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Positive Riesz distributions on homogeneous cones

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Abstract. Riesz distributions are relatively invariant distributions supported by the closure $\overline{\Omega}$ of a homogeneous cone Ω. In this paper, we clarify the positivity condition of Riesz distributions by relating it to the orbit structure of $\overline{\Omega}$. Moreover each of the positive Riesz distributions is described explicitly as a measure on an orbit in $\overline{\Omega}$.

Introduction.

The study of Riesz distributions originates from the paper [**9**] of Marcel Riesz, where he introduces a family of convolution operators as a generalization of the classical Riemann–Liouville operators on the positive reals to the Lorentz cone in $\boldsymbol{R}^n$. Riesz's distributions are the distributions determined by the homogeneous functions $r_\alpha(x) := (x_1^2 - x_2^2 - \cdots - x_n^2)^{\alpha/2}$ $(\alpha \in \boldsymbol{C})$ divided by Γ-factors (see for example the book [**13**, (II.3;31)] or the article [**2**] for details). His aim was to solve the wave equation, and a systematic use of analytic continuation is made. Among subsequent works carried out by various mathematicians, Gindikin's paper [**5**] was the first to consider a generalization of Riesz's distribution on an arbitrary homogeneous cone Ω. Gindikin replaces the above r_α by functions Δ_s $(s = (s_1, s_2, \ldots, s_r) \in \boldsymbol{C}^r)$ on Ω which are relatively invariant under a split solvable Lie group H acting linearly and simply transitively on Ω. When Ω is the cone of positive definite real symmetric matrices, Δ_s is the product $D_1^{s_1-s_2} \cdots D_{r-1}^{s_{r-1}-s_r} D_r^{s_r}$ of the principal minors $D_1, \ldots, D_{r-1}, D_r = \det$. He also investigates the gamma function Γ_Ω associated to the cone Ω playing the role of the Γ-factors in Riesz's distribution. Such being the case, we call the distributions $\mathscr{R}_s$ determined by $\Gamma_\Omega(s)^{-1}\Delta_s\, d\mu$ *the Riesz distributions on* Ω, where $d\mu$ is the H-invariant measure on Ω.

Our major concern for the Riesz distributions $\mathscr{R}_s$ is to determine the set Ξ of the parameter s for which $\mathscr{R}_s$ is a positive measure. The set Ξ plays an important role in the study of Hilbert spaces of holomorphic functions on Siegel domains as is done by Vergne and Rossi [**14**] to deal with the analytic continuation of the holomorphic discrete series representations of semisimple Lie groups (see also [**3**], [**16**] for this). For general cones, the set Ξ is described in Gindikin's paper [**6**]. Since the intersection of Ξ with the set $\{(\alpha, \alpha, \ldots, \alpha) \mid \alpha \in \boldsymbol{C}\}$ is seen to be identical with the Wallach set in the case of symmetric cones (cf. [**3**]), we shall call Ξ *the Gindikin–Wallach set*. However, only a very rough sketch is given in [**6**], and the complete details seem to be unpublished until now. Proofs available to us as of now are for the case of symmetric cones treated in [**3**, Chapter VII, Section 3], where a fine structure of Euclidean Jordan algebras is thoroughly utilized. The motivation of the present work was to improve this situation.

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But more is done in this paper. In fact, not only do we have a clearer description of Ξ by relating it to the H-orbit structure of $\overline{\Omega}$, but also we give explicitly a positive measure with its support specified for each of the elements in Ξ. Moreover, a simple algorithm for the specification is supplied.

Let us explain our results in more detail. The basic tool with which we work in place of Jordan algebras is the structure theory of normal j-algebras developed in [**8**]. Based on the fact that the normal j-algebras are in one-to-one correspondence with the Siegel domains, we shall start with a normal j-algebra $\mathfrak{g}$ such that the corresponding Siegel domain is of tube type. Thus $\mathfrak{g}$ is graded as $\mathfrak{g} = \mathfrak{g}(0) \oplus \mathfrak{g}(1)$ and our cone Ω is sitting in $V := \mathfrak{g}(1)$. Moreover, $\mathfrak{h} := \mathfrak{g}(0)$ is the Lie algebra of the group H acting simply transitively on Ω. The structure theorem of normal j-algebras (see Theorem 1.1 for details) says that there is a basis $\alpha_1, \ldots, \alpha_r$ for the roots of $\mathfrak{g}$, so that $\mathfrak{h}$ and V are respectively direct sums of the root spaces:

$$\mathfrak{h} = \mathfrak{a} \oplus \left(\bigoplus_{1 \le k < m \le r} \mathfrak{g}_{(\alpha_m - \alpha_k)/2} \right), \quad V = \left(\bigoplus_{k=1}^{r} \mathfrak{g}_{\alpha_k} \right) \oplus \left(\bigoplus_{1 \le k < m \le r} \mathfrak{g}_{(\alpha_m + \alpha_k)/2} \right),$$

see (1.3), (1.7) and (1.8). Let $E_k \in \mathfrak{g}_{\alpha_k}$ $(k = 1, 2, \ldots, r)$ be the root vectors such that $[jE_k, E_l] = \delta_{kl} E_l$. Our first theorem is the following.

THEOREM A. *The 2^r elements $E_\varepsilon := \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$ $(\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r)$ form a complete set of representatives of the H-orbits in $\overline{\Omega}$:*

$$\overline{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon \quad (\mathcal{O}_\varepsilon := H \cdot E_\varepsilon).$$

In order to state our second theorem, we put for every $\varepsilon \in \{0,1\}^r$

$$p_k(\varepsilon) := \sum_{i<k} \varepsilon_i \dim \mathfrak{g}_{(\alpha_k - \alpha_i)/2},$$

$$\Xi(\varepsilon) := \{ s \in \boldsymbol{R}^r \mid s_k = p_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 0), \ s_k > p_k(\varepsilon)/2 \ (\text{if } \varepsilon_k = 1) \}.$$

THEOREM B. *The Riesz distribution $\mathcal{R}_s$ is a relatively invariant positive measure on $\mathcal{O}_\varepsilon$ under the action of H if and only if $s \in \Xi(\varepsilon)$. Thus the Gindikin–Wallach set Ξ is the disjoint union*

$$\Xi = \bigsqcup_{\varepsilon \in \{0,1\}^r} \Xi(\varepsilon).$$

It is easy to verify that our description of Ξ reduces to that of Faraut and Korány [**3**, p. 138] for the case of symmetric cones (see Remark given after Theorem 6.2).

To describe the measure that $\mathcal{R}_s$ $(s \in \Xi(\varepsilon))$ induces on $\mathcal{O}_\varepsilon$ we will pick up a subgroup $H(\mathcal{O}_\varepsilon)$ of H which is diffeomorphic to $\mathcal{O}_\varepsilon$ (see (3.4) or (3.5) for an explicit description of $H(\mathcal{O}_\varepsilon)$). Then transferring the left Haar measure on $H(\mathcal{O}_\varepsilon)$ to $\mathcal{O}_\varepsilon$ by the orbit map $\Psi_\varepsilon : t \mapsto t \cdot E_\varepsilon$, we have an $H(\mathcal{O}_\varepsilon)$-invariant measure μ_ε on $\mathcal{O}_\varepsilon$. The measure μ_ε turns out to be relatively invariant under H (Proposition 4.1). Then the measure $d\mathcal{R}_s$ $(s \in \Xi(\varepsilon))$ is a positive number multiple of $\Delta_{\varepsilon \cdot s}^{\varepsilon} d\mu_\varepsilon$ with the density $\Delta_{\varepsilon \cdot s}^{\varepsilon}$ which is the

transfer of a character of $H(\mathcal{O}_\varepsilon)$ to $\mathcal{O}_\varepsilon$, where $\varepsilon \cdot s := (\varepsilon_1 s_1, \ldots, \varepsilon_r s_r)$. Furthermore, the positive constant is the reciprocal of the gamma function $\Gamma_{\mathcal{O}_\varepsilon}$ associated to the orbit $\mathcal{O}_\varepsilon$ (see (4.8) for the definition of $\Gamma_{\mathcal{O}_\varepsilon}$).

THEOREM C. *The measure $d\mathcal{R}_s$ $(s \in \Xi(\varepsilon))$ is given by*

$$d\mathcal{R}_s = \Gamma_{\mathcal{O}_\varepsilon}(\varepsilon \cdot s)^{-1} \Delta^\varepsilon_{\varepsilon \cdot s}\, d\mu_\varepsilon. \tag{0.1}$$

Theorem C is essentially proved in Section 5. The function $\Gamma_{\mathcal{O}_\varepsilon}$ itself is a generalization of the function Γ_Ω studied by Gindikin [**5**]. We show in Theorem 4.2 that it is expressed as a product of the usual gamma functions.

THEOREM D. *Up to a positive multiple depending on the normalization of the measure concerned, one has*

$$\Gamma_{\mathcal{O}_\varepsilon}(s) = \prod_{\varepsilon_i = 1} \Gamma\left(s_i - \frac{p_i(\varepsilon)}{2}\right).$$

We now describe the organization of this paper. In Section 1, we summarize the basic structure of normal j-algebras. Section 2 is devoted to giving various explicit formulas. This is done by expressing the elements in H by lower triangular matrices with vector entries (see (2.4)). The idea of such expression is due to Vinberg [**15**] (see also [**5**]), where the theory is based on the left symmetric algebras. Our formulation is inspired by the book [**3**, Chapter VI, Section 3] of Faraut and Korány in which the framework of Jordan algebra is developed.

In Section 3, we study the H-orbit structure and an inductive structure of $\overline{\Omega}$. The explicit formulas obtained in Section 2 play a fundamental role here. We remark that the inductive description of homogeneous cones is first given by Vinberg [**15**] and further pursued by Rothaus [**12**] and Dorfmeister [**1**]. Theorem A above is proved in Theorem 3.5.

The purpose of Section 4 is to introduce and investigate the gamma function $\Gamma_{\mathcal{O}_\varepsilon}$ associated to $\mathcal{O}_\varepsilon$ for every $\varepsilon \in \{0,1\}^r$ mentioned above. The definition of $\Gamma_{\mathcal{O}_\varepsilon}$ requires a function and a measure on $\mathcal{O}_\varepsilon$ which are relatively invariant under the action of H. These two relative invariants are defined through the above-mentioned diffeomorphism $\Psi_\varepsilon : H(\mathcal{O}_\varepsilon) \to \mathcal{O}_\varepsilon$. It is essentially from the convergence condition of the defining integral (4.8) of $\Gamma_{\mathcal{O}_\varepsilon}$ that the condition for $\Xi(\varepsilon)$ is derived.

In Section 5, we consider the analytic continuations of the distributions $\mathcal{R}^\varepsilon_s$ obtained from the relatively invariant measures on $\mathcal{O}_\varepsilon$ defined by the right-hand side of (0.1). This is done in Theorem 5.1, and comparing the Laplace transforms of these distributions, we see in Theorem 5.2 that these distributions coincide with the Riesz distributions in each of their definition domains of the parameter s.

In the last section, Section 6, the Gindikin–Wallach set Ξ is determined. Properly speaking, we define Ξ as the disjoint union of $\Xi(\varepsilon)$'s and prove that Ξ is exactly the positivity set for the Riesz distribution. The proof is carried out by induction on the rank of cones, and in this regard, the inductive description of $\overline{\Omega}$ in Section 3 is of substantial importance. The algorithm mentioned above is given after Proposition 6.1.

Let us fix some general notations used in this paper. Let U be a real vector space and U^* its dual vector space. For $A \in \mathrm{End}(U)$, A^* denotes the adjoint operator of A, that is, $A^*\xi := \xi \circ A$ $(\xi \in U^*)$. Moreover, if U is endowed with an inner product, we denote by $\mathscr{S}(U)$ the Schwartz space of rapidly decreasing functions on U.

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§1. Preliminaries.

Let V be a finite-dimensional vector space over $\boldsymbol{R}$ and Ω an open convex cone in V containing no line. We assume that Ω is homogeneous. This means that the group $\mathrm{GL}(\Omega)$ of linear automorphisms of the cone Ω acts on Ω transitively. Let $T_\Omega := V + i\Omega$ be the corresponding tube domain in the complexification $V_{\boldsymbol{C}}$ of V. Then T_Ω is homogeneous under the affine transformations

$$x + iy \mapsto a + t \cdot x + it \cdot y \quad (a \in V,\ t \in \mathrm{GL}(\Omega)).$$

The tube domain T_Ω is also called *a Siegel domain of type* I, a special case of *Siegel domains of type* II (see [**8**] for definition). Let D be a Siegel domain of type II on which the group $\mathrm{Hol}(D)$ of holomorphic automorphisms of D acts transitively. By [**7**] there is a split solvable closed subgroup G of $\mathrm{Hol}(D)$ acting simply transitively on D, and the Lie algebra $\mathfrak{g}$ of G has a structure of normal j-algebra. Conversely any normal j-algebra gives rise to a homogeneous Siegel domain of type II as described in [**8**, Chapter 2, Section 5], [**11**, Section 4A]. These observations ensure that we lose no generality by beginning the present paper with a normal j-algebra and assuming our homogeneous cone to be defined through the normal j-algebra.

Now let $\mathfrak{g}$ be a real split solvable Lie algebra, j a linear automorphism on $\mathfrak{g}$ such that $j^2 = -\mathrm{id}_\mathfrak{g}$, and ω_0 a linear form on $\mathfrak{g}$. The triple $(\mathfrak{g}, j, \omega_0)$ is called *a normal j-algebra* if the following three conditions are satisfied:

(i) $[Y, Y'] + j[Y, jY'] + j[jY, Y'] - [jY, jY'] = 0$ for all $Y, Y' \in \mathfrak{g}$,

(ii) $\langle [jY, jY'], \omega_0 \rangle = \langle [Y, Y'], \omega_0 \rangle$ for all $Y, Y' \in \mathfrak{g}$,

(iii) $\langle [Y, jY], \omega_0 \rangle > 0$ for all non-zero $Y \in \mathfrak{g}$.

By (ii) and (iii), we define an inner product $(\cdot|\cdot)_{\omega_0}$ on $\mathfrak{g}$ as

$$(Y|Y')_{\omega_0} := \langle [Y, jY'], \omega_0 \rangle \quad (Y, Y' \in \mathfrak{g}). \tag{1.1}$$

Let $\mathfrak{a}$ be the orthogonal complement of $[\mathfrak{g}, \mathfrak{g}]$ relative to this inner product. It is known that $\mathfrak{a}$ is a commutative subalgebra of $\mathfrak{g}$ and $\mathrm{ad}(\mathfrak{a})$ is a commutative family of self-adjoint operators on $\mathfrak{g}$. The dimension $r := \dim \mathfrak{a}$ is called *the rank of* $\mathfrak{g}$. For a linear form α on $\mathfrak{a}$, we set

$$\mathfrak{g}_\alpha := \{ Y \in \mathfrak{g} \,|\, [C, Y] = \langle C, \alpha \rangle Y \text{ for all } C \in \mathfrak{a} \}.$$

Then $[\mathfrak{g}_\alpha, \mathfrak{g}_\beta] \subset \mathfrak{g}_{\alpha+\beta}$, and if $\alpha \neq \beta$, we have $\mathfrak{g}_\alpha \perp \mathfrak{g}_\beta$ with respect to the inner product (1.1). If $\alpha \neq 0$ and $\mathfrak{g}_\alpha \neq \{0\}$, we call α *a root* and $\mathfrak{g}_\alpha$ *the root space* corresponding to α. Concerning normal j-algebras, the following theorem is fundamental.

THEOREM 1.1 (Pyatetskii-Shapiro [**8**]). (i) *There is a linear basis* $\{A_1, \dots, A_r\}$ *of* $\mathfrak{a}$ *such that if one puts* $E_l := -jA_l$, *then* $[A_k, E_l] = \delta_{kl}E_l$ $(1 \le k, l \le r)$.

(ii) *Let* $\alpha_1, \dots, \alpha_r$ *be the basis of* $\mathfrak{a}^*$ *dual to* $A_1, \dots, A_r$. *Then the possible roots are of the following forms*:

$$\alpha_k, \qquad \alpha_k/2 \qquad (1 \le k \le r),$$

$$(\alpha_m - \alpha_k)/2, \qquad (\alpha_m + \alpha_k)/2 \qquad (1 \le k < m \le r).$$

(iii) *The root spaces* $\mathfrak{g}_{\alpha_k}$ $(1 \le k \le r)$ *are one-dimensional:* $\mathfrak{g}_{\alpha_k} = \boldsymbol{R}E_k$.

(iv) *If* $m > k$, *then* $j\mathfrak{g}_{(\alpha_m-\alpha_k)/2} = \mathfrak{g}_{(\alpha_m+\alpha_k)/2}$, *and the action of* j *is given by*

$$jY = -[Y, E_k] \quad (Y \in \mathfrak{g}_{(\alpha_m-\alpha_k)/2}). \tag{1.2}$$

Moreover $j\mathfrak{g}_{\alpha_k/2} = \mathfrak{g}_{\alpha_k/2}$.

Let us put

$$\begin{aligned} \mathfrak{g}(1) &:= \left(\bigoplus_{k=1}^{r} \mathfrak{g}_{\alpha_k}\right) \oplus \left(\bigoplus_{1 \le k < m \le r} \mathfrak{g}_{(\alpha_m+\alpha_k)/2}\right), \quad \mathfrak{g}(1/2) := \bigoplus_{k=1}^{r} \mathfrak{g}_{\alpha_k/2}, \\ \mathfrak{g}(0) &:= \mathfrak{a} \oplus \left(\bigoplus_{1 \le k < m \le r} \mathfrak{g}_{(\alpha_m-\alpha_k)/2}\right). \end{aligned} \tag{1.3}$$

Then we have $\mathfrak{g} = \mathfrak{g}(1) \oplus \mathfrak{g}(1/2) \oplus \mathfrak{g}(0)$. We also put

$$E := E_1 + \cdots + E_r, \quad A := A_1 + \cdots + A_r \tag{1.4}$$

for simplicity. Then $\mathrm{ad}(A)Y = \mu Y$ for $Y \in \mathfrak{g}(\mu)$. Furthermore

$$[\mathfrak{g}(\mu), \mathfrak{g}(\nu)] \subset \mathfrak{g}(\mu + \nu) \quad (\mu, \nu = 0, 1/2, 1), \tag{1.5}$$

where we understand $\mathfrak{g}(\mu) = \{0\}$ for $\mu > 1$. It is immediately seen from Theorem 1.1 that $j\mathfrak{g}(0) = \mathfrak{g}(1)$ with

$$jY = -[Y, E] \quad (Y \in \mathfrak{g}(0)) \tag{1.6}$$

and $j\mathfrak{g}(1/2) = \mathfrak{g}(1/2)$.

It is known that normal j-algebras corresponding to tube domains are those for which $\mathfrak{g}(1/2) = \{0\}$. Henceforth, let $\mathfrak{g}$ be a normal j-algebra such that $\mathfrak{g}(1/2) = \{0\}$. Put $V = \mathfrak{g}(1)$ and $\mathfrak{h} = \mathfrak{g}(0)$. By (1.5), $\mathfrak{g}$ is a semidirect product $V \rtimes \mathfrak{h}$ with V a commutative ideal. Let G be the simply connected Lie group corresponding to the Lie algebra $\mathfrak{g}$. Since $\mathfrak{g}$ is split solvable, the exponential mapping is a diffeomorphism from $\mathfrak{g}$ onto G. Put $H := \exp \mathfrak{h} \subset G$. The adjoint action of H leaves V invariant, and we write $t \cdot x$ for $\mathrm{Ad}(t)x$ $(t \in H, x \in V)$. By [**11**, Theorem 4.15] the H-orbit Ω through E is an open convex cone containing no line. Moreover, H acts on Ω simply transitively. In order to make the notation clearer, we shall put

$$V_{mk} := \mathfrak{g}_{(\alpha_m+\alpha_k)/2} \quad (1 \le k \le m \le r), \quad \mathfrak{h}_{mk} := \mathfrak{g}_{(\alpha_m-\alpha_k)/2} \quad (1 \le k < m \le r).$$

Note that $V_{kk} = \boldsymbol{R}E_k$ $(1 \le k \le r)$. We have by (1.3)

$$V = \left(\bigoplus_{k=1}^{r} \boldsymbol{R}E_k\right) \oplus \left(\bigoplus_{1 \le k < m \le r} V_{mk}\right), \tag{1.7}$$

$$\mathfrak{h} = \left(\bigoplus_{k=1}^{r} \boldsymbol{R}A_k\right) \oplus \left(\bigoplus_{1 \le k < m \le r} \mathfrak{h}_{mk}\right). \tag{1.8}$$

Before closing this preliminary section, we shall make a brief observation about the dual cones. The dual cone Ω^* of Ω is by definition

$$\Omega^* := \{\xi \in V^* \mid \langle x, \xi \rangle > 0 \text{ for all } x \in \overline{\Omega} \backslash \{0\}\}. \tag{1.9}$$

Then Ω^* is also an open convex cone in V^* containing no line. It is a fundamental fact that $\Omega^{**} := (\Omega^*)^*$ equals Ω under the canonical identification of V^{**} with V. Let $H^* := \{t^* | t \in H\} \subset \mathrm{GL}(V^*)$. It is known that H^* acts simply transitively on Ω^* (see [**15**, Proposition 9]). The following lemma is a consequence of this observation.

LEMMA 1.2. *Take* $\xi_0 \in \Omega^*$. *One has*

$$\begin{aligned} \overline{\Omega} &= \{x \in V \mid \langle x, \xi \rangle \ge 0 \text{ for all } \xi \in \Omega^*\} \\ &= \{x \in V \mid \langle t \cdot x, \xi_0 \rangle \ge 0 \text{ for all } t \in H\}. \end{aligned}$$

§2. Explicit description of H-action on V.

In this section, retaining the notation of the previous section, we describe the action of H on V explicitly by expressing the elements of H as lower triangular matrices with vector entries. This expression visualizes not only the multiplication of the elements of H but also the action of H on V, and makes the resultant formulas easier to understand.

We start by expressing every $T \in \mathfrak{h}$, according to (1.8), uniquely by

$$T = \sum_{k=1}^{r} t_{kk} A_k + \sum_{m>k} T_{mk} \quad (t_{kk} \in \boldsymbol{R},\ T_{mk} \in \mathfrak{h}_{mk}). \tag{2.1}$$

For the element $T \in \mathfrak{h}$ in (2.1), we put

$$L_k := \sum_{m>k} T_{mk} \quad (1 \le k \le r-1). \tag{2.2}$$

Given $T \in \mathfrak{h}$, the symbols T_{mk} $(m > k)$ as well as L_k will be used to denote the elements in (2.1) and (2.2) without any comments in this paper. Let Π be the open subset of $\mathfrak{h}$ defined by

$$\Pi := \{T \in \mathfrak{h} \mid t_{kk} > 0 \text{ for all } k = 1, 2, \ldots, r\}.$$

Putting $T_{kk} := (2 \log t_{kk}) A_k$ $(1 \le k \le r)$ for $T \in \Pi$, we set

$$\gamma(T) := \exp T_{11} \cdot \exp L_1 \cdot \exp T_{22} \cdots \exp L_{r-1} \cdot \exp T_{rr}. \tag{2.3}$$

Then $\gamma(T) \in H$, and we shall write it in the following form of lower triangular matrix:

$$\begin{pmatrix} t_{11} & & & \\ T_{21} & t_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & & t_{rr} \end{pmatrix}. \tag{2.4}$$

PROPOSITION 2.1. (i) *For* $T, T' \in \Pi$, *one has*

$$\begin{pmatrix} t_{11} & & & \\ T_{21} & t_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & & t_{rr} \end{pmatrix} \begin{pmatrix} t'_{11} & & & \\ T'_{21} & t'_{22} & & \\ \vdots & & \ddots & \\ T'_{r1} & T'_{r2} & & t'_{rr} \end{pmatrix} = \begin{pmatrix} t''_{11} & & & \\ T''_{21} & t''_{22} & & \\ \vdots & & \ddots & \\ T''_{r1} & T''_{r2} & & t''_{rr} \end{pmatrix} \quad \textit{with}$$

$$t''_{kk} = t_{kk} t'_{kk} \quad (1 \le k \le r), \tag{2.5}$$

$$T''_{mk} = t_{mm} T'_{mk} + \sum_{k<l<m} [T_{ml}, T'_{lk}] + t'_{kk} T_{mk} \quad (1 \le k < m \le r). \tag{2.6}$$

(ii) *The map* γ *is a diffeomorphism from* Π *onto* H.

PROOF. (i) Let $1 \le i \le k \le r$. We have

$$\exp L_k \cdot \exp L'_i = \exp \mathrm{Ad}(\exp L_k) L'_i \cdot \exp L_k = \exp e^{\mathrm{ad}\, L_k} L'_i \cdot \exp L_k. \tag{2.7}$$

In the same way, we get

$$\exp T_{kk} \cdot \exp L'_i = \exp e^{\mathrm{ad}\, T_{kk}} L'_i \cdot \exp T_{kk}, \tag{2.8}$$

$$\exp L_k \cdot \exp T'_{ii} = \begin{cases} \exp T'_{ii} \cdot \exp L_k & (i < k), \\ \exp T'_{kk} \cdot \exp e^{-\mathrm{ad}\, T'_{kk}} L_k & (i = k). \end{cases} \tag{2.9}$$

Using (2.9) and noting that $\exp \mathfrak{a}$ is commutative, we obtain

$$\gamma(T) \exp T'_{11} = \exp T_{11} \exp T'_{11} \cdot \exp e^{-\mathrm{ad}\, T'_{11}} L_1 \cdot \exp T_{22} \cdots \exp L_{r-1} \exp T_{rr}.$$

Then owing to (2.7) and (2.8), we see that $\gamma(T) \exp T'_{11} \exp L'_1$ is equal to

$$\exp T_{11} \exp T'_{11} \cdot \exp e^{-\mathrm{ad}\, T'_{11}} L_1 \exp(e^{\mathrm{ad}\, T_{22}} e^{\mathrm{ad}\, L_2} \cdots e^{\mathrm{ad}\, L_{r-1}} e^{\mathrm{ad}\, T_{rr}}) L'_1 \cdot$$
$$\cdot \exp T_{22} \cdots \exp L_{r-1} \exp T_{rr}.$$

Repetition of this argument yields $\gamma(T)\gamma(T') = \gamma(T'')$, where

$$T''_{kk} = T_{kk} + T'_{kk}, \tag{2.10}$$

$$L''_k = e^{-\mathrm{ad}\, T'_{kk}} L_k + e^{\mathrm{ad}\, T_{k+1,\, k+1}} e^{\mathrm{ad}\, L_{k+1}} \cdots e^{\mathrm{ad}\, L_{r-1}} e^{\mathrm{ad}\, T_{rr}} L'_k. \tag{2.11}$$

Thus (2.5) follows immediately from (2.10). To show (2.6) we observe that (1.8), (2.2) and Theorem 1.1 imply $\mathrm{ad}\,(L_k)^2 L'_i = 0$, so that if $i < k$,

$$e^{\mathrm{ad}\, L_k} L'_i = L'_i + [L_k, L'_i] = L'_i + [L_k, T'_{ki}]. \tag{2.12}$$

Similarly,

$$e^{\mathrm{ad}\, T_{kk}} L_i' = t_{kk} T_{ki}' + \sum_{l \neq k,\, l>i} T_{li}' \quad (\text{if } i<k), \quad e^{-\mathrm{ad}\, T_{kk}'} L_k = t_{kk}' L_k. \tag{2.13}$$

Making use of (2.12) and (2.13), we calculate the right-hand side of (2.11) to arrive at the right-hand side of (2.6).

(ii) Let $\tilde{H}$ be the image of γ. By (i), $\tilde{H}$ is a subgroup of H. Recall the element A in (1.4). Then $\gamma(A)$ equals the unit element of H. Since

$$\left.\frac{d}{dh}\right|_{h=0} \gamma(A+hT) = T \quad (T \in \mathfrak{h})$$

as is easily seen, the map γ is a local diffeomorphism at A, so that the subgroup $\tilde{H}$ is open. Since H is connected, we have $H = \tilde{H}$. It remains to show that γ is injective. Here we need the following simple lemma.

LEMMA 2.2. *For two subalgebras* $\mathfrak{h}_1$ *and* $\mathfrak{h}_2$ *of* $\mathfrak{h}$ *such that* $\mathfrak{h}_1 \cap \mathfrak{h}_2 = \{0\}$, *the map* $\exp \mathfrak{h}_1 \times \exp \mathfrak{h}_2 \ni (t_1, t_2) \mapsto t_1 t_2 \in H$ *is injective.*

PROOF. Suppose that $t_1 t_2 = t_1' t_2'$ $(t_i, t_i' \in \exp \mathfrak{h}_i,\ i = 1, 2)$. Then there exists $T \in \mathfrak{h}$ such that $\exp T = t_1^{-1} t_1' = t_2 t_2'^{-1}$. This implies $T \in \mathfrak{h}_1 \cap \mathfrak{h}_2$, so that $T = 0$. Hence $t_1 = t_1'$ and $t_2 = t_2'$. □

We now prove the injectivity of γ. Let $\mathfrak{n}_i$ $(1 \leq i \leq r-1)$ be the commutative subalgebras of $\mathfrak{h}$ given by

$$\mathfrak{n}_i := \bigoplus_{m>i} \mathfrak{h}_{mi}. \tag{2.14}$$

We shall apply Lemma 2.2 repeatedly to the pairs of the subalgebras

$$\mathfrak{h}^{(l)} := \left(\bigoplus_{i=1}^{l} \boldsymbol{R} A_i\right) \oplus \left(\bigoplus_{i=1}^{l} \mathfrak{n}_i\right), \quad \tilde{\mathfrak{h}}^{(l)} := \left(\bigoplus_{i=1}^{l} \boldsymbol{R} A_i\right) \oplus \left(\bigoplus_{i=1}^{l-1} \mathfrak{n}_i\right)$$

for $l = 1, \ldots, r-1$. Suppose $\gamma(T) = \gamma(T')$ for $T, T' \in \Pi$. Then Lemma 2.2 with $\mathfrak{h}_1 := \mathfrak{h}^{(r-1)}$ and $\mathfrak{h}_2 := \boldsymbol{R} A_r$ gives $t_{rr} = t_{rr}'$ and $\gamma(T)\exp(-T_{rr}) = \gamma(T')\exp(-T_{rr}') \in \exp \mathfrak{h}^{(r-1)}$. Next, putting $\mathfrak{h}_1 := \tilde{\mathfrak{h}}^{(r-1)}$ and $\mathfrak{h}_2 := \mathfrak{n}_{r-1}$ in Lemma 2.2, we get $L_{r-1} = L_{r-1}'$. Repeating these arguments, we obtain $T = T'$. □

We next observe the action of H on V. To do so we express every $x \in V$ as

$$x = \sum_{k=1}^{r} x_{kk} E_k + \sum_{m>k} X_{mk} \quad (x_{kk} \in \boldsymbol{R},\ X_{mk} \in V_{mk}). \tag{2.15}$$

We put $X_{kk} := x_{kk} E_k$ $(1 \leq k \leq r)$. We shall use the symbol X_{mk} $(m \geq k)$ to denote the V_{mk}-component of a given $x \in V$ without mentioning it. The action of $\exp \mathfrak{a}$ is diagonal and easy to describe:

$$\left(\exp \sum_{l=1}^{r} (2 \log t_{ll}) A_l\right) \cdot x = \sum_{m \geq k} t_{mm} t_{kk} X_{mk}. \tag{2.16}$$

In order to see the action of the other elements of H, we introduce the lexicographic order $\succeq$ in the index set $\Lambda := \{(m,k) \mid 1 \le k \le m \le r\}$ by defining $(m,l) \succeq (k,i)$ if either "$m > k$" or "$m = k,\ l \ge i$". Since Theorem 1.1 leads us to

$$\mathrm{ad}(\mathfrak{h})V_{ki} \subset \bigoplus_{(m,l)\succeq(k,i)} V_{ml} \quad ((k,i) \in \Lambda),$$

we see that the action of H on V is "lower triangular" with respect to the order $\succeq$. Let us set

$$V^{[k]} := \bigoplus_{m \ge l > k} V_{ml}. \tag{2.17}$$

Since the summation range can be interpreted as $(m,l) \succeq (k+1, k+1)$, $V^{[k]}$ is H-invariant.

We now investigate the action of H on the specific elements more closely. For this purpose, we first need to introduce a new inner product on $\mathfrak{g}$. Let E^* be the linear form on $\mathfrak{g}$ defined by

$$\langle x + T,\ E^* \rangle := \sum_{k=1}^{r} x_{kk} \quad (x \in V,\ T \in \mathfrak{h}).$$

LEMMA 2.3. *The bilinear form* $(Y|Y') := (1/2)\langle [jY, Y'], E^* \rangle$ *defines an inner product on* $\mathfrak{g}$.

PROOF. We begin the proof by noting that if ω_0 is as in (1.1), then

$$\langle x + T, \omega_0 \rangle = -\sum_{k=1}^{r} \eta_k x_{kk} \quad (x \in V,\ T \in \mathfrak{h}), \tag{2.18}$$

where $\eta_k := -\langle E_k, \omega_0 \rangle > 0$ for all k (see [**11**, (4.6)]). Now let $x, x' \in V$ and $T, T' \in \mathfrak{h}$ and express them as in (2.15) and (2.1) respectively. By Theorem 1.1, we have $[jX_{mk}, X'_{mk}],\ [jT_{mk}, T'_{mk}] \in \boldsymbol{R}E_m$. Thus (1.1) and (2.18) tell us that for $m > k$,

$$[jX_{mk}, X'_{mk}] = \frac{1}{\eta_m}(X_{mk}|X'_{mk})_{\omega_0} E_m, \quad [jT_{mk}, T'_{mk}] = \frac{1}{\eta_m}(T_{mk}|T'_{mk})_{\omega_0} E_m.$$

This together with $[jE_k, E_k] = E_k$ and $[jA_k, A_k] = E_k$ gives

$$\begin{aligned} 2(x + T \,|\, x' + T') = &\sum_{k=1}^{r} \left(x_{kk}x'_{kk} + \sum_{m>k} \frac{1}{\eta_m}(X_{mk}|X'_{mk})_{\omega_0} \right) \\ &+ \sum_{k=1}^{r} \left(t_{kk}t'_{kk} + \sum_{m>k} \frac{1}{\eta_m}(T_{mk}|T'_{mk})_{\omega_0} \right). \end{aligned} \tag{2.19}$$

Therefore the bilinear form $(\cdot|\cdot)$ is symmetric and positive definite. □

The inner product $(\cdot|\cdot)$ in Lemma 2.3 will be called *the standard inner product* in what follows. By (2.19), the change to the standard inner product does not affect the orthogonality of the decomposition of V and $\mathfrak{h}$ in (1.7) and (1.8) respectively.

COROLLARY 2.4. $[T_{mk}, [T'_{mk}, E_k]] = 2(T'_{mk}|T_{mk})E_m$.

PROOF. Immediate from $[T'_{mk}, E_k] = -jT'_{mk}$ by (1.2). □

We set

$$T_{mk} \circ T'_{lk} := [T_{mk}, [T'_{lk}, E_k]] \in \begin{cases} V_{ml} & (m > l), \\ V_{lm} & (l > m). \end{cases} \tag{2.20}$$

Since $[T_{mk}, T'_{lk}] = 0$ $(m \neq l)$ by Theorem 1.1, Jacobi's identity gives the commutativity $T_{mk} \circ T'_{lk} = T'_{lk} \circ T_{mk}$. With these preparations, we are now able to describe explicitly the action of H on the elements

$$E_\varepsilon := \sum_{k=1}^{r} \varepsilon_k E_k \in V \quad (\varepsilon = (\varepsilon_1, \ldots, \varepsilon_r) \in \{0,1\}^r). \tag{2.21}$$

PROPOSITION 2.5. *Let* $x := \gamma(T) \cdot E_\varepsilon$ $(T \in \Pi)$. *Then*

$$x_{kk} = \varepsilon_k (t_{kk})^2 + \sum_{i<k} \varepsilon_i \|T_{ki}\|^2 \quad (1 \leq k \leq r), \tag{2.22}$$

$$X_{mk} = \varepsilon_k t_{kk} [T_{mk}, E_k] + \sum_{i<k} \varepsilon_i T_{mi} \circ T_{ki} \quad (1 \leq k < m \leq r), \tag{2.23}$$

where $\|\cdot\|$ *stands for the norm defined by the standard inner product on* $\mathfrak{g}$.

PROOF. Let $1 \leq i \leq r$. If $l > i$, then Theorem 1.1 implies that $\exp L_l \cdot E_i = E_i$ and $\exp T_{ll} \cdot E_i = E_i$. Thus

$$\gamma(T) \cdot E_i = \exp T_{11} \cdots \exp L_{i-1} \cdot (\exp T_{ii} \exp L_i \cdot E_i).$$

Since E_i lies in the H-invariant subspace $V^{[i-1]}$ (see (2.17)), it holds that $\exp T_{ii} \exp L_i \cdot E_i \in V^{[i-1]}$. Furthermore, if $h \leq i-1$, Theorem 1.1 shows that $\exp L_h$ as well as $\exp T_{hh}$ acts trivially on $V^{[i-1]}$. It follows that

$$\gamma(T) \cdot E_i = \exp T_{ii} \exp L_i \cdot E_i.$$

Noting $\mathrm{ad}(L_i)^3 E_i = 0$ and using (2.16), we have

$$\begin{aligned} \exp T_{ii} \exp L_i \cdot E_i &= \exp T_{ii} \cdot (E_i + [L_i, E_i] + (1/2)[L_i, [L_i, E_i]]) \\ &= (t_{ii})^2 E_i + t_{ii}[L_i, E_i] + (1/2)[L_i, [L_i, E_i]]. \end{aligned} \tag{2.24}$$

Substituting $L_i = \sum_{m>i} T_{mi}$ in the last term, we obtain by Corollary 2.4 and (2.20)

$$\gamma(T) \cdot E_i = (t_{ii})^2 E_i + \sum_{m>i} t_{ii}[T_{mi}, E_i] + \sum_{k>i} \|T_{ki}\|^2 E_k + \sum_{m>k>i} T_{mi} \circ T_{ki},$$

from which the formulas (2.22) and (2.23) follow immediately. □

REMARK. If one expresses $x \in V$ by the symmetric matrix

$$\begin{pmatrix} x_{11} & X_{21} & \cdots & X_{r1} \\ X_{21} & x_{22} & & X_{r2} \\ \vdots & & \ddots & \\ X_{r1} & X_{r2} & & x_{rr} \end{pmatrix},$$

then Proposition 2.5 may be better understood. The elements E_ε being expressed by the diagonal matrices $D_\varepsilon = \mathrm{diag}[\varepsilon_1, \ldots, \varepsilon_r]$, the formulas (2.22) and (2.23) are the matrix multiplication rule for $x = \gamma(T) \cdot D_\varepsilon \cdot {}^t\gamma(T)$ under an appropriate interpretation.

§3. Orbit decomposition of $\overline{\Omega}$.

From now on, we shall regard H as a subgroup of $\mathrm{GL}(V)$ by the adjoint representation. Let $\overline{H}$ denote the closure of H in $\mathrm{End}(V)$. Then $\overline{H}$ is a semigroup. Recalling (1.7), we denote by P_{ml} $(1 \leq l \leq m \leq r)$ the orthogonal projection $V \to V_{ml}$. For $k = 1, \ldots, r$, we define a map $\psi_k : [0, \infty) \to \mathrm{End}(V)$ by

$$\psi_k(t) := t^2 P_{kk} + t\left(\sum_{i<k} P_{ki} + \sum_{m>k} P_{mk}\right) + \sum_{\substack{m>i \\ m\neq k,\, i\neq k}} P_{mi}. \tag{3.1}$$

The formula (2.16) says that $\psi_k(t) = \exp((2\log t)A_k) \in H$ for $t > 0$. Thus $\psi_k(t)$ belongs to $\overline{H}$ for any $t \geq 0$. Let $\overline{\Pi}$ be the closure of Π in $\mathfrak{h}$:

$$\overline{\Pi} = \{T \in \mathfrak{h} \mid t_{kk} \geq 0 \text{ for all } k = 1, 2, \ldots, r\}.$$

In view of (2.3), we define a map $\overline{\gamma} : \overline{\Pi} \to \overline{H}$ by

$$\overline{\gamma}(T) := \psi_1(t_{11}) \cdot \exp L_1 \cdot \psi_2(t_{22}) \cdots \exp L_{r-1} \cdot \psi_r(t_{rr}). \tag{3.2}$$

Evidently $\overline{\gamma}$ extends γ, so that we express $\overline{\gamma}(T)$ still by the matrix (2.4). We remark that the formulas in Propositions 2.1 (i) and 2.5 remain valid for $\overline{\gamma}(T)$ by continuity.

PROPOSITION 3.1. (i) *Let $\overline{\Omega}$ be the closure of Ω in V. Then $\overline{H} \cdot E = \overline{\Omega}$.*
(ii) $\overline{\gamma}(\overline{\Pi}) = \overline{H}$.

PROOF. (i) Clearly $\overline{H} \cdot E \subset \overline{\Omega}$. To prove the converse inclusion we take $\tilde{x} \in \overline{\Omega}$. Then there exists a sequence $\{x^{(\nu)}\}_{\nu \in \boldsymbol{N}}$ in Ω such that $x^{(\nu)} \to \tilde{x}$ as $\nu \to \infty$. According to Proposition 2.1 (ii), we take $T^{(\nu)} \in \Pi$ for which $x^{(\nu)} = \gamma(T^{(\nu)}) \cdot E$. By (2.22) with $\varepsilon = (1, \ldots, 1)$, we have

$$x_{kk}^{(\nu)} = (t_{kk}^{(\nu)})^2 + \sum_{i<k} \|T_{ki}^{(\nu)}\|^2 \quad (1 \leq k \leq r). \tag{3.3}$$

Since (3.3) says that all the sequences $\{t_{kk}^{(\nu)}\}_{\nu \in \boldsymbol{N}}$ and $\{T_{ki}^{(\nu)}\}_{\nu \in \boldsymbol{N}}$ are bounded, we take a strictly increasing sequence $\{\nu_n\}$ of positive integers such that all of the limits $\tilde{t}_{kk} := \lim t_{kk}^{(\nu_n)}$ and $\tilde{T}_{ki} := \lim T_{ki}^{(\nu_n)}$ exist. Put $\tilde{T} := \sum \tilde{t}_{kk} A_{kk} + \sum \tilde{T}_{ki}$. Then $T^{(\nu_n)} \to \tilde{T} \in \overline{\Pi}$ as $n \to \infty$. Therefore $\tilde{x} = \overline{\gamma}(\tilde{T}) \cdot E \in \overline{H} \cdot E$, whence (i) follows.

(ii) For $\tilde{t} \in \overline{H}$, we take a sequence $\{t^{(\nu)}\}_{\nu \in \boldsymbol{N}}$ in H such that $t^{(\nu)} \to \tilde{t}$ as $\nu \to \infty$. Put $\tilde{x} := \tilde{t} \cdot E \in \overline{\Omega}$ and $x^{(\nu)} := t^{(\nu)} \cdot E \in \Omega$. Then $x^{(\nu)} \to \tilde{x}$ as $\nu \to \infty$, and repeating the argument in (i), we find $\tilde{T} \in \overline{\Pi}$ for which $\tilde{t} = \overline{\gamma}(\tilde{T})$. Hence (ii) is proved. □

We denote by $\mathcal{O}_\varepsilon$ the H-orbit in V through $E_\varepsilon \in \overline{\Omega}$. Then $\mathcal{O}_\varepsilon$ is contained in $\overline{\Omega}$. Note that $\mathcal{O}_{(1,\ldots,1)} = \Omega$ and $\mathcal{O}_\varepsilon \subset \partial\Omega$ if $\varepsilon \neq (1, \ldots, 1)$. For $x \in V$, let $H_x := \{t \in H \mid t \cdot x = x\}$ be the stabilizer at x in H.

PROPOSITION 3.2. (i) *Let* $x := \gamma(T)\cdot E_\varepsilon$ *with* $T \in \Pi$. *If* $x \in \bigoplus_{k=1}^r \boldsymbol{R}E_k$, *then* $\varepsilon_k T_{mk} = 0$ *for all* $1 \le k < m \le r$ *and* $x = \sum_{k=1}^r \varepsilon_k (t_{kk})^2 E_k$.

(ii) *If* $\varepsilon \neq \varepsilon'$, *then the two orbits* $\mathcal{O}_\varepsilon$ *and* $\mathcal{O}_{\varepsilon'}$ *are distinct*.

(iii) *One has*

$$H_{E_\varepsilon} = \{\gamma(T) \in H \mid \text{if } \varepsilon_i = 1, \text{ then } t_{ii} = 1 \text{ and } T_{ki} = 0\,(k > i)\}.$$

PROOF. (i) We shall prove $\varepsilon_k T_{mk} = 0$ by induction on k. By (2.23) and (1.2), we have

$$0 = X_{m1} = \varepsilon_1 t_{11}[T_{m1}, E_1] = -\varepsilon_1 t_{11} j T_{m1}.$$

Since $t_{11} \neq 0$, we obtain $\varepsilon_1 T_{m1} = 0$. Assume next that $\varepsilon_i T_{mi} = 0$ $(m > i)$ for $i = 1, \ldots, k-1$. Then, again by (2.23) and (1.2) we have for $m > k$,

$$0 = \varepsilon_k t_{kk}[T_{mk}, E_k] + \sum_{i<k} \varepsilon_i T_{mi} \circ T_{ki} = -\varepsilon_k t_{kk} j T_{mk}.$$

Hence $\varepsilon_k T_{mk} = 0$. The second assertion follows from this and (2.22).

(ii) Clear from (i) by considering the intersection with $\bigoplus_{k=1}^r \boldsymbol{R}E_k$.

(iii) The assertion follows from (i) and Proposition 2.5. □

Let $\mathfrak{h}_{E_\varepsilon}$ be the Lie algebra of H_{E_ε}. Then Proposition 3.2 (iii) tells us that

$$\mathfrak{h}_{E_\varepsilon} = \bigoplus_{\varepsilon_i = 0} (\boldsymbol{R}A_i \oplus \mathfrak{n}_i),$$

see (2.14) for the definition of $\mathfrak{n}_i$. Let $\mathfrak{h}(\mathcal{O}_\varepsilon)$ be the orthogonal complement of $\mathfrak{h}_{E_\varepsilon}$ in $\mathfrak{h}$. Then

$$\mathfrak{h}(\mathcal{O}_\varepsilon) = \bigoplus_{\varepsilon_i = 1} (\boldsymbol{R}A_i \oplus \mathfrak{n}_i).$$

Note that $\mathfrak{h}(\mathcal{O}_\varepsilon)$ coincides with $\mathfrak{h}_{E_{\mathbf{1}-\varepsilon}}$, where $\mathbf{1} := (1, \ldots, 1) \in \{0,1\}^r$. In particular, $\mathfrak{h}(\mathcal{O}_\varepsilon)$ is a subalgebra of $\mathfrak{h}$. Put $H(\mathcal{O}_\varepsilon) := \exp \mathfrak{h}(\mathcal{O}_\varepsilon)$. Then $H(\mathcal{O}_\varepsilon) = H_{E_{\mathbf{1}-\varepsilon}}$, so that

$$H(\mathcal{O}_\varepsilon) = \{\gamma(T) \in H \mid \text{if } \varepsilon_i = 0, \text{ then } t_{ii} = 1 \text{ and } T_{ki} = 0\,(k > i)\}. \tag{3.4}$$

Define a map $\pi_\varepsilon : H \to H(\mathcal{O}_\varepsilon)$ by

$$\begin{pmatrix} t_{11} & & & \\ T_{21} & t_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & & t_{rr} \end{pmatrix} \mapsto \begin{pmatrix} (t_{11})^{\varepsilon_1} & & & \\ \varepsilon_1 T_{21} & (t_{22})^{\varepsilon_2} & & \\ \vdots & & \ddots & \\ \varepsilon_1 T_{r1} & \varepsilon_2 T_{r2} & & (t_{rr})^{\varepsilon_r} \end{pmatrix}. \tag{3.5}$$

Clearly π_ε is surjective.

LEMMA 3.3. (i) *For* $t \in H$, *one has* $t \cdot E_\varepsilon = \pi_\varepsilon(t) \cdot E_\varepsilon$.

(ii) *The map* $\Psi_\varepsilon : H(\mathcal{O}_\varepsilon) \ni t \mapsto t \cdot E_\varepsilon \in \mathcal{O}_\varepsilon$ *is bijective*.

PROOF. (i) If $\varepsilon_i = 0$, the right-hand sides of (2.22) and (2.23) do not contain t_{ii} nor T_{ki} $(k > i)$. Hence the assertion holds.

(ii) The surjectivity of Ψ_ε follows from (i). Since $\mathfrak{h}_{E_\varepsilon} \cap \mathfrak{h}(\mathcal{O}_\varepsilon) = \{0\}$, the injectivity is clear. □

We set for $k = 1, 2, \ldots, r-1$,

$$[k] := (\underbrace{0, \ldots, 0}_{k \text{ times}}, 1, \ldots, 1) \in \{0,1\}^r. \tag{3.6}$$

Then $E_{[k]} = \sum_{m>k} E_m$ belongs to the H-invariant subspace $V^{[k]}$ defined in (2.17), and $\mathcal{O}_{[k]}$ is a homogeneous cone in $V^{[k]}$ of rank $r-k$, where *the rank of a cone* is the rank of the corresponding normal j-algebra. In fact, $\mathcal{O}_{[k]}$ is the homogeneous cone corresponding to the normal j-subalgebra $\mathfrak{h}(\mathcal{O}_{[k]}) \oplus V^{[k]}$. Thus it is natural to denote the orbit $\mathcal{O}_{[k]}$ by $\Omega^{[k]}$. Accordingly, we write $H(\Omega^{[k]})$ instead of $H(\mathcal{O}_{[k]})$. Let $P_{V^{[k]}}$ be the orthogonal projection $V \to V^{[k]}$. Define $E^*_{[k]} \in V^*$ by

$$\langle x, E^*_{[k]} \rangle := \sum_{m>k} x_{mm}. \tag{3.7}$$

By (2.22), we have

$$\langle \bar{\gamma}(T) \cdot E_{[k]},\, E^*_{[k]} \rangle = \sum_{m>k} \left\{ (t_{mm})^2 + \sum_{m>l>k} \|T_{ml}\|^2 \right\} \quad (T \in \overline{\Pi}). \tag{3.8}$$

If $\bar{\gamma}(T) \cdot E_{[k]} \neq 0$, then the right-hand side of (3.8) is strictly positive owing to Proposition 2.5. Thus Proposition 3.1 and (1.9) tell us that $E^*_{[k]}|_{V^{[k]}}$ belongs to $(\Omega^{[k]})^*$, the dual cone of $\Omega^{[k]}$.

LEMMA 3.4. (i) *For* $t \in H(\Omega^{[k]})$ *and* $x \in V$, *one has* $t \cdot P_{V^{[k]}}(x) = P_{V^{[k]}}(t \cdot x)$.
(ii) *For* $1 \leq k \leq r-1$, *one has* $P_{V^{[k]}}(\overline{\Omega}) = \overline{\Omega}^{[k]}$.

PROOF. (i) Let $(V^{[k]})^\perp$ be the orthogonal complement of $V^{[k]}$ in V. Then by (2.17),

$$(V^{[k]})^\perp = \bigoplus_{i=1}^{k} \bigoplus_{l=i}^{r} V_{li}.$$

If $i \leq k < m$, then Theorem 1.1 ensures that $\bigoplus_{l=i}^{r} V_{li}$ and hence $(V^{[k]})^\perp$ are stable under the adjoint action of $\mathfrak{n}_m$. Since $\mathfrak{h}(\Omega^{[k]}) = \bigoplus_{m>k} (\boldsymbol{R} A_m \oplus \mathfrak{n}_m)$, it follows that $(V^{[k]})^\perp$ is stable under $H(\Omega^{[k]})$. The $H(\Omega^{[k]})$-stability of $V^{[k]}$ being evident, the assertion (i) holds.

(ii) Let $x \in \overline{\Omega}$. According to Proposition 3.1 (i), we take $\tilde{t} \in \overline{H}$ for which $x = \tilde{t} \cdot E$. If $t_0 \in H(\Omega^{[k]})$, we obtain by (i), (3.7) and (2.22)

$$\langle t_0 \cdot P_{V^{[k]}}(x), E^*_{[k]} \rangle = \langle P_{V^{[k]}}(t_0 \cdot x),\, E^*_{[k]} \rangle = \langle t_0 \tilde{t} \cdot E,\, E^*_{[k]} \rangle \geq 0.$$

Therefore, applying Lemma 1.2 to the cone $\Omega^{[k]}$ in the vector space $V^{[k]}$, we obtain $P_{V^{[k]}}(x) \in \overline{\Omega}^{[k]}$. Hence $P_{V^{[k]}}(\overline{\Omega}) \subset \overline{\Omega}^{[k]}$. The converse inclusion is clear because $\overline{\Omega}^{[k]} \subset \overline{\Omega}$. □

We now arrive at the main result of this section.

THEOREM 3.5. *The H-orbit decomposition of* $\overline{\Omega}$ *is given as*

$$\overline{\Omega} = \bigsqcup_{\varepsilon \in \{0,1\}^r} \mathcal{O}_\varepsilon.$$

PROOF. By Proposition 3.2 (ii) and Lemma 3.3 (ii), it suffices to show

CLAIM. *Given* $x \in \overline{\Omega}$, *there exist* $\varepsilon \in \{0,1\}^r$ *and* $t \in H(\mathcal{O}_\varepsilon)$ *such that* $x = t \cdot E_\varepsilon$.

We shall prove the claim by induction on the rank r of cones. The rank one case is obvious. Assume that the claim holds when the rank is $r-1$. In particular, the claim holds for $\Omega^{[1]}$. Let $x \in \overline{\Omega}$. According to Proposition 3.1, we take $\tilde{T} \in \overline{\Pi}$ such that $x = \overline{\gamma}(\tilde{T}) \cdot E$ (though $\tilde{T}$ is not necessarily unique). We assume first that $x_{11} = 0$. Then $\tilde{t}_{11} = (x_{11})^{1/2} = 0$ by (2.22), so that $X_{m1} = \tilde{t}_{11}[\tilde{T}_{m1}, E_1] = 0$ $(m > 1)$ by (2.23). Thus $x = P_{V^{[1]}}(x) \in \overline{\Omega}^{[1]}$ by Lemma 3.4 (ii). Then the claim holds in this case by the induction hypothesis. Next, let us consider the case $x_{11} > 0$. Put

$$\tilde{t}^1 := \begin{pmatrix} \tilde{t}_{11} & & & \\ \tilde{T}_{21} & 1 & & \\ \vdots & & \ddots & \\ \tilde{T}_{r1} & 0 & & 1 \end{pmatrix}, \quad \tilde{t}^{[1]} := \begin{pmatrix} 1 & & & \\ 0 & \tilde{t}_{22} & & \\ \vdots & & \ddots & \\ 0 & \tilde{T}_{r2} & & \tilde{t}_{rr} \end{pmatrix}.$$

Then $\overline{\gamma}(\tilde{T}) = \tilde{t}^1 \tilde{t}^{[1]} \in \overline{H}$ by Proposition 2.1, and we have $\tilde{t}_{11} = (x_{11})^{1/2} > 0$ by (2.22), so that $\tilde{t}^1 \in H$. On the other hand, we have $\tilde{t}^{[1]} \cdot E_{[1]} \in \overline{\Omega}^{[1]}$. By the induction hypothesis, there exist unique $\varepsilon' := (0, \varepsilon_2, \ldots, \varepsilon_r) \in \{0,1\}^r$ and $t' = \gamma(T') \in H(\mathcal{O}_{\varepsilon'})$ $(T' \in \Pi)$ such that $\tilde{t}^{[1]} \cdot E_{[1]} = t' \cdot E_{\varepsilon'}$. Put $\varepsilon := (1, \varepsilon_2, \ldots, \varepsilon_r)$ and $t := \tilde{t}^1 t' \in H(\mathcal{O}_\varepsilon)$. Then by Proposition 2.5 and Lemma 3.3 (i), we get

$$\begin{aligned} x &= \tilde{t}^1 \tilde{t}^{[1]} \cdot (E_1 + E_{[1]}) = \tilde{t}^1 \cdot E_1 + \tilde{t}^{[1]} \cdot E_{[1]} \\ &= \tilde{t}^1 \cdot E_1 + t' \cdot E_{\varepsilon'} = \tilde{t}^1 t' (E_1 + E_{\varepsilon'}) = t \cdot E_\varepsilon, \end{aligned}$$

which means that the claim holds when the rank is r. Therefore the theorem is proved. □

The second half of this section is devoted to an inductive description of $\overline{\Omega}$ which will be needed in the proof of Theorem 6.2.

We set

$$\Pi(\mathcal{O}_\varepsilon) := \{T \in \mathfrak{h}(\mathcal{O}_\varepsilon) \mid t_{ii} > 0 \text{ for all } i \text{ such that } \varepsilon_i = 1\}. \tag{3.9}$$

By (3.4), the map $\Pi(\mathcal{O}_\varepsilon) \ni T \mapsto \gamma(A_{\mathbf{1}-\varepsilon} + T) \in H(\mathcal{O}_\varepsilon)$ is bijective, where

$$A_\varepsilon := jE_\varepsilon = \sum_{k=1}^r \varepsilon_k A_k \in \mathfrak{a}.$$

PROPOSITION 3.6. *Let* $\varepsilon, \varepsilon' \in \{0,1\}^r$. *If* $\varepsilon + \varepsilon' \in \{0,1\}^r$, *the map*

$$\mathcal{O}_\varepsilon \times \mathcal{O}_{\varepsilon'} \ni (x, x') \mapsto x + x' \in V \tag{3.10}$$

gives a diffeomorphism from $\mathcal{O}_\varepsilon \times \mathcal{O}_{\varepsilon'}$ *onto* $\mathcal{O}_{\varepsilon+\varepsilon'}$.

PROOF. Given $x \in \mathcal{O}_\varepsilon$, we take unique elements $t \in H(\mathcal{O}_\varepsilon)$ and $T \in \Pi(\mathcal{O}_\varepsilon)$ for which $x = t \cdot E_\varepsilon$ and $t = \gamma(A_{\mathbf{1}-\varepsilon} + T)$. Similarly, we take $t' \in H(\mathcal{O}_{\varepsilon'})$ and $T' \in \Pi(\mathcal{O}_{\varepsilon'})$ for $x' \in \mathcal{O}_{\varepsilon'}$ to have $x' = t' \cdot E_{\varepsilon'}$. Put

$$\tilde{t} := \gamma(A_{\mathbf{1}-(\varepsilon+\varepsilon')} + T + T') \in H(\mathcal{O}_{\varepsilon+\varepsilon'}).$$

Then it is clear that $\pi_\varepsilon(\tilde{t}) = t$ and $\pi_{\varepsilon'}(\tilde{t}) = t'$ by (3.5). Thus, by Lemma 3.3 (i),

$$x + x' = t \cdot E_\varepsilon + t' \cdot E_{\varepsilon'} = \tilde{t} \cdot (E_\varepsilon + E_{\varepsilon'}) = \tilde{t} \cdot E_{\varepsilon+\varepsilon'} \in \mathcal{O}_{\varepsilon+\varepsilon'},$$

showing that the image of the map in (3.10) is contained in $\mathcal{O}_{\varepsilon+\varepsilon'}$. Since the map $H(\mathcal{O}_\varepsilon) \times H(\mathcal{O}_{\varepsilon'}) \ni (t, t') \mapsto \tilde{t} \in H(\mathcal{O}_{\varepsilon+\varepsilon'})$ is visibly bijective by (3.5), our map in (3.10) is bijective from the above discussion. □

We define bilinear maps $Q^{[i]} : \mathfrak{n}_i \times \mathfrak{n}_i \to V^{[i]}$ $(i = 1, 2, \ldots, r-1)$ by

$$Q^{[i]}(L, L') := \frac{1}{2}\,[L, [L', E_i]]. \tag{3.11}$$

LEMMA 3.7. *For every $i = 1, 2, \ldots, r-1$, one has $Q^{[i]}(L, L) \in \overline{\Omega}^{[i]}$ for any $L \in \mathfrak{n}_i$, and if $L_i = \sum_{m>i} T_{mi}$, then*

$$Q^{[i]}(L_i, L_i) = \sum_{m>i} \|T_{mi}\|^2 E_m + \sum_{m>k>i} T_{mi} \circ T_{ki}. \tag{3.12}$$

PROOF. By (2.24) we have $\exp L \cdot E_i = E_i + [L, E_i] + Q^{[i]}(L, L)$ $(L \in \mathfrak{n}_i)$. Therefore

$$Q^{[i]}(L, L) = P_{V^{[i]}}(\exp L \cdot E_i) \in P_{V^{[i]}}(\overline{\Omega}) = \overline{\Omega}^{[i]},$$

where we have used Lemma 3.4 to get the last equality. The second assertion follows from this and Proposition 2.5. □

Let $x_{[1]} := P_{V^{[1]}}(x)$ for $x \in V$. By (2.15) we have $x = x_{11}E_1 + \sum_{m=2}^{r} X_{m1} + x_{[1]}$.

PROPOSITION 3.8. *One has*

$$\overline{\Omega} = \overline{\Omega}^{[1]} \cup \left\{ x \in V \,|\, x_{11} > 0,\ \ x_{[1]} - \frac{1}{x_{11}}\, Q^{[1]}\Big(\sum jX_{m1}, \sum jX_{m1}\Big) \in \overline{\Omega}^{[1]} \right\}.$$

PROOF. If $x \in \Omega$, then Proposition 3.6 for $\varepsilon = (1, 0, \ldots, 0)$, $\varepsilon' = [1]$ tells us that there exist unique $t_{11} > 0$, $L_1 \in \mathfrak{n}_1$, and $y \in \Omega^{[1]}$ such that $x = \psi_1(t_{11}) \exp L_1 \cdot E_1 + y$. Using (2.24) and (1.2), we get

$$x = (t_{11})^2 E_1 - t_{11} j L_1 + Q^{[1]}(L_1, L_1) + y. \tag{3.13}$$

Hence $x_{[1]} = Q^{[1]}(L_1, L_1) + y$. On the other hand, comparing (3.13) with (2.15), we obtain $x_{11} = (t_{11})^2$ and $X_{m1} = -t_{11} j T_{m1}$. Therefore

$$L_1 = \sum_{m>1} T_{m1} = (x_{11})^{-1/2} \sum_{m>1} jX_{m1},$$

so that

$$\Omega^{[1]} \ni y = x_{[1]} - Q^{[1]}(L_1, L_1) = x_{[1]} - \frac{1}{x_{11}}\, Q^{[1]}\Big(\sum jX_{m1}, \sum jX_{m1}\Big).$$

Thus, for $a > 0$, we see that the set $\overline{\Omega} \cap \{x \in V \mid x_{11} = a\}$ equals

$$\left\{ aE_1 + \sum X_{m1} + x_{[1]} \in V \,\middle|\, x_{[1]} - \frac{1}{a} Q^{[1]}\Big(\sum jX_{m1}, \sum jX_{m1}\Big) \in \overline{\Omega}^{[1]} \right\}.$$

We also obtain $\overline{\Omega} \cap \{x \in V \mid x_{11} = 0\} = \overline{\Omega}^{[1]}$ as was shown in the proof of Theorem 3.5. Hence the proposition is proved. □

§4. Gamma integrals on H-orbits.

In this section, we shall define and evaluate Γ-type integrals on the H-orbits $\mathcal{O}_\varepsilon$ after introducing functions and measures on $\mathcal{O}_\varepsilon$ relatively invariant under the action of H.

For any $s = (s_1, \ldots, s_r) \in \boldsymbol{C}^r$, let χ_s be the character of H given by

$$\chi_s : \begin{pmatrix} t_{11} & & & \\ T_{21} & t_{22} & & \\ \vdots & & \ddots & \\ T_{r1} & T_{r2} & & t_{rr} \end{pmatrix} \mapsto (t_{11})^{2s_1} (t_{22})^{2s_2} \cdots (t_{rr})^{2s_r}.$$

Then $\chi_{s+s'}(t) = \chi_s(t)\chi_{s'}(t)$. For each $\varepsilon \in \{0,1\}^r$, we set

$$C(\varepsilon) := \{s \in \boldsymbol{C}^r \mid s_i = 0 \text{ for all } i \text{ such that } \varepsilon_i = 0\}. \tag{4.1}$$

When $s \in C(\varepsilon)$, we have $\chi_s(t) = 1$ for all $t \in H_{E_\varepsilon}$ by Proposition 3.2 (iii). Thus for every $s \in C(\varepsilon)$, we can define a function Δ_s^ε on $\mathcal{O}_\varepsilon$ by

$$\Delta_s^\varepsilon(t \cdot E_\varepsilon) := \chi_s(t) \quad (t \in H).$$

Clearly we have

$$\Delta_s^\varepsilon(t \cdot x) = \chi_s(t)\Delta_s^\varepsilon(x) \quad (t \in H,\ x \in \mathcal{O}_\varepsilon). \tag{4.2}$$

In other words, Δ_s^ε is relatively invariant under the action of H.

REMARK. If $s \in \boldsymbol{Z}^r \cap C(\varepsilon)$, one can show that Δ_s^ε is a rational function. We do not give the details here, however.

We put

$$n_{ki} := \dim \mathfrak{h}_{ki} \quad (1 \le i < k \le r), \tag{4.3}$$

and define $p(\varepsilon) = (p_1(\varepsilon), \ldots, p_r(\varepsilon)) \in \boldsymbol{Z}^r$ by

$$p_k(\varepsilon) := \sum_{i<k} \varepsilon_i n_{ki} = \sum_{i<k,\, \varepsilon_i = 1} n_{ki} \quad (1 \le k \le r). \tag{4.4}$$

For $s \in \boldsymbol{C}^r$ and $\varepsilon \in \{0,1\}^r$, let

$$\varepsilon \cdot s := (\varepsilon_1 s_1, \varepsilon_2 s_2, \ldots, \varepsilon_r s_r). \tag{4.5}$$

It is clear that $\varepsilon \cdot s \in C(\varepsilon)$.

When $\varepsilon \neq \mathbf{0} := (0, \ldots, 0)$, we make use of $\Pi(\mathcal{O}_\varepsilon)$ in (3.9) as coordinates for $\mathcal{O}_\varepsilon$ and define a measure μ_ε on $\mathcal{O}_\varepsilon$ by

$$d\mu_\varepsilon(t \cdot E_\varepsilon) = \chi_{-\varepsilon\cdot(\mathbf{1}+p(\varepsilon))/2}(t) \prod_{\varepsilon_i=1} dt_{ii} \prod_{\varepsilon_i=1,\, k>i} dT_{ki}, \tag{4.6}$$

where $t = \gamma(A_{\mathbf{1}-\varepsilon} + T) \in H(\mathcal{O}_\varepsilon)$ $(T \in \Pi(\mathcal{O}_\varepsilon))$ and dT_{ki} stands for the Euclidean measure on $\mathfrak{h}_{ki}$ normalized by the standard inner product on $\mathfrak{g}$. Let $\mu_{\mathbf{0}}$ be the Dirac measure at $x = 0$. Recalling the $H(\mathcal{O}_\varepsilon)$-equivariant diffeomorphism $\Psi_\varepsilon : H(\mathcal{O}_\varepsilon) \to \mathcal{O}_\varepsilon$ in Lemma 3.3, we see from the following proposition that μ_ε is the transfer of the left Haar measure on $H(\mathcal{O}_\varepsilon)$ by Ψ_ε.

PROPOSITION 4.1. *The measure μ_ε is relatively invariant under H:*

$$d\mu_\varepsilon(t_0 \cdot x) = \chi_{(\mathbf{1}-\varepsilon)\cdot p(\varepsilon)/2}(t_0)\, d\mu_\varepsilon(x) \quad (t_0 \in H). \tag{4.7}$$

In particular, μ_ε is $H(\mathcal{O}_\varepsilon)$-invariant.

PROOF. The case $\varepsilon = \mathbf{0}$ is trivial. Assume that $\varepsilon \neq \mathbf{0}$. Given $t_0 \in H$, we take $T^0 \in \Pi$ for which $t_0 = \gamma(T^0)$. Consider the map $H(\mathcal{O}_\varepsilon) \ni t \mapsto t' := \pi_\varepsilon(t_0 t) \in H(\mathcal{O}_\varepsilon)$. In view of (2.5) and (2.6), differentiation of this map gives

$$dt'_{ii} = t^0_{ii}\, dt_{ii},$$

$$dT'_{ki} = t^0_{kk}\, dT_{ki} + (\text{terms including } dt_{ii} \text{ or } dT_{li} \ (i < l < k)).$$

Considering the lexicographic order $\succeq$ introduced in Section 2, we obtain

$$\begin{aligned} \prod_{\varepsilon_i=1} dt'_{ii} \prod_{\varepsilon_i=1,k>i} dT'_{ki} &= \prod_{\varepsilon_i=1} t^0_{ii}\, dt_{ii} \prod_{\varepsilon_i=1,k>i} (t^0_{kk})^{n_{ki}}\, dT_{ki} \\ &= \chi_{(\varepsilon+p(\varepsilon))/2}(t_0) \prod_{\varepsilon_i=1} dt_{ii} \prod_{\varepsilon_i=1,k>i} dT_{ki}. \end{aligned}$$

Now, if $x = t \cdot E_\varepsilon$, then $t_0 \cdot x = t' \cdot E_\varepsilon$ by Lemma 3.3 (i). Therefore

$$\begin{aligned} d\mu_\varepsilon(t_0 \cdot x) &= \chi_{-\varepsilon\cdot(\mathbf{1}+p(\varepsilon))/2}(t') \prod dt'_{ii} \prod dT'_{ki} \\ &= \chi_{-\varepsilon\cdot(\mathbf{1}+p(\varepsilon))/2}(t_0 t)\chi_{(\varepsilon+p(\varepsilon))/2}(t_0) \prod dt_{ii} \prod dT_{ki} \\ &= \chi_{(\mathbf{1}-\varepsilon)\cdot p(\varepsilon)/2}(t_0)\, d\mu_\varepsilon(x). \end{aligned}$$

Hence (4.7) is proved. For the last assertion, note $(\mathbf{1} - \varepsilon) \cdot p(\varepsilon)/2 \in C(\mathbf{1} - \varepsilon)$ and $H(\mathcal{O}_\varepsilon) = H_{E_{\mathbf{1}-\varepsilon}}$. □

We now study the integral

$$\Gamma_{\mathcal{O}_\varepsilon}(s) := \int_{\mathcal{O}_\varepsilon} e^{-\langle x, E^* \rangle} \Delta_s^\varepsilon(x)\, d\mu_\varepsilon(x) \quad (s \in C(\varepsilon)). \tag{4.8}$$

THEOREM 4.2. *The integral* (4.8) *converges if and only if $s \in C(\varepsilon)$ satisfies the following condition*:

$$\Re s_i > p_i(\varepsilon)/2 \quad \textit{for all } i \textit{ such that } \varepsilon_i = 1. \tag{4.9}$$

Moreover, when this condition is satisfied, one has

$$\Gamma_{\mathcal{O}_\varepsilon}(s) = 2^{-|\varepsilon|}\pi^{|p(\varepsilon)|/2}\prod_{\varepsilon_i=1}\Gamma\left(s_i - \frac{p_i(\varepsilon)}{2}\right), \tag{4.10}$$

where $|\varepsilon| := \sum_{i=1}^r \varepsilon_i$ *and* $|p(\varepsilon)| := \sum_{i=1}^r p_i(\varepsilon)$.

PROOF. If $\varepsilon = \mathbf{0}$, the integral (4.8) reduces to 1. Thus (4.9) and (4.10) hold trivially. Assume now that $\varepsilon \neq \mathbf{0}$. By (2.22), we have for $T \in \Pi(\mathcal{O}_\varepsilon)$

$$\langle \overline{\gamma}(T)\cdot E, E^*\rangle := \sum_{\varepsilon_i=1}(t_{ii})^2 + \sum_{\varepsilon_i=1,\, k>i}\|T_{ki}\|^2 = \sum_{\varepsilon_i=1}\{(t_{ii})^2 + \|L_i\|^2\}.$$

Since (2.22) and (2.23) lead us to

$$\overline{\gamma}(T)\cdot E = \overline{\gamma}(T)\cdot E_\varepsilon = \gamma(A_{\mathbf{1}-\varepsilon} + T)\cdot E_\varepsilon, \tag{4.11}$$

we get from (4.6) by setting $t = \gamma(A_{\mathbf{1}-\varepsilon} + T)$ and $dL_i = \prod_{k>i} dT_{ki}$,

$$\begin{aligned}\Gamma_{\mathcal{O}_\varepsilon}(s) &= \int_{\Pi(\mathcal{O}_\varepsilon)} \exp\left(-\sum_{\varepsilon_i=1}\{(t_{ii})^2 + \|L_i\|^2\}\right)\chi_{s-\varepsilon\cdot(\mathbf{1}+p(\varepsilon))/2}(t)\prod_{\varepsilon_i=1} dt_{ii}dL_i \\ &= \prod_{\varepsilon_i=1}\int_0^\infty e^{-(t_{ii})^2}(t_{ii})^{2s_i-1-p_i(\varepsilon)}\,dt_{ii}\prod_{\varepsilon_i=1}\int_{\mathfrak{n}_i} e^{-\|L_i\|^2}dL_i \\ &= \prod_{\varepsilon_i=1}\left(\frac{1}{2}\int_0^\infty e^{-u}u^{s_i-1-p_i(\varepsilon)/2}\,du\right)\prod_{\varepsilon_i=1}\pi^{(\dim \mathfrak{n}_i)/2}.\end{aligned}$$

Therefore the convergence argument is reduced to the one for the ordinary gamma functions. Since (2.14), (4.3) and (4.4) imply

$$\sum_{\varepsilon_i=1}\dim \mathfrak{n}_i = \sum_{\varepsilon_i=1,\, k>i} n_{ki} = \sum_{k=1}^r p_k(\varepsilon) = |p(\varepsilon)|,$$

we obtain (4.10). □

We introduce the following subsets in $\boldsymbol{C}^r$:

$$D(\varepsilon) := \{s \in \boldsymbol{C}^r \mid s_i = p_i(\varepsilon)/2 \text{ for all } i \text{ such that } \varepsilon_i = 0\}, \tag{4.12}$$

$$\Xi_{\boldsymbol{C}}(\varepsilon) := \{s \in D(\varepsilon) \mid \Re s_i > p_i(\varepsilon)/2 \text{ for all } i \text{ such that } \varepsilon_i = 1\}. \tag{4.13}$$

Clearly we have

$$D(\varepsilon) = (\mathbf{1} - \varepsilon)\cdot p(\varepsilon)/2 + C(\varepsilon).$$

For every $s \in \boldsymbol{C}^r$, let

$$\tilde{s} := s - (\mathbf{1} - \varepsilon)\cdot p(\varepsilon)/2. \tag{4.14}$$

If $s \in D(\varepsilon)$, then we have $\tilde{s} \in C(\varepsilon)$, so that $\Delta_{\tilde{s}}^\varepsilon$ is defined. Proposition 4.1 and (4.2) tell us that if $t \in H$ and $s \in D(\varepsilon)$, then

$$\Delta_{\tilde{s}}^\varepsilon(t\cdot x)\,d\mu_\varepsilon(t\cdot x) = \chi_s(t)\Delta_{\tilde{s}}^\varepsilon(x)\,d\mu_\varepsilon(x). \tag{4.15}$$

PROPOSITION 4.3. *Let* $s \in \Xi_{\boldsymbol{C}}(\varepsilon)$ *and* $t \in H$. *Then* $\tilde{s}$ *satisfies* (4.9) *and*

$$\int_{\mathcal{O}_\varepsilon} e^{-\langle x, t^* \cdot E^* \rangle} \Delta_{\tilde{s}}^{\varepsilon}(x)\, d\mu_\varepsilon(x) = \Gamma_{\mathcal{O}_\varepsilon}(\tilde{s}) \chi_{-s}(t).$$

PROOF. Observe that $\Re \tilde{s}_i = \Re s_i$ provided $\varepsilon_i = 1$. Hence $\tilde{s}$ satisfies (4.9), and the proposition follows from (4.8) and (4.15). □

§5. Riesz distributions.

We begin this section by considering the map $\Theta(T) := \overline{\gamma}(T) \cdot E$ from $\overline{\Pi}$ to V. By Proposition 2.5 with $\varepsilon = \mathbf{1}$, we see that Θ is a polynomial map. Therefore we consider from now on that the definition domain of Θ is the whole space $\mathfrak{h}$. Thus, if $x = \Theta(T)$ $(T \in \mathfrak{h})$ with the expressions (2.15) and (2.1), we have

$$x_{kk} = (t_{kk})^2 + \sum_{i<k} \|T_{ki}\|^2, \quad X_{mk} = t_{kk}[T_{mk}, E_k] + \sum_{i<k} T_{mi} \circ T_{ki}. \tag{5.1}$$

For $c = (c_1, \ldots, c_r) \in \{-1, 1\}^r$, let g_c be the linear automorphism on $\mathfrak{h}$ defined by

$$g_c\Big(\sum t_{kk} A_k + \sum T_{mk}\Big) := \sum c_k t_{kk} A_k + \sum c_k T_{mk}.$$

Then by (5.1), Θ is g_c-invariant:

$$\Theta(g_c T) = \Theta(T) \quad (T \in \mathfrak{h}). \tag{5.2}$$

Since $\Theta(\overline{\Pi}) = \overline{\Omega}$ by Proposition 3.1, the equality (5.2) tells us that $\Theta(\mathfrak{h}) = \overline{\Omega}$. For every $\varepsilon \in \{0, 1\}^r$, we set

$$R(\varepsilon) := C(\varepsilon) \cap \boldsymbol{R}^r = \{s \in \boldsymbol{R}^r \mid s_i = 0 \text{ for all } i \text{ such that } \varepsilon_i = 0\}, \tag{5.3}$$

$$R_+(\varepsilon) := \{s \in R(\varepsilon) \mid s_i > 0 \text{ for all } i \text{ such that } \varepsilon_i = 1\}. \tag{5.4}$$

THEOREM 5.1. (i) *Let* $s \in \Xi_{\boldsymbol{C}}(\varepsilon)$ *and* $\tilde{s}$ *be as in* (4.14). *Then, for* $\varphi \in \mathcal{S}(V)$, *the following integral converges*:

$$\langle \mathscr{R}_s^{\varepsilon}, \varphi \rangle := \frac{1}{\Gamma_{\mathcal{O}_\varepsilon}(\tilde{s})} \int_{\mathcal{O}_\varepsilon} \varphi(x) \Delta_{\tilde{s}}^{\varepsilon}(x)\, d\mu_\varepsilon(x). \tag{5.5}$$

(ii) $\langle \mathscr{R}_s^{\varepsilon}, \varphi \rangle$ *admits an analytic continuation as a holomorphic function of* $s \in D(\varepsilon)$, *and defines a tempered distribution.*

PROOF. (i) Put $x = \Theta(T)$ in the integral (5.5). We have (5.1) and, proceeding as in the proof of Theorem 4.2, we get

$$\langle \mathscr{R}_s^{\varepsilon}, \varphi \rangle = \frac{1}{\Gamma_{\mathcal{O}_\varepsilon}(\tilde{s})} \int_{\Pi(\mathcal{O}_\varepsilon)} \varphi(\Theta(T)) \prod_{\varepsilon_i = 1} dL_i \prod_{\varepsilon_i = 1} (t_{ii})^{2s_i - p_i(\varepsilon) - 1}\, dt_{ii}. \tag{5.6}$$

Here putting $\mathfrak{n}_\varepsilon := \bigoplus_{\varepsilon_i = 1} \mathfrak{n}_i$ for simplicity, we define $\tilde{I}_\varepsilon \varphi \in C^\infty(R(\varepsilon))$ and $I_\varepsilon \varphi \in C^\infty(R_+(\varepsilon))$ by

$$\tilde{I}_\varepsilon\varphi(t_{11},\dots,t_{rr}) := \int_{\mathfrak{n}_\varepsilon} \varphi\left(\Theta\left(\sum_{\varepsilon_i=1}(t_{ii}A_i + L_i)\right)\right)\prod_{\varepsilon_i=1} dL_i, \tag{5.7}$$

$$I_\varepsilon\varphi(u_1,\dots,u_r) := 2^{-|\varepsilon|}\tilde{I}_\varepsilon\varphi((u_1)^{1/2},\dots,(u_r)^{1/2}). \tag{5.8}$$

Since $|t_{kk}|^2$, $\|T_{ki}\|^2 \le x_{kk}$ by (5.1), the integral in (5.7) converges and $\tilde{I}_\varepsilon\varphi \in \mathscr{S}(V)$. By (5.2), we see that $\tilde{I}_\varepsilon\varphi$ is an even function in each variable. Thus, all the derivatives of $I_\varepsilon\varphi$ can be extended to $\overline{R_+(\varepsilon)}$ as continuous functions. By (5.6), (5.7) and (5.8), we have

$$\langle \mathscr{R}_s^\varepsilon, \varphi\rangle = \frac{1}{\Gamma_{\mathscr{O}_\varepsilon}(\tilde{s})}\int_{R_+(\varepsilon)} I_\varepsilon\varphi(u_1,\dots,u_r)\prod_{\varepsilon_i=1}(u_i)^{s_i-1-p_i(\varepsilon)/2}\,du_i. \tag{5.9}$$

If $s \in \Xi_{\boldsymbol{C}}(\varepsilon)$, then the last integral converges, which proves (i).

(ii) Take $\beta = (\beta_1,\dots,\beta_r) \in \boldsymbol{Z}^r \cap R_+(\varepsilon)$. Using the obvious identity

$$(u_i)^{s_i-1-p_i(\varepsilon)/2} = \left\{\prod_{l=0}^{\beta_i-1}\left(s_i + l - \frac{p_i(\varepsilon)}{2}\right)^{-1}\right\}\left(\frac{\partial}{\partial u_i}\right)^{\beta_i}(u_i)^{s_i+\beta_i-1-p_i(\varepsilon)/2},$$

we have by (4.10) and (5.9),

$$\langle \mathscr{R}_s^\varepsilon, \varphi\rangle = \frac{2^{|\varepsilon|}\pi^{-|p(\varepsilon)|/2}}{\prod_{\varepsilon_i=1}\Gamma(s_i+\beta_i-p_i(\varepsilon)/2)} \times$$

$$\times \int_{R_+(\varepsilon)}\prod_{\varepsilon_i=1}\left(-\frac{\partial}{\partial u_i}\right)^{\beta_i}[I_\varepsilon\varphi](u_1,\dots,u_r)\prod_{\varepsilon_i=1}(u_i)^{s_i+\beta_i-1-p_i(\varepsilon)/2}\,du_i.$$

The right-hand side is holomorphic for $s \in -\beta + \Xi_{\boldsymbol{C}}(\varepsilon)$. Hence $\langle \mathscr{R}_s^\varepsilon, \varphi\rangle$ can be continued analytically to $-\beta + \Xi_{\boldsymbol{C}}(\varepsilon)$. Since $\beta \in \boldsymbol{Z}^r \cap R_+(\varepsilon)$ is arbitrary, the assertion (ii) holds. □

We simply write $\mathscr{R}_s$ for $\mathscr{R}_s^{\mathbf{1}}$, and call it *the Riesz distribution on* Ω. We note that $\tilde{s} = s$ if $\varepsilon = \mathbf{1}$ and that since $D(\mathbf{1}) = \boldsymbol{C}^r$, $\mathscr{R}_s$ is defined for all $s \in \boldsymbol{C}^r$.

THEOREM 5.2. (i) *Let* $s \in D(\varepsilon)$ *and* $\xi \in \Omega^*$. *Then the Laplace transform of* $\mathscr{R}_s^\varepsilon$ *is given as* $\langle \mathscr{R}_s^\varepsilon, e^{-\langle x,\xi\rangle}\rangle_x = \chi_{-s}(t)$, *where* $t \in H$ *is taken so that* $\xi = t^*\cdot E^*$.

(ii) *The distribution* $\mathscr{R}_s^\varepsilon$ *coincides with* $\mathscr{R}_s$ *for any* $\varepsilon \in \{0,1\}^r$ *and* $s \in D(\varepsilon)$. *In particular,* $\operatorname{supp}\mathscr{R}_s \subset \overline{\mathscr{O}_\varepsilon}$ *if* $s \in D(\varepsilon)$.

(iii) *For* $s, s' \in \boldsymbol{C}^r$, *one has* $\mathscr{R}_s * \mathscr{R}_{s'} = \mathscr{R}_{s+s'}$.

PROOF. (i) If $s \in \Xi_{\boldsymbol{C}}(\varepsilon)$, the claim follows from Proposition 4.3. By analytic continuation, we see that the assertion holds for any $s \in D(\varepsilon)$.

(ii) The assertion is clear from (i) and the injectivity of the Laplace transform.

(iii) The assertion also follows from the injectivity of the Laplace transform. □

We obtain the following proposition from (4.15) by analytic continuation.

PROPOSITION 5.3. *For* $s \in \boldsymbol{C}^r$, $\varphi \in \mathscr{S}(V)$ *and* $t \in H$, *one has*

$$\langle \mathscr{R}_s, \varphi(t^{-1}\cdot x)\rangle_x = \chi_s(t)\langle \mathscr{R}_s, \varphi(x)\rangle_x.$$

We conclude this section by deriving some formulas which will be used in the next section. Fix k such that $1 \le k \le r-1$. For $s = (0, \dots, 0, s_{k+1}, \dots, s_r) \in D([k])$, put $\sigma := (s_{k+1}, \dots, s_r) \in \boldsymbol{C}^{r-k}$. Since $\mathcal{O}_{[k]} = \Omega^{[k]} \subset V^{[k]}$, the definition (5.5) of $\mathcal{R}_s^{[k]}$ $(= \mathcal{R}_s)$ tells us that

$$\langle \mathcal{R}_s, \varphi \rangle = \langle \mathcal{R}(\Omega^{[k]})_\sigma, \varphi|_{V^{[k]}} \rangle \quad (\varphi \in \mathcal{S}(V)), \tag{5.10}$$

where $\mathcal{R}(\Omega^{[k]})_\sigma$ stands for the Riesz distribution on $\Omega^{[k]}$. Setting

$$\delta^k = (0, \dots, 0, \overset{(k)}{1}, 0, \dots, 0) \in \{0,1\}^r, \tag{5.11}$$

we put

$$M^k(z) := z \cdot \delta^k + p(\delta^k)/2 \quad (z \in \boldsymbol{C}). \tag{5.12}$$

Then it is easy to check that $M^k(z) \in D(\delta^k)$ (see (4.12)). Since

$$p(\delta^k) = (0, \dots, 0, n_{k+1,k}, \dots, n_{rk}), \tag{5.13}$$

we see that $|p(\delta^k)| = \sum_{m>k} n_{mk} = \dim \mathfrak{n}_k$ by (2.14). Assume that $\Re z > 0$. Then $M^k(z) \in \Xi_{\boldsymbol{C}}(\delta^k)$, and we have by (5.6) and (4.10)

$$\langle \mathcal{R}_{M^k(z)}, \varphi \rangle = \frac{2\pi^{-(\dim \mathfrak{n}_k)/2}}{\Gamma(z)} \int_{\Pi_k} \varphi(\Theta(T))\, dL_k (t_{kk})^{2z-1}\, dt_{kk}, \tag{5.14}$$

where $\Pi_k := \Pi(\mathcal{O}_{\delta^k})$ for simplicity. Now if $T \in \Pi_k$, we have by (4.11)

$$\Theta(T) = \overline{\gamma}(T) \cdot E = \gamma(A_{\mathbf{1}-\delta^k} + t_{kk} A_k + L_k) \cdot E_k = \psi_k(t_{kk}) \exp L_k \cdot E_k.$$

Thus by (2.24), (1.2) and (3.11), we get

$$\Theta(T) = (t_{kk})^2 E_k - t_{kk} j L_k + Q^{[k]}(L_k, L_k).$$

Then, changing the variable $t_{kk} = \sqrt{u}$, we see that the right-hand side of (5.14) is

$$\frac{\pi^{-(\dim \mathfrak{n}_k)/2}}{\Gamma(z)} \int_0^\infty \int_{\mathfrak{n}_k} \varphi(uE_k - \sqrt{u} j L_k + Q^{[k]}(L_k, L_k))\, dL_k\, u^{z-1}\, du.$$

Let ρ_z denote the Riesz distribution on $(0, +\infty)$ given by

$$\langle \rho_z, \psi \rangle := \frac{1}{\Gamma(z)} \int_0^{+\infty} \psi(u) u^{z-1}\, du \quad (\psi \in \mathcal{S}(\boldsymbol{R})). \tag{5.15}$$

Then $\langle \mathcal{R}_{M^k(z)}, \varphi \rangle$ is rewritten as

$$\pi^{-(\dim \mathfrak{n}_k)/2} \left\langle \rho_z, \int_{\mathfrak{n}_k} \varphi(uE_k - \sqrt{u} j L_k + Q^{[k]}(L_k, L_k))\, dL_k \right\rangle_u. \tag{5.16}$$

By analytic continuation, we see that this expression for $\langle \mathcal{R}_{M^k(z)}, \varphi \rangle$ is valid for any $z \in \boldsymbol{C}$. Since ρ_0 is the Dirac measure at $u = 0$, we get by putting $z = 0$,

$$\langle \mathcal{R}_{p(\delta^k)/2}, \varphi \rangle = \pi^{-(\dim \mathfrak{n}_k)/2} \int_{\mathfrak{n}_k} \varphi(Q^{[k]}(L_k, L_k))\, dL_k. \tag{5.17}$$

Combining (5.17) with (5.10), we obtain for $\phi \in \mathscr{S}(V^{[k]})$,

$$\int_{\mathfrak{n}_k} \phi(Q^{[k]}(L_k, L_k))\, dL_k = \pi^{(\dim \mathfrak{n}_k)/2} \langle \mathscr{R}(\Omega^{[k]})_{(n_{k+1,k}/2, \ldots, n_{rk}/2)}, \phi \rangle. \tag{5.18}$$

§6. Gindikin–Wallach sets.

We now investigate the positivity set (Gindikin–Wallach set) of the parameter s of the Riesz distribution $\mathscr{R}_s$ and describe it in connection with the H-orbit structure in $\overline{\Omega}$.

Let us set

$$\Xi(\varepsilon) := \Xi_{\boldsymbol{C}}(\varepsilon) \cap \boldsymbol{R}^r. \tag{6.1}$$

By (4.13) and (5.4) it is evident that

$$\Xi(\varepsilon) = \frac{1}{2} p(\varepsilon) + R_+(\varepsilon). \tag{6.2}$$

For $s \in \Xi(\varepsilon)$ and $1 \leq k \leq r$, let

$$s^{(k)} := s - \frac{1}{2} \sum_{i=1}^{k-1} \varepsilon_i p(\delta^i). \tag{6.3}$$

We note here that (4.4) and (5.13) give

$$p(\varepsilon) = \sum_{i=1}^{r-1} \varepsilon_i p(\delta^i). \tag{6.4}$$

Since $p_k(\delta^i) = 0$ for $i \geq k$, we have $s_k^{(k)} = s_k^{(r)}$. Moreover (6.2), (6.3) and (6.4) imply $s^{(r)} \in R_+(\varepsilon)$, so that

$$s_k^{(k)} > 0 \quad (\text{if } \varepsilon_k = 1), \quad s_k^{(k)} = 0 \quad (\text{if } \varepsilon_k = 0).$$

This together with (6.3) gives us

$$s^{(k)} = \begin{cases} s^{(k-1)} - p(\delta^{k-1})/2 & (\text{if } s_{k-1}^{(k-1)} > 0), \\ s^{(k-1)} & (\text{if } s_{k-1}^{(k-1)} = 0). \end{cases} \tag{6.5}$$

PROPOSITION 6.1. *For $s \in \boldsymbol{R}^r$, define $\sigma^{(1)}, \ldots, \sigma^{(r)} \in \boldsymbol{R}^r$ inductively by $\sigma^{(1)} := s$ and, for $2 \leq k \leq r$,*

$$\sigma^{(k)} := \begin{cases} \sigma^{(k-1)} - p(\delta^{k-1})/2 & (\textit{if } \sigma_{k-1}^{(k-1)} > 0), \\ \sigma^{(k-1)} & (\textit{if } \sigma_{k-1}^{(k-1)} \leq 0). \end{cases} \tag{6.6}$$

Then $s \in \Xi(\varepsilon)$ if and only if $\sigma^{(r)} \in R_+(\varepsilon)$.

PROOF. If $s \in \Xi(\varepsilon)$, then the argument preceding Proposition 6.1 says that $\sigma^{(r)} = s^{(r)} \in R_+(\varepsilon)$. Suppose conversely that $\sigma^{(r)} \in R_+(\varepsilon)$. If $m \geq k-1$, we have $p_{k-1}(\delta^m) = 0$, so that $\sigma_{k-1}^{(m)} = \sigma_{k-1}^{(m+1)}$ by (6.6). Thus $\sigma_{k-1}^{(k-1)} = \sigma_{k-1}^{(r)}$, and (6.6) is rewritten as

$$\sigma^{(k-1)} = \frac{1}{2} \varepsilon_{k-1} p(\delta^{k-1}) + \sigma^{(k)}.$$

Therefore by (5.13) and (6.2) we arrive at

$$s = \sigma^{(1)} = \frac{1}{2}\sum_{k=1}^{r-1} \varepsilon_k p(\delta^k) + \sigma^{(r)} \in \frac{1}{2} p(\varepsilon) + R_+(\varepsilon) = \Xi(\varepsilon),$$

completing the proof of Proposition 6.1. □

Let $\varepsilon, \varepsilon' \in \{0,1\}^r$. If $\varepsilon \neq \varepsilon'$, then clearly $R_+(\varepsilon) \cap R_+(\varepsilon') = \varnothing$, so that we have $\Xi(\varepsilon) \cap \Xi(\varepsilon') = \varnothing$ by Proposition 6.1. Set

$$\Xi := \bigsqcup_{\varepsilon \in \{0,1\}^r} \Xi(\varepsilon).$$

Based on Theorem 6.2 which we are going to prove, we shall call Ξ *the Gindikin–Wallach set* for the Riesz distributions $\mathscr{R}_s$. We note here that Proposition 6.1 supplies us a simple algorithm for determining whether $s \in \Xi$ or not for a given $s \in \boldsymbol{R}^r$. Moreover, if $s \in \Xi$, the algorithm tells us the $\varepsilon \in \{0,1\}^r$ for which $s \in \Xi(\varepsilon)$. In fact, given $s \in \boldsymbol{R}^r$, we compute $\sigma^{(k)}$ $(k = 1, \ldots, r)$ by the formula (6.6). If once $\sigma_k^{(k)} < 0$ for some k, we then conclude that $s \notin \Xi$. If $\sigma_k^{(k)} \geq 0$ for all $k = 1, 2, \ldots, r$, then putting

$$\varepsilon_k := 1 \quad (\text{if } \sigma_k^{(k)} > 0), \quad \varepsilon_k := 0 \quad (\text{if } \sigma_k^{(k)} = 0),$$

we have $s \in \Xi(\varepsilon)$.

Before stating Theorem 6.2, we note that if $s \in \Xi(\varepsilon)$, then we have $\tilde{s} = \varepsilon \cdot s$ by (4.5), (4.12), (4.13), (4.14) and (6.1).

THEOREM 6.2. *The Riesz distribution $\mathscr{R}_s$ is positive if and only if $s \in \Xi$. Moreover, if $s \in \Xi(\varepsilon)$, then $\mathscr{R}_s$ is a measure on the H-orbit $\mathcal{O}_\varepsilon$ and one has*

$$d\mathscr{R}_s = \Gamma_{\mathcal{O}_\varepsilon}(\varepsilon \cdot s)^{-1} \Delta_{\varepsilon \cdot s}^{\varepsilon}\, d\mu_\varepsilon. \tag{6.7}$$

PROOF. If $s \in \Xi(\varepsilon)$, then Theorems 5.1 and 5.2 say that $\mathscr{R}_s$ is a positive measure given by (6.7) on the H-orbit $\mathcal{O}_\varepsilon$. To show the converse assertion, we assume that $\mathscr{R}_s$ is positive. Since $e^{-\langle x, t^* \cdot E^* \rangle} > 0$ for $x \in \overline{\Omega}$ and $t \in H$, we have $\chi_{-s}(t) = \langle \mathscr{R}_s, e^{-\langle x, t^* \cdot E^* \rangle} \rangle \geq 0$ by Theorem 5.2. Hence s is necessarily real. We shall prove $s \in \Xi$ by induction on the rank r of the cone Ω.

Let us consider first the case $r = 1$. In this case $\mathscr{R}_s$ coincides with ρ_s given by (5.15). Take $\phi_\nu \in \mathscr{S}(\boldsymbol{R})$ $(\nu = 1, 2, \ldots)$ such that $\phi_\nu(x) = (x + (1/\nu))e^{-x}$ $(x \geq 0)$ and $\phi_\nu \geq 0$. Then $0 \leq \langle \rho_s, \phi_\nu \rangle = s + (1/\nu)$. Letting $\nu \to +\infty$, we get $s \geq 0$, so that the claim holds in this case. Assume next that the claim holds for cones of rank $r - 1$. In particular, the claim holds for the cone $\Omega^{[1]} \subset V^{[1]}$. For $\sigma = (\sigma_2, \ldots, \sigma_r) \in \boldsymbol{C}^{r-1}$, let $\check{\mathscr{R}}_\sigma$ be the Riesz distribution $\mathscr{R}(\Omega^{[1]})_\sigma$ on $\Omega^{[1]}$. Let $\check{\Xi}$ be the Gindikin–Wallach set for $\check{\mathscr{R}}_\sigma$. We have $\check{\Xi} = \bigsqcup_{\check{\varepsilon} \in \{0,1\}^{r-1}} \check{\Xi}(\check{\varepsilon})$ with $\check{\Xi}(\check{\varepsilon})$ $(\check{\varepsilon} = (\varepsilon_2, \ldots, \varepsilon_r))$ having a description similar to (6.2): $\check{\Xi}(\check{\varepsilon}) = \check{p}(\check{\varepsilon})/2 + \check{R}_+(\check{\varepsilon})$ for the obvious definition of $\check{R}_+(\check{\varepsilon})$ (see (5.4)) and $\check{p}(\check{\varepsilon}) = (\check{p}_2(\check{\varepsilon}), \ldots, \check{p}_r(\check{\varepsilon}))$ with

$$\check{p}_k(\check{\varepsilon}) := \sum_{2 \leq i < k} \varepsilon_i n_{ki}. \tag{6.8}$$

Put $\check{s} := (s_2, \dots, s_r) \in \boldsymbol{R}^{r-1}$, $\check{n} := (n_{21}, \dots, n_{r1})$ and

$$M(s_1) := M^1(s_1) = (s_1, n_{21}/2, \dots, n_{r1}/2) \in D(\delta^1),$$

see (5.12). Then $s - M(s_1) = (0, \check{s} - \check{n}/2) \in D([1])$. Since $\mathscr{R}_s = \mathscr{R}_{M(s_1)} * \mathscr{R}_{s-M(s_1)}$ by Theorem 5.2 (iii), we have, thanks to (5.16) and (5.10),

$$\begin{aligned} \langle \mathscr{R}_s, \varphi \rangle &= \pi^{-(\dim \mathfrak{n}_1)/2} \times \\ &\times \left\langle \rho_{s_1}, \int_{\mathfrak{n}_1} \langle \check{\mathscr{R}}_{\check{s}-\check{n}/2}, \varphi(uE_1 - \sqrt{u} jL_1 + Q^{[1]}(L_1, L_1) + y) \rangle_y \, dL_1 \right\rangle_u \end{aligned} \tag{6.9}$$

for $\varphi \in \mathscr{S}(V)$. We consider the functions of the form

$$\varphi(x) := \varphi_1(x_{11})\varphi_2(x_{[1]}) \quad \left(x = x_{11}E_1 + \sum_{m>1} X_{m1} + x_{[1]} \in V \right)$$

with $\varphi_1 \in C_c^\infty(\boldsymbol{R})$, $\varphi_2 \in C_c^\infty(V^{[1]})$. By virtue of Proposition 3.8, we see that $\operatorname{supp} \varphi \cap \overline{\Omega}$ is compact, so that $\mathscr{R}_s$ can be applied to φ. Now by (6.9), we have

$$\begin{aligned} \langle \mathscr{R}_s, \varphi \rangle &= \pi^{-(\dim \mathfrak{n}_1)/2} \langle \rho_{s_1}, \varphi_1 \rangle \int_{\mathfrak{n}_1} \langle \check{\mathscr{R}}_{\check{s}-\check{n}/2}, \varphi_2(Q^{[1]}(L_1, L_1) + y) \rangle_y \, dL_1 \\ &= \langle \rho_{s_1}, \varphi_1 \rangle \langle \check{\mathscr{R}}_{\check{n}/2} * \check{\mathscr{R}}_{\check{s}-\check{n}/2}, \varphi_2 \rangle = \langle \rho_{s_1}, \varphi_1 \rangle \langle \check{\mathscr{R}}_{\check{s}}, \varphi_2 \rangle, \end{aligned} \tag{6.10}$$

where we have used (5.18) for the second equality. Take a positive integer N such that $s_1 + N > 0$. Since $\langle \rho_{s_1}, (u+N)e^{-u} \rangle_u = s_1 + N > 0$, a suitable choice of non-negative $\varphi_1 \in C_c^\infty(\boldsymbol{R})$ approximating $(u+N)e^{-u}$ on $[0, +\infty)$ yields $\langle \rho_{s_1}, \varphi_1 \rangle > 0$. If $\varphi_2 \geq 0$, the positivity of $\mathscr{R}_s$ tells us $\langle \check{\mathscr{R}}_{\check{s}}, \varphi_2 \rangle = \langle \rho_{s_1}, \varphi_1 \rangle^{-1} \langle \mathscr{R}_s, \varphi \rangle \geq 0$ by (6.10). Thus $\check{\mathscr{R}}_{\check{s}}$ is positive and the induction hypothesis ensures that $\check{s} \in \check{\Xi}(\check{\varepsilon})$ for some $\check{\varepsilon} = (\varepsilon_2, \dots, \varepsilon_r) \in \{0, 1\}^{r-1}$. We next fix a non-negative φ_2 such that $\langle \check{\mathscr{R}}_{\check{s}}, \varphi_2 \rangle$ is strictly positive. Then using (6.10) again, we get $\langle \rho_{s_1}, \varphi_1 \rangle \geq 0$ for any $\varphi_1 \geq 0$ in turn. Therefore ρ_{s_1} is positive, so that $s_1 \geq 0$. If $s_1 = 0$, then putting $\varepsilon := (0, \varepsilon_2, \dots, \varepsilon_r)$, we get $s \in \Xi(\varepsilon)$.

It now remains the case $s_1 > 0$. Before proceeding we remark that the map $(0, +\infty) \times \mathfrak{n}_1 \times V^{[1]} \to \{x \in V \mid x_{11} > 0\}$ given by

$$(u, L_1, y) \mapsto uE_1 - \sqrt{u} jL_1 + Q^{[1]}(L_1, L_1) + y$$

is a diffeomorphism. In fact, as is suggested by the proof of Proposition 3.8 and can be checked directly, the inverse map is given by

$$x \mapsto \left(x_{11}, (x_{11})^{-1/2} \sum jX_{m1}, \, x_{[1]} - \frac{1}{x_{11}} Q^{[1]}\Big(\sum jX_{m1}, \sum jX_{m1} \Big) \right).$$

Now let us consider the functions $\varphi \in C_c^\infty(V)$ given by

$$\varphi(x) := \begin{cases} \phi_1(u)\phi_2(L_1)\phi_3(y) & (x_{11} > 0), \\ 0 & (x_{11} \leq 0), \end{cases}$$

with $\phi_1 \in C_c^\infty(0, +\infty)$, $\phi_2 \in C_c^\infty(\mathfrak{n}_1)$, $\phi_3 \in C_c^\infty(V_{[1]})$. Then we have by (6.9)

$$\langle \mathscr{R}_s, \varphi \rangle = \pi^{-(\dim \mathfrak{n}_1)/2} \langle \rho_{s_1}, \phi_1 \rangle \langle \check{\mathscr{R}}_{\check{s}-\check{n}/2}, \phi_3 \rangle \int_{\mathfrak{n}_1} \phi_2(L_1) \, dL_1.$$

Since $s_1 > 0$, the positivity assumption of $\mathscr{R}_s$ yields that $\check{\mathscr{R}}_{\check{s}-\check{n}/2}$ is positive. Hence by the induction hypothesis we have $\check{s}-\check{n}/2 \in \check{\Xi}(\check{\varepsilon})$ for some $\check{\varepsilon} = (\varepsilon_2, \dots, \varepsilon_r) \in \{0,1\}^{r-1}$. Put $\varepsilon := (1, \varepsilon_2, \dots, \varepsilon_r)$. Then by (4.4) and (6.8) we have $p_k(\varepsilon) = \check{p}_k(\check{\varepsilon}) + n_{k1}$ for $k = 2, \dots, r$. Thus $\check{s}-\check{n}/2 \in \check{\Xi}(\check{\varepsilon})$ is equivalent to

$$s_i > p_i(\varepsilon)/2 \quad (\text{if } \varepsilon_i = 1), \quad s_i = p_i(\varepsilon)/2 \quad (\text{if } \varepsilon_i = 0)$$

for $i = 2, \dots, r$. Since we have $s_1 > 0 = p_1(\varepsilon)/2$, we see that $s \in \Xi(\varepsilon)$. Hence the theorem is completely proved. □

REMARK. For every $t \geq 0$, we set

$$\varepsilon(t) = 1 \quad (\text{if } t > 0), \quad \varepsilon(t) = 0 \quad (\text{if } t = 0).$$

Then we have

$$R_+(\varepsilon) = \{u \in \boldsymbol{R}^r \mid u_i \geq 0,\ \varepsilon(u_i) = \varepsilon_i \text{ for all } i = 1, 2, \dots, r\}.$$

Making $s, u \in \boldsymbol{R}^r$ correspond to each other by $s = u + p(\varepsilon)/2$, we see by (6.2) that

$$s \in \Xi(\varepsilon) \quad \Leftrightarrow \quad u \in R_+(\varepsilon).$$

Moreover if Ω is symmetric and simple, then $d := \dim \mathfrak{h}_{ki} = \dim V_{ki}$ is independent of k, i, so that (4.3) and (4.4) imply

$$p_k(\varepsilon) = d(\varepsilon_1 + \cdots + \varepsilon_{k-1}).$$

From these observations follows the description of Ξ for the case of symmetric cones given in [**3**, p. 138].

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