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On embeddedness of area-minimizing disks, and an application to constructing complete minimal surfaces

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Abstract. Let α be a polygonal Jordan curve in $\boldsymbol{R}^3$. We show that if α satisfies certain conditions, then the least-area Douglas–Radó disk in $\boldsymbol{R}^3$ with boundary α is unique and is a smooth graph. As our conditions on α are not included amongst previously known conditions for embeddedness, we are enlarging the set of Jordan curves in $\boldsymbol{R}^3$ which are known to be spanned by an embedded least-area disk.

As an application, we consider the conjugate surface construction method for minimal surfaces. With our result we can apply this method to a wider range of complete catenoid-ended minimal surfaces in $\boldsymbol{R}^3$.

1. Introduction.

Much investigation has been made on the Plateau problem, i.e. to show that any rectifiable Jordan curve in $\boldsymbol{R}^3$ bounds a minimal surface of least area. The first results were by Douglas and Radó in the early 1930's, when they proved existence of a smooth least-area disk for any given boundary curve [**Os**]. This disk is often called the Douglas–Radó solution. Osserman [**Os**] later showed that the Douglas–Radó solution has no true branch points in its interior, and Gulliver [**Gu**] showed that it also has no false branch points in its interior. Hildebrandt showed regularity at the boundary of the Douglas–Radó solution wherever its boundary is real-analytic. Then, along real-analytic portions of the boundary, Gulliver and Lesley [**GuLe**] showed nonexistence of branch points. Putting all of these results together, we have the following theorem.

Theorem 1.1. *Let α be a rectifiable Jordan curve in $\boldsymbol{R}^3$. Then there exists a map*

$$h : D \to \boldsymbol{R}^3,$$

where D is the closed unit disk in $\boldsymbol{R}^2$, satisfying

1. *h is continuous in D;*
2. *h maps the boundary of D, ∂D, bijectively to α;*
3. *$h \in C^\infty$ (in fact, h is harmonic) in the interior of D, and is a regular conformal minimal immersion in the interior of D;*
4. *the image of D under h has minimum area among all maps $D \to \boldsymbol{R}^3$ which are piecewise smooth in the interior and satisfy the conditions* 1 *and* 2 *above.*

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5. *If σ is a closed subarc of ∂D that is mapped by h into the interior of some real-analytic subarc γ of α, then h can be analytically continued across σ (as a minimal surface), and h has no branch points on σ.*

In the case that the γ above is a straight line or a planar geodesic, an even stronger conclusion is known, and is called the Schwarz reflection principle ([**Ka2**], Sec. 1.3.2):

Theorem 1.2. *The union of a minimal surface with its reflection (resp. rotation by 180 degrees) across (resp. about) a plane containing a boundary planar geodesic (resp. a line segment in the boundary of the surface) is a smooth minimal surface.*

The results above go far to solve the problem of existence and regularity of least-area disks for a given curve, as well as to show nonexistence of branch points in least-area disks. However, the question of embeddedness is only partly answered. Almgren and Thurston showed that there exist unknotted Jordan curves that cannot bound any embedded minimal disk [**MeYa1**]. It seems difficult to find conditions on a curve that imply its Douglas–Radó solutions are embedded, but some partial results have been found. Radó proved in 1932 [**MeYa2**] that if α is an embedded rectifiable curve in $\boldsymbol{R}^3$ whose vertical projection to the x_1x_2-plane (or a central projection from a certain point) is one-to-one and convex, then the Douglas–Radó solution is the unique least-area surface bounded by α and is a graph over the x_1x_2-plane (or a graph with respect to central projection). Meeks and Yau [**MeYa1**] generalized this to the case that α is extremal, i.e. lies on the boundary of its convex hull. They showed that any Douglas–Radó solution for an extremal curve α is embedded. They later generalized this to show the same conclusion even when α only lies in the boundary $\partial\hat{M}$ of a closed region $\hat{M} \subseteq \boldsymbol{R}^3$ such that $\partial\hat{M}$ has nonnegative mean curvature with respect to the interior of $\hat{M}$ [**MeYa2**].

We shall show (Theorem 2.1) that for certain types of polygonal Jordan curves in $\boldsymbol{R}^3$, the Douglas–Radó solution is an embedded graph. We can apply Theorem 2.1 in some cases where the results of Meeks and Yau do not apply. The original motivation for considering these types of polygonal Jordan curves is their usefulness in the conjugate surface construction method for minimal surfaces in $\boldsymbol{R}^3$ ([**BeRo**], [**Ka1**], [**Ka2**], [**Ka3**], [**Ka4**], [**Ro**]). Some examples of this construction are shown in Section 5.

Theorem 2.1 allows us to extend the conjugate surface construction to more cases. The strategy is roughly as follows: We wish to prove existence of complete catenoid-ended minimal surfaces M with symmetry, where the symmetries of M are generated by a discrete set of reflections in $\boldsymbol{R}^3$. We consider the smallest portion of M that will generate the entire surface under the action of the symmetry group, and we call it the *fundamental piece* of M. We choose the fundamental piece so that it is bounded by planar geodesics. It is then enough to show existence of the fundamental piece only, since the entire surface M can be produced from the fundamental piece by reflection (Theorem 1.2). Furthermore, we can show existence of the fundamental piece by showing existence of the conjugate surface M' to the fundamental piece. (We define the conjugate surface in Section 3.) The advantage of considering the conjugate surface M' is that it is bounded by straight lines. We prove the existence of M' by showing it exists as the limit of a sequence of compact embedded stable minimal disks M_i bounded by Jordan polygonal curves α_i.

We will use the term stable in the following sense: Minimal surfaces are critical for the first variation formula. A minimal surface $\mathscr{S}$ (possibly with boundary $\partial\mathscr{S}$) is *stable* if the second derivative of area is nonnegative at $\mathscr{S}$ for all smooth variations of the surface with compact support (and fixing $\partial\mathscr{S}$).

So the first step is to demonstrate the existence of M_i bounded by α_i. For the minimal surfaces M we are considering, α_i can be chosen to satisfy all the conditions of Theorem 2.1. Thus the Douglas–Radó surfaces M_i for α_i are smooth graphs in $\boldsymbol{R}^3$. In particular, M_i are smoothly embedded and stable. Once we have stability, we can show that $\{M_i\}_{i=1}^{\infty}$ has a convergent subsequence (Lemma 4.1). M' is the limit surface. (The question of minimal graphs over unbounded planar domains has been investigated in [**EaRo**], [**BeRo**], and [**Ro**].) We then show that M' is connected in the cases we consider.

In the case that M may have some unwanted periodicity, we need to show that M' can be constructed so that M does not have this periodicity. Lemma 4.2 is useful for this.

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2. The main result.

THEOREM 2.1. *Let $\alpha = \ell_1 \cup \cdots \cup \ell_m$ be a closed embedded polygonal curve in $\boldsymbol{R}^3$ consisting of straight line segments ℓ_i and vertices $\ell_1 \cap \ell_2, \dots, \ell_{m-1} \cap \ell_m, \ell_m \cap \ell_1$. Let P be a polygonal region in the x_1x_2-plane $\{x_3 = 0\}$ bounded by the polygon $\partial P = \rho_1 \cup \cdots \cup \rho_{m-1}$ consisting of edges ρ_i and vertices $\rho_1 \cap \rho_2, \dots, \rho_{m-2} \cap \rho_{m-1}, \rho_{m-1} \cap \rho_1$. Suppose the following*:

1. *ℓ_m is vertical; and for each $i = 1, \dots, m-1$, ℓ_i is not vertical.*
2. *ℓ_{m-1} and ℓ_1 are horizontal, and α lies entirely between the two horizontal planes containing ℓ_{m-1} and ℓ_1. That is, there exist $a, b \in \boldsymbol{R}$ such that $\ell_{m-1} \subset \{x_3 = a\}$ and $\ell_1 \subset \{x_3 = b\}$ and $\alpha \subset \{\min(a,b) \le x_3 \le \max(a,b)\}$.*
3. *Denoting the boundary of the convex hull of P in the x_1x_2-plane by $\partial\,\mathrm{Conv}(P)$, we have $\partial\,\mathrm{Conv}(P) \cap \partial P = \rho_2 \cup \cdots \cup \rho_{m-2}$.*
4. *Each ℓ_i, $i = 1, \dots, m-1$ is mapped bijectively to ρ_i by the vertical projection $\mathscr{P}$: $(x_1, x_2, x_3) \to (x_1, x_2, 0)$, and $\mathscr{P}(\ell_m) = \rho_{m-1} \cap \rho_1$.*

Then the Douglas–Radó solution with boundary α is unique and embedded, and its interior is a graph over the interior of P.

REMARK. It is clear from the proof below that this theorem could be generalized somewhat. For example, we could easily adapt the proof to include cases where ℓ_{m-1} and ℓ_1 are not horizontal, or where α has portions that are not polygonal. However, as the statement above is sufficient for the applications in Section 5, for simplicity we do not consider any generalizations here.

EXAMPLE 2.1. Let $\lambda_1, \lambda_2, \beta_1, \beta_2, \gamma$ be any positive numbers. Let α be the polygonal curve from $(0,0,1)$ to $(0,0,2)$ to $(-\lambda_1, \beta_1, 2)$ to $(-\lambda_1, -\gamma, 2)$ to $(\lambda_2, -\gamma, 1)$ to $(\lambda_2, \beta_2, 1)$

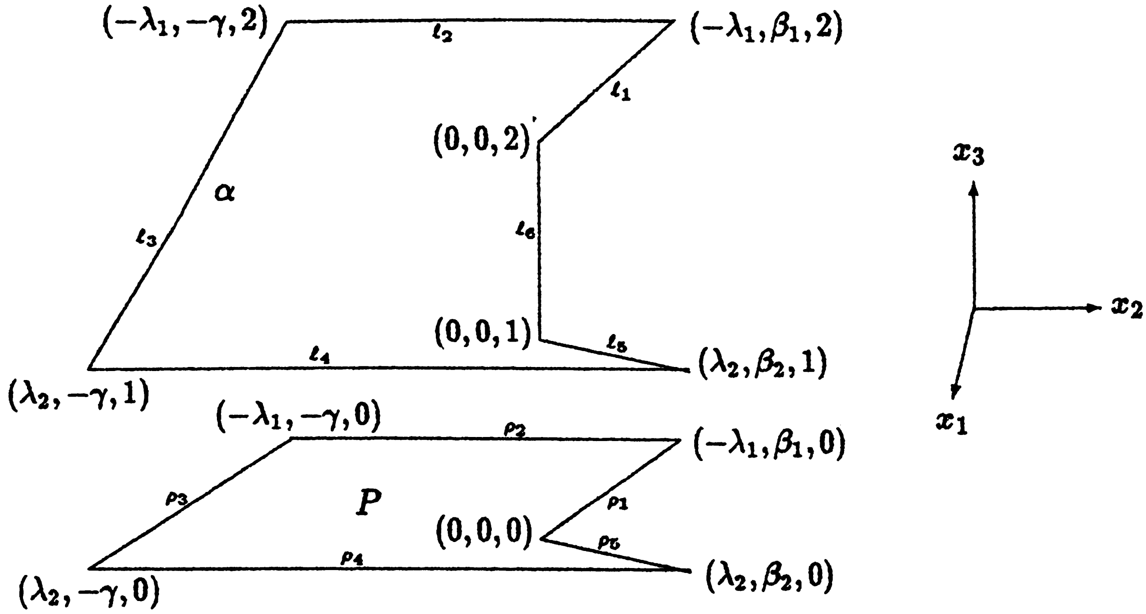

Figure 1: A curve α satisfying all the conditions of Theorem 2.1.

and back to $(0,0,1)$. Let ∂P be the 5-gon in the x_1x_2-plane with vertices $(0,0,0)$, $(\lambda_2,\beta_2,0)$, $(-\lambda_1,\beta_1,0)$, $(-\lambda_1,-\gamma,0)$, and $(\lambda_2,-\gamma,0)$, so that $\partial P = \mathscr{P}(\alpha)$ (see Figure 1). By Theorem 2.1, the Douglas–Radó solution for α is unique and is a graph over P.

In the case $\lambda_1 = \lambda_2$ and $\beta_1 = \beta_2$, it was already known that α bounds a smoothly embedded minimal disk that is stable in $\boldsymbol{R}^3$. Consider the polygonal curve $\tilde{\alpha}$ from $(0,0,1)$ to $(\lambda_2,\beta_2,1)$ to $(\lambda_2,-\gamma,1)$ to $(0,-\gamma,3/2)$ to $(0,0,3/2)$ and back to $(0,0,1)$. The least-area surface $\tilde{M}$ spanning $\tilde{\alpha}$ is unique and is a graph over the region $P \cap \{x_1 \geq 0\}$, by Nitsche's theorem [**BeRo**], [**Ro**]. Let Rot: $\boldsymbol{R}^3 \to \boldsymbol{R}^3$ be rotation by 180 degrees about the line through $(0,-\gamma,3/2)$ and $(0,0,3/2)$. Then $\tilde{M} \cup \mathrm{Rot}(\tilde{M})$ is a smoothly embedded minimal graph with boundary α, by Theorem 1.2. Since the image of the Gauss map on $\tilde{M} \cup \mathrm{Rot}(\tilde{M})$ is contained in a hemisphere, $\tilde{M} \cup \mathrm{Rot}(\tilde{M})$ is stable [**BdC**].

However, Theorem 2.1 shows that $\tilde{M} \cup \mathrm{Rot}(\tilde{M})$ is also the unique least-area surface with boundary α. In fact, Example 2.1 shows existence of a unique least-area surface of disk type with boundary α in the nonsymmetric cases $\lambda_1 \neq \lambda_2$ or $\beta_1 \neq \beta_2$ as well, where it was not previously known if there were even stable embedded minimal disks with boundary α.

Remark. Theorem 2.1 is not true without the fourth condition. For example, let α be the polygonal curve consisting of line segments from $(0,0,0)$ to $(2,0,0)$ to $(2,\delta,0)$ to $(2,0,\delta)$ to $(0,0,\delta)$ to $(0,1,\delta)$ to $(1/2,\varepsilon,\delta)$ to $(1/2,\varepsilon,0)$ to $(0,1,0)$ and back to $(0,0,0)$. Let ∂P be the polygonal 5-gon in the x_1x_2-plane with vertices $(0,0,0)$, $(2,0,0)$, $(2,\delta,0)$, $(0,1,0)$, and $(1/2,\varepsilon,0)$, so that $\mathscr{P}(\alpha) \subset \partial P$. If $0 < \varepsilon \ll \delta \ll 1$, then the least-area surface bounded by α is not embedded, since its interior will intersect α along the line segment from $(1/2,\varepsilon,0)$ to $(1/2,\varepsilon,\delta)$.

We now state two lemmas following from Theorem 4 and Lemmas 2 and 3 of [**MeYa1**]. We use these two lemmas in the proof of Theorem 2.1. Let $B_\varepsilon(p) := \{q \in \boldsymbol{R}^3 \mid \mathrm{dist}(p,q) < \varepsilon\}$.

Lemma 2.1. *If the self-intersection set* $S(h) = \{p \in D \mid \exists q \neq p \in D \text{ with } h(p) = h(q)\}$ *is disjoint from* ∂D, *then* h *is an embedding.*

Lemma 2.2. *Let* $g : D \to \boldsymbol{R}^3$ *and* $f : D \to \boldsymbol{R}^3$ *be regular minimal embeddings that intersect at a point* $p \in \boldsymbol{R}^3$ *such that* $p \notin g(\partial D) \cup f(\partial D)$. *Assume that the images of* g *and* f *do not coincide in a neighborhood of* p. *Then for some small* ε, *the intersection set* $f(D) \cap g(D) \cap B_\varepsilon(p)$ *consists of a finite number of curves through* p *and the intersection is transverse at points other than* p. *The intersection set cannot be a point, and cannot contain a curve with an endpoint in* $\mathrm{Int}(B_\varepsilon(p))$, *and cannot have nonempty interior.*

In particular, this holds for the intersection of a nonflat minimal immersion with any of its tangent planes.

The proof of Theorem 2.1 relies on properties of the Gauss map. Let M be the image of a conformal minimal immersion $h : D \to \boldsymbol{R}^3$. The Gauss map $G : D \to S^2$ for the conformal minimal immersion $h : D \to M \subseteq \boldsymbol{R}^3$ maps each point $p \in D$ to the unit normal of M at $h(p)$ (considered as a point in the standard unit sphere S^2). G is a holomorphic map from the complex coordinate $z = x_1 + ix_2 \in D$ to S^2 with the standard complex structure. Therefore, if G is not constant, it must map open sets in the interior of D to open sets in S^2. We now prove Theorem 2.1.

Proof. Let $h : D \to M \subset \boldsymbol{R}^3$ be any Douglas–Radó solution for α, as in Theorem 1.1. Thus h is a C^∞ harmonic conformal minimal immersion on $D \backslash \{h^{-1}(\ell_m \cap \ell_1), h^{-1}(\ell_1 \cap \ell_2), \dots, h^{-1}(\ell_{m-1} \cap \ell_m)\}$, and G is holomorphic on this same set. Also, G is well-defined and continuous at the vertices $h^{-1}(\ell_m \cap \ell_1), h^{-1}(\ell_1 \cap \ell_2), \dots, h^{-1}(\ell_{m-1} \cap \ell_m)$; in fact, the unit normal at $h^{-1}(\ell_i \cap \ell_{i+1})$ (resp. $h^{-1}(\ell_m \cap \ell_1)$) must be perpendicular to the plane containing $\ell_i \cup \ell_{i+1}$ (resp. $\ell_m \cup \ell_1$) ([**DHKW**], Section 8.3).

We now give the proof in five steps.

Step 1: $G(h^{-1}(\ell_2 \cup \cdots \cup \ell_{m-2} \cup \ell_m)) \subset S^2 \cap \{x_3 \geq 0\}$.

Since $\mathscr{P}$ is a bijection from $\ell_2 \cup \cdots \cup \ell_{m-2}$ to $\partial \mathrm{Conv}(P) \cap \partial P$, $\ell_2 \cup \cdots \cup \ell_{m-2}$ is contained in the boundary of the convex hull of α. Since $h(D)$ is contained in the convex hull of α (see, for example, [**DHKW**], Section 6.1), the boundary point maximum principle [**Scn**] implies that G is never horizontal on $\ell_2 \cup \cdots \cup \ell_{m-2}$, except possibly at corner points $\ell_i \cap \ell_{i+1}, i = 1, \dots, m-2$. However, for $i = 1, \dots, m-2$, neither ℓ_i nor ℓ_{i+1} is vertical, and ρ_i and ρ_{i+1} are not parallel, and $\mathscr{P}(\ell_i) = \rho_i$ and $\mathscr{P}(\ell_{i+1}) = \rho_{i+1}$, hence the normal vector at $\ell_i \cap \ell_{i+1}$ is not horizontal. Thus $G(h^{-1}(\ell_2 \cup \cdots \cup \ell_{m-2})) \cap \{x_3 = 0\} = \varnothing$. We choose the orientation of M so that $G(h^{-1}(\ell_2 \cup \cdots \cup \ell_{m-2})) \subset S^2 \cap \{x_3 > 0\}$. Since ℓ_m is vertical, $G(h^{-1}(\ell_m)) \subseteq S^2 \cap \{x_3 = 0\}$. This shows Step 1.

Step 2: *There exists a horizontal vector* $\vec{v} \in S^2$ *such that* $\vec{v} \notin G(D)$.

By the conditions on P and α, there exists a horizontal vector $\vec{v} = (v_1, v_2, 0)$ so that any plane perpendicular to $\vec{v}$ intersects α in at most two components. We can choose $\vec{v}$ so that for any plane H perpendicular to $\vec{v}$ satisfying $H \cap \alpha \neq \varnothing$, one component of $H \cap \alpha$ is a single point, and the other component is either empty or a single point or ℓ_m.

We claim that there cannot be any point in the interior of M with normal $\pm\vec{v}$. Suppose there is such a point $p \in \mathrm{Int}(M)$. Let $S(T_p(M)) = \{z \in D \mid h(z) \in T_p(M)\}$. By

Lemma 2.2, $S(T_p(M))$ is a plane embedded graph in D, and each vertex of $S(T_p(M))$ contained in $\mathrm{Int}(D)$ is connected to at least four edges. Note that $S(T_p(M)) \cap \mathrm{Int}(D)$ has at least one vertex, at $h^{-1}(p)$.

Since $T_p(M) \cap \alpha$ has at most two components, and the map $h|_{\partial D} : \partial D \to \alpha$ is bijective and continuous (Theorem 1.1), we know that $S(T_p(M)) \cap \partial D$ also has at most two components.

It follows from elementary graph theory that $S(T_p(M))$ contains a closed loop β. $h(\beta) \subseteq M \cap T_p(M)$ and $h(\beta)$ must bound a subdisk of M; and this subdisk must be contained in $T_p(M)$, since h is a harmonic map. Thus $M \subset T_p(M)$, since h is harmonic. But $\partial M = \alpha \not\subset T_p(M)$, thus there cannot be any point in $\mathrm{Int}(M)$ with normal $\pm\vec{v}$.

We now claim that there is at most one point in α with normal $\pm\vec{v}$. Suppose there are two distinct points $p, q \in \alpha$ with normal $\pm\vec{v}$. By our choice of $\vec{v}$, the points p, q must be contained in the interior of ℓ_m. Let H be the plane perpendicular to $\vec{v}$ and containing ℓ_m. Let $S(H) = \{z \in D | h(z) \in H\}$. $S(H) \cap \partial D$ has two components, one of which is $h^{-1}(\ell_m)$. There are (at least) two edges of $S(H)$ in $\mathrm{Int}(D)$ with endpoints in $h^{-1}(\ell_m)$, meeting $h^{-1}(\ell_m)$ at the vertices $h^{-1}(p)$ and $h^{-1}(q)$. Thus again we see $S(H)$ must contain a closed loop, and we have a contradiction.

Therefore either $\vec{v} \notin G(D)$ or $-\vec{v} \notin G(D)$. Changing $\vec{v}$ to $-\vec{v}$ if necessary, we have $\vec{v} \notin G(D)$. This shows Step 2.

Step 3: *$G(D) \subset S^2 \cap \{x_3 \geq 0\}$, and for any point $z \in \mathrm{Int}(D)$ there is an open neighborhood $U \subset \mathrm{Int}(D)$ of z so that $h(U)$ is a graph over $\{x_3 = 0\}$.*

Since $h(D)$ is contained in the convex hull of α, and since ℓ_{m-1} and ℓ_1 are contained in the boundary of the convex hull of α, $G(h^{-1}(\ell_{m-1})) \subseteq \sigma_{m-1}$ and $G(h^{-1}(\ell_1)) \subseteq \sigma_1$, where σ_{m-1} and σ_1 are 180 degree arcs of great circles in S^2 from $(0,0,1)$ to $(0,0,-1)$. Furthermore, the boundary point maximum principle [**Scn**] implies $(0,0,-1) \notin G(h^{-1}(\ell_{m-1} \cup \ell_1))$.

We saw in Step 1 that $G(h^{-1}(\ell_2 \cup \cdots \cup \ell_{m-2} \cup \ell_m)) \subset S^2 \cap \{x_3 \geq 0\}$ and $G(h^{-1}(\ell_m)) \subseteq S^2 \cap \{x_3 = 0\}$. This and the preceding paragraph imply that

$$\mathscr{X} = (S^2 \cap \{x_3 \leq 0\}) \backslash \{G(h^{-1}(\ell_{m-1} \cup \ell_1))\}$$

is a connected set.

We will show that

$$G(D) \subset S^2 \cap \{x_3 \geq 0\}.$$

Suppose $G(D) \not\subset S^2 \cap \{x_3 \geq 0\}$. Then, by Step 1, there is some point $p \in \mathrm{Int}(D) \cup h^{-1}(\ell_1) \cup h^{-1}(\ell_{m-1})$ such that $G(p) \in S^2 \cap \{x_3 < 0\}$. Hence some open neighborhood U of p in D satisfies $G(U) \subset S^2 \cap \{x_3 < 0\}$. Since the Gauss map G maps open sets to open sets, we have

$$\mathrm{Int}(\mathscr{X}) \cap G(D) \neq \varnothing.$$

Since $\mathscr{X} \cap G(D)$ is both open and closed in $\mathscr{X}$, and since $\mathscr{X}$ is connected, we have

$$\mathscr{X} \cap G(D) = \mathscr{X}.$$

($\mathscr{X} \cap G(D)$ is closed in $\mathscr{X}$, since G is holomorphic, and D is closed; $\mathscr{X} \cap G(D)$ is open in $\mathscr{X}$, since G is holomorphic and so $\partial G(D) \subset G(\partial D)$, and also since $G(\partial D) \cap \mathrm{Int}(\mathscr{X}) = \varnothing$.) Therefore $\mathscr{X} \subset G(D)$ and so $S^2 \cap \{x_3 \leq 0\} \subset G(D)$, contradicting Step 2. We conclude that $G(D) \subset S^2 \cap \{x_3 \geq 0\}$.

Finally, by the holomorphicity of G, $G(\mathrm{Int}(D)) \subset S^2 \cap \{x_3 > 0\}$. Thus at each point in $\mathrm{Int}(M)$, M is locally a graph over the x_1x_2-plane. This shows Step 3.

Step 4: *M is embedded and $\mathscr{P}(\mathrm{Int}(M)) \subseteq \mathrm{Int}(P)$.*

Let $\partial\mathscr{P}(M)$ be the boundary of $\mathscr{P}(M)$ in $\{x_3 = 0\}$. Suppose there exists a point $p \in \mathrm{Int}(M)$ such that $\mathscr{P}(p) \notin \mathrm{Int}(P)$, then $(\partial\mathscr{P}(M))\backslash P$ is not empty. Let ℓ be a vertical line intersecting $(\partial\mathscr{P}(M))\backslash P$. The line ℓ must make a tangential intersection with some point $q \in \mathrm{Int}(M)$. Thus $T_q(M)$ is a vertical tangent plane. This contradicts Step 3, hence $\mathscr{P}(\mathrm{Int}(M)) \subseteq \mathrm{Int}(P)$. We conclude that M is embedded at its boundary. Thus, by Lemma 2.1, M is embedded. This shows Step 4.

Step 5: *$h|_{\mathrm{Int}(D)}$ is a graph over $\mathrm{Int}(P)$, and is the unique Douglas–Radó solution with boundary α.*

The arguments in the next two paragraphs are similar to the proof of Theorem 1 in [**Scn**], except that our projection domain P is not convex, and we use a family of translations instead of Schoen's family of reflections. Hence we only outline the arguments here.

First we show $h : \mathrm{Int}(D) \to \mathrm{Int}(M)$ is a graph over $\mathrm{Int}(P)$. Let $h_\lambda : D \to M$ be defined by $h_\lambda(p) = h(p) + (0, 0, \lambda)$. Choose $\lambda_0 \geq 0$ to be the smallest value so that for any $\lambda \geq \lambda_0$, $h_\lambda(\mathrm{Int}(D))$ and $h(\mathrm{Int}(D))$ have no points of transverse intersection. If $\lambda_0 > 0$, then $h(D)$ and $h_\lambda(D)$ must violate the maximum principle [**Scn**], either at an interior point or at a boundary point. Thus $\lambda_0 = 0$, which implies that $\mathrm{Int}(M)$ is a graph. (Note that we are using $h_\lambda(\mathrm{Int}(D))$ and $h(\mathrm{Int}(D))$ to define λ_0, and we are not using $h_\lambda(D)$ and $h(D)$. This distinction is important, as an intersection of $h_\lambda(D)$ and $h(D)$ at a point in $h_\lambda(\partial D) \cap h(\partial D)$ does not necessarily constitute a contradiction to the maximum principle.)

Finally, suppose there exist two Douglas–Radó solutions $h : D \to \boldsymbol{R}^3$ and $g : D \to \boldsymbol{R}^3$. As we have shown, they must both be embedded graphs over P. Let $g_\lambda(p) = g(p) + (0, 0, \lambda)$. Choose $\lambda_0 \geq 0$ to be the smallest value so that for any $\lambda \geq \lambda_0$, $g_\lambda(\mathrm{Int}(D))$ and $h(\mathrm{Int}(D))$ have no points of transverse intersection. If $\lambda_0 > 0$, the maximum principle is violated. Thus $\lambda_0 = 0$, which implies that $g(D)$ lies above $h(D)$. Similarly, $h(D)$ lies above $g(D)$. Therefore $g(D) = h(D)$, and the Douglas–Radó solution is unique. This shows Step 5. $\square$

3. The conjugate surface construction.

The Weierstrass representation is a principal tool used for the construction of minimal surfaces in $\boldsymbol{R}^3$. Given a compact Riemann surface Σ, a set of points $\{p_j\}$ in Σ, a meromorphic function $g : \Sigma\backslash\{p_j\} \to \boldsymbol{C}$, and a holomorphic one-form ω on $\Sigma\backslash\{p_j\}$, the mapping $X : \Sigma\backslash\{p_j\} \to \boldsymbol{R}^3$ defined by

$$X(z)=\operatorname{Re}\int_p^z\left(\frac{1}{2}(g^{-1}-g)\omega,\frac{i}{2}(g^{-1}+g)\omega,\omega\right) \tag{3.1}$$

is a conformal minimal immersion, where $p\in\Sigma$ is fixed. X is regular away from poles and zeroes of g provided ω is nonzero there, and X is regular at a pole or zero of g of order m provided ω has a zero there of order m. For X to be well-defined on $\Sigma\backslash\{p_j\}$, we must have

$$\operatorname{Re}\oint_\gamma\left(\frac{1}{2}(g^{-1}-g)\omega,\frac{i}{2}(g^{-1}+g)\omega,\omega\right)=0 \tag{3.2}$$

for any representative γ of any non-trivial homotopy class.

The Riemann surface $\Sigma\backslash\{p_j\}$, meromorphic function g, and one-form ω are referred to as the *Weierstrass data*. Here g is the Gauss map G of X composed with stereographic projection to the complex plane.

The conjugate surface X' of X is the minimal surface with the same underlying Riemann surface $\Sigma\backslash\{p_j\}$, and the same meromorphic function g, but with holomorphic one-form $i\omega$. (Note that $(X')'=-X$.) The parametrization $X'(p)$ may only be well-defined on a covering of $\Sigma\backslash\{p_j\}$, since equation (3.2) can hold for the Weierstrass data $\{g,\omega\}$ on $\Sigma\backslash\{p_j\}$ without holding for the Weierstrass data $\{g,i\omega\}$ on $\Sigma\backslash\{p_j\}$.

Thus we have the maps $z\to X(z)$ and $z\to X'(z)$ from simply connected domains of $\Sigma\backslash\{p_j\}$ to X and X', respectively. This induces a covering map $\phi: X'(z)\to z\to X(z)$, the *conjugate map*, from X' to X. The following lemma is proven in [**Ka1**], [**Ka3**], [**Ka4**].

Lemma 3.1. *The conjugate map ϕ has the following properties*:

1) *ϕ is an isometry*;
2) *ϕ preserves the Gauss map G*;
3) *ϕ maps planar principal curves in X' to planar asymptotic curves in X, and maps planar asymptotic curves in X' to planar principal curves in X; that is to say, ϕ maps non-straight planar geodesics to straight lines, and vice versa.*

It follows from the second and third properties of ϕ that a planar geodesic in X' contained in a plane H is mapped by ϕ to a line in X that is perpendicular to H.

4. Limit surface lemma and period removal lemma.

We use Lemma 4.1 to produce stable noncompact embedded minimal surfaces from compact embedded least-area disks. It is a slight variation of a lemma in [**Ro**], and the proof in [**Ro**] applies to this case as well.

Lemma 4.1. *Let $\{\alpha_i\}_{i=1}^\infty$ be a sequence of compact Jordan contours in $\boldsymbol{R}^3$ so that the following conditions hold*:

1) *There is a positive integer n so that, for all i, α_i is a polygonal Jordan curve consisting of at most n line segments*;
2) *Each α_i bounds a least-area minimal disk M_i*;
3) *$\{\alpha_i\}_{i=1}^\infty$ converges (in the topology of compact uniform convergence) to a non-compact polygonal curve α (not necessarily connected), and α consists of a finite number of line segments, rays, and complete lines.*

Then a subsequence of $\{M_i\}_{i=1}^{\infty}$ converges to a nonempty stable minimal surface M (possibly disconnected) with boundary α. Furthermore, if each M_i is embedded, then M is embedded.

Lemma 4.2 is useful for solving a period problem at a catenoid end of a minimal surface. Consider a minimal surface M (with boundary ∂M) with an end that is a 180 degree arc of a helicoid end. Denote a neighborhood of this end by E. Suppose that outside a compact ball in $\boldsymbol{R}^3$ the boundary ∂E is a pair of straight rays r_1, r_2. These two rays are necessarily parallel and pointing in opposite directions. The conjugate surface E' of E is a surface with a 180 degree arc of a catenoid end that, outside a compact ball in $\boldsymbol{R}^3$, is bounded by two infinite planar geodesics s_1, s_2 asymptotic to catenaries. The curves s_1, s_2 lie in parallel planes, and these planes are perpendicular to r_1 and r_2. For this situation, we have the following lemma. A proof can be found in [**Ro**].

Lemma 4.2. *The two planar geodesics $s_1, s_2 \subset \partial E'$ lie in the same plane if and only if the plane containing the two conjugate straight boundary rays $r_1, r_2 \subset \partial E$ is parallel to the normal vector at the helicoid end of E.*

5. Complete minimal surfaces.

The conjugate surface construction method described in the introduction has been successful in many cases of minimal surfaces M of the following type:

- M has catenoid ends;
- each end is invariant under some plane of reflective symmetry of M;
- the conjugate surface M' of the fundamental piece of M is embedded;
- all period problems that do not occur at an end of M can be simultaneously removed by comparison arguments.

Many previously known examples fit this description. Among them are the Jorge–Meeks n-oids [**JoMe**], the genus-1 n-oids [**BeRo**], the Platonoids [**Xu**] [**Kat**] [**UmYa**], the higher-genus Platonoids [**BeRo**], the $\mathscr{AW}_0(2n, w)$ surfaces [**Ro**], the prismoids [**Ro**] [**Kat**], the higher-genus prismoids [**Ro**], and the Jorge–Meeks fence [**Ro**].

Some of the examples below have been shown to exist by other methods. Wohlgemuth [**Wo**] has made similar periodic examples, by adding handles to a catenoid. For his examples, he constructs the Weierstrass data. In [**Ka3**], Karcher shows how to deal with examples similar to the first three examples below, by directly constructing the Weierstrass data. One might also be able to construct the last three examples using Weierstrass data, using the methods of [**Ka1**], [**Kat**], and [**Wo**].

The purpose of the examples here is to demonstrate that we can apply the conjugate surface construction to cases where it couldn't be applied before, by using Theorem 2.1.

Example 5.1. Choose any real number $w > 0$ and any integer $n \geq 3$. For each positive integer i, let α_i be the polygonal curve with line segments from $(-1, w, 0)$ to $(i, w, 0)$ to $(i, 0, i)$ to $(0, 0, i)$ to $(0, 0, -i)$ to $(-1, -i\tan(\pi/n), -i)$ and back to $(-1, w, 0)$. Let ∂P_i be the 5-gon in the x_1x_2-plane such that $\mathscr{P}(\alpha_i) = \partial P_i$. By Theorem 2.1, the Douglas–Radó solution M_i for α_i is unique and is a graph over P_i. By Lemma 4.1,

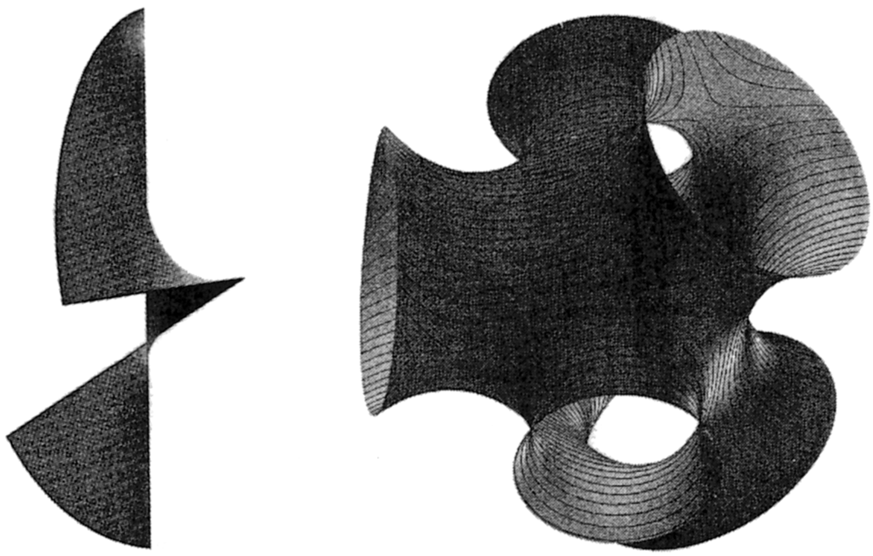

Figure 2: The limit surface M described in Example 5.1 and the resulting complete minimal surface, with $n = 3$.

there exists an embedded limit surface M for some subsequence of $i \to \infty$. (See Figure 2.)

The boundary of M consists of a ray with endpoint $(-1, w, 0)$ pointing in the direction of the positive x_1-axis, a ray with endpoint $(-1, w, 0)$ containing the point $(-1, w - \tan(\pi/n), -1)$, and the x_3-axis. By construction, $\mathscr{P}(M) \subseteq \lim_{i \to \infty} P_i$. Since each M_i is a graph over P_i, we may also conclude that the image of the Gauss map G on M, $G(M) \subseteq S^2 \cap \{x_3 \geq 0\}$. (Here we are making a convenient abuse of notation by considering G to be defined directly on the minimal surface, rather than on some immersion of the surface.) Since each M_i is a disk, M is either a single simply-connected surface, or the union $M = M_A \cup M_B$ of two disjoint simply-connected surfaces M_A and M_B, with ∂M_A being the two rays extending from $(-1, w, 0)$, and ∂M_B being the x_3-axis. However the second case $M = M_A \cup M_B$ is not possible, and we defer to the Appendix for a proof of this. Thus M is connected. The argument in the Appendix also shows that M has finite total curvature.

The conjugate surface M' to M is bounded by three planar geodesics, none of which lie in parallel planes. M' has one end which is a 90 degree arc of a catenoid end, and another end which is a $180/n$ degree arc of a catenoid end. Thus we can extend M' using Theorem 1.2 to a complete minimal surface. This surface has no period problems, and therefore is nonperiodic and of finite total curvature. It consists of $4n$ copies of M', and has $n + 2$ catenoid ends. Amongst the $n + 2$ ends, n of them have equal weight, and the other two have equal weight. By a homothety of $\boldsymbol{R}^3$ if necessary, we may assume that n ends have weight 1 and two ends have weight $r = cw$ for some positive constant c depending only on n. This surface is known to exist by other methods [**Kat**], [**Xu**]. It was also proven to exist by the conjugate surface construction in Theorem 1.3 of [**Ro**]. However, in [**Ro**] the additional assumption was made that r is larger than some given positive constant. Here, due to Theorem 2.1, we can show existence of the surface for any $r > 0$. (See Figure 2.)

We remark that this example can also be constructed, as above, for the case $n = 2$. For $n = 2$, we need only replace the vertex $(-1, -i\tan(\pi/n), -i)$ of α_i with $(-1, -i, 0)$ instead.

Example 5.2. Choose any integer $n \geq 2$, and any real numbers $w > 0$ and $s > w/\sin(\pi/n)$. For each positive integer i, let α_i be the polygonal curve from $(-1, w, 0)$ to $(i, w, 0)$ to $(i, 0, i)$ to $(0, 0, i)$ to $(0, 0, -i)$ to $(i(s \cdot \sin(\pi/n) - w), -i, -i)$ to $(((s \cdot \sin(\pi/n) - w)/i) - 1, w - s \cdot \sin(\pi/n) - (1/i), i)$ to $(-1, w - s \cdot \sin(\pi/n), -s \cdot \cos(\pi/n))$ and back to $(-1, w, 0)$. Let ∂P_i be the 7-gon in the x_1x_2-plane so that $\mathscr{P}(\alpha_i) = \partial P_i$. By Theorem 2.1, the Douglas–Radó solution M_i for α_i is unique and is a graph over P_i. Again some subsequence converges to an embedded surface M as $i \to \infty$. As in the last example, one can argue that M is simply connected and of finite total curvature (see the Appendix).

Let M' be the conjugate surface to M. There are two planar geodesics in $\partial M'$ that lie in parallel planes. In order to extend using Theorem 1.2 to a complete minimal surface with finite total curvature, these two parallel planes must be the same plane. Thus there is one period problem to solve at a catenoid end of the surface. By Lemma 4.2, these two parallel planes are equal if the vertical ray and complete vertical line in ∂M lie in a common plane perpendicular to the 180 degree helicoid end of M. This is the case, since we constructed α_i so that this would be so. Thus the period problem is solved, and using Theorem 1.2, M' extends to an immersed complete minimal surface of finite total curvature with catenoid ends. The complete surface consists of $4n$ copies of M', and is a "prismoid" with 3 layers of ends, and its symmetry group is $D_n \times \boldsymbol{Z}_2$. It may be placed in $\boldsymbol{R}^3$ so that it has n ends with horizontal normal vectors, all of equal weight, and has n ends with normal vectors pointing upward making an angle θ with a horizontal plane, and has n ends with normal vectors pointing downward making the same angle θ with a horizontal plane, for any $\theta \in (0, \pi/2)$. All of these last $2n$ ends have equal weight. The ratio between the weight of the first n ends and the weight of the last $2n$ ends can be any positive value. Thus, for each n, we have a two-parameter family of these surfaces. This example has been shown to exist by a different method in [**Kat**]. (See Figure 3.)

And we can produce examples that were previously unknown, as in the examples below.

Example 5.3. Choose any real numbers $w > 0$ and $\lambda > 0$, and choose any integer $n \geq 2$. Choose $y > \cot(\pi/n)$ to be the unique value so that the distance from the point $(1, y, 0)$ to the plane $\{x_2 = \cot(\pi/n) \cdot x_1\}$ is w. For each positive integer i, let α_i be the polygonal curve from $(0, 0, -\lambda)$ to $(-1/i, 1/i, \lambda)$ to $(0, i^2, \lambda + i)$ to $(1, y, \lambda + i)$ to $(1, y, -\lambda - i)$ to $(i^2 \sin(\pi/n), i^2 \cos(\pi/n), -\lambda - i)$ and back to $(0, 0, -\lambda)$. Let ∂P_i be the 5-gon in the x_1x_2-plane so that $\mathscr{P}(\alpha_i) = \partial P_i$. By Theorem 2.1, for all i sufficiently large, the Douglas–Radó solution for α_i is unique and is a graph over P_i. Again, some subsequence converges to a limit surface M. As in the previous examples, we can show that M is connected (see the Appendix). The conjugate M' of M can be extended by reflection to a complete minimal surface in $\boldsymbol{R}^3$. In this case there is one period problem that is not at an end. But here we do not solve the period problem, as we wish to

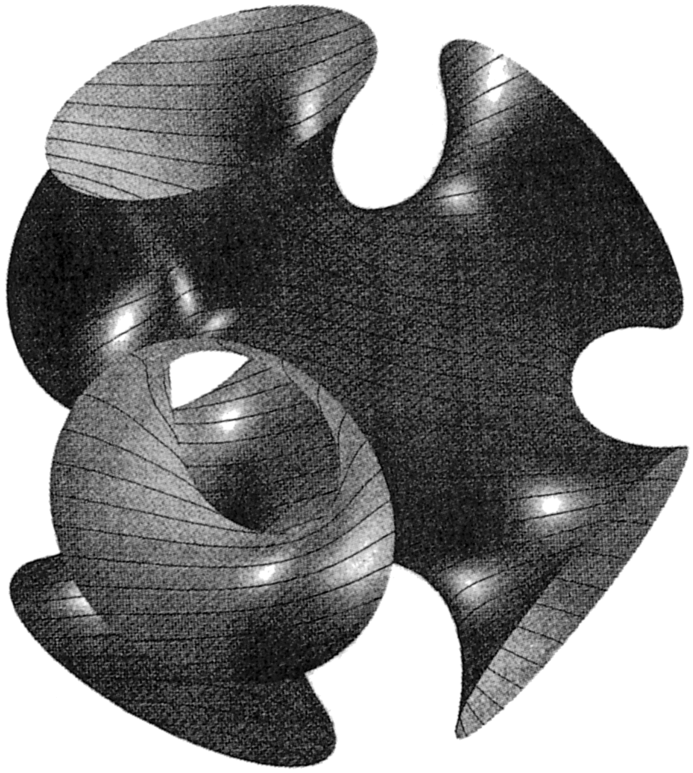

Figure 3: A prismoid with two layers of ends and with $n = 3$. The 3-layered prismoid is similar, but has an additional n ends along the horizontal plane of symmetry.

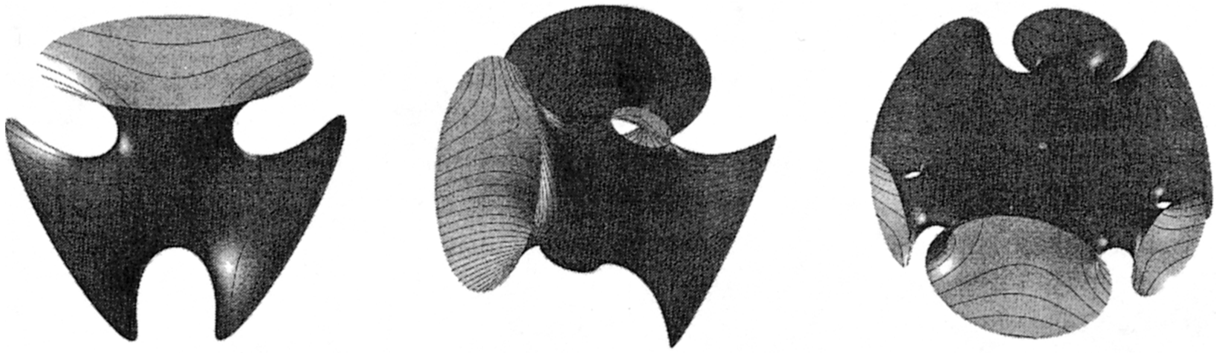

Figure 4: The Jorge–Meeks surface for $n = 3$, the Jorge–Meeks fence for $n = 3$, and an $\mathscr{A}\mathscr{W}_0(2n, w)$ surface for $n = 3$. Example 5.3 shows that $\mathscr{A}\mathscr{W}_0(2n, w)$ surfaces can be put together to make a periodic fence, just as Jorge–Meeks surfaces can be put together to make a Jorge–Meeks fence.

produce a periodic surface. The resulting surface is a periodic fence of $\mathscr{A}\mathscr{W}_0(2n, w)$ surfaces. The $\mathscr{A}\mathscr{W}_0(2n, w)$ surfaces are described in [**Ro**], [**Kat**], and they essentially look like Jorge–Meeks surfaces with $2n$ ends, but the ends have weights that alternate between two positive values. We have a 1-parameter family of these $\mathscr{A}\mathscr{W}_0(2n, w)$ fences, given by the parameter $\lambda > 0$. (See Figure 4.)

Example 5.4. Let $n \geq 2$ be any integer, and let θ and w be any real numbers such that $0 < \theta < \pi/n$ and $0 < w < \sin(\pi/n)/\cos(\theta)$. For each positive integer i, let α_i be the polygonal curve from $(0, 0, 0)$ to $((1-(1/i))\sin(\pi/n), -(1-(1/i)-(1/i^2))\cos(\pi/n), 0)$ to

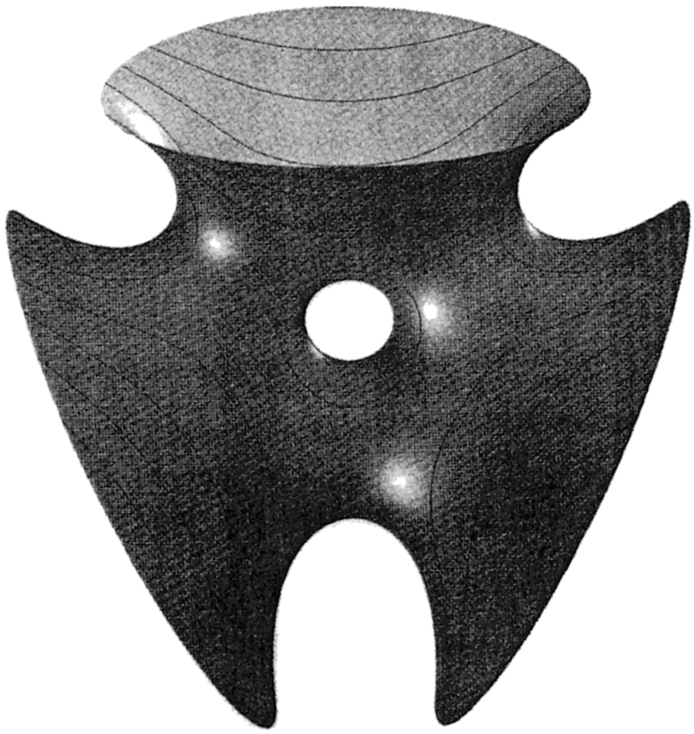

Figure 5: The genus-1 n-oid for $n = 3$.

$(\sin(\pi/n), -\cos(\pi/n), -i)$ to $(i\sin(\theta), -i\cos(\theta), i)$ to $(\sin(\pi/n) - w\cos(\theta), -\cos(\pi/n) - w\sin(\theta), i)$ to $(\sin(\pi/n) - w\cos(\theta), -\cos(\pi/n) - w\sin(\theta), -i)$ to $(\sin(\pi/n) - w\cos(\theta), -i^2, -i)$ and back to $(0,0,0)$. Let ∂P_i be the 6-gon so that $\mathscr{P}(\alpha_i) = \partial P_i$. By Theorem 2.1, for any i sufficiently large, the Douglas–Radó solution for α_i is unique and is a graph over P_i. As in the previous examples, we can create a complete minimal surface with catenoid ends. There is one period problem at an end which is solved by Lemma 4.2.

The resulting surface has a circle of ends, all of which are symmetric across the same plane of reflective symmetry. There are $3n$ ends. Up to a homothety, we may assume that n of the ends have weight 1, and that the other $2n$ ends have weight r. We may choose r to be any positive number. As one travel around this circle of ends, the weights of the ends follow a pattern of $1, r, r, 1, r, r, \ldots, 1, r, r$. The angle between any two adjacent ends with different weights is θ. The angle between any two adjacent ends both of weight r is $2((\pi/n) - \theta)$. Thus, for each n, we have a 2-parameter family of these surfaces, with parameters θ and r.

Example 5.5. One can also produce a genus-1 counterpart to the last example, just as the genus-1 n-oid is a genus-1 counterpart to the genus-0 Jorge–Meeks n-oid. (See Figure 5.) The author has verified that one can construct finite contours α_i so that Theorem 2.1 can be applied to the genus-1 case as well. As before, we have a connected limit surface M and a conjugate fundamental piece M'. In this case M' has two period problems. One of them is at a catenoid end and is solved by Lemma 4.2. The other is not at an end, and we can solve this by a comparison argument using a portion of a helicoid. We do not include the comparison argument here, as it is similar to arguments in [**Ka4**], [**BeRo**], and [**Ro**].

6. Appendix.

In this Appendix, we will show that the limit surface M is connected and of finite total curvature in each of Examples 5.1, 5.2, 5.3, 5.4, and 5.5.

For Example 5.1: In this example, the surfaces M_i have boundaries ∂M_i consisting of six line segments and six vertices. At five of the vertices the exterior angle is $\pi/2$ radians, and at the other vertex the exterior angle approaches $(\pi/2)+(\pi/n)$ as $i \to \infty$. The Gauss–Bonnet theorem then implies that

$$\int_{M_i} |K| dA = -\int_{M_i} K dA \to \pi + \frac{\pi}{n}$$

as $i \to \infty$. (Note that dA is the area form on M_i induced as a submanifold of $\boldsymbol{R}^3$, and that $|K| = -K$ on a minimal surface.) Thus the limit surface M has finite total curvature at most $\pi + (\pi/n)$.

Suppose the second case $M = M_A \cup M_B$ described in Example 5.1 occurs; that is, suppose M is the union of two simply-connected minimal surfaces M_A and M_B. M_A is embedded, of finite total curvature, and is bounded by two rays as described in Example 5.1. Furthermore, $\mathscr{P}(M_A) \subseteq \lim_{i\to\infty} P_i$, since $\mathscr{P}(M) \subseteq \lim_{i\to\infty} P_i$. We will show that such an M_A cannot exist, deriving a contradiction that implies M is connected.

The conjugate surface M_A' of M_A has boundary $\partial M_A'$ consisting of two planar geodesics: one contained in the plane $\{x_1 = c_1\}$ for some constant $c_1 \in \boldsymbol{R}$, the other contained in $\{x_3 + \tan(\pi/n)x_2 = c_2\}$ for some constant $c_2 \in \boldsymbol{R}$. Let $\mathrm{Ref}_1 : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be reflection across the plane $\{x_1 = c_1\}$, and let $\mathrm{Ref}_2 : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be reflection across the plane $\{x_3 + \tan(\pi/n)x_2 = c_2\}$. Then $\hat{M}_A' := M_A' \cup \mathrm{Ref}_1(M_A') \cup \mathrm{Ref}_2(M_A' \cup \mathrm{Ref}_1(M_A'))$ is a complete minimal surface with finite total curvature, and is simply connected with a single end. By Theorem 9.5 of [**Os2**], the Gauss map G extends continuously across the end of $\hat{M}_A'$. Since the Gauss map is preserved by conjugation, G extends continuously across the end of M_A as well. Thus the normal vector at the end of M_A is well defined. This normal vector must be perpendicular to both of the rays in ∂M_A, hence it is $(0, -1, \tan(\pi/n))$. However, with this limiting normal vector at the end, it is clear that $\mathscr{P}(M_A) \not\subseteq \lim_{i\to\infty} P_i = \{(0,0,0)\} \cup \{(x_1, x_2, 0) | x_1 \geq 0, x_2 \in (0, w]\} \cup \{(x_1, x_2, 0) | x_1 \in [-1, 0), x_2 \leq w\}$, a contradiction.

For Example 5.2: Suppose the M in Example 5.2 is not connected. It then consists of two disjoint embedded simply-connected minimal surfaces M_A and M_B. Let M_A be the component bounded by the ray pointing in the direction of the positive x_1-axis with endpoint $(-1, w, 0)$, the ray pointing in the direction of the positive x_3-axis with endpoint $(-1, w - s\sin(\pi/n), -s\cos(\pi/n))$, and the line segment from $(-1, w, 0)$ to $(-1, w - s\sin(\pi/n), -s\cos(\pi/n))$. Since $\mathscr{P}(M_i) \subseteq P_i$, we have $\mathscr{P}(M_A) \subseteq \lim_{i\to\infty} P_i$. Using the Gauss–Bonnet theorem just like for Example 5.1, we conclude that M and M_A have finite total curvature.

The conjugate surface M_A' of M_A is bounded by three planar geodesics, one of finite length, another of infinite length contained in the plane $\{x_1 = c_1\}$ for some constant $c_1 \in \boldsymbol{R}$, and the third of infinite length contained in the plane $\{x_3 = c_2\}$ for some constant $c_2 \in \boldsymbol{R}$. Let $\mathrm{Ref}_1 : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be reflection across the plane $\{x_1 = c_1\}$, and let $\mathrm{Ref}_2 : \boldsymbol{R}^3 \to \boldsymbol{R}^3$ be reflection across the plane $\{x_3 = c_2\}$. Then $\hat{M}'_A := M'_A \cup \mathrm{Ref}_1(M'_A) \cup \mathrm{Ref}_2(M'_A \cup \mathrm{Ref}_1(M'_A))$ is an annular minimal surface of finite total curvature with a single compact boundary loop and a single end.

Unlike the case of Example 5.1, in this case the boundary $\partial\hat{M}'_A \neq \emptyset$; however, $\partial\hat{M}'_A$ is a compact loop and hence we may still apply Theorem 9.5 of [**Os2**] to conclude that the Gauss map G extends across the end of $\hat{M}'_A$. Hence G extends to the end of M_A. As in the case of Example 5.1, we see that $\mathscr{P}(M_A) \notin \lim_{i\to\infty} P_i$. This contradiction implies M is connected.

For Examples 5.3, 5.4, and 5.5: In these final three examples it can be shown, in the same way as for Example 5.2, that M is connected and has finite total curvature.

References

[BdC] L. Barbosa and M. do Carmo. On the Size of a Stable Minimal Surface in $\boldsymbol{R}^3$. American Journal of Mathematics, **98** (2) (1976), 515–528.

[BeRo] J. Berglund and W. Rossman. Minimal Surfaces with Catenoid Ends. Pacific J. Math., **171** (2) (1995), 353–371.

[DHKW] U. Dierkes, S. Hildebrandt, A. Kuster and O. Wohlrab. Minimal Surfaces I, II. A Series of Comprehensive Studies in Mathematics 295–296, Springer-Verlag, 1992.

[EaRo] Sa Earp and H. Rosenberg. The Dirichlet Problem for the Minimal Surface Equation on Unbounded Planar Domains. Journal de Mathématiques Pures et Appl., **68** (1989), 163–183.

[Evol] K. Brakke. Evolver software. Geometry Center, University of Minnesota, Minneapolis, MN.

[Gu] R. Gulliver. Regularity of Minimizing Surfaces of Prescribed Mean Curvature. Ann. Math., **97** (1973), 275–305.

[GuLe] R. Gulliver and F. D. Lesley. On Boundary Branch Points of Minimizing Surfaces. Arch. Rational Mech. Anal., **52** (1973), 20–25.

[Hi] S. Hildebrandt. Boundary Behavior of Minimal Surfaces. Arch. Rational Mech. Anal., **35** (1969), 47–82.

[JoMe] L. P. M. Jorge and W. H. Meeks III. The Topology of Complete Minimal Surfaces of Finite Total Gaussian Curvature. Topology, **22**(2) (1983), 203–221.

[Ka1] H. Karcher. Embedded Minimal Surfaces Derived From Scherk's Surfaces. Manuscripta math., **62** (1988), 83–114.

[Ka2] H. Karcher. The Triply Periodic Minimal Surfaces of Alan Schoen and their Constant Mean Curvature Companions. Manuscripta math. **64** (1989), 291–357.

[Ka3] H. Karcher. Construction of Minimal Surfaces. Surveys in Geometry, 1–96, University of Tokyo, 1989 (Also: Lecture Notes No. 12, SFB256, Bonn, 1989).

[Ka4] H. Karcher. Construction of Higher Genus Embedded Minimal Surfaces, Geom. and Top. of Submanifolds III, p. 174–191, World Sc. 1990.

[Kat] S. Kato. Construction of n-end Catenoids with Prescribed Flux. Kodai Math. J., **18** (1995), 86–98.

[MESH] J. Hoffman. Software for Constructing Minimal Surfaces Using Weierstrass Data. G.A.N.G. Lab, University of Massachusetts, Amherst, MA.

[MeYa1] W. H. Meeks and S-T Yau. The Classical Plateau Problem and the Topology of Three-Dimensional Manifolds. Topology **21**(4) (1982), 409–442.

[MeYa2] W. H. Meeks and S-T Yau. The Existence of Embedded Minimal Surfaces and the Problem of Uniqueness. Math. Z., **179** (1982), 151–168.

[Os] R. Osserman. A Proof of the Regularity Everywhere of the Classical Solution to Plateau's Problem. Ann. of Math., **91** (1970), 550–569.

[Os2] R. Osserman. A Survey of Minimal Surfaces. Dover Publications, Inc., New York, NY, 1986.

[Ro] W. Rossman. Minimal Surfaces in R^3 with Dihedral Symmetry. Tohoku Math. J., **47** (1995), 31–54.

[Scn] R. Schoen. Uniqueness, Symmetry, and Embeddedness of Minimal Surfaces. Journal of Differential Geometry, **18** (1982), 791–809.

[UmYa] M. Umehara and K. Yamada. Surfaces of Constant Mean Curvature c in $\boldsymbol{H}^3(-c^2)$ with Prescribed Hyperbolic Gauss Map. Math. Ann., **304** (1996), 203–224.

[Wo] M. Wohlgemuth. Higher Genus Minimal Surfaces by Growing Handles Out of a Catenoid. Manuscripta Math., **70** (1991), 397–428.

[Xu] Y. Xu. Symmetric Minimal Surfaces in $\boldsymbol{R}^3$. Pacific J. Math., **171**(1) (1995), 275–296.

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