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The slice determined by moduli equation $x = \bar{y}$ in the deformation space of once punctured tori

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Abstract. In the deformation space of once punctured tori, we investigate the slice determined by moduli equation such that the first and the second moduli are complex conjugate. We show the figure of the slice to some extent.

§1. Introduction.

We recall some terminology from [**1**] and [**5**]. As usual, we identify $PSL(2, \boldsymbol{C})$ with the group of all Möbius transformations. Let A and B be loxodromic elements of $PSL(2, \boldsymbol{C})$ with no common fixed point. Let $G = \langle A, B \rangle$ be the group generated by A and B. Let $x = tr(A)$, $y = tr(B)$ and $z = tr(AB)$, where $tr(*)$ is the trace of $*$. The triple (x, y, z) is called a moduli triple of G. A triple determines a group G uniquely up to conjugation such that the moduli triple of G is identical with the original one. So, G is identified with its moduli triple. If G is a quasi-Fuchsian group and if the moduli triple (x, y, z) of G satisfies the equation

$$(*) \qquad x^2 + y^2 + z^2 = xyz,$$

then G represents a pair of once punctured tori. Let $A_0 = \begin{pmatrix} \sqrt{2}+1 & 0 \\ 0 & \sqrt{2}-1 \end{pmatrix}$ and $B_0 = \begin{pmatrix} \sqrt{2} & 1 \\ 1 & \sqrt{2} \end{pmatrix}$ and put $G_0 = \langle A_0, B_0 \rangle$. Then G_0 is a Fuchsian group and $(\sqrt{8}, \sqrt{8}, 4)$ is a moduli triple of G_0 satisfying $(*)$. The deformation space $\boldsymbol{D}(G_0)$ is the set of all quasi-Fuchsian groups which are quasiconformal deformations of G_0. Let

$$\boldsymbol{T}^* = \{(x, y, z) \mid x^2 + y^2 + z^2 = xyz\} \subset \boldsymbol{C}^3.$$

Then, by the stability of quasi-Fuchsian groups, $\boldsymbol{D}(G_0)$ is an open subset of $\boldsymbol{T}^*$. There are a lot of studies of the Bers slice of $\boldsymbol{D}(G_0)$, so called the Teichmüller space of once punctured tori. On the other hand, there are a little of studies of $\boldsymbol{D}(G_0)$ from a point of view of moduli equations. From the latter point of view Keen studied symmetric Riemann surfaces each of which is either a rectangle or a rhombus in [**1**]. The case of rectangle is studied also in [**6**] as a slice of $\boldsymbol{D}(G_0)$.

In this article we shall investigate $\boldsymbol{D}(G_0)$ by means of a slice which is given by the equation

$$(**) \qquad x = \bar{y}.$$

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This is an extension of the case of rhombi. We put

$$S = \{(x, y, z) \in \boldsymbol{D}(G_0) \mid x = \bar{y}\}.$$

We shall call S *the slice determined by* $(**)$. Under the equation $(**)$ the equation $(*)$ turns to

$$(*)' \qquad z^2 - |x|^2 z + x^2 + \bar{x}^2 = 0.$$

Putting

$$D = |x|^4 - 4(x^2 + \bar{x}^2) = |x^2 - 4|^2 - 16,$$

we see that the third module z is one of the following two.

$$(|x|^2 \pm \sqrt{D})/2.$$

So, S is a double cover of its projection into the first module with branch curve over $D = 0$. The projection of the branch curve to x-plane is a lemniscate

$$(1.1) \qquad |x + 2|\,|x - 2| = 4.$$

In the outside or the boundary of the lemniscate (1.1), the third module z is real. Note that this is equivalent to $D \geq 0$. For such a real z the following is known in [**1**] and [**4**].

THEOREM 1.1 ([**1**], [**4**]). *Let (x, y, z) be a moduli triple satisfying $(*)$ and let G be a group associated to it. Assume that $x = \bar{y}$ and z is real. Then G is a quasi-Fuchsian group if and only if*

$$(1.2) \qquad z > 2 \quad \textit{and} \quad |x|^2 > z + 2.$$

Inequalities (1.2) mean that both solutions of $(*)'$ are greater than 2. So, it is equivalent to

$$(|x|^2 - \sqrt{D})/2 > 2$$

or

$$|x| > 2 \quad \text{and} \quad 2|x|^2 - (x^2 + \bar{x}^2) < 4.$$

The last inequality means that $|\mathrm{Im}(x)| < 1$, where $\mathrm{Im}(x)$ means the imaginary part of x. Therefore, putting

$$E = \{(x, \bar{x}, (|x|^2 \pm \sqrt{D})/2) \mid |x^2 - 4| \geq 4, |x| > 2, |\mathrm{Im}(x)| < 1\},$$

we have the following.

COROLLARY 1. $S \cap \{(x, y, z) \mid |x^2 - 4| \geq 4\} = E$.

See Fig. 1 for the projection of E to the first coordinate.

So, the remaining place to investigate is the inside of the lemniscate (1.1). That is, the region determined by $D < 0$. In a case that x is real, the following is known.

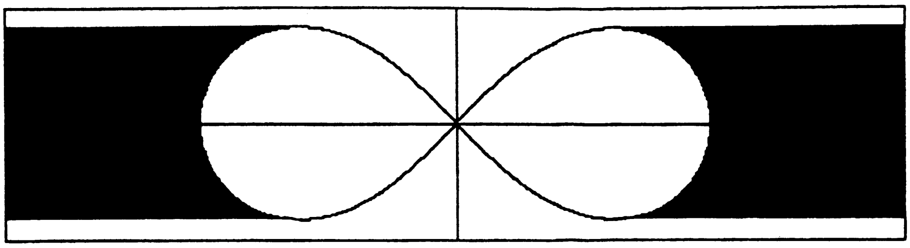

Figure 1

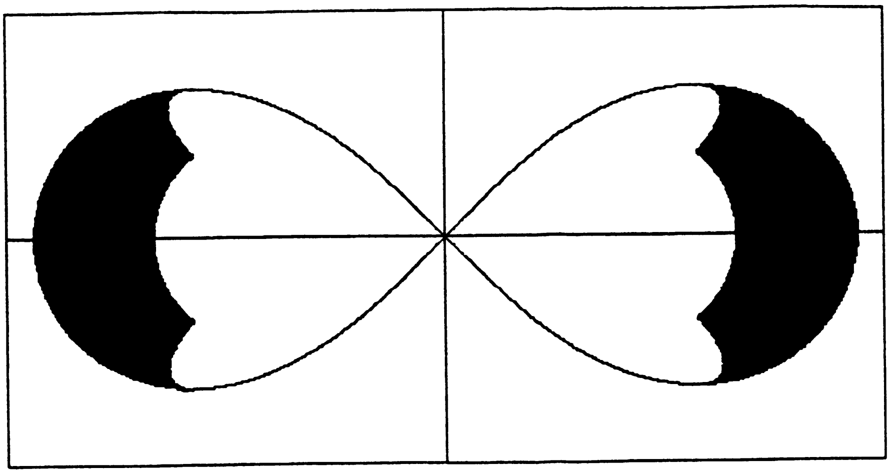

Figure 2

THEOREM 1.2 ([**2**]). *Let* (x, y, z) *be a moduli triple satisfying* $(*)$. *If*

$$x > 2 \quad \textit{and} \quad y > 2,$$

then the group G determined by (x, y, z) *is a quasi-Fuchsian group.*

Theorem 1.2 says that if (x, y, z) satisfies $(*)$ and $(**)$ and if x lies in the interval $(2, \sqrt{8})$, which lies in the inside of the lemniscate (1.1), then G is a quasi-Fuchsian group so that the lift of $(2, \sqrt{8})$ into $\{(x, y, z) \in \boldsymbol{T}^* \mid x = \bar{y}\}$ is contained in S. We shall extend these facts to the following.

THEOREM 1.3. *Let* (x, y, z) *be a moduli triple satisfying* $(*)$ *and* $(**)$. *If*

$$|x^2 - 4| < 4, \tag{1.3}$$

$$2|z|^2 < |x|^4 \quad \textit{and} \tag{1.4}$$

$$\sqrt{(4|z|^2 - |x|^4)(4|x|^4 - |z|^4)} < |x|^2|z|^2 + 4|z|^2 - 8|x|^2, \tag{1.5}$$

then the group G determined by (x, y, z) *is a quasi-Fuchsian group.*

See Fig. 2 for the region of x satisfying the assumption of Theorem 1.3. In contrast to Corollary 1, Theorem 1.3 gives us a partial view of S lying over the inside of the lemniscate (1.1). Let $i = \sqrt{-1}$ and let

$$I = \{(x, \bar{x}, (|x|^2 \pm i\sqrt{-D})) \mid x \text{ satisfies } (1.3), (1.4), (1.5)\}.$$

Corollary 2. $S \cap \{(x, y, z) \mid |x^2 - 4| < 4\} \supset I$.

Our strategy of the proof of Theorem 1.3 is to check an infinite number of inequalities which appear in the following theorem.

Theorem 1.4 ([3]). *Let* $A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}$ *and* $B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$, $bc \neq 0$, *be loxodromic elements of* $PSL(2, \boldsymbol{C})$ *such that* $ABA^{-1}B^{-1}$ *is parabolic and let* $G = \langle A, B \rangle$. *If, for each integer* n, *the inequality*

$$\frac{|\alpha^n a| + |\beta^n d|}{|\alpha^n a + \beta^n d|} < \frac{|\alpha| + |\beta|}{|\alpha - \beta|} \tag{1.6}$$

holds, then G *is a quasi-Fuchsian group and represents a pair of once punctured tori.*

It is well known that the assumption of Theorem 1.4 that $ABA^{-1}B^{-1}$ is parabolic is equivalent to $(*)$. It is shown in §4 that (1.6) for $n = 0$ and $n = \pm 1$ correspond to (1.4) and (1.5), respectively. The rest of this article constitutes of the proof of Theorem 1.3. In §2 we make a normalization and then derive some equalities and inequalities in §3. In §4 we shall check (1.6) for n satisfying $|n| \leq 2$. In §5 the cases of $|n| \geq 4$ are checked. Lastly, we check the cases of $|n| = 3$ in §6.

§2. Normalization.

We assume that five conditions for (x, y, z), that is, $(*)$, $(**)$, (1.3), (1.4) and (1.5) hold. If the first module x is real then $z = (x^2 \pm i\sqrt{8x^2 - x^4})/2$ and $|z|^2 = 2x^2$. Then condition (1.4) implies that $4 < x^2$ so that A is neither elliptic nor parabolic. Hence G is generated by loxodromic elements A and B. In order to use Theorem 1.4, we normalize A and B as follows:

$$A = \begin{pmatrix} \alpha & 0 \\ 0 & \beta \end{pmatrix}, \quad \alpha\beta = 1, \quad |\alpha| > 1$$

and

$$B = \begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

Then

$$\alpha + \beta = x, \quad a + d = y, \quad \alpha a + \beta d = z. \tag{2.1}$$

The moduli equation $(*)$ implies that

$$ad = \left(\frac{\alpha + \beta}{\alpha - \beta}\right)^2. \tag{2.2}$$

Let

$$x = x_1 + ix_2,$$

where x_1 and x_2 are real numbers. By the isomorphism of $PSL(2, \boldsymbol{C})$ to the Möbius transformation group, we may assume that

$$x_1 \geq 0. \tag{2.3}$$

If $x_2 < 0$, then $(**)$ implies that $y = x_1 - ix_2$ so that both real and imaginary parts of y are non-negative. Then we change our normalization of A and B to each other. Of course, this change keeps five conditions invariant. Hence hereafter we assume

$$x_2 \geq 0. \tag{2.4}$$

We shall write up α, β, a, d and z by x. Solving $\alpha^2 + 1 = (x_1 + ix_2)\alpha$ under the condition $|\alpha| > 1$, one obtains

$$\alpha = \frac{1}{2}(x_1 + X_1 + i(x_2 + X_2)), \tag{2.5}$$

where

$$X_1 = \sqrt{\frac{x_1^2 - x_2^2 - 4 + \sqrt{(x_1^2 - x_2^2 - 4)^2 + 4x_1^2 x_2^2}}{2}} \quad \text{and}$$
$$X_2 = \sqrt{\frac{-(x_1^2 - x_2^2 - 4) + \sqrt{(x_1^2 - x_2^2 - 4)^2 + 4x_1^2 x_2^2}}{2}}. \tag{2.6}$$

We put

$$X = \alpha - \beta. \tag{2.7}$$

Then, since $\beta = (x_1 - X_1 + i(x_2 - X_2))/2$, we have

$$X = X_1 + iX_2 \quad \text{and} \quad X^2 = x^2 - 4. \tag{2.8}$$

We also have

$$|X|^4 = (x_1^2 - x_2^2 - 4)^2 + 4x_1^2 x_2^2. \tag{2.9}$$

Now, $(|x|^2 \pm i\sqrt{-D})/2$ are the solutions of $(*)'$, so if we choose

$$z = (|x|^2 + i\sqrt{-D})/2, \tag{2.10}$$

then another solution of $(*)'$ is $\bar{z}$, where $D = |x|^4 - 4(x^2 + \bar{x}^2)$. So, we have

$$|z|^2 = x^2 + \bar{x}^2 = 2(x_1^2 - x_2^2). \tag{2.11}$$

Remark. If $\langle A, B\rangle$ is a quasi-Fuchsian group, then $\langle A, B^{-1}\rangle$ is so, too. The converse is also true. If $(x, \bar{x}, z)$ is the moduli triple of $\langle A, B\rangle$, then $(x, \bar{x}, \bar{z})$ is that of $\langle A, B^{-1}\rangle$, too. So, we may only consider the triples of the form $(x, \bar{x}, z)$.

Proposition 2.1. $4|z|^2 - |x|^4 = -D > 0$.

Proof. Since $D = |x^2 - 4|^2 - 16$, inequality (1.3) implies that $-D > 0$. Hence it suffices to show

$$4|z|^2 - |x|^4 = -(|x|^4 - 4(x^2 + \bar{x}^2)).$$

This follows from (2.11). □

We shall write z such that

$$z = z_1 + iz_2, \quad z_1 = \frac{|x|^2}{2} \quad \text{and} \quad z_2 = \frac{\sqrt{4|z|^2 - |x|^4}}{2}. \tag{2.12}$$

By (2.9) and (2.11), the equality (2.6) can be written as follows:

$$X_1 = \frac{\sqrt{2|X|^2 - 8 + |z|^2}}{2} \quad \text{and} \quad X_2 = \frac{\sqrt{2|X|^2 + 8 - |z|^2}}{2} \tag{2.13}$$

Also by (2.9) and (2.11) we have

$$|X|^4 = |x|^4 - 4|z|^2 + 16. \tag{2.14}$$

By (2.8), (2.13) and (2.14) we have

$$4x_1x_2 = 4X_1X_2 = \sqrt{4|x|^4 - |z|^4}. \tag{2.15}$$

Solving the equations

$$\begin{cases} a + d = x_1 - ix_2 = \bar{x} \\ ad = \left(\dfrac{\alpha + \beta}{\alpha - \beta}\right)^2 = \dfrac{x^2}{X^2}, \end{cases}$$

we have

$$a = \frac{\bar{x}}{2} \pm \frac{i\sqrt{4|z|^2 - |x|^4}}{2X} = \frac{\bar{x}}{2} \pm \frac{iz_2}{X}.$$

The third equality of (2.1) insists that

$$a = \frac{\bar{x}}{2} + \frac{iz_2}{X} \quad \text{and} \quad d = \frac{\bar{x}}{2} - \frac{iz_2}{X}. \tag{2.16}$$

§3. Equalities and inequalities.

In this section we compute some quantities related to $|\alpha^n a| + |\beta^n d|$, $|\alpha^n a + \beta^n d|$ and $|\alpha| + |\beta|$ which appear in (1.6). By (2.5), (2.7) and (2.8) a straight forward calculation shows that

$$\begin{aligned} |\alpha|^2 &= \frac{1}{4}(|x|^2 + |X|^2 + 2(x_1X_1 + x_2X_2)) \quad \text{and} \\ |\beta|^2 &= \frac{1}{4}(|x|^2 + |X|^2 - 2(x_1X_1 + x_2X_2)). \end{aligned} \tag{3.1}$$

So, we have

$$|\alpha|^2 + |\beta|^2 = \frac{|x|^2 + |X|^2}{2}. \tag{3.2}$$

By (2.16) and (2.12) we have

$$(3.3)\qquad \begin{aligned} |a|^2 &= \frac{|x|^2|X|^2 + 4(x_1X_2 - x_2X_1)z_2 + 4|z|^2 - |x|^4}{4|X|^2} \quad \text{and} \\ |d|^2 &= \frac{|x|^2|X|^2 - 4(x_1X_2 - x_2X_1)z_2 + 4|z|^2 - |x|^4}{4|X|^2}. \end{aligned}$$

So, we have

$$(3.4)\qquad |a|^2 + |d|^2 = \frac{|x|^2|X|^2 + 4|z|^2 - |x|^4}{2|X|^2} = \frac{|x|^2|X|^2 + 16 - |X|^4}{2|X|^2}.$$

PROPOSITION 3.1. $|a| \geq |d|$.

PROOF. By (3.3) we may show that $(x_1X_2 - x_2X_1)z_2 \geq 0$. By (2.3), (2.4), (2.6), (2.12) and Proposition 2.1 we see that five numbers x_1, x_2, X_1, X_2, z_2 are non negative. Hence it suffices to show that

$$x_1^2X_2^2 \geq x_2^2X_1^2.$$

By (2.13) this is written as

$$x_1^2(2|X|^2 + 8 - |z|^2) \geq x_2^2(2|X|^2 - 8 + |z|^2).$$

Since $|z|^2 = 2(x_1^2 - x_2^2)$ by (2.11), this is written as

$$8|x|^2 \geq (|x|^2 - |X|^2)|z|^2.$$

Because of $|z|^2 = 2(x_1^2 - x_2^2) \leq 2|x|^2$, it suffices to show

$$4 \geq |x|^2 - |X|^2.$$

By (2.8) we have $||x|^2 - |X|^2| \leq |x^2 - X^2| = 4$, so that we have the desired inequality. □

Here we note that by (2.12) and (2.15) we have

$$\sqrt{(4|z|^2 - |x|^4)(4|x|^4 - |z|^4)} = 8x_1x_2z_2,$$

which is the left hand side of (1.5). We shall put

$$(3.5)\qquad t = \sqrt{(4|z|^2 - |x|^4)(4|x|^4 - |z|^4)} = 8x_1x_2z_2.$$

PROPOSITION 3.2.

$$|\alpha|^2|a|^2 + |\beta|^2|d|^2 = \frac{|z|^2|X|^2 + 4|x|^2 + t}{2|X|^2}$$

and

$$|\beta|^2|a|^2 + |\alpha|^2|d|^2 = \frac{|z|^2|X|^2 + 4|x|^2 - t}{2|X|^2}.$$

PROOF. Writing temporarily

$$A = \frac{|x|^2 + |X|^2}{4}, \quad B = \frac{x_1 X_1 + x_2 X_2}{2}, \quad C = \frac{|x|^2|X|^2 + 4|z|^2 - |x|^4}{4|X|^2}$$

$$\text{and} \quad D = \frac{(x_1 X_2 - x_2 X_1) z_2}{|X|^2},$$

we see by (3.1) and (3.3) that

$$|\alpha|^2 = A + B, \quad |\beta|^2 = A - B, \quad |a|^2 = C + D \quad \text{and} \quad |d|^2 = C - D$$

so that

$$|\alpha|^2|a|^2 + |\beta|^2|d|^2 = 2(AC + BD).$$

Making use of (2.14), we have

$$AC = \frac{|x|^2(|X|^4 - |x|^4) + 4|z|^2(|x|^2 + |X|^2)}{16|X|^2} = \frac{|z|^2|X|^2 + 4|x|^2}{4|X|^2}.$$

Making use of (2.11), (2.13) and (2.15), we have

$$BD = \frac{((x_1^2 - x_2^2)X_1 X_2 - x_1 x_2 (X_1^2 - X_2^2)) z_2}{2|X|^2} = \frac{(|z|^2 X_1 X_2 + x_1 x_2 (8 - |z|^2)) z_2}{4|X|^2}$$

$$= \frac{8 x_1 x_2 z_2}{4|X|^2} = \frac{t}{4|X|^2}.$$

Thus we have the first equality. The second is shown similarly. □

Since $2|x|^2/|X|^2 = 2|ad|$, we have by Proposition 3.2

$$0 \le (|\beta|\,|a| - |\alpha|\,|d|)^2 = |\beta|^2|a|^2 + |\alpha|^2|d|^2 - 2|ad| = \frac{|z|^2|X|^2 - t}{2|X|^2}.$$

Hence we have

$$t \le |z|^2|X|^2. \tag{3.6}$$

PROPOSITION 3.3.

$$|\alpha^2 a + \beta^2 d|^2 = \frac{|z|^2|x|^2 + 2|x|^2 + t}{2}$$

and

$$|\beta^2 a + \alpha^2 d|^2 = \frac{|z|^2|x|^2 + 2|x|^2 - t}{2}.$$

PROOF. Since

$$\alpha^2 a + \beta^2 d = (\alpha + \beta)(\alpha a + \beta d) - (a + d) = xz - \bar{x},$$

we have by (2.11) and (2.12)

$$\begin{aligned}|\alpha^2 a + \beta^2 d|^2 &= |z|^2|x|^2 + |x|^2 - (x^2 z + \bar{x}^2\bar{z}) \\ &= |z|^2|x|^2 + |x|^2 - 2(x_1^2 - x_2^2)z_1 + 4x_1x_2z_2 \\ &= |z|^2|x|^2 + |x|^2 - |z|^2|x|^2/2 + t/2.\end{aligned}$$

Thus we have the first equality. The proof of the second equality is similar. □

For later uses, we note by (2.14) that (1.4) is equivalent to the following.

$$16 < |x|^4 + |X|^4. \tag{1.4$'$}$$

Also for later uses we prepare the following lemma. Of course we are assuming five conditions, especially, (1.3), (1.4) and (1.5).

LEMMA 3.4. $|x|^2 > 2\sqrt{5} - 2$, $x_2^2 < 1$ *and* $2 < x_1^2$.

PROOF. By (1.5) we have

$$\frac{8|x|^2}{|x|^2 + 4} < |z|^2.$$

Then (1.4) insists that

$$|x|^4 + 4|x|^2 - 16 > 0. \tag{3.7}$$

So, we have $|x|^2 > 2\sqrt{5} - 2$.

In terms of x_1 and x_2, (1.3) is written as

$$x_2^4 + 2(x_1^2 + 4)x_2^2 + x_1^4 - 8x_1^2 < 0.$$

A manipulation shows that

$$x_2^2 + \left(\sqrt{x_1^2 + 1} - 2\right)^2 < 1.$$

This implies that $x_2^2 < 1$.

We shall prove the last inequality by dividing into two cases.

Case I: $x_2^2 \geq 2\sqrt{5} - 4$. By (1.5) we have

$$|x|^2|z|^2 + 4|z|^2 - 8|x|^2 > 0.$$

Then by (2.11) we have

$$x_1^4 - x_2^4 - 8x_2^2 > 0.$$

Hence by the assumption that $x_2^2 \geq 2\sqrt{5} - 4$ we have

$$x_1^4 > x_2^4 + 8x_2^2 \geq 4,$$

so that $x_1^2 > 2$.

Case II: $x_2^2 < 2\sqrt{5} - 4$. By the first inequality we have

$$x_1^2 + x_2^2 > 2\sqrt{5} - 2.$$

Hence we have

$$x_1^2 > 2\sqrt{5} - 2 - x_2^2 > 2. \qquad \square$$

§4. Check for the cases of $|n| \leq 2$.

Under the assumption of Theorem 1.3, we prove (1.6) of Theorem 1.4 in this and subsequent two sections. In this section we treat the cases of $|n| \leq 2$.

We shall put

$$L(n) = \frac{|\alpha^n a| + |\beta^n d|}{|\alpha^n a + \beta^n d|} \quad \text{and} \quad R = \frac{|\alpha| + |\beta|}{|\alpha - \beta|}.$$

Then (1.6) is written as

$$L(n) < R. \tag{4.1}$$

By (2.7) and (3.2) we have

$$R^2 = \left(\frac{|\alpha| + |\beta|}{|\alpha - \beta|}\right)^2 = \frac{|x|^2 + |X|^2 + 4}{2|X|^2}. \tag{4.2}$$

Proposition 4.1. *Inequality $L(0) < R$ is equivalent to* (1.4).

Proof. By (2.2) and (3.4) we have

$$L^2(0) = \left(\frac{|a| + |d|}{|a + d|}\right)^2 = \frac{|a|^2 + |d|^2 + 2|ad|}{|x|^2} = \frac{|x|^2|X|^2 + 4|z|^2 - |x|^4 + 4|x|^2}{2|x|^2|X|^2}. \tag{4.3}$$

Hence by (4.2) and (4.3) we see that the inequality $L(0) < R$ is equivalent to

$$4|z|^2 - |x|^4 < |x|^4.$$

Clearly this is equivalent to (1.4). $\square$

Thus we have proved that (1.6) holds for $n = 0$.

Proposition 4.2. *Inequality $L(1) < R$ is equivalent to* (1.5).

Proof. By (2.1), (2.2) and Proposition 3.2 we have

$$L^2(1) = \frac{|\alpha|^2|a|^2 + |\beta|^2|d|^2 + 2|ad|}{|\alpha a + \beta d|^2} = \frac{|z|^2|X|^2 + 8|x|^2 + t}{2|z|^2|X|^2}.$$

Making use of (3.5) and (4.2), we see easily that $L(1) < R$ is equivalent to (1.5). $\square$

Thus we have proved that (1.6) holds for $n = 1$.

Proposition 4.3. $L(-1) \leq L(1)$.

PROOF. By (2.12) we have $z + \bar{z} = |x|^2$. Hence we have

$$|\beta a + \alpha d| = |tr(AB^{-1})| = |tr(A)tr(B) - tr(AB)| = |\,|x|^2 - z| = |\bar{z}| = |\alpha a + \beta d|.$$

Since

$$|\alpha a| + |\beta d| - |\beta a| - |\alpha d| = (|\alpha| - |\beta|)(|a| - |d|) \geq 0,$$

we have

$$L(-1) = \frac{|\beta a| + |\alpha d|}{|\beta a + \alpha d|} \leq \frac{|\alpha a| + |\beta d|}{|\alpha a + \beta d|} = L(1). \qquad \square$$

Propositions 4.2 and 4.3 imply that (1.6) holds for $n = -1$.

PROPOSITION 4.4. $L(-2) < R$.

PROOF. By Proposition 3.3 and by the equality

$$(|\alpha^2 a| + |\beta^2 d|)^2 = (|\alpha|^2 + |\beta|^2)(|\alpha|^2|a|^2 + |\beta|^2|d|^2) - (|a|^2 + |d|^2) + 2|ad|$$

it is not difficult to see that $L(-2) < R$ is equivalent to

$$t < |z|^4 - |x|^4 + |z|^2|x|^2 - 2|z|^2. \tag{4.4}$$

By (1.5) it suffices to show that

$$|x|^2|z|^2 + 4|z|^2 - 8|x|^2 < |z|^4 - |x|^4 + |z|^2|x|^2 - 2|z|^2$$

or

$$(|x|^2 - 4)^2 < (|z|^2 - 3)^2 + 7. \tag{4.5}$$

By Lemma 3.4 we have $-\sqrt{7} < 2\sqrt{5} - 6 < |x|^2 - 4$. Hence we may show this only for those which satisfy $|x|^2 > 6$. For those x, we have by Lemma 3.4

$$|z|^2 - 3 = 2(x_1^2 - x_2^2) - 3 = 2|x|^2 - 4x_2^2 - 3 > 2|x|^2 - 7 > 5.$$

Hence, in order to show (4.5), it suffices to show

$$(|x|^2 - 4)^2 < (2|x|^2 - 7)^2 + 7$$

or

$$0 < 3|x|^4 - 20|x|^2 + 40.$$

Clearly this inequality is true. Thus we have (4.5) so that (4.4) holds. $\square$

PROPOSITION 4.5. $L(2) < R$.

PROOF. By Proposition 3.3 it is not difficult to see that $L(2) < R$ is equivalent to

$$-t < |z|^4 - |x|^4 + |z|^2|x|^2 - 2|z|^2. \tag{4.6}$$

Since $t \geq 0$, we have (4.6) by (4.4). $\square$

Propositions 4.4 and 4.5 imply that (1.6) holds for $|n| = 2$. Therefore we have shown that (1.6) holds for each integer n satisfying $|n| \leq 2$.

§5. Check for the cases of $|n| \geq 4$.

In this section we prove that (1.6) holds for the cases of $|n| \geq 4$. Instead of (4.1) we shall use another inequality. For each integer n, we set

$$A(n) = |\alpha|^{2n}|a|^2 + |\beta|^{2n}|d|^2.$$

Proposition 5.1. *If the inequality*

$$8|x|^2|X|^2 < (|x|^2 - |X|^2 + 4)(|X|^2 A(n) - 2|x|^2) \tag{5.1}$$

holds for an integer n, then (1.6) *holds for n.*

Proof. Since

$$L^2(n) \leq \frac{A(n) + 2|ad|}{A(n) - 2|ad|} \quad \text{and} \quad |ad| = \left|\frac{x}{X}\right|^2,$$

in order to show $L(n) < R$, it suffices to show that

$$\frac{A(n) + 2|x/X|^2}{A(n) - 2|x/X|^2} < R^2.$$

Since R^2 is given by (4.2), a calculation shows that this is equivalent to (5.1). □

Proposition 5.2. *Let n_0 be a negative integer. If* (5.1) *holds for $n = n_0$, then* (1.6) *holds for each integer n satisfying $|n| \geq -n_0$.*

Proof. By Proposition 5.1 it suffices to show that $A(n_0) \leq A(n)$ for $|n| \geq -n_0$. It also suffices to show that

$$A(n_0) \leq A(-n) \leq A(n)$$

for each positive integer $n \geq -n_0$. By Proposition 3.1 we have

$$A(n) - A(-n) = (|\alpha|^{2n} - |\beta|^{2n})(|a|^2 - |d|^2) \geq 0.$$

Hence we have $A(-n) \leq A(n)$ for each positive integer n.

Next we shall show that $A(-n) \leq A(-n-1)$ for each positive integer n. Since

$$\begin{aligned} A(-n-1) - A(-n) &= (|\alpha|^2 + |\beta|^2)A(-n) - A(-n+1) - A(-n) \\ &= (|\alpha| - |\beta|)^2 A(-n) + A(-n) - A(-n+1) \\ &> A(-n) - A(-n+1), \end{aligned}$$

it suffices to show that $A(-1) \leq A(-2)$. Since

$$A(-2) - A(-1) = (|\alpha|^2 + |\beta|^2 - 1)(|\beta|^2|a|^2 + |\alpha|^2|d|^2) - (|a|^2 + |d|^2),$$

we see by (3.2), (3.4) and Proposition 3.2 that, in order to show $A(-1) \le A(-2)$, it suffices to show

$$2(|x|^2|X|^2 + 16 - |X|^4) < (|x|^2 + |X|^2 - 2)(|z|^2|X|^2 + 4|x|^2 - t).$$

Since $t \le |z|^2|X|^2$ by (3.6), it also suffices to show

$$|x|^2|X|^2 + 16 - |X|^4 < 2|x|^2(|x|^2 + |X|^2 - 2)$$

or

$$16 < |x|^4 + |X|^4 + |x|^2(|x|^2 + |X|^2 - 4).$$

By $(1.4)'$ we see easily that $|x|^2 + |X|^2 > 4$. Hence the above inequality holds by $(1.4)'$. Therefore, we have $A(-1) \le A(-2)$, so that $A(-n) \le A(-n-1)$ for each positive integer n. Hence we have shown that $A(n_0) \le A(-n)$ for $n > -n_0$. Thus we have shown that $A(n_0) \le A(-n) \le A(n)$ for each positive integer $n \ge -n_0$. □

Now, we shall show that (5.1) holds for $n = -4$. To do this we need some inequalities. By (3.2) we have

$$|\alpha|^4 + |\beta|^4 = (|\alpha|^2 + |\beta|^2)^2 - 2 = \frac{|x|^4 + |X|^4 + 2|x|^2|X|^2 - 8}{4}.$$

Then by $(1.4)'$ we have

$$|\alpha|^4 + |\beta|^4 > \frac{|x|^2|X|^2 + 4}{2}. \tag{5.2}$$

Also by (3.4) and $(1.4)'$ we have

$$|a|^2 + |d|^2 < \frac{|x|^2(|x|^2 + |X|^2)}{2|X|^2}. \tag{5.3}$$

Since $\alpha\beta = 1$, we have

$$|\beta|^2|a|^2 + |\alpha|^2|d|^2 \ge 2|ad| = \frac{2|x|^2}{|X|^2}. \tag{5.4}$$

Making use of (5.2), (5.3) and (5.4), we have

$$\begin{aligned} A(-4) &= ((|\beta|^2|a|^2 + |\alpha|^2|d|^2)(|\alpha|^2 + |\beta|^2) - (|a|^2 + |d|^2))(|\alpha|^4 + |\beta|^4) - (|a|^2 + |d|^2) \\ &> \frac{|x|^2(|x|^2 + |X|^2)(|x|^2|X|^2 + 2)}{4|X|^2}. \end{aligned}$$

Therefore, in order to show (5.1) for $n = -4$, it suffices to show the following.

$$32|X|^2 < (|x|^2 - |X|^2 + 4)(2(|x|^2 + |X|^2 - 4) + |x|^2|X|^2(|x|^2 + |X|^2)).$$

Since $|x|^2 + |X|^2 > 4$ by $(1.4)'$, it suffices to show the following.

$$8 < |x|^2(|x|^2 - |X|^2 + 4)$$

or

$$|x|^2|X|^2 < |x|^4 + 4|x|^2 - 8.$$

Squaring both sides of the above inequality and using (2.14), we have

$$(2|x|^2 + |z|^2 - 4)|x|^4 - 16|x|^2 + 16 > 0.$$

Since $2|x|^2 + |z|^2 - 4 = 4(x_1^2 - 1)$ by (2.11), the last inequality is written as

$$(x_1^2 - 1)|x|^4 - 4|x|^2 + 4 > 0.$$

Lemma 3.4 assures that this inequality is true. Thus we have proved that (5.1) holds for $n = -4$. Then Proposition 5.2 implies that (1.6) holds for each integer n satisfying $|n| \geq 4$.

§6. Check for the cases of $|n| = 3$.

In this section we prove (6.1) for the cases of $|n| = 3$. First we treat the case of $n = -3$. There are two cases to consider which are divided by the use of inequalities (4.1) or (5.1).

Case I: $x_1^2 \geq 3$. In this case we shall prove (5.1). By (3.2), (3.4) and Proposition 3.2 we have

$$\begin{aligned} 8|X|^2A(-3) = {} & ((|x|^2 + |X|^2)^2 - 4)(|z|^2|X|^2 + 4|x|^2 - t) \\ & - 2(|x|^2 + |X|^2)(|x|^2|X|^2 - |X|^4 + 16). \end{aligned}$$

Since $|z|^2|X|^2 \geq t$ and $|x|^4 + |X|^4 > 16$ by (3.6) and $(1.4)'$, respectively, we have

$$\begin{aligned} 8|X|^2A(-3) &> 4((|x|^2 + |X|^2)^2 - 4)|x|^2 - 2(|x|^2 + |X|^2)^2|x|^2 \\ &= 2((|x|^2 + |X|^2)^2 - 8)|x|^2 \\ &> 4(4 + |x|^2|X|^2)|x|^2. \end{aligned}$$

Thus, in order to show (5.1) for $n = -3$, it suffices to show (5.1) with the replace of $|X|^2A(-3)$ by $(4 + |x|^2|X|^2)|x|^2/2$. It is written as

$$|x|^2|X|^2 < |x|^4 + 4|x|^2 - 16. \tag{6.1}$$

Since $|x|^4 + 4|x|^2 - 16 > 0$ by (3.7), squaring the both sides of (6.1) and making use of (2.14), we see that it suffices to show that

$$|x|^6 - 4|x|^4 + (|z|^2/2)|x|^4 - 16|x|^2 + 32 > 0.$$

By (2.11) we see that the above inequality is equivalent to

$$(x_1^2 - 2)|x|^4 - 8|x|^2 + 16 > 0.$$

Since $x_1^2 \geq 3$ and since if $x_1^2 = 3$ then $|x|^2 < 4$ by Lemma 3.4, the above inequality holds. Therefore, we have proved (5.1) so that in Case I (1.6) holds for $n = -3$ by Proposition 5.2.

Case II: $x_1^2 < 3$. In this case we prove (4.1). Making use of (2.12), (3.5) and the equality $\beta^3 a + \alpha^3 d = \bar{z}(x^2 - 2) - z$, we have

$$|\beta^3 a + \alpha^3 d|^2 = (|z|^2|x|^4 + 4|x|^4 - 2|z|^4 + 2|z|^2 - |x|^2 t)/2. \tag{6.2}$$

On the other hand, we have by Proposition 3.2 and (3.2)

$$\begin{aligned} 2|X|^2(|\beta^3 a| + |\alpha^3 d|)^2 = {} & (|z|^2|X|^2 + 4|x|^2)((|x|^2 + |X|^2)^2 - 8) \\ & - 4|z|^2|X|^2 - ((|x|^2 + |X|^2)^2 - 4)t)/4. \end{aligned} \tag{6.3}$$

Inserting (6.2) and (6.3) into $L(-3)^2 < R^2$ and making use of the identity

$$\begin{aligned} & 2(|z|^2|x|^4 + 4|x|^4 - 2|z|^4 + 2|z|^2) - (|z|^2|X|^2 + 4|x|^2)(|x|^2 + |X|^2 - 4) \\ & = (|x|^2 - |X|^2 + 4)(|x|^2|z|^2 - 4|z|^2 + 4|x|^2) + 4|z|^2, \end{aligned}$$

we have

$$\begin{aligned} & 4(2|x|^2 + |z|^2 - 3)t - 4(|z|^2|x|^2 + 4|z|^2 - 8|x|^2) \\ & < (|x|^2 + |X|^2 + 4)(|x|^2 - |X|^2 + 4)(|x|^2|z|^2 - 4|z|^2 + 4|x|^2) \end{aligned} \tag{6.4}$$

Since

$$(|x|^2 + |X|^2 + 4)(|x|^2 - |X|^2 + 4) = 4(2|x|^2 + |z|^2)$$

by (2.14) and

$$t \le |z|^2|x|^2 + 4|z|^2 - 8|x|^2$$

by (1.5) and (3.5), in order to show (6.4), we may show the following.

$$(2|x|^2 + |z|^2 - 4)(|z|^2|x|^2 + 4|z|^2 - 8|x|^2) < (2|x|^2 + |z|^2)(|x|^2|z|^2 - 4|z|^2 + 4|x|^2)$$

or

$$3|x|^4 - |z|^4 + 2|z|^2 - 4|x|^2 > 0. \tag{6.5}$$

We shall put $x_1^2 = u$ and $x_2^2 = v$. Then $|x|^2 = u + v$ and $|z|^2 = 2(u - v)$ by (2.11). Note that $2 < u < 3$ by Lemma 3.4 and by our assumption $x_1^2 < 3$. Now, (6.5) is written as

$$7u - 4 - 2\sqrt{12u^2 - 14u + 4} < v. \tag{6.6}$$

Thus, in order to show (4.1) for $n = -3$, it suffices to show (6.6) for $2 < u < 3$. It is not difficult to see by (1.4) that

$$7u - 4 - 2\sqrt{12u^2 - 14u + 4} < 17 - 2\sqrt{70} < 2\sqrt{7} - 5 < v.$$

Hence we have shown that (4.1) holds for $n = -3$. Therefore, we have shown that in Case II (1.6) holds for $n = -3$.

Thus we have shown that (1.6) holds for $n = -3$.

Lastly, we treat the case of $n = 3$. By a similar calculation to (6.2) and (6.3) we have

$$|\alpha^3 a + \beta^3 d|^2 = |\beta^3 a + \alpha^3 d|^2 + |x|^2 t$$

and

$$2|X|^2(|\alpha^3 a| + |\beta^3 d|)^2 = 2|X|^2(|\beta^3 a| + |\alpha^3 d|)^2 + ((|x|^2 + |X|^2)^2 - 4)t/2.$$

Then we see that $L(3)^2 < R^2$ is equivalent to (6.4) with the replace of t by $-t$. Since (6.4) is true, it clearly holds. Hence we have proved that (4.1) holds for $n = 3$ so that (1.6) holds for $n = 3$.

Thus we have completed the check of (1.6) for the cases of $|n| = 3$.

In conclusion, we have completed the proof of Theorem 1.3 by Theorem 1.4.

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