	−1	1	3
[3240, 9]	3240_z	−3	−1	1	1	3	5	7	9
[3240, 31]	$3240'_z$	−9	−7	−5	−3	−1	−1	1	3
[525, 12]	525_x	−5	−1	1	1	5	5	7	11
[525, 36]	$525'_x$	−11	−7	−5	−5	−1	−1	1	5
[4536, 13]	4536_z	−3	−1	−1	1	1	3	3	7
[4536, 23]	$4536'_z$	−7	−3	−3	−1	−1	1	1	3
[2835, 14]	2835_x	−3	−1	−1	1	1	3	3	5
[2835, 22]	$2835'_x$	−5	−3	−3	−1	−1	1	1	3
[6075, 14]	6075_x	−5	−3	−1	1	1	3	5	7
[6075, 22]	$6075'_x$	−7	−5	−3	−1	−1	1	3	5
[4200, 15]	4200_z	−5	−1	−1	1	1	1	5	5
[4200, 21]	$4200'_z$	−5	−5	−1	−1	−1	1	1	5
[2100, 20]	2100_y	−7	−5	−1	−1	1	1	5	7

2-element family ($\mathfrak{S}_2$): −1 1 7

[4096, 11]	4096_z	$(1,1)$	$(t-2)(t-4)(t+8)(t+4)(t+2)$
[4096, 12]	4096_x	$(1,\varepsilon)$	$(t-2)(t-4)(t+8)(t+4)(t+2)$

2-element family ($\mathfrak{S}_2$): −7 −1 1

[4096, 26]	$4096'_x$	$(1,1)$	$(t-2)(t-4)(t-8)(t+4)(t+2)$
[4096, 27]	$4096'_z$	$(1,\varepsilon)$	$(t-2)(t-4)(t-8)(t+4)(t+2)$

3-element family ($\mathfrak{S}_2$): −1 1 7 11 13 17

[112, 3]	112_z	$(1,1)$	$t^2+12t+17$
[84, 4]	84_x	$(g_2,1)$	$t^2+12t+47$
[28, 8]	28_x	$(1,\varepsilon)$	$t^2+12t-73$

3-element family ($\mathfrak{S}_2$): −17 −13 −11 −7 −1 1

[112, 63]	$112'_z$	$(1,1)$	$t^2-12t+17$
[84, 64]	$84'_x$	$(g_2,1)$	$t^2-12t+47$
[28, 68]	$28'_x$	$(1,\varepsilon)$	$t^2-12t-73$

Table 6. (continued)

3-element family ($\mathfrak{S}_2$): -1 1 5 7 11 13

$[210, 4]$	210_x	$(1, 1)$	$(t + 13)(t - 1)$
$[50, 8]$	50_x	$(g_2, 1)$	$t^2 + 12t + 83$
$[160, 7]$	160_z	$(1, \varepsilon)$	$t^2 + 12t - 43$

3-element family ($\mathfrak{S}_2$): -13 -11 -7 -5 -1 1

$[210, 52]$	$210'_x$	$(1, 1)$	$(t + 1)(t - 13)$
$[50, 56]$	$50'_x$	$(g_2, 1)$	$t^2 - 12t + 83$
$[160, 55]$	$160'_z$	$(1, \varepsilon)$	$t^2 - 12t - 43$

3-element family ($\mathfrak{S}_2$): -1 1 1 5 7 11

$[700, 6]$	700_x	$(1, 1)$	$t^2 + 12t + 23$
$[400, 7]$	400_z	$(g_2, 1)$	$t^2 + 12t + 41$
$[300, 8]$	300_x	$(1, \varepsilon)$	$t^2 + 12t - 1$

3-element family ($\mathfrak{S}_2$): -11 -7 -5 -1 -1 1

$[700, 42]$	$700'_x$	$(1, 1)$	$t^2 - 12t + 23$
$[400, 43]$	$400'_z$	$(g_2, 1)$	$t^2 - 12t + 41$
$[300, 44]$	$300'_x$	$(1, \varepsilon)$	$t^2 - 12t - 1$

3-element family ($\mathfrak{S}_2$): -3 -1 1 3 7 9

$[2268, 10]$	2268_x	$(1, 1)$	$t^2 + 4t - 17$
$[972, 12]$	972_x	$(g_2, 1)$	$t^2 + 4t + 23$
$[1296, 13]$	1296_z	$(1, \varepsilon)$	$t^2 + 4t - 47$

3-element family ($\mathfrak{S}_2$): -9 -7 -3 -1 1 3

$[2268, 30]$	$2268'_x$	$(1, 1)$	$t^2 - 4t - 17$
$[972, 32]$	$972'_x$	$(g_2, 1)$	$t^2 - 4t + 23$
$[1296, 33]$	$1296'_z$	$(1, \varepsilon)$	$t^2 - 4t - 47$

3-element family ($\mathfrak{S}_2$): -1 -1 1 1 5 7

$[2240, 10]$	2240_x	$(1, 1)$	$t^2 + 6t - 1$
$[1400, 11]$	1400_{zz}	$(g_2, 1)$	$t^2 + 6t + 17$
$[840, 13]$	840_z	$(1, \varepsilon)$	$t^2 + 6t - 31$

3-element family ($\mathfrak{S}_2$): -7 -5 -1 -1 1 1

$[2240, 28]$	$2240'_x$	$(1, 1)$	$t^2 - 6t - 1$
$[1400, 29]$	$1400'_{zz}$	$(g_2, 1)$	$t^2 - 6t + 17$
$[840, 31]$	$840'_z$	$(1, \varepsilon)$	$t^2 - 6t - 31$

3-element family ($\mathfrak{S}_2$): -1 -1 1 1 5 7

$[4200, 12]$	4200_x	$(1, 1)$	$t^2 - 13$
$[840, 14]$	840_x	$(g_2, 1)$	$t^2 + 11$
$[3360, 13]$	3360_z	$(1, \varepsilon)$	$t^2 - 19$

Table 6. (continued)

3-element family ($\mathfrak{S}_2$): -7 -5 -1 -1 1 1

$[4200, 24]$	$4200'_x$	$(1, 1)$	$t^2 - 13$
$[840, 26]$	$840'_x$	$(g_2, 1)$	$t^2 + 11$
$[3360, 25]$	$3360'_z$	$(1, \varepsilon)$	$t^2 - 19$

3-element family ($\mathfrak{S}_2$): -5 -1 1 1 5 7

$[2800, 13]$	2800_z	$(1, 1)$	$t^2 + 4t - 23$
$[700, 16]$	700_{xx}	$(g_2, 1)$	$t^2 + 4t + 31$
$[2100, 16]$	2100_x	$(1, \varepsilon)$	$t^2 + 4t - 41$

3-element family ($\mathfrak{S}_2$): -7 -5 -1 -1 1 5

$[2800, 25]$	$2800'_z$	$(1, 1)$	$t^2 - 4t - 23$
$[700, 28]$	$700'_{xx}$	$(g_2, 1)$	$t^2 - 4t + 31$
$[2100, 28]$	$2100'_x$	$(1, \varepsilon)$	$t^2 - 4t - 41$

3-element family ($\mathfrak{S}_2$): -5 -1 -1 1 5 7

$[5600, 15]$	5600_z	$(1, 1)$	$t^2 - 13$
$[3200, 16]$	3200_x	$(g_2, 1)$	$t^2 + 5$
$[2400, 17]$	2400_z	$(1, \varepsilon)$	$t^2 - 37$

3-element family ($\mathfrak{S}_2$): -7 -5 -1 1 1 5

$[5600, 21]$	$5600'_z$	$(1, 1)$	$t^2 - 13$
$[3200, 22]$	$3200'_x$	$(g_2, 1)$	$t^2 + 5$
$[2400, 23]$	$2400'_z$	$(1, \varepsilon)$	$t^2 - 37$

5-element family ($\mathfrak{S}_3$): -1 1 7 11

$[1400, 7]$	1400_z	$(1, 1)$	$(t + 5)(t^2 + 6t - 19)(t + 1)$
$[1344, 8]$	1344_x	$(g_2, 1)$	$t^4 + 12t^3 + 28t^2 - 48t - 353$
$[1008, 9]$	1008_z	$(1, r)$	$(t^2 + 10t + 1)(t^2 + 2t - 23)$
$[448, 9]$	448_z	$(g_3, 1)$	$(t^2 + 12t + 47)(t^2 + 11)$
$[56, 19]$	56_z	$(1, \varepsilon)$	$(t - 7)(t + 13)(t^2 + 6t - 67)$

5-element family ($\mathfrak{S}_3$): -11 -7 -1 1

$[1400, 37]$	$1400'_z$	$(1, 1)$	$(t - 5)(t^2 - 6t - 19)(t - 1)$
$[1344, 38]$	$1344'_x$	$(g_2, 1)$	$t^4 - 12t^3 + 28t^2 + 48t - 353$
$[1008, 39]$	$1008'_z$	$(1, r)$	$(t^2 - 10t + 1)(t^2 - 2t - 23)$
$[448, 39]$	$448'_z$	$(g_3, 1)$	$(t^2 + 11)(t^2 - 12t + 47)$
$[56, 49]$	$56'_z$	$(1, \varepsilon)$	$(t - 13)(t + 7)(t^2 - 6t - 67)$

5-element family ($\mathfrak{S}_3$): -1 1 5 7

$[1400, 8]$	1400_x	$(1, 1)$	$(t^2 + 6t - 19)(t + 7)(t - 1)$
$[1050, 10]$	1050_x	$(g_2, 1)$	$t^4 + 12t^3 + 34t^2 - 12t - 611$
$[1575, 10]$	1575_x	$(1, r)$	$(t^2 + 8t - 17)(t^2 + 4t - 29)$
$[175, 12]$	175_x	$(g_3, 1)$	$(t^2 + 23)(t^2 + 12t + 59)$
$[350, 14]$	350_x	$(1, \varepsilon)$	$(t + 11)(t - 5)(t^2 + 6t - 43)$

Table 6. (continued)

5-element family ($\mathfrak{S}_3$): -7 -5 -1 1

[1400, 32]	$1400'_x$	$(1,1)$	$(t^2-6t-19)(t+1)(t-7)$
[1050, 34]	$1050'_x$	$(g_2,1)$	$t^4-12t^3+34t^2+12t-611$
[1575, 34]	$1575'_x$	$(1,r)$	$(t^2-4t-29)(t^2-8t-17)$
[175, 36]	$175'_x$	$(g_3,1)$	$(t^2+23)(t^2-12t+59)$
[350, 38]	$350'_x$	$(1,\varepsilon)$	$(t-11)(t+5)(t^2-6t-43)$

17-element family ($\mathfrak{S}_5$): -1 1

[4480, 16]	4480_y	$(1,1)$	$(t^2-7)(t^2-19)(t^2-13)$
[3150, 18]	3150_y	$(g_3,1)$	$(t-5)(t+5)(t^4-2t^2+193)$
[4200, 18]	4200_y	$(g'_2,1)$	$(t-5)(t+5)(t^2+11)(t^2-13)$
[4536, 18]	4536_y	$(1,\nu)$	$(t-3)(t+3)(t^2-29)(t^2-21)$
[5670, 18]	5670_y	$(1,\lambda')$	$(t+3)(t-3)(t^2-17)(t^2-33)$
[420, 20]	420_y	$(g_5,1)$	$(t^2+23)(t^4-2t^2+577)$
[1134, 20]	1134_y	(g_3,ε)	$(t^2-41)(t-3)^2(t+3)^2$
[1400, 20]	1400_y	$(1,\nu')$	$(t-5)(t+5)(t^2-13)(t^2-37)$
[2688, 20]	2688_y	(g'_2,ε'')	$(t^2+11)(t^2-31)(t^2-19)$
[1680, 22]	1680_y	$(1,\lambda^2)$	$(t-7)(t+7)(t^2-13)(t^2-37)$
[168, 24]	168_y	(g'_2,ε')	$(t^2+11)(t^4+10t^2-1451)$
[70, 32]	70_y	$(1,\lambda^3)$	$(t-7)(t+7)(t^2-73)(t^2-97)$
[7168, 17]	7168_w	$(g_2,1)$	$(t^2-19)(t^4-20t^2-71)$
[1344, 19]	1344_w	$(g_4,1)$	$(t^2-19)(t^2+11)(t-1)(t+1)$
[2016, 19]	2016_w	$(g_6,1)$	$t^6-9t^4-81t^2-5671$
[5600, 19]	5600_w	(g_2,r)	$(t-5)(t+5)(t^2-31)(t-1)(t+1)$
[448, 25]	448_w	(g_2,ε)	$(t+7)(t-7)(t^2-19)(t^2-61)$

Table 7.

E_6:

[30, 3]	30_p	$(4,-1)$	$(5,-2)$	$(2,1)$
[30, 15]	$30'_p$	$(1,-4)$	$(2,-5)$	$(-1,-2)$

E_7:

[56, 3]	$56'_a$	$(7,-1)$	$(8,-2)$	$(4,2)$
[56, 30]	56_a	$(1,-7)$	$(2,-8)$	$(-2,-4)$
[120, 4]	120_a	$(5,1)$	$(8,-2)$	$(4,2)$
[120, 25]	$120'_a$	$(-1,-5)$	$(2,-8)$	$(-2,-4)$
[405, 8]	405_a	$(4,-2)$	$(5,-3)$	$(2,0)$
[405, 15]	$405'_a$	$(2,-4)$	$(3,-5)$	$(0,-2)$
[420, 10]	420_a	$(2,-2)$	$(4,-4)$	$(1,-1)$
[420, 13]	$420'_a$	$(2,-2)$	$(4,-4)$	$(1,-1)$

Table 7. (continued)

E_8:

$[112, 3]$	112_z	$(13, -1)$	$(14, -2)$	$(8, 4)$
$[112, 63]$	$112'_z$	$(1, -13)$	$(2, -14)$	$(-8, -4)$
$[210, 4]$	210_x	$(10, 2)$	$(14, -2)$	$(7, 5)$
$[210, 52]$	$210'_x$	$(-2, -10)$	$(2, -14)$	$(-7, -5)$
$[700, 6]$	700_x	$(10, 2)$	$(11, 1)$	$(8, 4)$
$[700, 42]$	$700'_x$	$(-2, -10)$	$(-1, -11)$	$(-4, -8)$
$[2268, 10]$	2268_x	$(6, -2)$	$(8, -4)$	$(3, 1)$
$[2268, 30]$	$2268'_x$	$(2, -6)$	$(4, -8)$	$(-1, -3)$
$[2240, 10]$	2240_x	$(7, -1)$	$(8, -2)$	$(4, 2)$
$[2240, 28]$	$2240'_x$	$(1, -7)$	$(2, -8)$	$(-2, -4)$
$[4200, 12]$	4200_x	$(2, -2)$	$(4, -4)$	$(1, -1)$
$[4200, 24]$	$4200'_x$	$(2, -2)$	$(4, -4)$	$(1, -1)$
$[2800, 13]$	2800_z	$(5, -1)$	$(8, -4)$	$(2, 2)$
$[2800, 25]$	$2800'_z$	$(1, -5)$	$(4, -8)$	$(-2, -2)$
$[5600, 15]$	5600_z	$(4, -4)$	$(5, -5)$	$(2, -2)$
$[5600, 21]$	$5600'_z$	$(4, -4)$	$(5, -5)$	$(2, -2)$