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Homogeneous surfaces in the three-dimensional projective geometry

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Abstract. Using an algebraic analog of Cartan's method of moving frames and the complete classification of two-dimensional subalgebras in $\mathfrak{sl}(4, \boldsymbol{R})$, we describe all locally homogeneous surfaces in $\boldsymbol{R}P^3$.

1. Introduction

The main aim of this work is to describe all locally homogeneous two-dimensional surfaces in three-dimensional projective geometry. In general, a submanifold L of a homogeneous space M with transitive transformation group G is called locally homogeneous if for all p and q in L there is a transformation $g \in G$ such that $g(p) = q$ and g maps an open neighborhood of p in L into L. In the first part of this paper we give alternative definitions for locally homogeneous submanifolds. In particular, we give a direct proof that a locally homogeneous submanifold is always an open part of an orbit of some, not necessarily closed, subgroup of G. This will be done in the first part of the paper.

In the second part of the paper we recall the techniques developed in [**3**] and [**8**] for classifying homogeneous submanifolds. Then we apply these techniques to classify projectively homogeneous surfaces of $\boldsymbol{R}P^3$. The classification of projectively homogeneous surfaces was considered with some restrictions in the works of Sophus Lie [**4**] (over the field of complex numbers), by Nomizu–Sasaki [**7**] (only surfaces with non-vanishing Pick invariant) and Dillen–Sasaki–Vrancken [**2**] (surfaces with vanishing Pick-invariant and degenerate surfaces). For corresponding classification in unimodular and affine three-dimensional geometries see [**1**], [**3**], [**5**], [**6**]. The techniques in [**7**], [**2**] are purely differential geometric: special adapted frames are constructed and the classification is done by integrating the fundamental equations, using the fact that all geometric invariants are constant. Here we give a different proof, based on the language of Lie algebras.

We divide our problem into the following two parts:

1. Classification of all surfaces whose symmetry algebra has dimension 3 or higher.
2. Classification of those surfaces whose symmetry algebra is two-dimensional.

To solve the first of these two smaller problems, we use an algebraic analog of the

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method of moving frames developed in [**3**]. The second problem consists in describing all two-dimensional subalgebras in $\mathfrak{sl}(4, \boldsymbol{R})$ and their two-dimensional orbits. Here we shall draw upon the classification methods of linear Lie algebras described in [**8**].

The main result of this paper is the following theorem:

THEOREM 1. *Every locally homogeneous surface in* $\boldsymbol{R}P^3$ *is either an open subset of a cylinder or a quadric, or is equivalent to an open subset of one of the following surfaces*:

1. $z = x^a y^b$.
2. $z = (x^2 + y^2)^a e^{b \arg(x+iy)}$.
3. $\arg(x + iy) = a \ln \dfrac{x^2 + y^2}{1 + z^2} + b \operatorname{arctg} z$.
4. $z = \ln x + a \ln y$.
5. $z = a \arg(x + iy) + b \ln(x^2 + y^2)$, $a = 1, b \geq 0$ *or* $a = 0, b = 1$.
6. $z = y^2 \pm x^a$.
7. $z = xy + x^a$.
8. $z = xy + e^{a \operatorname{arctg} x}(1 + x^2)$.
9. $\left(z - xy + \dfrac{1}{3}x^3\right)^2 = a\left(y - \dfrac{1}{2}x^2\right)^3$.
10. $z = xy + x \ln x$.
11. $z = xy + (1 + x^2)\operatorname{arctg} x$.
12. $z = xy + \ln x$.
13. $z = y^2 \pm x \ln x$.
14. $z = y^2 \pm \ln x$.
15. $z = y^2 \pm e^x$.
16. $z = xy + e^x$.

REMARK 1. Every locally homogeneous cylinder is equivalent to a cylinder whose base is a locally homogeneous curve in the plane (for a list of all locally homogeneous plane curves, see for instance [**2**]), while every quadric which is not a cylinder is equivalent to one of the two quadrics $z = x^2 \pm y^2$.

REMARK 2. For certain values of parameters, the above surfaces may turn into quadrics or even cylinders. Moreover, in each case there is an equivalence relation on the set of parameters. This information is given in Tables 1 and 2.

REMARK 3. The complex version of this theorem is obtained by excluding the surfaces 2, 3, 5, 8, 11 and changing the signs $\pm$ to simply $+$ (or $-$).

Below we list the symmetry groups of surfaces given in Theorem 1. We consider them as subgroups in $GL(4, \boldsymbol{R})$, even though the action of $GL(4, \boldsymbol{R})$ on $\boldsymbol{R}P^3$ is not effective and these symmetry groups are defined up to multiplication by scalar matrices. In all cases except for 3, 8, and 11 the symmetry group preserves an affine chart, in which the equation of the surface is written. In all these cases we present the symmetry group in such a way that it lies in the standard embedding of the group of affine motions $\mathrm{Aff}(\boldsymbol{R}^3)$ into $GL(4, \boldsymbol{R})$.

1. $$\left\{ \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & q & 0 & 0 \\ 0 & 0 & p^a q^b & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

2. $$\left\{ \begin{pmatrix} e^q \cos p & -e^q \sin p & 0 & 0 \\ e^q \sin p & e^q \cos p & 0 & 0 \\ 0 & 0 & e^{2aq+bp} & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

3. $$\left\{ \begin{pmatrix} e^q \cos p & -e^q \sin p & 0 & 0 \\ e^q \sin p & e^q \cos p & 0 & 0 \\ 0 & 0 & e^{-q} \cos(\alpha q + \beta p) & -e^{-q} \sin(\alpha q + \beta p) \\ 0 & 0 & e^{-q} \sin(\alpha q + \beta p) & e^{-q} \cos(\alpha q + \beta p) \end{pmatrix} \right\};$$

4. $$\left\{ \begin{pmatrix} e^p & 0 & 0 & 0 \\ 0 & e^q & 0 & 0 \\ 0 & 0 & 1 & p + aq \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

5. $$\left\{ \begin{pmatrix} e^q \cos p & -e^q \sin p & 0 & 0 \\ e^q \sin p & e^q \cos p & 0 & 0 \\ 0 & 0 & 1 & ap + 2bq \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

6. $$\left\{ \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^{a/2} & 0 & q \\ 0 & 2p^{a/2} q & p^a & q^2 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

7. $$\left\{ \begin{pmatrix} p & 0 & 0 & 0 \\ 0 & p^{a-1} & 0 & q \\ pq & 0 & p^a & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\};$$

8. $$\left\{ \begin{pmatrix} e^{-\alpha p} \cos p & -e^{-\alpha p} \sin p & 0 & 0 \\ e^{-\alpha p} \sin p & e^{-\alpha p} \cos p & 0 & 0 \\ \cos p & -q \sin p & e^{\alpha p} \cos p & -e^{\alpha p} \sin p \\ q \sin p & q \cos p & e^{\alpha p} \sin p & e^{\alpha p} \cos p \end{pmatrix} \right\};$$

9. $\left\{\begin{pmatrix} p & 0 & 0 & q \\ pq & p^2 & 0 & q^2/2 \\ \frac{1}{2}pq^2 & p^2q & p^3 & q^3/6 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

10. $\left\{\begin{pmatrix} p & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ p(q+\ln p) & 0 & p & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

11. $\left\{\begin{pmatrix} \cos p & -\sin p & 0 & 0 \\ \sin p & \cos p & 0 & 0 \\ q\cos p - p\sin p & -q\sin p - p\cos p & \cos p & -\sin p \\ q\sin p + p\cos p & q\cos p - p\sin p & \sin p & \cos p \end{pmatrix}\right\};$

12. $\left\{\begin{pmatrix} e^p & 0 & 0 & 0 \\ 0 & e^{-p} & 0 & p \\ qe^p & 0 & 1 & q \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

13. $\left\{\begin{pmatrix} e^p & 0 & 0 & 0 \\ 0 & e^{p/2} & 0 & q \\ \pm pe^p & 2qe^{p/2} & e^p & q^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

14. $\left\{\begin{pmatrix} e^p & 0 & 0 & 0 \\ 0 & 1 & 0 & q \\ 0 & 2q & 1 & q^2 \pm p \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

15. $\left\{\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & e^{p/2} & 0 & q \\ 0 & 2qe^{p/2} & e^p & q^2 \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\};$

16. $\left\{\begin{pmatrix} 1 & 0 & 0 & p \\ 0 & e^p & 0 & q \\ q & pe^p & e^p & pq \\ 0 & 0 & 0 & 1 \end{pmatrix}\right\}.$

2. Jets of submanifolds.

Let M be an m-dimensional manifold, and let L_1, L_2 be two submanifolds of dimension l containing a point $p \in M$.

Definition. The submanifolds L_1 and L_2 are said to be *k-equivalent* (or also to have *contact of order* at least k) at p if there exists a local coordinate system $(x^1, \dots, x^l, y^1, \dots, y^{m-l})$ with origin at p such that

1. the submanifolds L_i $(i = 1, 2)$ have, in this coordinate system, equations of the form $y_i^j = f_i^j\ (x_1, \dots, x_l)$, $j = 1, \dots, m-l$, respectively;
2. for each j, the functions f_1^j and f_2^j have the same partial derivatives of order $\le k$ at the point $(0, \dots, 0)$.

It is easily verified that the relation of being k-equivalent is well defined and is indeed an equivalence relation. The k-equivalence class of a submanifold L at a point $p \in L$, denoted by $[L]_p^k$, is called the *k-jet* of L. The totality $J^k(M, l)$ of all k-jets at all points of the manifold M may be endowed with a smooth manifold structure and is called the *space of k-jets of l-dimensional submanifolds of the manifold M*.

For any k, n such that $1 \le k \le n$, we define a smooth mapping

$$\pi_{n,k} : J^n(M, l) \to J^k(M, l), \quad [L]_p^n \mapsto [L]_p^k,$$

and for any $k \ge 1$ we define a mapping

$$\pi_k : J^k(M, l) \to M, \quad [L]_p^k \mapsto p.$$

If L is an arbitrary l-dimensional submanifold in M, then the subset $L^{(k)} = \{[L]_p^k | p \in L\}$ is a submanifold in $J^k(M, l)$, called the *k-th prolongation* of L. It is clear that the mapping π_k maps $L^{(k)}$ diffeomorphically onto L.

The following result, which we shall need later, is given here without proof, as it follows immediately from the definitions.

Lemma 1. *If two submanifolds L_1 and L_2 are k-equivalent at a point $p \in M$, then the following two conditions are equivalent*:

1. *L_1 and L_2 are $(k+1)$-equivalent at the point p*;
2. *the tangent spaces to the submanifolds $L_1^{(k)}$ and $L_2^{(k)}$ at the point $q = [L_1]_p^k = [L_2]_p^k \in J^k(M, l)$ coincide.*

Suppose that a certain Lie group G acts on M smoothly. Then this action may be in a natural way extended to the action of G on $J^k(M, l)$:

$$g.[L]_p^k = [g.L]_{g.p}^k.$$

3. Invariant foliations on homogeneous spaces.

Let M be a homogeneous space of a Lie group $\bar{G}$, o a point in M, and G the stationary subgroup of the point o. Then M can be identified with the set of left cosets $\bar{G}/G$, the point o being identified with the coset eG.

Invariant foliations on M are in a one-to-one correspondence with invariant completely integrable distributions on M. Our goal in this sections is to prove the following auxiliary result.

Lemma 2. *Every invariant foliation on M has the form $\{(gH)G\}_{g \in \bar{G}}$, where $H \subset \bar{G}$ is some virtual (i.e. not necessarily closed) Lie subgroup containing G. The fiber of this foliation at a point o is formed by the orbit of o under the action of H^o on M.*

PROOF. Suppose E is the invariant completely integrable distribution corresponding to a certain invariant foliation on M. Let now $\bar{\mathfrak{g}}$ be the Lie algebra of $\bar{G}$, which we shall identify with $T_e\bar{G}$, and let $\mathfrak{g}$ be a subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup G. We consider the natural projection $\pi : \bar{G} \to M, g \mapsto gG$ and identify the space T_oM with $\bar{\mathfrak{g}}/\mathfrak{g}$ by means of the surjection $d_o\pi : \bar{\mathfrak{g}} \to T_oM$. Let $\mathfrak{h} = d_o\pi^{-1}(E_o)$. Since E is completely integrable, it follows that $\mathfrak{h}$ is a subalgebra of $\bar{\mathfrak{g}}$. Let H_0 be the connected virtual Lie subgroup of the Lie group $\bar{G}$, corresponding to the subalgebra $\mathfrak{h}$.

From the invariance of the distribution E it follows that the subalgebra $\mathfrak{h}$ is stable under $\operatorname{Ad} G$. Hence the subgroup H_0 is stable under the conjugation by the elements of G. Put $H = H_0G$. Since H_0 contains the identity component of G, we see that H is a virtual subgroup of $\bar{G}$ whose identity component coincides with H_0. It is now easy to show that $\pi(H) = \pi(H_0)$ is a maximal integral manifold of the distribution E and has the form $HG = H_0G$. Since the foliation is invariant, the rest of its fibers have the form $(gH)G$ with $g \in \bar{G}$.

4. Homogeneous submanifolds.

Let M be a homogeneous space of a Lie group $\bar{G}$, and let L be an immersed submanifold in M. We assume that the action of $\bar{G}$ on M is locally effective and identify the Lie algebra $\bar{\mathfrak{g}}$ of $\bar{G}$ with a subalgebra of the Lie algebra of all vector fields on M.

DEFINITION. The *symmetry algebra of a submanifold L* is the subalgebra $\operatorname{sym}(L)$ in $\bar{\mathfrak{g}}$ defined by

$$\operatorname{sym}(L) = \{X \in \bar{\mathfrak{g}} \mid X_p \in T_pL \text{ for all } p \in L\}.$$

For any subalgebra $\mathfrak{h} \subset \bar{\mathfrak{g}}$ and arbitrary point $p \in M$, let $\mathfrak{h}_p$ denote the subspace T_pM of the form $\{X_p \mid X \in \mathfrak{h}\}$. For all points $p \in L$ we obviously have $\operatorname{sym}(L)_p \subset T_pL$.

THEOREM 2. *If L is a connected immersed submanifold in M, then the following four conditions are equivalent*:

1. *for any point $p \in L$ there exist a neighborhood U of p in L and a smooth mapping $\phi : U \to \bar{G}$ such that $\phi(q).p = q$ and $\phi(q).U \subset L$ for all $q \in U$;*
2. *given two arbitrary points $p, q \in L$, there always exist a neighborhood U of p in L and an element $g \in \bar{G}$, such that $g.p = q$ and $g.U \subset L$;*
3. *the submanifold L is an open subset in some orbit of a connected virtual subgroup $H \subset \bar{G}$;*
4. *for any point $p \in L$ the following equality holds:* $\operatorname{sym}(L)_p = T_pL$.

PROOF. (1) $\Rightarrow$ (2). We fix an arbitrary point $p \in L$ and denote by N the set of all points q in L for which condition (2) is satisfied (i.e., there exist a neighborhood of p in L and an element $g \in \bar{G}$ such that $g.p = q$ and $g.U \subset L$). The set N is nonempty and is open in L. Indeed, it contains the point p itself, and it follows from condition (1) that together with each point q this set contains a certain neighborhood of q. By a similar reasoning, the complement of N in L is also open. Since L is connected, we have $N = L$ and L satisfies condition (2).

$(2) \Rightarrow (3)$. Consider the natural lift of the action of the Lie group $\bar{G}$ to $J^k(M,l)$. Fix a point $p \in L$ and let $p_k = [L]_p^k$. Let G_k denote the stationary subgroup of p_k. It is clear that $G_{k+1} \subset G_k$ for all $k \geq 0$. Hence for some sufficiently large n we shall have $\dim G_{n-1} = \dim G_n$.

Denote by $\mathscr{E}_k$ the orbit of p_k under the action of $\bar{G}$ on $J^k(M,l)$. It immediately follows from (2) that $L^{(k)} \subset \mathscr{E}_k$. Define

$$\pi = \pi_{n,n-1}|_{\mathscr{E}_n} : \mathscr{E}_n \to \mathscr{E}_{n-1}.$$

Since the dimensions of the stationary subgroups G_n and G_{n-1} are the same, π is a local diffeomorphism.

Consider the subspace $E_{p_k} = T_{p_k}L^{(k)}$ of the space $T_{p_k}\mathscr{E}_k$ for all $k \geq 0$. We shall now show that E_{p_n} is invariant under the isotropic action of G_n on $T_{p_n}\mathscr{E}_n$. Indeed, we have

$$d_{p_n}g(E_{p_n}) = T_{p_n}(g.L)^{(n)}, \quad \text{for all } g \in G_n.$$

Since the submanifolds L and $g.L$ are n-equivalent at p, we have

$$T_{p_{n-1}}(g.L)^{(n-1)} = E_{p_{k-1}}.$$

But the mapping $d_{p_n}\pi$ is an isomorphism of the spaces $T_{p_n}\mathscr{E}_n$ and $T_{p_{n-1}}\mathscr{E}_{n-1}$ and transforms each of the subspaces E_{p_n} and $T_{p_n}(g.L)^{(n)}$ into the subspace $E_{p_{n-1}}$. Hence $d_{p_n}g(E_{p_n}) = E_{p_n}$ for all $g \in G_n$. This allows to define a $\bar{G}$-invariant distribution E on $\mathscr{E}_n$ by the formula

$$E_q = d_{p_n}g(E_{p_n}), \quad \text{for all } g \in \bar{G}, g.p_n = q.$$

We show that E is completely integrable. It will actually suffice to show that $L^{(n)}$ is an integral manifold of M. Indeed, let $q = \pi_n(q_n)$ for any $q_n \in L^{(n)}$. Now let g be an element of $\bar{G}$ such that $g.p = q$ and $g.U \subset L$ for some neighborhood U of p in L. Then $g.p_n = q_n$ and $g.U^{(n)} \subset L^{(n)}$. It follows that

$$T_{q_n}L^{(n)} = d_{p_n}g(T_{p_n}L^{(n)}) = d_{p_n}g(E_{p_n}) = E_{q_n}.$$

Suppose N is the maximal integral manifold of E passing through p. By Lemma 2, there exists a connected Lie subgroup $H \subset \bar{G}$ such that N is the orbit of p_n under the action of H on $J^n(M,l)$. In addition, $L^{(n)}$ is an open subset in N. Thus L is an open subset of the orbit of p under the action of H on p.

$(3) \Rightarrow (1)$. Let p be an arbitrary point in the submanifold L, and let N be the orbit of p under the action of H on M. Consider the natural projection $\pi : H \to N$, $g \mapsto g.p$. Then π is a fibration, and in some neighborhood U of p there exists a section $\phi : U \to H$ of π. We may assume without loss of generality that $U \subset L$. But then the neighborhood U and the mapping ϕ satisfy condition (1).

$(3) \Rightarrow (4)$. Let $\mathfrak{h}$ be a subalgebra of $\bar{\mathfrak{g}}$ corresponding to the subgroup H. It is clear that $\mathfrak{h} \subset \mathrm{sym}(L)$, and $\mathfrak{h}_p = T_pL$ for all $p \in L$. It now easily follows that $\mathrm{sym}(L)_p = T_pL$ for all $p \in L$.

$(4) \Rightarrow (3)$. Let H be the connected virtual subgroup of $\bar{G}$ corresponding to the subalgebra $\mathrm{sym}(L) \subset \bar{\mathfrak{g}}$, and let p be an arbitrary point of L. It is easy to show that the orbit of p under the action of H on M contains L as an open subset.

DEFINITION. A connected immersed submanifold L in M is called (*locally*) *homogeneous* if it satisfies one of the equivalent conditions of Theorem 2.

5. Symmetry algebras of homogeneous surfaces.

Consider a smooth manifold M with a transitive action of a Lie group $\bar{G}$. Let $\mathfrak{h}$ be an arbitrary subalgebra of $\bar{\mathfrak{g}}$, and H the corresponding connected virtual subgroup of $\bar{G}$. Then the orbits of H can be considered as embedded submanifolds of M. We say that L is an orbit of $\mathfrak{h}$ through a point $p \in M$ if L is a connected open submanifold (in the internal topology) of the orbit of H through the point p, and $p \in L$. Of course, an orbit of $\mathfrak{h}$ through $p \in M$ is not unique, but any two orbits coincide in a certain neighborhood of p.

Let L be an orbit of $\mathfrak{h}$. Then, obviously, $\mathfrak{h} \subset \operatorname{sym}(L)$. It is easy to see that L is locally homogeneous. But, generally speaking, $\operatorname{sym}(L)$ is bigger than $\mathfrak{h}$. The next theorem determines the whole symmetry algebra of L.

THEOREM 3 ([**3**]). *Let a be a fixed point on M, and let L be an orbit of $\mathfrak{h}$ through a. Define a subalgebra $\mathfrak{g} \subset \bar{\mathfrak{g}}$ as follows:* $\mathfrak{g} = \{X \in \bar{\mathfrak{g}} \mid X_a = 0\}$.

1. *The subalgebra* $\operatorname{sym}(L)$ *is the greatest subalgebra* $\mathfrak{a}$ *such that* $\mathfrak{h} \subset \mathfrak{a} \subset \mathfrak{g} + \mathfrak{h}$.
2. *Consider the series of subalgebras* $\mathfrak{g} = \mathfrak{g}_0 \supset \mathfrak{g}_1 \supset \cdots$ *such that*

$$\mathfrak{g}_0 = \mathfrak{g}, \quad \mathfrak{g}_{n+1} = \{x \in \mathfrak{g}_n \mid [x, \mathfrak{h}] \supset \mathfrak{g}_n + \mathfrak{h}\},$$

and let $\mathfrak{g}_\infty = \bigcap_{n=0}^{\infty} \mathfrak{g}_n$. *Then*

$$\operatorname{sym}(L) = \mathfrak{g}_\infty + \mathfrak{h}.$$

COROLLARY. *The subalgebra $\mathfrak{h} \subset \bar{\mathfrak{g}}$ is a symmetry algebra of some homogeneous submanifold L containing a if and only if $\mathfrak{h}$ satisfies one of the following equivalent conditions*:

(1) *every subalgebra of $\bar{\mathfrak{g}}$ that contains $\mathfrak{h}$ and lies in $\mathfrak{g} + \mathfrak{h}$, coincides with $\mathfrak{h}$*;
(2) $\mathfrak{g}_\infty \subset \mathfrak{h}$.

DEFINITION. We call a subalgebra $\mathfrak{h} \subset \bar{\mathfrak{g}}$ *admissible* (*with respect to the pair* $(\bar{\mathfrak{g}}, \mathfrak{g})$) if it satisfies the condition of the corollary.

Let us now develop a method for the classification of all admissible subalgebras $\mathfrak{h} \subset \bar{\mathfrak{g}}$. Suppose that $\mathfrak{h}$ is such a subalgebra. Define the following vector spaces and linear mappings between them:

– let $V_n = (\mathfrak{h} + \mathfrak{g}_n)/\mathfrak{g}_n \subset \bar{\mathfrak{g}}/\mathfrak{g}_n$ for all $n \geq 0$, and $V_\infty = (\mathfrak{h} + \mathfrak{g}_\infty)/\mathfrak{g}_\infty \subset \bar{\mathfrak{g}}/\mathfrak{g}_\infty$;
– let $\pi_n : \bar{\mathfrak{g}} \to \bar{\mathfrak{g}}/\mathfrak{g}_n$ and $\tau_n : \bar{\mathfrak{g}}/\mathfrak{g}_{n+1} \to \bar{\mathfrak{g}}/\mathfrak{g}_n$ be the natural projections for all $n \geq 0$, and in a similar way we define π_∞.

We have the following commutative diagram (here vertical arrows correspond to the natural embeddings):

$$\begin{array}{ccccccccc}
\mathfrak{h} & \longrightarrow & V_\infty & \longrightarrow & \cdots & \longrightarrow & V_1 & \longrightarrow & V_0 \\
\downarrow & & \downarrow & & \downarrow & & \downarrow & & \downarrow \\
\bar{\mathfrak{g}} & \xrightarrow{\pi_\infty} & \bar{\mathfrak{g}}/\mathfrak{g}_\infty & \longrightarrow & \cdots & \xrightarrow{\tau_1} & \bar{\mathfrak{g}}/\mathfrak{g}_1 & \xrightarrow{\tau_0} & \bar{\mathfrak{g}}/\mathfrak{g}_0
\end{array}$$

This diagram is finite, i.e. $\mathfrak{g}_n = \mathfrak{g}_{n+1} = \cdots = \mathfrak{g}_\infty$ and $V_n = V_{n+1} = \cdots$ for sufficiently large n.

It is easy to see that these objects satisfy the following conditions:

1. $\mathfrak{g}_\infty = \mathfrak{h} \cap \mathfrak{g}_n$ for all $n \geq 0$.
2. $\dim V_n = \dim V_\infty$ for all $n \geq 0$.
3. $\tau_n|_{V_{n+1}}$ is an isomorphism of V_{n+1} to V_n for all $n \geq 0$.
4. $\mathfrak{g}_{n+1} = \{x \in \mathfrak{g}_n \mid x.V_n \subset V_n\}$. (Here we consider $\bar{\mathfrak{g}}/\mathfrak{g}_n$ as a natural $\mathfrak{g}_n$-module.)
5. $\mathfrak{h} = \pi_\infty^{-1}(V_\infty)$.
6. The subspace V_n for all $n \geq 1$ satisfies the following condition:

CS: for every two elements $x + \mathfrak{g}_n, y + \mathfrak{g}_n \in V_n$, the element $[x, y] + \mathfrak{g}_{n-1}$ lies in V_{n-1}.

We say that two subalgebras of $\bar{\mathfrak{g}}$ are equivalent if they can be transformed into each other by means of the elements of $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$. For a given subspace $V_n \subset \bar{\mathfrak{g}}/\mathfrak{g}_n$, denote by G_{n+1} the subgroup in $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$ consisting of all automorphisms that preserve the subalgebra $\mathfrak{g}_n$ and induce on $\bar{\mathfrak{g}}/\mathfrak{g}_n$ a transformation preserving V_n. From the definition it follows that G_{n+1} also preserves $\mathfrak{g}_{n+1}$ and, therefore, induces a group of linear transformations on $\bar{\mathfrak{g}}/\mathfrak{g}_{n+1}$.

From the above considerations we obtain the following algorithm for the description (up to equivalence) of all admissible subalgebras in $\bar{\mathfrak{g}}$ corresponding to locally homogeneous submanifolds of a given dimension m.

1. Describe all subspaces V_0 of dimension m in $\bar{\mathfrak{g}}/\mathfrak{g}$ (up to the transformations induced by $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$).

2. For each subspace V_n found before, find the subalgebra $\mathfrak{g}_{n+1}$, the group G_{n+1}, and the subspace $W = \tau_n^{-1}(V_n)$ in $\bar{\mathfrak{g}}/\mathfrak{g}_{n+1}$. If $\mathfrak{g}_{n+1} \neq \mathfrak{g}_n$, then describe (up to G_{n+1}) all subspaces V_{n+1} in W such that

- V_{n+1} is complementary to $\ker \tau_n = \mathfrak{g}_n/\mathfrak{g}_{n+1}$;
- V_{n+1} satisfies the condition **CS**.

Then repeat this step again.

3. If $\mathfrak{g}_n = \mathfrak{g}_{n+1}$, then find the subspace $\mathfrak{h} = \pi_n^{-1}(V_n)$ in $\bar{\mathfrak{g}}$. If $\mathfrak{h}$ is a subalgebra, then Theorem 3 implies that it is a symmetry algebra of a certain homogeneous submanifold. Moreover, all admissible algebras can be obtained in this way.

6. Three-dimensional projective geometry.

In this section we apply the above algorithm to the case of three-dimensional projective geometry, so that $\bar{\mathfrak{g}} = \mathfrak{sl}(4, \boldsymbol{R})$ and $\mathbf{M} = \boldsymbol{R}P^3$. Fix the subspace $V \subset \boldsymbol{R}^4$ of the form

$$V = \left\{ \begin{pmatrix} x \\ 0 \\ 0 \\ 0 \end{pmatrix} \,\middle|\, x \in \boldsymbol{R} \right\}.$$

The subspace V will also be considered as a point $o \in \boldsymbol{R}P^3$; the projective coordinates of this point are $o = [1 : 0 : 0 : 0]$. Then $\mathfrak{g} = \mathfrak{g}_V$ is the algebra of the stabilizer of V. It may be found from the formula $\mathfrak{g} = \{X \in \bar{\mathfrak{g}} \mid X.V \subset V\}$ and has the following form:

$$\mathfrak{g} = \left\{ \begin{pmatrix} -\operatorname{tr} A & B \\ 0 & A \end{pmatrix} \,\middle|\, A \in \mathfrak{gl}(3, \boldsymbol{R}),\ B \in \mathrm{Mat}_{1\times 3}(\boldsymbol{R}) \right\}.$$

To describe the group $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$, consider the group

$$\mathrm{Aut}(\bar{\mathfrak{g}}, V) = \{a_g,\ g \in GL(4, \boldsymbol{R}) \mid g.V = V\},$$

where, as usual, $a_g \in \mathrm{Aut}(\bar{\mathfrak{g}})$ is the conjugation by the element g, that is, $a_g : A \mapsto gAg^{-1}$. It is easily verified that $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g}) = \mathrm{Aut}(\bar{\mathfrak{g}}, V)$.

In order to get rid of homotheties, we identify $\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$ with the subgroup

$$G_0 = \left\{ \begin{pmatrix} 1 & Y \\ 0 & X \end{pmatrix} \,\middle|\, X \in GL(3, \boldsymbol{R}),\ Y \in \mathrm{Mat}_{1\times 3}(\boldsymbol{R}) \right\} \subset GL(4, \boldsymbol{R}).$$

Consider the action of G_0 on $\bar{\mathfrak{g}}/\mathfrak{g}$. To this end, we write the elements of $\bar{\mathfrak{g}}/\mathfrak{g}$ in the form

$$u = \begin{pmatrix} 0 & 0 \\ A & 0 \end{pmatrix} + \mathfrak{g}, \quad A \in \boldsymbol{R}^3.$$

Then the action of G_0 on $\bar{\mathfrak{g}}/\mathfrak{g}$ establishes the following equivalence relation: $A \sim XA$, $X \in GL(3, \boldsymbol{R})$. It follows that any two one-dimensional or two-dimensional subspaces of $\bar{\mathfrak{g}}/\mathfrak{g}$ are equivalent up to the group G_0.

To simplify the notation, we shall identify the quotient spaces $\bar{\mathfrak{g}}/\mathfrak{g}_n$ with certain complements of $\mathfrak{g}_n$ in $\bar{\mathfrak{g}}$.

7. Surfaces whose symmetry algebras are at the least three-dimensional.

7.1. PRELIMINARY STAGE. In this section we shall focus on symmetry algebras of surfaces. Then, in accordance with our algorithm, $\dim V_n = 2$ for all n. Here we shall not consider cases in which $\mathfrak{g}_n = \{0\}$. Since all two-dimensional subspaces of $\bar{\mathfrak{g}}/\mathfrak{g}$ are equivalent up to the action of G_0, we may assume that

$$V_0 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \,\middle|\, x, y \in \boldsymbol{R} \right\}.$$

Using the definition of $\mathfrak{g}_1$ and G_1, we find that

$$\mathfrak{g}_1 = \left\{ \begin{pmatrix} -d - \operatorname{tr} A & B & e \\ 0 & A & C \\ 0 & 0 & d \end{pmatrix} \,\middle|\, A \in \mathfrak{gl}(2, \boldsymbol{R}), B, {}^tC \in \mathrm{Mat}_{1\times 2}(\boldsymbol{R}), d, e \in \boldsymbol{R} \right\},$$

$$G_1 = \left\{ \begin{pmatrix} 1 & Y & u \\ 0 & X & Z \\ 0 & 0 & t \end{pmatrix} \,\middle|\, X \in GL(2, \boldsymbol{R}), Y, {}^tZ \in \mathrm{Mat}_{1\times 2}(\boldsymbol{R}), t \in \boldsymbol{R}^*, u \in \boldsymbol{R} \right\}.$$

Every subspace V_1 satisfying the condition $\tau_0(V_1) = V_0$ may be identified with a subspace of the form

$$V_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 \\ A & 0 & 0 \\ 0 & {}^tA \cdot P & 0 \end{pmatrix} \middle| A \in \boldsymbol{R}^2 \right\}.$$

where P is a 2×2 matrix. It follows from **CS** that the matrix P is symmetric, and the group G_1 defines the following equivalence relation: $P \sim XP\,{}^tX$, $X \in GL(2, \boldsymbol{R})$. Hence, up to this equivalence relation, P has one of the forms

0. $P = 0$;
1. $P = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}$;
2. $P = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$;
3. $P = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

Now for each of these cases we find the subalgebra $\mathfrak{g}_2$:

2.0. $\mathfrak{g}_2 = \mathfrak{g}_1$. In this case the algorithm terminates, and the corresponding admissible subalgebra has the form

$$\mathfrak{h}_1 = \left\{ \begin{pmatrix} A & B \\ 0 & -\operatorname{tr} A \end{pmatrix} \middle| A \in \mathfrak{gl}(3, \boldsymbol{R}),\ B \in \boldsymbol{R}^3 \right\}. \tag{1}$$

It is clear that $\mathfrak{h}_1$ corresponds to the plane $z = 0$.

2.1.

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 2a - b & c & d & e \\ 0 & -3a & f & g \\ 0 & 0 & a & h \\ 0 & 0 & 0 & b \end{pmatrix} \middle| a, b, c, d, e, f, g, h \in \boldsymbol{R} \right\}.$$

In the following this case will be called parabolic.

2.2.

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} -a & c & d & e \\ 0 & 0 & b & f \\ 0 & -b & 0 & g \\ 0 & 0 & 0 & a \end{pmatrix} \middle| a, b, c, d, e, f, g \in \boldsymbol{R} \right\}.$$

In the following this case will be called elliptic.

2.3.

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} -a & c & d & e \\ 0 & b & 0 & f \\ 0 & 0 & -b & g \\ 0 & 0 & 0 & a \end{pmatrix} \middle| \, a, b, c, d, e, f, g \in \boldsymbol{R} \right\}.$$

In the following this case will be called hyperbolic. Consider as an example the parabolic case.

7.2. PARABOLIC CASE. Here we have

$$V_1 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_2 = \left\{ \begin{pmatrix} 2a-b & c & d & e \\ 0 & -3a & f & g \\ 0 & 0 & a & h \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d, e, f, g, h \in \boldsymbol{R} \right\},$$

$$G_2 = \left\{ \begin{pmatrix} 1 & x_3 & x_4 & x_5 \\ 0 & x_1 & x_6 & x_7 \\ 0 & 0 & x_2 & x_8 \\ 0 & 0 & 0 & x_2^2 \end{pmatrix} \middle| \, x_1, x_2 \in \boldsymbol{R}^*, x_3, x_4, x_5, x_6, x_7, x_8 \in \boldsymbol{R} \right\}.$$

Then any subspace V_2 may be identified with a subspace of the form

$$V_2 = \left\{ \begin{pmatrix} -\alpha x - \gamma y & 0 & 0 & 0 \\ x & \alpha x + \gamma y & 0 & 0 \\ y & \beta x + \delta y & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

From **CS** we have $\beta = 0$ and $\alpha = \delta$. Up to the action of the group G_2, we may assume that $\alpha = \gamma = 0$. In accordance with our algorithm, we find $\mathfrak{g}_3$ and G_3:

$$V_2 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_3 = \left\{ \begin{pmatrix} 2a-b & 0 & d & e \\ 0 & -3a & c & f \\ 0 & 0 & a & d \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d, e, f \in \boldsymbol{R} \right\},$$

$$G_3 = \left\{ \begin{pmatrix} 1 & 0 & t & u \\ 0 & x & z & v \\ 0 & 0 & y & yt \\ 0 & 0 & 0 & y^2 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R}^*, z, t, u, v \in \boldsymbol{R} \right\}.$$

Then all possible subspaces V_3 may be written as

$$V_3 = \left\{ \begin{pmatrix} 0 & \alpha x + \beta y & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & \gamma x + \delta y \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

Using the condition **CS**, we find that $\alpha = 0$ and $\beta = -3\gamma$. The subspaces V_3, viewed up to the action of the group G_3, have the form:

2.1.1
$$V_3 = \left\{ \begin{pmatrix} 0 & -3y & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & x \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_4 = \left\{ \begin{pmatrix} a & 0 & c & -2b \\ 0 & 3a & b & d \\ 0 & 0 & -a & c \\ 0 & 0 & 0 & -3a \end{pmatrix} \middle| \, a, b, c, d \in \boldsymbol{R} \right\};$$

2.1.2
$$V_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_4 = \left\{ \begin{pmatrix} 2a - b & 0 & d & 0 \\ 0 & -3a & c & e \\ 0 & 0 & a & d \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d, e \in \boldsymbol{R} \right\}.$$

Consider the case 2.1.1. The group G_4 has the form

$$G_4 = \left\{ \begin{pmatrix} 1 & 0 & y & -2xz + y^2/2 \\ 0 & 1/x & z & t \\ 0 & 0 & x & xy \\ 0 & 0 & 0 & x^2 \end{pmatrix} \middle| \, x \in \boldsymbol{R}^*, \; y, z, t \in \boldsymbol{R} \right\}.$$

We write V_4 in the form

$$V_4 = \left\{ \begin{pmatrix} \alpha x + \gamma y & -3y & 0 & \beta x + \delta y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & x \\ 0 & 0 & y & -\alpha x - \gamma y \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

From **CS** it follows that $\alpha = 0$ and $\beta = -\gamma$. Then all subspaces $V_4^{(\beta,\delta)}$ are equivalent up to the action of G_4. Consequently,

$$V_4 = \left\{ \begin{pmatrix} 0 & -3y & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & x \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_5 = \left\{ \begin{pmatrix} a & 0 & 0 & -2b \\ 0 & 3a & b & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -3a \end{pmatrix} \middle| \, a, b \in \boldsymbol{R} \right\},$$

$$G_5 = \left\{ \begin{pmatrix} 1 & 0 & 0 & -2xy \\ 0 & 1/x & y & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x^2 \end{pmatrix} \middle| \, x \in \boldsymbol{R}^*, \, y \in \boldsymbol{R} \right\}.$$

Every subspace V_5 may in its turn be identified with a subspace of the form

$$V_5 = \left\{ \begin{pmatrix} 0 & -3y & \alpha x + \gamma y & 0 \\ x & 0 & 0 & \beta x + \delta y \\ y & 0 & 0 & (\alpha + 1)x + \gamma y \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

Using **CS**, we obtain that $\alpha = -4$ and $\gamma = -5\beta$. Up to the action of the group G_5, we may assume that $\beta = 0$ and $\delta = 0$ or $\delta = \pm 1$. However the subalgebra $\mathfrak{g}_6$ is nontrivial only when $\delta = 0$. Hence the only interesting case is

$$V_5 = \left\{ \begin{pmatrix} 0 & -3y & -4x & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & -3x \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_6 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & 3a & 0 & 0 \\ 0 & 0 & -a & 0 \\ 0 & 0 & 0 & -3a \end{pmatrix} \middle| \, a \in \boldsymbol{R} \right\}.$$

The subspace V_6 may be written as

$$V_6 = \left\{ \begin{pmatrix} 0 & -3y & -4x & -2(\alpha x + \beta y) \\ x & 0 & \alpha x + \beta y & 0 \\ y & 0 & 0 & -3x \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

A straightforward calculation shows that $\mathfrak{g}_7 \neq \{0\}$ if and only if $\alpha = \beta = 0$. Thus the case 2.1.1 yields a single subalgebra $\mathfrak{h}_2$ which may be conveniently written as

$$\mathfrak{h}_2 = \left\{ \begin{pmatrix} 3z & -x & 0 & 0 \\ -3y & z & 4x & 0 \\ 0 & y & -z & 3x \\ 0 & 0 & y & -3z \end{pmatrix} \middle| \, x, y, z \in \boldsymbol{R} \right\}, \tag{2}$$

and the orbit must be taken to pass through the point $o' = [0:1:0:0] \in \boldsymbol{RP}^3$.

Notice that the subalgebra $\mathfrak{h}_2$ is isomorphic to $\mathfrak{sl}(2, \boldsymbol{R})$ and its natural representation of $\mathfrak{h}_2$ in $\boldsymbol{R}^4$ is irreducible. Therefore the corresponding virtual subgroup $H_2 \subset SL(4, \boldsymbol{R})$ will be isomorphic to $SL(2, \boldsymbol{R})$ and will act irreducibly on $\boldsymbol{R}^4$. It is well known that this action is equivalent to the following action of $SL(2, \boldsymbol{R})$ on the space $\boldsymbol{R}_3[t_1, t_2]$ of all homogeneous polynomials in t_1, t_2 of degree 3:

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix} . p(t_1, t_2) = p(at_1 + ct_2, bt_1 + dt_2) \quad \forall p \in \boldsymbol{R}_3[t_1, t_2].$$

Let us introduce the non-homogeneous coordinates in the projective space, associated with $\boldsymbol{R}_3[t_1, t_2]$, in such way that the point (x, y, z) corresponds to the class of polynomials which are proportional to $t_1^3 + 3xt_1^2t_2 + 6yt_1t_2^2 + 6zt_2^3$. The corresponding action of $SL(2, \boldsymbol{R})$ on the projective space $\boldsymbol{RP}_3[t_1, t_2]$ has exactly three orbits:

1. a one-dimensional orbit O_1, consisting of all polynomials of the form $(\alpha_1 t_1 + \alpha_2 t_2)^3$. In the non-homogeneous coordinates this orbit has the form: $(x, 1/2x^2, 1/6x^3)$, $x \in \boldsymbol{R}$;
2. a two-dimensional orbit, consisting of all polynomials having a linear multiplier of multiplicity 2. It is easy to verify that this orbit is the developable of O_1, excluding O_1 itself. In the non-homogeneous coordinates it is given by the following equation:

$$\left(z - xy + \frac{x^3}{3}\right)^2 = -\frac{8}{9}\left(y - \frac{x^2}{2}\right)^3; \tag{3}$$

3. an open three-dimensional orbit O_3, consisting of all polynomials without multiple multipliers.

Hence, the required orbit of the subgroup H_2 up to projective transformations is equivalent to that one given by (3).

Consider now the case 2.1.2. We have:

$$V_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_4 = \left\{ \begin{pmatrix} 2a - b & 0 & d & 0 \\ 0 & -3a & c & e \\ 0 & 0 & a & d \\ 0 & 0 & 0 & b \end{pmatrix} \middle| \, a, b, c, d, e \in \boldsymbol{R} \right\},$$

$$G_4 = \left\{ \begin{pmatrix} 1 & 0 & t & t^2/2 \\ 0 & x & z & u \\ 0 & 0 & y & yt \\ 0 & 0 & 0 & y^2 \end{pmatrix} \middle| x, y \in \boldsymbol{R}^*, z, t, u \in \boldsymbol{R} \right\}.$$

All possible subspaces V_4 may be written in the form

$$V_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & \alpha x + \beta y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| x, y \in \boldsymbol{R} \right\}.$$

From **CS** it follows that $\alpha = 0$. Up to the action of the group G_4 we may assume that $\beta = 1$ or $\beta = 0$. Thus V_4 has one of the following forms:

2.1.2.1
$$V_4 = \left\{ \begin{pmatrix} 0 & 0 & 0 & y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_5 = \left\{ \begin{pmatrix} a & 0 & c & 0 \\ 0 & -3a & b & d \\ 0 & 0 & a & c \\ 0 & 0 & 0 & a \end{pmatrix} \middle| a, b, c, d \in \boldsymbol{R} \right\};$$

2.1.2.2
$$V_3 = \left\{ \begin{pmatrix} 0 & 0 & 0 & 0 \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \middle| x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_5 = \mathfrak{g}_4 = \left\{ \begin{pmatrix} 2a-b & 0 & d & 0 \\ 0 & -3a & c & e \\ 0 & 0 & a & d \\ 0 & 0 & 0 & b \end{pmatrix} \middle| a, b, c, d, e \in \boldsymbol{R} \right\}.$$

In the latter case we at once get an admissible subalgebra which is equivalent to the subalgebra

(4)
$$\mathfrak{h}_3 = \left\{ \begin{pmatrix} -3z & u & v & w \\ 0 & t+z & x & 0 \\ 0 & y & z & x \\ 0 & 0 & y & z-t \end{pmatrix} \middle| x, y, z, t, u, v, w \in \boldsymbol{R} \right\},$$

and the orbit must be taken to pass through the point $o' = [0 : 1 : 0 : 0]$.

Consider the subalgebra

$$\mathfrak{h}_3' = \left\{ \begin{pmatrix} 0 & u & v & w \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \middle| \, u, v, w \in \boldsymbol{R} \right\},$$

in $\mathfrak{h}_3$. The corresponding subgroup H_3' has the form:

$$H_3' = \left\{ \begin{pmatrix} 1 & x & y & z \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \right\},$$

and the set of its orbits consists of the point $o = [1:0:0:0]$ and lines of the form

$$l_{[a_1:a_2:a_3]} = \{[t : a_1 : a_2 : a_3] \mid t \in \boldsymbol{R}\}, \tag{5}$$

where $[a_1 : a_2 : a_3]$ is an arbitrary point in $\boldsymbol{RP}^2$.

Now let H_3 be the connected subgroup in $SL(4, \boldsymbol{R})$ corresponding to the subalgebra $\mathfrak{h}_3$, and let L be an orbit of H_3 through o'. Notice, that the point o is also stable with respect to the action of H_3, and, therefore, is not contained in L. Hence, L is fibered by lines of the form (5), i.e. it is a cylinder whose base is a curve in $\boldsymbol{RP}^2$ which is an orbit of the subalgebra

$$\tilde{\mathfrak{h}}_3 = \left\{ \begin{pmatrix} t+z & x & 0 \\ y & z & x \\ 0 & y & z-t \end{pmatrix} \middle| \, x, y, z, t \in \boldsymbol{R} \right\}$$

through the point $[1:0:0]$. As in the case 2.1.1, considering the three-dimensional irreducible representations of the Lie algebra $\mathfrak{sl}(2, \boldsymbol{R})$ and the Lie group $SL(2, \boldsymbol{R})$, we get that this curve is a parabola $y = x^2$. Hence, the surface L is the cylinder with the parabola $y = x^2$ as its base. (See also Theorem 7 for more details.)

Consider the case 2.1.2.1. The group G_5 has the form

$$G_5 = \left\{ \begin{pmatrix} 1 & 0 & t & t^2/2 \\ 0 & x & y & z \\ 0 & 0 & 1 & t \\ 0 & 0 & 0 & 1 \end{pmatrix} \middle| \, x \in \boldsymbol{R}^*, \, y, z, t \in \boldsymbol{R} \right\}.$$

We identify V_5 with a subspace of the form

$$V_5 = \left\{ \begin{pmatrix} \alpha x + \beta y & 0 & 0 & y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & -\alpha x - \beta y \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

Using the condition **CS**, we find that $\alpha = 0$. Up to the action of G_5, we let $\beta = 0$.

Then we have

$$V_5 = \left\{ \begin{pmatrix} 0 & 0 & 0 & y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & 0 \\ 0 & 0 & y & 0 \end{pmatrix} \;\middle|\; x, y \in \boldsymbol{R} \right\},$$

$$\mathfrak{g}_6 = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & -3a & b & c \\ 0 & 0 & a & 0 \\ 0 & 0 & 0 & a \end{pmatrix} \;\middle|\; a, b, c \in \boldsymbol{R} \right\},$$

$$G_6 = \left\{ \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & x & y & z \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix} \;\middle|\; x \in \boldsymbol{R}^*,\ y, z \in \boldsymbol{R} \right\}.$$

All possible subspaces V_6 may be written in the form

$$V_6 = \left\{ \begin{pmatrix} 0 & 0 & \alpha x + \beta y & y \\ x & 0 & 0 & 0 \\ y & 0 & 0 & \alpha x + \beta y \\ 0 & 0 & y & 0 \end{pmatrix} \;\middle|\; x, y \in \boldsymbol{R} \right\}.$$

From **CS** it follows that $\alpha = 0$. It is easy to see that in this case the action of the group G_6 is trivial. In addition, $\mathfrak{g}_7 = \mathfrak{g}_6$ for all β. Thus we obtain a new subalgebra, which up to conjugation has the form

$$\mathfrak{h}_4 = \left\{ \begin{pmatrix} -3y & u & v & w \\ 0 & y & \alpha x & x \\ 0 & x & y & \alpha x \\ 0 & 0 & x & y \end{pmatrix} \;\middle|\; x, y, u, v, w \in \boldsymbol{R} \right\}. \tag{6}$$

As in the case 2.1.2.2, we obtain that the corresponding orbits are cylinders whose bases are formed by locally homogeneous curves in $\boldsymbol{R}P^2$ with one-dimensional symmetry algebras of the form:

$$\tilde{\mathfrak{h}}_4 = \left\{ \begin{pmatrix} 0 & \alpha x & x \\ x & 0 & \alpha x \\ 0 & x & 0 \end{pmatrix} \;\middle|\; x \in \boldsymbol{R} \right\}.$$

See also Theorem 7 for more details.

This completes the discussion of the parabolic case. The elliptic and the hyperbolic cases are discussed in much the same way, and we omit here all computations in these cases.

In the elliptic case we get the only surface with at least three-dimensional symmetry algebra. It is the quadric $z = x^2 + y^2$. In the hyperbolic case we get the quadric $z =$

$y^2 - z^2$ and the Cayley surface $z = xy - x^3$ (see also [**3**] for similar computation in the case of affine geometry).

As a result, we obtain the following proposition:

Proposition. *Up to projective equivalence, every surface in $\boldsymbol{RP}^3$ whose symmetry algebra is at least three-dimensional is one of the following*:

(1) *cylinders with plane locally homogeneous curves as their bases*;
(2) *quadrics of the form $z = x^2 \pm y^2$*;
(3) *Cayley surface: $z = xy - x^3/3$*;
(4) *developable surface of the curve $(t, t^2/2, t^3/6)$*:

$$\left(z - xy + \frac{x^3}{3}\right)^2 = -\frac{8}{9}\left(y - \frac{x^2}{2}\right)^3.$$

8. Another approach to the classification of homogeneous manifolds.

8.1. Classification algorithm. In the previous section we regarded submanifolds as orbits of certain subalgebras passing through a fixed point $o \in \boldsymbol{RP}^3$. Now we shall no longer fix any point. Instead we shall consider pairs $(\mathfrak{h}, W)$, where $\mathfrak{h}$ is a subalgebra in $\mathfrak{sl}(4, \boldsymbol{R})$ and W is a one-dimensional subspace of $\boldsymbol{R}^4$. The action of the group $GL(4, \boldsymbol{R})$ on pairs $(\mathfrak{h}, W)$ is defined in the obvious way: $X.(\mathfrak{h}, W) = (X\mathfrak{h}X^{-1}, XW)$. The orbit of $GL(4, \boldsymbol{R})$ passing through $(\mathfrak{h}, W)$ will be denoted by $[(\mathfrak{h}, W)]_{GL}$.

A submanifold in $\boldsymbol{RP}^3$ will be regarded as an orbit of a subalgebra $\mathfrak{h} \subset \bar{\mathfrak{g}}$ through a one-dimensional subspace $W \subset \boldsymbol{R}^4$, viewed as a point in $\boldsymbol{RP}^3$. Then two submanifolds are projectively equivalent if the corresponding pairs $(\mathfrak{h}, W)$ are equivalent up to the action of $GL(4, \boldsymbol{R})$. This approach is justified by the next proposition.

Proposition. *The orbits of the action of the group $GL(4, \boldsymbol{R})$ on pairs $(\mathfrak{h}, W)$ are in a one-to-one correspondence with the orbits of the action of $G_0 = \mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$ on subalgebras $\mathfrak{h} \subset \mathfrak{sl}(4, \boldsymbol{R})$.*

Proof. The correspondence may be defined as follows. Let $[(\mathfrak{h}, W)]_{GL}$ be the orbit of $GL(4, \boldsymbol{R})$ through the pair $(\mathfrak{h}, W)$. It is clear that $[(\mathfrak{h}, W)]_{GL}$ contains an element of the form $(\tilde{\mathfrak{h}}, V)$. To each orbit $[(\mathfrak{h}, W)]_{GL}$ we assign the orbit $\pi([(\mathfrak{h}, W)]_{GL}) = [\tilde{\mathfrak{h}}]_{\mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})}$. That the mapping π is well defined follows from the formula $\mathrm{Aut}(\bar{\mathfrak{g}}, V) = \mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{g})$, which was proved in §3. The fact that π is surjective and injective is obvious. □

Since we no longer fix any points, we must modify accordingly the concept of admissibility. We shall say that a pair $(\mathfrak{h}, W)$ is admissible if every subalgebra of $\bar{\mathfrak{g}}$ that contains $\mathfrak{h}$ and is contained in $\mathfrak{g}_W + \mathfrak{h}$, coincides with $\mathfrak{h}$. Based on the above, we suggest the following algorithm for the classification of projectively homogeneous submanifolds:

1. Classify subalgebras $\mathfrak{h} \subset \bar{\mathfrak{g}}$ up to the conjugation by the elements of $GL(4, \boldsymbol{R})$.
2. For every $\mathfrak{h} \subset \bar{\mathfrak{g}}$, compute the group $G_{\mathfrak{h}} = \mathrm{Aut}(\bar{\mathfrak{g}}, \mathfrak{h})$ and classify, up to this group, all one-dimensional subspaces $W \subset \boldsymbol{R}^4$.

3. Find the connected virtual subgroup $H \subset SL(4, \boldsymbol{R})$ corresponding to the subalgebra $\mathfrak{h}$. Then find the orbit H through a subspace W, viewed as a point in $\boldsymbol{R}P^3$. The resulting submanifold L will be homogeneous since $\mathrm{Sym}(L) \supset H$. Hence the corresponding pair $(\mathfrak{h}, W)$ will not be admissible if and only if $\dim \mathrm{Sym}(L) > \dim \mathfrak{h}$. In this case L is projectively equivalent to one of the manifolds obtained in Section 7.

8.2. SUBALGEBRAS OF $\mathfrak{gl}(V)$. In this subsection we develop mathematical tools necessary to classify all subalgebras of $\mathfrak{gl}(V)$, where V is a finite-dimensional vector space over a field of characteristic 0. After we have done this, to classify subalgebras in $\mathfrak{sl}(V)$ we need only add the natural condition on the trace of matrices. The idea underlying this approach consists in the assignment to each subalgebra of a certain collection of invariants with respect to the conjugation by the elements of $GL(V)$.

Here we outline only the main results used in the classification. For complete proofs and further details see [**8**]. We begin by introducing several new concepts.

DEFINITION. Let $\mathfrak{g}$ be a subalgebra of $\mathfrak{gl}(V)$. The greatest ideal of $\mathfrak{g}$ consisting of nilpotent endomorphisms is called the *greatest nilpotency ideal of the identical representation*.

We shall denote this ideal by $\mathfrak{n}(\mathfrak{g})$.

DEFINITION. A subalgebra $\mathfrak{g} \subset \mathfrak{gl}(V)$ is said to be *separating* if it contains the semisimple and the nilpotent part of each of its elements.

It is clear that the intersection of separating subalgebras is again a separating subalgebra, and hence the following concept is well defined.

DEFINITION. If $\mathfrak{g}$ is a subalgebra of $\mathfrak{gl}(V)$, the *separating envelope* of $\mathfrak{g}$ is the smallest separating subalgebra of $\mathfrak{gl}(V)$ containing $\mathfrak{g}$. The separating envelope of $\mathfrak{g}$ will be denoted by $e(\mathfrak{g})$.

It is easy to see that if two subalgebras of $\mathfrak{gl}(V)$ are conjugate, then so are their separating envelopes and the greatest nilpotency ideals of the identical representation. Taking this into account, we can classify subalgebras of $\mathfrak{gl}(V)$ by implementing the following algorithm:

1. Describe all subalgebras $\mathfrak{n} \subset \mathfrak{gl}(V)$ that consist of nilpotent elements.
2. For each subalgebra $\mathfrak{n}$ found in 1, describe all separating subalgebras $\mathfrak{e} \subset \mathfrak{gl}(V)$ such that $\mathfrak{n}(\mathfrak{e}) = \mathfrak{n}$.
3. For each subalgebra $\mathfrak{e}$ obtained in 2, find all subalgebras $\mathfrak{g} \subset \mathfrak{e}$ such that $e(\mathfrak{g}) = \mathfrak{e}$.

The first step of this algorithm presents no serious problems. By the Engel theorem, the subalgebra $\mathfrak{n} \subset \mathfrak{gl}(V)$ may be brought into the strictly upper triangular form, which means that we must classify subalgebras $\mathfrak{n}$ up to conjugation.

The next two theorems enable us to implement step 2 of our algorithm.

THEOREM 4. *Let $\mathfrak{g}$ be a separating subalgebra of the Lie algebra $\mathfrak{gl}(V)$. Then there exists a subalgebra $\mathfrak{m} \subset \mathfrak{g}$ such that $\mathfrak{m}$ is reductive in $\mathfrak{gl}(V)$ and $\mathfrak{g}$ is the semidirect product of $\mathfrak{m}$ and the ideal $\mathfrak{n}(\mathfrak{g})$, that is, $\mathfrak{g} = \mathfrak{m} \leftthreetimes \mathfrak{n}(\mathfrak{g})$.*

Let $\mathbf{N}$ denote the normalizer of $\mathfrak{n}$ in $\mathfrak{gl}(V)$. It is clear that $\mathbf{N}$ is a separating subalgebra, and hence, in view of Theorem 4, we have the following decomposition: $\mathbf{N} = \bar{\mathfrak{m}} \rightthreetimes \mathfrak{n}(\mathbf{N})$.

Let $\mathscr{A}(\mathfrak{n}) = \{\phi \in GL(V) \mid \phi\mathfrak{n}\phi^{-1} = \mathfrak{n}\}$. If $\phi \in \mathscr{A}(\mathfrak{n})$, then obviously $\phi\mathbf{N}(\mathfrak{n})\phi^{-1} = \mathbf{N}(\mathfrak{n})$. The following theorem provides a method for the classification of subalgebras discussed in Theorem 4.

Theorem 5.

1) *Every subalgebra* $\mathfrak{m} \subset \mathbf{N}(\mathfrak{n})$ *which is reductive in* $\mathfrak{gl}(V)$ *is conjugate* (*with respect to the group* $\mathscr{A}(\mathfrak{n})$) *to a subalgebra of* $\bar{\mathfrak{m}}$.
2) *If two subalgebras of* $\bar{\mathfrak{m}}$ *which are reductive in* $\mathfrak{gl}(V)$ *are conjugate with respect to* $\mathscr{A}(\mathfrak{n})$, *then they are conjugate with respect to* $\mathscr{A}(\mathfrak{n}, \bar{\mathfrak{m}})$.

Theorems 4 and 5 provide the following algorithm for the classification of the separating subalgebras $\mathfrak{e} \subset \mathfrak{gl}(V)$ such that $\mathfrak{n}(\mathfrak{e}) = \mathfrak{n}$:

1. Find a subalgebra $\bar{\mathfrak{m}} \subset \mathbf{N}(\mathfrak{n})$ which is reductive in $\mathfrak{gl}(V)$ and satisfies the following condition:

$$\mathbf{N}(\mathfrak{n}) = \bar{\mathfrak{m}} \rightthreetimes \mathfrak{n}(\mathbf{N}(\mathfrak{n})).$$

2. Describe (up to the group $\mathscr{A}(\mathfrak{n}, \bar{\mathfrak{m}})$) all subalgebras $\mathfrak{m} \subset \bar{\mathfrak{m}}$ which are reductive in $\mathfrak{gl}(V)$. The desired separating subalgebras will then arise as subalgebras of the form $\mathfrak{e} = \mathfrak{m} \rightthreetimes \mathfrak{n}$.

The implementation of the third step of our algorithm is simplified by the following theorem:

Theorem 6 [**8**]. *Let* $\mathfrak{g}$ *be an arbitrary subalgebra of* $\mathfrak{gl}(V)$. *For a separating subalgebra* $\mathfrak{e} \subset \mathfrak{gl}(V)$ *to be the separating envelope of* $\mathfrak{g}$, *it is necessary and sufficient that the following three conditions are satisfied*:

(i) $[\mathfrak{e}, \mathfrak{e}] \subset \mathfrak{g}$;
(ii) $\mathfrak{g} + \mathfrak{m} = \mathfrak{e}$;
(iii) $\mathfrak{g} + \mathfrak{n} = \mathfrak{e}$.

Moreover, in this case we have $[\mathfrak{e}, \mathfrak{e}] = [\mathfrak{g}, \mathfrak{g}]$.

Thus, to implement step 3, we must classify all subspaces in $\mathfrak{e}$ satisfying conditions (i)–(iii) with respect to the action of the group $\mathscr{A}(\mathfrak{e}) \subset \mathscr{A}(\mathfrak{n})$.

9. Homogeneous submanifolds with two-dimensional symmetry algebras.

9.1. Symmetry algebras of dimension 2. In this section we classify all locally homogeneous surfaces in $\boldsymbol{R}P^3$ whose symmetry algebras are two-dimensional. We should also be able to determine whether a given pair $(\mathfrak{h}, W)$ is admissible. This may be done with the help of Theorems 7 and 8.

The first of these theorems allows to verify whether the surface corresponding to the pair $(\mathfrak{h}, W)$ is a cylinder. To state this theorem, we shall need some extra notation. If $U \subset \boldsymbol{R}^4$ is an $\mathfrak{h}$-invariant subspace, consider the subspace $\tilde{V} = \boldsymbol{R}^4/U$ and the subspace $\tilde{W}$ which is defined to be the image of W on passing to the quotient $\boldsymbol{R}^4 \to \tilde{V}$. One can also consider the subalgebra $\tilde{\mathfrak{h}} \subset \mathfrak{gl}(\tilde{V})$ which arises when the action of $\mathfrak{h}$ is induced to $\tilde{V}$.

Theorem 7. *The surface given by the pair* $(\mathfrak{h}, W)$ *is a cylinder if and only if one of the following two conditions is satisfied*:

1) *There exists a three-dimensional* $\mathfrak{h}$*-invariant subspace containing* W.
2) *There exists a one-dimensional* $\mathfrak{h}$*-invariant subspace* $U \subset \boldsymbol{R}^4$ *such that the pair* $(\tilde{\mathfrak{h}}, \tilde{W})$ *defines a curve in* $P(\tilde{V})$.

Proof. Assume that the orbit L of $\mathfrak{h}$ passing through W is a cylinder. If L is a plane, then the pair $(\mathfrak{h}, W)$ satisfies condition 1). If, however, L is a nondegenerate cylinder, the corresponding group H contains translations along a certain one-dimensional subspace U. Since the cylinder is non-degenerate, this subspace is uniquely defined. It follows that U is $\mathfrak{h}$-invariant. Then, on passing to the quotient, we obtain a pair $(\tilde{\mathfrak{h}}, \tilde{W})$ which defines a curve in $\boldsymbol{R}P^2$—the generator of our cylinder.

On the other hand, if a surface L corresponds to a pair $(\mathfrak{h}, W)$ satisfying condition 1) or 2), then it is easily seen that L is a cylinder. □

Condition 2) of Theorem 7 may be written in the algebraic language. To do this, consider the mapping

$$\phi : \tilde{\mathfrak{h}} \otimes \tilde{W} \to \tilde{V}/\tilde{W}, \quad x \otimes w \mapsto x.w + \tilde{W}.$$

Then condition 2) is equivalent to the following one: $\dim \operatorname{im} \phi = 1$.

To prove this fact, consider the orbit of the group $\tilde{H} \subset GL(\tilde{V})$ through the point $\tilde{W}$ on the Grassmannian manifold $\mathrm{Gr}_1(\tilde{V})$. In order to find the dimension of $\tilde{H}.\tilde{W}$ in $\mathrm{Gr}_1(\tilde{V})$, it suffices to find the dimension of the tangent space to our orbit at the point $\tilde{W}$. Using the isomorphism $T_W \mathrm{Gr}_1(\tilde{V}) \cong W \otimes \left(\tilde{V}/\tilde{W}\right)^*$, we find that this dimension is equal to $\dim \operatorname{im} \phi = 1$.

Theorem 8 allows to find out if the surface L is equivalent to a quadric.

Theorem 8. *The surface corresponding to a pair* $(\mathfrak{h}, W)$ *is a quadric if and only if there exists a nondegenerate symmetric bilinear form* θ *with the following two properties*:

1) *The subalgebra* $\mathfrak{h}$ *preserves* θ.
2) *The subspace* W *is isotropic for* θ*, that is,* $\theta(w, w) = 0$ *for all* $w \in W$.

Proof. Suppose that the orbit of $\mathfrak{h}$ through W is a quadric. Then $\mathfrak{h}$ may be embedded into one of the symmetry algebras corresponding to quadrics (subalgebras $\mathfrak{so}(3,1)$ and $\mathfrak{so}(2,2)$). However for each of these subalgebras there exists a non-degenerate symmetric bilinear form θ' satisfying conditions 1) and 2) of the theorem (here W may be chosen to be the original one-dimensional subspace V). Knowing θ', we can easily construct a form θ for the pair $(\mathfrak{h}, W)$.

Now assume that $(\mathfrak{h}, W)$ is a pair such that there exists a nondegenerate symmetric bilinear form θ satisfying conditions 1) and 2). Then the subgroup $H \subset GL(4, \boldsymbol{R})$ corresponding to the subalgebra $\mathfrak{h}$ also preserves the form θ; in other words, ${}^tX\theta X = \theta$ for any $X \in H$. Then, given a subspace $U = X.W$ in the orbit of W, in view of condition 2 we obtain ${}^tU\theta U = {}^tW\,{}^tX\theta XW = {}^tW\theta W = 0$. Hence $H.W$ is a quadric. □

It is clear that in order to classify subalgebras of dimension 2, we need only those subalgebras of nilpotent elements whose dimension does not exceed 2. To simplify the notation, we shall omit the curly brackets, assuming that x and y run over $\boldsymbol{R}$ for all subalgebras.

0. $\dim \mathfrak{n} = 0: \quad \mathfrak{n} = \{0\}$.

$$\dim \mathfrak{n} = 1:$$

$$1.1 \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 1.2 \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 1.3 \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 1.4 \begin{pmatrix} 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$\dim \mathfrak{n} = 2:$$

$$2.1 \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.1' \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & -y & x \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.2 \begin{pmatrix} 0 & 0 & y & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.3 \begin{pmatrix} 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2.4 \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.5 \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.6 \begin{pmatrix} 0 & 0 & x & y \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.7 \begin{pmatrix} 0 & 0 & x & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

$$2.8 \begin{pmatrix} 0 & x & 0 & y \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.9 \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & x & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.10 \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & 0 \end{pmatrix} \quad 2.10' \begin{pmatrix} 0 & x & y & 0 \\ 0 & 0 & 0 & x \\ 0 & 0 & 0 & y \\ 0 & 0 & 0 & 0 \end{pmatrix}$$

Consider the case 0. It is obvious that in this case $\mathbf{N} = \mathbf{N}(\mathfrak{n}) = \bar{\mathfrak{g}}$, $\mathfrak{n}(\mathbf{N}) = \{0\}$. Then, in accordance with our algorithm, we must describe all subalgebras $\mathfrak{m}$ of dimension 2 which are reductive in $\mathfrak{sl}(4, \boldsymbol{R})$. Since semisimple Lie algebras are at least three-dimensional, the subalgebra $\mathfrak{m}$ is toral.

Then $\mathfrak{m}$ is the linear envelope of semisimple commuting endomorphisms A_1 and A_2, and one of the following two cases takes place:

0.2a The eigenvalues of any element $X \in \mathfrak{m}$ are all real.

0.2b Not all elements of $\mathfrak{m}$ have only real eigenvalues, but there exists a one-dimensional subspace which is invariant with respect to $\mathfrak{m}$.

0.2c No one-dimensional subspace is invariant with respect to $\mathfrak{m}$.

In the case 0.2a, the subalgebra $\mathfrak{h} = \mathfrak{m} = \langle A_1, A_2 \rangle_{\boldsymbol{R}}$ may be brought into the form

$$\mathfrak{h}_{2.1} = \left\{ \begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & \alpha x + \beta y & 0 \\ 0 & 0 & 0 & -(1+\alpha)x - (1+\beta)y \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}.$$

Notice, that conjugation of $\mathfrak{h}_{2.1}$ by monomial matrices, corresponding to the elements of the symmetric group S_4, gives us the subalgebras of the same form and induces a finite transformation group on the set of parameters (α, β). Explicitly, this group is generated by the following mappings:

$$(\alpha,\beta) \mapsto (\beta,\alpha), \left(\frac{1}{\alpha}, -\frac{\beta}{\alpha}\right), \left(-\frac{\alpha}{1+\alpha}, \frac{\beta-\alpha}{\alpha+1}\right). \tag{7}$$

Since in generic case the subalgebra $\mathfrak{g}_{\mathfrak{h}} = \{X \in \mathfrak{gl}(4,\boldsymbol{R}) \mid [X,\mathfrak{h}] \subset \mathfrak{h}\}$ has the form

$$\mathfrak{g}_{\mathfrak{h}} = \left\{ \begin{pmatrix} a & 0 & 0 & 0 \\ 0 & b & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \middle| \, a,b,c,d \in \boldsymbol{R} \right\}, \tag{8}$$

we see that two subalgebras $\mathfrak{h}_{2.1}$, corresponding to different sets of parameters, are conjugate if and only if these sets of parameters belong to the one orbit of the S_4 action determined by (7). The equality (8) is not satisfied when

$$(\alpha,\beta) = (1,0), (0,1), (2,-1), (-1,2), \text{ and } (1/2,1/2).$$

In all these cases $\alpha+\beta=1$, and the corresponding surfaces are cylinders. So, these cases can be excluded without loss of generality.

Taking into account the remark we have made above concerning W, we may assume that up to $G_{\mathfrak{h}}$ the subspace $W \subset \boldsymbol{R}^4$ is equal to

$$W = \left\{ \begin{pmatrix} x \\ x \\ x \\ x \end{pmatrix} \middle| \, x \in \boldsymbol{R} \right\}.$$

Applying Theorem 7, we get that the corresponding surface in a cylinder if and only if one of the following equations is satisfied:

$$\alpha+\beta=1, \quad \alpha+\beta=-3, \quad \alpha-3\beta=1, \quad -3\alpha+\beta=1,$$

and from Theorem 8 we obtain that this surface is a non-degenerate quadric if and only if (α,β) equals to $(-1,0)$ or $(0,-1)$.

In affine coordinates, the surface corresponding to the pair $(\mathfrak{h}, W)$ is defined by the equation

$$z = x^a y^b. \qquad (2.1),$$

where $a = (1+3\alpha-\beta)/(3+\alpha+\beta)$, $b = (1-\alpha+3\beta)/(3+\alpha+\beta)$.

The equivalence relation on (a,b) has the form: $(a,b) \sim (b,a) \sim (1/a, -b/a) \sim (-a-b+1, b)$. Singular values of parameters a, b, when this surface degenerates to a cylinder or a quadric, lie on the straight lines

$$a=0, \quad b=0, \quad a+b=1$$

and at the points $(1,-1)$, $(-1,1)$, and $(1,1)$.

Consider now the case 0.2b. In this case there exists a two-dimensional subspace $U \subset \boldsymbol{R}^4$ with the following two properties:

1) U is invariant with respect to $\mathfrak{m}$;

2) no one-dimensional subspace of U is invariant with respect to $\mathfrak{m}$.

Accordingly, the restriction of $\mathfrak{m}$ to U may have dimension 1 or 2. Hence the subalgebra $\mathfrak{h} = \mathfrak{m}$ has one of the following forms:

$$\mathfrak{h}_{2.2} = \left\{ \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & \alpha x + \beta y & 0 \\ 0 & 0 & 0 & -(\alpha+2)x - \beta y \end{pmatrix} \right\} \text{ or } \mathfrak{h}'_{2.2} = \left\{ \begin{pmatrix} \gamma x & -x & 0 & 0 \\ x & \gamma x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -2\gamma x - y \end{pmatrix} \right\},$$

where x, y run over $\boldsymbol{R}$.

The subalgebra $\mathfrak{g}_{\mathfrak{h}} = \mathbf{N}_{\mathfrak{gl}}(\mathfrak{h})$ is in both cases equal to

$$\mathfrak{g}_{\mathfrak{h}} = \left\{ \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & 0 \\ 0 & 0 & 0 & d \end{pmatrix} \middle| \, a, b, c, d \in \boldsymbol{R} \right\}.$$

Up to the action of $G_{\mathfrak{h}}$ the subspace $W \subset \boldsymbol{R}^4$ has the form

$$W = \left\{ \begin{pmatrix} x \\ 0 \\ x \\ x \end{pmatrix} \middle| \, x \in \boldsymbol{R} \right\}.$$

Again, considering conjugations by monomial matrices that preserve the form of $\mathfrak{h}_{2.2}$ and $\mathfrak{h}'_{2.2}$, we get the following equivalence relation on parameter sets: $(\alpha, \beta) \sim (\alpha, -\beta) \sim (-\alpha - 2, -\beta)$, for the subalgebra $\mathfrak{h}_{2.2}$; and $\gamma \sim -\gamma$ for the subalgebra $\mathfrak{h}'_{2.2}$.

The singular values for $\mathfrak{h}_{2.2}$ derived from Theorems 7 and 8 are given by the straight lines $\alpha = 1$ and $\alpha = -3$ and by the point $(-1, 0)$. The only singular value for $\mathfrak{h}'_{2.2}$ is $\gamma = 0$.

A straightforward computation shows that the family of surfaces corresponding to the subalgebras $\mathfrak{h}_{2.2}$ and $\mathfrak{h}'_{2.2}$ is given by the equation

$$z = (x^2 + y^2)^a e^{b \arg(x+iy)}, \tag{2.2}$$

where $a = (\alpha+1)/(\alpha+3)$, $b = (4\beta)/(\alpha+3)$ for the subalgebra $\mathfrak{h}_{2.2}$; and $a = 1$, $b = -4\gamma$ for $\mathfrak{h}'_{2.2}$.

The equivalence relation on (a, b) for this surface may be written as follows: $(a, b) \sim (a, -b) \sim (a/(2a-1), b/(2a-1))$. The singular values for this surface are $a = 1/2$, $(a, b) = (0, 0)$; $(1, 0)$.

In the case 0.2c we obtain the following subalgebra $\mathfrak{h}$:

$$\mathfrak{h}_{2.3} = \left\{ \begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & -(\alpha x + \beta y) \\ 0 & 0 & \alpha x + \beta y & -x \end{pmatrix} \middle| \, x, y \in \boldsymbol{R} \right\}$$

Then, as usual, we find the subalgebra $\mathfrak{g}_{\mathfrak{h}}$ and the subspace W:

$$\mathfrak{g}_{\mathfrak{h}} = \left\{ \begin{pmatrix} a & -b & 0 & 0 \\ b & a & 0 & 0 \\ 0 & 0 & c & -d \\ 0 & 0 & d & c \end{pmatrix} \middle| a, b, c, d \in \boldsymbol{R} \right\}, \quad W = \left\{ \begin{pmatrix} x \\ 0 \\ x \\ 0 \end{pmatrix} \middle| x \in \boldsymbol{R} \right\}.$$

The surface corresponding to the pair $(\mathfrak{h}, W)$ is given by the equation

$$\arg(x + iy) = a \ln \frac{x^2 + y^2}{1 + z^2} + b \operatorname{arctg} z. \tag{2.3}$$

Singular points for both the subalgebra $\mathfrak{h}_{2.3}$ and the surface are $(0,0)$, $(0,1)$, and $(0,-1)$. The equivalence relation on parameters is as follows: $(a,b) \sim (a,-b) \sim (-a,-b) \sim (a/b, 1/b)$.

The case that $\dim \mathfrak{n} = 0$ and $\dim \mathfrak{m} = 2$ is now fully considered.

All other cases may be considered in a similar way. As a result we get the following list of two-dimensional subalgebras in the Lie algebra $\mathfrak{sl}(4, \boldsymbol{R})$, that contain non-nilpotent elements. In this list we always assume that x, y, z run through $\boldsymbol{R}$. Parameters are denoted by α, β, γ and belong to $\boldsymbol{R}$, if it is not stated otherwise.

1. $\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & \alpha x + \beta y & 0 \\ 0 & 0 & 0 & -(\alpha+1)x - (\beta+1)y \end{pmatrix}$;

2. $\begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & \alpha x + \beta y & 0 \\ 0 & 0 & 0 & -(\alpha+2)x - \beta y \end{pmatrix}$;

3. $\begin{pmatrix} \alpha x & -x & 0 & 0 \\ x & \alpha x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -2\alpha x - y \end{pmatrix}$; 4. $\begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & -\alpha x - \beta y \\ 0 & 0 & \alpha x + \beta y & -x \end{pmatrix}$;

5. $\begin{pmatrix} 0 & -x & 0 & 0 \\ x & 0 & 0 & 0 \\ 0 & 0 & 0 & -y \\ 0 & 0 & y & 0 \end{pmatrix}$; 6. $\begin{pmatrix} \alpha x & y & 0 & 0 \\ 0 & \beta x & 0 & 0 \\ 0 & 0 & \gamma x & 0 \\ 0 & 0 & 0 & -(\alpha+\beta+\gamma)x \end{pmatrix}$, $[\alpha : \beta : \gamma] \in \boldsymbol{R}P^2$;

7. $\begin{pmatrix} \beta x & y & 0 & 0 \\ 0 & -(2\alpha+\beta)x & 0 & 0 \\ 0 & 0 & \alpha x & -x \\ 0 & 0 & x & \alpha x \end{pmatrix}$;

8. $\begin{pmatrix} x & \alpha x+\beta y & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -2x-y \end{pmatrix}$, $[\alpha:\beta] \in \boldsymbol{RP}^1$;

9. $\begin{pmatrix} -x & \alpha x+\beta y & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}$, $[\alpha:\beta] \in \boldsymbol{RP}^1$;

10. $\begin{pmatrix} (\alpha+\beta)x & y & 0 & 0 \\ 0 & \alpha x & y & 0 \\ 0 & 0 & (\alpha-\beta)x & 0 \\ 0 & 0 & 0 & -3\alpha x \end{pmatrix}$, $[\alpha:\beta] \in \boldsymbol{RP}^1$;

11. $\begin{pmatrix} \alpha x & 0 & y & 0 \\ 0 & \beta x & 0 & y \\ 0 & 0 & -\beta x & 0 \\ 0 & 0 & 0 & -\alpha x \end{pmatrix}$, $[\alpha:\beta] \in \boldsymbol{RP}^1$; 12. $\begin{pmatrix} \alpha x & -x & y & 0 \\ x & \alpha x & 0 & y \\ 0 & 0 & -\alpha x & -x \\ 0 & 0 & x & -\alpha x \end{pmatrix}$;

13. $\begin{pmatrix} 3x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -3x \end{pmatrix}$; 14. $\begin{pmatrix} x & x & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -x \end{pmatrix}$;

15. $\begin{pmatrix} x+y & x & 0 & 0 \\ 0 & x+y & 0 & 0 \\ 0 & 0 & -x-y & y \\ 0 & 0 & 0 & -x-y \end{pmatrix}$; 16. $\begin{pmatrix} \alpha x & x & 0 & 0 \\ 0 & \alpha x & 0 & 0 \\ 0 & 0 & -(\alpha+1)x & y \\ 0 & 0 & 0 & (1-\alpha)x \end{pmatrix}$;

17. $\begin{pmatrix} 0 & x & x & y \\ -x & 0 & -y & x \\ 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{pmatrix}$; 18. $\begin{pmatrix} 0 & 0 & y & x \\ 0 & x & 0 & y \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}$;

19. $\begin{pmatrix} x & x & 0 & y \\ 0 & x & 0 & 0 \\ 0 & 0 & -x & x \\ 0 & 0 & 0 & -x \end{pmatrix}$; 20. $\begin{pmatrix} x & 0 & y & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix}$;

21. $\begin{pmatrix} (\alpha+1)x & 0 & y & 0 \\ 0 & (\alpha-1)x & x & 0 \\ 0 & 0 & (\alpha-1)x & 0 \\ 0 & 0 & 0 & (1-3\alpha)x \end{pmatrix}$; 22. $\begin{pmatrix} x & x & y & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix}$;

$$23.\ \begin{pmatrix} (\alpha-1)x & x & y & 0 \\ 0 & (\alpha-1)x & 0 & 0 \\ 0 & 0 & (\alpha+1)x & 0 \\ 0 & 0 & 0 & (1-3\alpha)x \end{pmatrix};\quad 24.\ \begin{pmatrix} x & x & 0 & y \\ 0 & x & x & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix};$$

$$25.\ \begin{pmatrix} 3x & x & y & 0 \\ 0 & 3x & 0 & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -5x \end{pmatrix};\quad 26.\ \begin{pmatrix} x & x & y & 0 \\ 0 & x & x & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix};$$

$$27.\ \begin{pmatrix} x & y & \pm x & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix};\quad 28.\ \begin{pmatrix} -3x & 0 & 0 & y \\ 0 & x & x & 0 \\ 0 & 0 & x & x \\ 0 & 0 & 0 & x \end{pmatrix};$$

$$29.\ \begin{pmatrix} 5x & y & 0 & 0 \\ 0 & x & 0 & y \\ 0 & 0 & -3x & x \\ 0 & 0 & 0 & -3x \end{pmatrix};\quad 30.\ \begin{pmatrix} x & x & y & 0 \\ 0 & x & 0 & y \\ 0 & 0 & -x & x \\ 0 & 0 & 0 & -x \end{pmatrix}.$$

Then according to the algorithm, we classify for each subalgebra $\mathfrak{h}$ listed above all one-dimensional subspaces $W \subset \boldsymbol{R}^4$ up to the group $G_{\mathfrak{h}}$, check if the resulting pair $(\mathfrak{h}, W)$ is admissible, and construct surfaces corresponding to admissible pairs.

As a result we get the following list of admissible pairs. As before we assume that x, y, z run through $\boldsymbol{R}$. Parameters are denoted by α, β and belong to $\boldsymbol{R}$, if it is not stated otherwise. As for surfaces, for some singular values of parameters these pairs may become non-admissible.

$$1.\ \left\{\begin{pmatrix} x & 0 & 0 & 0 \\ 0 & y & 0 & 0 \\ 0 & 0 & \alpha x+\beta y & 0 \\ 0 & 0 & 0 & -(1+\alpha)x-(1+\beta)y \end{pmatrix}\right\},\quad \left\{\begin{pmatrix} z \\ z \\ z \\ z \end{pmatrix}\right\};$$

$$2.\ \left\{\begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & \alpha x+\beta y & 0 \\ 0 & 0 & 0 & -(\alpha+2)x-\beta y \end{pmatrix}\right\},\quad \left\{\begin{pmatrix} z \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

$$2'.\ \left\{\begin{pmatrix} \gamma x & -x & 0 & 0 \\ x & \gamma x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -2\gamma x-y \end{pmatrix}\right\},\quad \left\{\begin{pmatrix} z \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

$$3.\ \left\{\begin{pmatrix} x & -y & 0 & 0 \\ y & x & 0 & 0 \\ 0 & 0 & -x & -(\alpha x+\beta y) \\ 0 & 0 & \alpha x+\beta y & -x \end{pmatrix}\right\},\quad \left\{\begin{pmatrix} z \\ 0 \\ z \\ 0 \end{pmatrix}\right\};$$

4. $$\left\{\begin{pmatrix} x & \alpha x+\beta y & 0 & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & y & 0 \\ 0 & 0 & 0 & -2x-y \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ z \\ z \\ z \end{pmatrix}\right\}, \ [\alpha:\beta]\in \boldsymbol{RP}^1;$$

5. $$\left\{\begin{pmatrix} -x & \alpha x+\beta y & 0 & 0 \\ 0 & -x & 0 & 0 \\ 0 & 0 & x & -y \\ 0 & 0 & y & x \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ z \\ z \\ 0 \end{pmatrix}\right\}, \ (\alpha,\beta)\in \boldsymbol{RP}^1;$$

6. $$\left\{\begin{pmatrix} (\alpha+1)x & y & 0 & 0 \\ 0 & (\alpha-1)x & y & 0 \\ 0 & 0 & (\alpha-3)x & 0 \\ 0 & 0 & 0 & 3(1-\alpha)x \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} \pm z \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

7. $$\left\{\begin{pmatrix} x & y & 0 & 0 \\ 0 & \alpha x & 0 & 0 \\ 0 & 0 & -\alpha x & y \\ 0 & 0 & 0 & -x \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} z \\ z \\ 0 \\ z \end{pmatrix}\right\};$$

8. $$\left\{\begin{pmatrix} \alpha x & y & -x & 0 \\ 0 & -\alpha x & 0 & -x \\ x & 0 & \alpha x & y \\ 0 & x & 0 & -\alpha x \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ z \\ z \\ 0 \end{pmatrix}\right\};$$

9. $$\left\{\begin{pmatrix} 3x & y & 0 & 0 \\ 0 & x & y & 0 \\ 0 & 0 & -x & y \\ 0 & 0 & 0 & -3x \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} z \\ \alpha z \\ 0 \\ z \end{pmatrix}\right\};$$

10. $$\left\{\begin{pmatrix} x+y & 0 & x & 0 \\ 0 & -x-y & 0 & y \\ 0 & 0 & x+y & 0 \\ 0 & 0 & 0 & -x-y \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

11. $$\left\{\begin{pmatrix} 0 & x & x & y \\ -x & 0 & -y & x \\ 0 & 0 & 0 & x \\ 0 & 0 & -x & 0 \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ 0 \\ 0 \\ z \end{pmatrix}\right\};$$

12. $$\left\{\begin{pmatrix} 0 & 0 & y & x \\ 0 & x & 0 & y \\ 0 & 0 & -x & 0 \\ 0 & 0 & 0 & 0 \end{pmatrix}\right\}, \qquad \left\{\begin{pmatrix} 0 \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

$$13.\quad \left\{\begin{pmatrix} x & x & y & 0 \\ 0 & x & 0 & 0 \\ 0 & 0 & -x/3 & y \\ 0 & 0 & 0 & -5x/3 \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 0 \\ \pm z \\ 0 \\ z \end{pmatrix}\right\};$$

$$14.\quad \left\{\begin{pmatrix} x & y & \pm x & 0 \\ 0 & x & y & 0 \\ 0 & 0 & x & 0 \\ 0 & 0 & 0 & -3x \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} 0 \\ 0 \\ z \\ z \end{pmatrix}\right\};$$

$$15.\quad \left\{\begin{pmatrix} 5x & 0 & y & 0 \\ 0 & -3x & 0 & x \\ 0 & 0 & x & y \\ 0 & 0 & 0 & -3x \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} \pm z \\ 0 \\ 0 \\ z \end{pmatrix}\right\};$$

$$16.\quad \left\{\begin{pmatrix} x & x & y & 0 \\ 0 & x & 0 & y \\ 0 & 0 & -x & x \\ 0 & 0 & 0 & -x \end{pmatrix}\right\}, \quad \left\{\begin{pmatrix} z \\ 0 \\ 0 \\ z \end{pmatrix}\right\}.$$

The corresponding list of surfaces (with the same numbering) is given in Theorem 1. Singular values of parameters and equivalence relations induced on sets of parameters are listed in Tables 1 and 2, and the correspondence between parameters in subalgebras and equations of surfaces is given in Table 3.

10. Tables

Table 1. Singular values of parameters

1.	$a = 0$, $b = 0$, $a + b = 1$, $(a, b) = (1, -1)$, $(-1, 1)$, $(1, 1)$.
2.	$a = 1/2$, $(a, b) = (0, 0)$, $(1, 0)$.
3.	$(a, b) = (0, 0)$, $(0, 1)$, $(0, -1)$.
4.	$a = 0$.
5.	$(a, b) = (0, 0)$.
6.	$a = 0,\ 1,\ 2$.
7.	$a = -1, 0, 1, 2, 3$.
8.	$a = 0$.
9.	$a = 0, -8/9$.

Table 2. Equivalence relation on parameters

1.	$(a,b) \mapsto (b,a),\ (1/a, -b/a),\ (-a-b+1, b).$
2.	$(a,b) \mapsto (a,-b),\ \left(\dfrac{a}{2a-1}, \dfrac{b}{2a-1}\right).$
3.	$(a,b) \mapsto (a,-b),\ (-a,-b),\ (a/b, 1/b).$
4.	$a \mapsto 1/a.$
5.	$(a,b) \mapsto (\lambda a, \pm\lambda b), \lambda \in \boldsymbol{R}^*.$
6.	no
7.	$a \mapsto 2-a.$
8.	$a \mapsto -a.$
9.	no

Table 3. Correspondence between parameters in subalgebras and surfaces

1.	$a = \dfrac{1+3\alpha-\beta}{3+\alpha+\beta},\ b = \dfrac{1-\alpha+3\beta}{3+\alpha+\beta}$
2.	$a = \dfrac{\alpha+1}{\alpha+3},\ b = \dfrac{4\beta}{\alpha+3}$
$2'$.	$a = 1,\ b = -4\gamma$
3.	$a = -\dfrac{\alpha}{4},\ b = \beta$
4.	$a = \dfrac{\alpha+\beta}{\alpha-3\beta}$
5.	$a = \beta,\ b = \dfrac{\alpha}{4}$
6.	$a = \dfrac{2}{3-2\alpha}$
7.	$a = \dfrac{2}{\alpha+1}$
8.	$a = 2\alpha$
9.	$a = \dfrac{1}{\alpha^3}$

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