J. Math. Soc. Japan
Vol. 52, No. 2, 2000

The group of one-dimensional bimodules arising from composition of subfactors

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(Received Apr. 30, 1998)
(Revised Aug. 13, 1998)

Abstract. We assume that a factor is equipped with outer actions of two finite groups, and consider a factor-subfactor pair consisting of the crossed product by one group and the fixed point algebra by the other. For this inclusion of factors, the group of one-dimensional bimodules appearing at even levels of the associated graph and that of non-strongly outer automorphisms for the subfactor (i.e., centrally trivial ones in the strongly amenable case) are determined.

1. Introduction.

Composition of subfactors is a useful method to construct new subfactors. For example, let $\mathscr{L}$ be a factor equipped with two outer actions $\alpha : H \to \mathrm{Aut}(\mathscr{L})$ and $\beta : K \to \mathrm{Aut}(\mathscr{L})$ of finite groups H and K respectively. Then, the crossed product by the H-action and the fixed point algebra by the K-action give rise to the inclusion

$$\mathscr{M} = \mathscr{L} \rtimes_{\alpha} H \supseteq \mathscr{N} = \mathscr{L}^{(\beta,K)}$$

of factors. It is easy to see that the Jones index ([**27**]) here is $[\mathscr{M};\mathscr{N}] = \sharp H \times \sharp K$ and that the irreducibility of the inclusion is equivalent to $\alpha_H \cap \beta_K = \{1\}$ in the quotient group $\mathrm{Out}(\mathscr{L}) = \mathrm{Aut}(\mathscr{L})/\mathrm{Int}(\mathscr{L})$. We remark that inclusions of index 4 with the Coxeter-Dynkin graphs $D_n^{(1)}$ (see [**25**], [**53**] for example) are known to arise in this way (with $H = K = \boldsymbol{Z}_2$).

Two actions α_H and β_K generate an infinite group in $\mathrm{Out}(\mathscr{L})$ if and only if $\mathscr{M} \supseteq \mathscr{N}$ is of infinite depth. Therefore, as long as two actions are "unrelated", the inclusion is of infinite depth, and in this way an abundance of inclusions of factors with various delicate growth conditions (of graphs) were constructed in [**2**]. On the other hand, when

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1991 *Mathematics Subject Classification.* Primary 46L37; Secondary 46L40.

Key Words and Phrases. bimodules, composition of subfactors, sectors, subfactors.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 09304017), Ministry of Education, Science and Culture, Japan.

α_H and β_K form a product group in $\mathrm{Out}(\mathscr{L})$, the above inclusion is of depth 2 and hence a Hopf algebra (more precisely a Kac algebra, [**28**]) of dimension $\sharp H \times \sharp K$ naturally appears thanks to Ocneanu's theorem ([**6**], [**21**], [**47**], [**60**], see also [**58**] for discussion based on the pentagon equation ([**1**])). A somewhat simple-minded but still quite important example is as follows: An outer action of the product group $H \cdot K$ on $\mathscr{L}$ is fixed and α, β are simply the restrictions to the subgroups H and K. In this case one actually obtains a bicrossed product of Majid-type ([**48**], [**61**], see also [**7**], [**57**] for an operator algebra theoretic treatment) as the resulting Hopf algebra. However, a situation is more subtle in the general depth 2 case. In fact, there are many unitaries around corresponding to inner perturbation, and those are sources for various cocycles. In this way we get "cocycle-twisted" Majid-type Hopf algebras, and it was the subject matter of the recent articles [**18**], [**26**].

In the Ocneanu approach ([**49**], [**50**]) on subfactor analysis, vertices of graphs arising from inclusions are regarded as irreducible bimodules. Hence, one can talk about the irreducible decomposition of the relative tensor product ([**59**]) of two bimodules (i.e., fusion rule), and hence one obtains a finer invariant for the subfactor in question. For example, bimodules at even levels in the dual principal graph are irreducible $\mathscr{M}$-$\mathscr{M}$ bimodules appearing in ${}_{\mathscr{M}}L^2(\mathscr{M}_k)_{\mathscr{M}}, k = 0, 1, 2, \dots$ (where $\{\mathscr{M}_k\}_{k=0,1,2,\dots}$ is the Jones tower, [**27**]). Note that in the sector approach for subfactors ([**19**], [**20**], [**45**]) these correspond to descendant sectors in the sense of [**19**]. One-dimensional bimodules (or equivalently sectors) among them are automorphisms of $\mathscr{M}$ (more precisely elements in $\mathrm{Out}(\mathscr{M})$). Importance of these automorphisms in general subfactor analysis was first pointed out by M. Izumi ([**19**]), and then they were characterized as non-strongly outer (or non-properly outer, [**55**]) automorphisms in [**3**], [**35**], [**36**] (see [**22**], [**24**], [**32**], [**38**], [**41**] for related topics). These automorphisms are known to play important roles in construction of orbifold subfactors ([**9**], [**12**], [**31**]) and in Goldman's type theorems ([**20**], [**23**], see also [**16**], [**17**] in a slightly different approach).

For the inclusion $\mathscr{M} = \mathscr{L} \rtimes_\alpha H \supseteq \mathscr{N} = \mathscr{L}^{(\beta, K)}$ (which is not necessarily of depth 2), in principle one should be able to write down all the relevant irreducible bimodules and their fusion rules. Indeed, what we need here is a cocycle-twisted version of the computation carried out in [**29**], [**40**], [**42**] (although it could be quite complicated). The purpose of the present article is to describe completely the group of one-dimensional bimodules described in the previous paragraph and that of the centrally trivial automorphisms (in the strongly amenable case) for the subfactor (see [**30**], [**54**], [**55**]). These groups are determined in §5 and §6 respectively after discussions in §3, §4 on descendant sectors (i.e., bimodules). Note that in the depth 2 case the two groups coincide. It is exactly the intrinsic group of the associated Hopf algebra, i.e., the group of all the group-like elements, and related results in a different method can be found in [**65**]. As an application of our analysis, in §7 Kawahigashi's relative χ group ([**30**]) of the inclusion with the Coxeter-Dynkin graph $D_{2n}^{(1)}$ is determined. Three appendices will be given. The first and third ones have nothing to do with the special inclusions considered here, and could be of independent interest.

The authors thank the referee for careful reading of the manuscript (and indeed simplifying proofs of Lemmas 2 and 9), and are grateful to JSPS for the support which made this joint work possible. Main results here were announced in [**37**]. It should be pointed out that closely related subjects are also studied in [**64**] based on tensor categories.

2. Preliminaries.

The Doplicher-Haag-Roberts theory ([**8**], see also [**56**]) on sectors originally occurred in QFT, and it was noticed by R. Longo that it is extremely useful to deal with subfactors. Here, we summarize basic facts on sectors ([**19**], [**45**], [**46**]) and relationship to bimodules needed for our later purpose. Details on bimodules can be found in [**4**], [**49**], [**50**], [**52**], [**59**], [**62**], [**63**]. Also quite a complete list of references on the index theory and related topics can be found in the recent book [**10**].

2.1. Sectors and bimodules. Let $\mathscr{M},\mathscr{N}$ be type III factors, and $\mathrm{End}(\mathscr{M},\mathscr{N})$ be the (unital normal) endomorphisms from $\mathscr{M}$ into $\mathscr{N}$. An $\mathscr{M}$-$\mathscr{N}$ bimodule $\mathscr{X}$ $(= {}_{\mathscr{M}}\mathscr{X}_{\mathscr{N}})$ is a Hilbert space equipped with commuting normal representations of $\mathscr{M}$ and the opposite algebra $\mathscr{N}^{\circ}$ (i.e., an $\mathscr{M}$-$\mathscr{N}$ correspondence in the sense of A. Connes, see Chap. V, Appendix B, [**4**]). For a given $\rho \in \mathrm{End}(\mathscr{M},\mathscr{N})$, let $\mathscr{H}_{\rho}$ be the standard Hilbert space $L^2(\mathscr{N})$ equipped with the $\mathscr{M}$-$\mathscr{N}$ action $m \cdot \xi \cdot n = \rho(m)J_{\mathscr{N}}n^*J_{\mathscr{N}}\xi$, where $J_{\mathscr{N}}$ is the modular conjugation. In this way, $\rho \in \mathrm{End}(\mathscr{M},\mathscr{N})$ naturally gives rise to the $\mathscr{M}$-$\mathscr{N}$ bimodule $\mathscr{H}_{\rho}$. For a given $\mathscr{M}$-$\mathscr{N}$ bimodule $\mathscr{X}$ the left $\mathscr{N}$-action on $\mathscr{X}$ and that on $L^2(\mathscr{N})$ (i.e., $n \in \mathscr{N} \to J_{\mathscr{N}}n^*J_{\mathscr{N}}$) are spatially implemented, since $\mathscr{N}$ is a type III factor. From this one easily proves that $\mathscr{X}$ is always of the form $\mathscr{H}_{\rho}$ (up to unitary equivalence) for some $\rho \in \mathrm{End}(\mathscr{M},\mathscr{N})$. Two $\mathscr{M}$-$\mathscr{N}$ bimodules $\mathscr{H}_{\rho_1},\mathscr{H}_{\rho_2}$ are unitarily equivalent if and only if ρ_1,ρ_2 are inner conjugate (i.e., $\rho_1(m) = u\rho_2(m)u^*, m \in \mathscr{M}$, for some unitary $u \in \mathscr{N}$). We set $\mathrm{Sect}(\mathscr{M},\mathscr{N}) = \mathrm{End}(\mathscr{M},\mathscr{N})/\mathrm{Int}(\mathscr{M},\mathscr{N})$ (and $\mathrm{Sect}(\mathscr{M},\mathscr{M}) = \mathrm{Sect}(\mathscr{M})$ for simplicity), the $\mathscr{M}$-$\mathscr{N}$ sectors. We will denote the class $[\rho]$ simply by ρ to ease the notation. The discussions so far show that $\mathrm{Sect}(\mathscr{M},\mathscr{N})$ is the same as the $\mathscr{M}$-$\mathscr{N}$ bimodules (up to unitary equivalence). However, we point out that one of advantages of the sector theory is that the relative tensor product for bimodules ([**59**]) corresponds to the ordinary composition of sectors

$$\mathscr{H}_\eta \otimes_{\mathscr{M}_2} \mathscr{H}_\zeta = \mathscr{H}_{\zeta\eta} \quad (\text{as } \mathscr{M}_1\text{-}\mathscr{M}_3 \text{ bimodules})$$

for $\eta \in \mathrm{Sect}(\mathscr{M}_1, \mathscr{M}_2)$ and $\zeta \in \mathrm{Sect}(\mathscr{M}_2, \mathscr{M}_3)$.

Note that the algebra of self-intertwiners is the relative commutant

$$\begin{aligned}\mathrm{Hom}(\mathscr{H}_\rho, \mathscr{H}_\rho) &= \{T \in B(\mathscr{H}_\rho); T(m \cdot \xi \cdot n) = m \cdot T\xi \cdot n\} \\ &= \mathscr{N} \cap \rho(\mathscr{M})'.\end{aligned}$$

In particular, $\mathscr{H}_\rho$ is irreducible if and only if $\mathscr{N} \supseteq \rho(\mathscr{M})$ is irreducible as an inclusion (i.e., $\mathscr{N} \cap \rho(\mathscr{M})' = \boldsymbol{C}1$). When $\mathscr{N} \supseteq \rho(\mathscr{M})$ is of finite index, then the relative commutant $\mathscr{N} \cap \rho(\mathscr{M})'$ is finite dimensional so that (as in the representation theory) one can perform the irreducible decomposition

$$\rho = \rho_1 \oplus \rho_2 \oplus \rho_3 \oplus \cdots \oplus \rho_n.$$

More precisely, let $\{p_i\}_{i=1,2,\dots,n}$ be minimal projections in $\mathscr{N} \cap \rho(\mathscr{M})'$ summing up to 1, and we choose and fix isometries v_i in $\mathscr{N}$ satisfying $v_i v_i^* = p_i$ $(i = 1, 2, \dots, n)$. We set $\rho_i = v_i^* \rho(\cdot) v_i \in \mathrm{Sect}(\mathscr{M}, \mathscr{N})$, and its irreducibility follows from the minimality of $p_i \in \mathscr{N} \cap \rho(\mathscr{M})'$. The intertwining property $v_i \rho_i(\cdot) = \rho(\cdot) v_i$ shows

$$\sum_{i=1}^{n} v_i \rho_i(m) v_i^* = \sum_{i=1}^{n} \rho(m) v_i v_i^* = \rho(m).$$

Actually this is the meaning of the above direct sum.

The square root of the minimal index ([**14**], [**15**], [**45**]) of the inclusion $\mathscr{N} \supseteq \rho(\mathscr{M})$ is called the statistical dimension

$$d\rho = \sqrt{[\mathscr{N}; \rho(\mathscr{M})]_0}\left(\in \left\{2\cos\left(\frac{\pi}{n}\right); n = 3, 4, \dots\right\} \cup [2, +\infty]\right),$$

which is by definition the dimension $\dim \mathscr{H}_\rho$ of the bimodule $\mathscr{H}_\rho$. As was shown in [**33**], [**39**], [**46**], the statistical dimension is additive and multiplicative:

$$d(\rho_1 \oplus \rho_2) = d\rho_1 + d\rho_2, \quad d(\rho_1\rho_2) = d\rho_1 d\rho_2.$$

Let

$$\gamma \ (= \gamma_{\mathscr{N} \supseteq \rho(\mathscr{M})}) = \mathrm{Ad}(J_{\rho(\mathscr{M})} J_{\mathscr{N}}) : \mathscr{N} \to \rho(\mathscr{M})$$

be the canonical endomorphism ([**44**]). We set $\bar{\rho} = \rho^{-1} \circ \gamma$ (whose class in $\mathrm{Sect}(\mathscr{N}, \mathscr{M})$

depends only on that of ρ), the conjugate sector of ρ. As another consequence of being type III, we see that an endomorphism ρ always admits an implementation. In fact, we choose and fix a faithful normal state ϕ on $\mathcal{M}$, and set $\psi = \phi \circ \rho^{-1} \in \rho(\mathcal{M})_*^+$. Let $\psi = \omega_{\xi_1}$ (resp. $\phi = \omega_{\xi_2}$) with a cyclic and separating vector $\xi_1 \in L^2(\mathcal{N})$ for $\rho(\mathcal{M})$ (resp. with a cyclic and separating vector $\xi_2 \in L^2(\mathcal{M})$ for $\mathcal{M}$). Then, the map $m\xi_2 \in \mathcal{M}\xi_2 \to \rho(m)\xi_1 \in \rho(\mathcal{M})\xi_1$ gives rise to the surjective isometry U from $L^2(\mathcal{M})$ onto $L^2(\mathcal{N})$ (due to $\psi \circ \rho = \phi$), and it is an easy exercise to see

$$\rho(m) = UmU^*(m \in \mathcal{M}), \quad \bar{\rho} = \mathrm{Ad}(U^* J_{\rho(\mathcal{M})} J_{\mathcal{N}}), \quad \text{and} \quad J_{\rho(\mathcal{M})} U = U J_{\mathcal{M}}.$$

In the bimodule picture, passing from ρ to $\bar{\rho}$ corresponds to considering the contragradient bimodule. Namely, the $\mathcal{N}$-$\mathcal{M}$ bimodule $\mathcal{H}_{\bar{\rho}}$ associated with $\bar{\rho} \in \mathrm{Sect}(\mathcal{N}, \mathcal{M})$ is unitarily equivalent to the opposite Hilbert space $\overline{\mathcal{H}_\rho}$ $(= \overline{L^2(\mathcal{N})})$ equipped with the $\mathcal{N}$-$\mathcal{M}$ action

$$n \cdot \bar{\xi} \cdot m = \overline{m^* \cdot \xi \cdot n^*} = \overline{\rho(m^*) J_{\mathcal{N}} n J_{\mathcal{N}} \xi}.$$

In fact, $\mathcal{H}_{\bar{\rho}}$ is the Hilbert space $L^2(\mathcal{M})$ together with the following $\mathcal{N}$-$\mathcal{M}$ action:

$$\begin{aligned}
\bar{\rho}(n) J_{\mathcal{M}} m^* J_{\mathcal{M}} \xi &= U^* J_{\rho(\mathcal{M})} J_{\mathcal{N}} n J_{\mathcal{N}} J_{\rho(\mathcal{M})} U J_{\mathcal{M}} m^* J_{\mathcal{M}} \xi \\
&= J_{\mathcal{M}} U^* J_{\mathcal{N}} n J_{\mathcal{N}} U m^* J_{\mathcal{M}} \xi \\
&= J_{\mathcal{M}} U^* J_{\mathcal{N}} n J_{\mathcal{N}} \rho(m^*) U J_{\mathcal{M}} \xi.
\end{aligned}$$

The operators $\rho(m^*) \in \rho(\mathcal{M}) (\subseteq \mathcal{N})$ and $J_{\mathcal{N}} n J_{\mathcal{N}}$ in the last part commute so that the surjective isometry $\xi \in \mathcal{H}_{\bar{\rho}} \to \overline{U J_{\mathcal{M}} \xi} \in \overline{\mathcal{H}_\rho}$ gives rise to the desired unitary equivalence.

A very useful fact that will be repeatedly used is the Frobenius reciprocity

$$\dim \mathrm{Hom}(\zeta\eta, \rho) = \dim \mathrm{Hom}(\zeta, \rho\bar{\eta}),$$

where $\eta \in \mathrm{Sect}(\mathcal{M}_1, \mathcal{M}_2), \zeta \in \mathrm{Sect}(\mathcal{M}_2, \mathcal{M}_3), \rho \in \mathrm{Sect}(\mathcal{M}_1, \mathcal{M}_3)$ are sectors of finite statistical dimension. Note that we also have

$$\begin{aligned}
\dim \mathrm{Hom}(\zeta\eta, \rho) &= \dim \mathrm{Hom}(\overline{\zeta\eta}, \bar{\rho}) = \dim \mathrm{Hom}(\bar{\eta}\bar{\zeta}, \bar{\rho}) \\
&= \dim \mathrm{Hom}(\bar{\eta}, \bar{\rho}\zeta) = \dim \mathrm{Hom}(\eta, \overline{\bar{\rho}\zeta}) = \dim \mathrm{Hom}(\eta, \bar{\zeta}\rho)
\end{aligned}$$

thanks to $\overline{\zeta\eta} = \bar{\eta}\bar{\zeta}$, $\bar{\bar{\zeta}} = \zeta$, etc.

2.2. Inclusions of factors. We now assume that $\mathcal{M} \supseteq \mathcal{N}$ is an inclusion of factors of finite index. We further assume $\mathcal{N} = \rho_0(\mathcal{M})$ with $\rho_0 \in \mathrm{End}(\mathcal{M}) = \mathrm{End}(\mathcal{M}, \mathcal{M})$ (which means that $\mathcal{M}$ and $\mathcal{N}$ should be isomorphic). Note that we have $\bar{\rho}_0 =$

$\rho_0^{-1} \circ \gamma \in \mathrm{Sect}(\mathscr{M})$ with $\gamma \ (= \gamma_{\mathscr{M} \supseteq \rho_0(\mathscr{M})}) = \mathrm{Ad}(J_{\mathscr{N}} J_{\mathscr{M}}) : \mathscr{M} \to \mathscr{N}$ and $d\rho_0 = \sqrt{[\mathscr{M};\mathscr{N}]_0}$. In the subfactor picture, considering $\bar{\rho}_0$ corresponds to passing to the basic extension

$$\mathscr{M}_1 = J_{\mathscr{M}} \mathscr{N}' J_{\mathscr{M}} \ \ (= \langle \mathscr{M}, e_{\mathscr{N}} \rangle'') \supseteq \mathscr{M}$$

(see the second paragraph in **2.4**).

In the index theory the Jones tower

$$\mathscr{N} \subseteq \mathscr{M} = \mathscr{M}_0 \subseteq \mathscr{M}_1 \subseteq \mathscr{M}_2 \subseteq \mathscr{M}_3 \subseteq \cdots$$

of successive basic extensions is of fundamental importance ([**27**]). Note that $\mathscr{M}_1$ also acts standardly on $L^2(\mathscr{M})$ and $J_{\mathscr{M}_1} = J_{\mathscr{M}} J_{\mathscr{N}'} J_{\mathscr{M}} = J_{\mathscr{M}} J_{\mathscr{N}} J_{\mathscr{M}}$. Hence, the $\mathscr{M}$-$\mathscr{M}$ bimodule ${}_{\mathscr{M}}L^2(\mathscr{M}_1)_{\mathscr{M}}$ can be realized as the Hilbert space $L^2(\mathscr{M})$ equipped with the $\mathscr{M}$-$\mathscr{M}$ action

$$m_1 J_{\mathscr{M}_1} m_2^* J_{\mathscr{M}_1} = m_1 J_{\mathscr{M}} J_{\mathscr{N}} J_{\mathscr{M}} m_2^* J_{\mathscr{M}} J_{\mathscr{N}} J_{\mathscr{M}}.$$

On the other hand, the $\mathscr{M}$-$\mathscr{M}$ bimodule corresponding to $\rho_0\bar{\rho}_0 \in \mathrm{Sect}(\mathscr{M})$ is the same Hilbert space $L^2(\mathscr{M})$ equipped with the $\mathscr{M}$-$\mathscr{M}$ action

$$\rho_0\bar{\rho}_0(m_1) J_{\mathscr{M}} m_2^* J_{\mathscr{M}} = J_{\mathscr{N}} J_{\mathscr{M}} m_1 J_{\mathscr{M}} J_{\mathscr{N}} J_{\mathscr{M}} m_2^* J_{\mathscr{M}}$$

thanks to $\rho_0\bar{\rho}_0 = \gamma = \mathrm{Ad}(J_{\mathscr{N}} J_{\mathscr{M}})$. Notice that the two bimodules are unitarily equivalent via the unitary $J_{\mathscr{M}} J_{\mathscr{N}}$, that is, ${}_{\mathscr{M}}L^2(\mathscr{M}_1)_{\mathscr{M}} = \mathscr{H}_{\rho_0\bar{\rho}_0}$. In this way (with a little bit more work) we see that the decomposition rule for

$$1 \to \rho_0 \to \rho_0\bar{\rho}_0 \to \rho_0\bar{\rho}_0\rho_0 \to \rho_0\bar{\rho}_0\rho_0\bar{\rho}_0 \to \rho_0\bar{\rho}_0\rho_0\bar{\rho}_0\rho_0 \to \cdots$$

$$1 \to \bar{\rho}_0 \to \bar{\rho}_0\rho_0 \to \bar{\rho}_0\rho_0\bar{\rho}_0 \to \bar{\rho}_0\rho_0\bar{\rho}_0\rho_0 \to \bar{\rho}_0\rho_0\bar{\rho}_0\rho_0\bar{\rho}_0 \to \cdots$$

is described by the Bratteli diagrams for the towers $\{\mathscr{M}_k \cap \mathscr{M}'\}_{k=0,1,2,\dots}$, $\{\mathscr{M}_k \cap \mathscr{N}'\}_{k=0,1,2,\dots}$ respectively, i.e., the dual principal graph and the principal graph respectively.

Vertices in the Bratteli diagrams are simply minimal projections, however in the sector (or bimodule) picture they are actually irreducible sectors (or bimodules). These $\mathscr{M}$ (i.e., $\mathscr{M}$-$\mathscr{M}$) sectors were called descendant ones in [**19**]. When $\zeta_1 \prec (\rho_0\bar{\rho}_0)^n$ and $\zeta_2 \prec (\rho_0\bar{\rho}_0)^m$, we have $\zeta_1\zeta_2 \prec (\rho_0\bar{\rho}_0)^{n+m}$. Therefore, $\sqcup_k (\rho_0\bar{\rho}_0)^k$ is closed under the product (i.e., the relative tensor product in the bimodule picture) and one can talk about the irreducible decomposition of $\zeta_1\zeta_2$ (i.e., the fusion rule).

2.3. One-dimensional sectors (automorphisms). Let $\mathscr{M} \supseteq \mathscr{N} = \rho_0(\mathscr{M})$ (with some $\rho_0 \in \mathrm{Sect}(\mathscr{M})$) be as in **2.2**. A sector a ($\in \mathrm{Sect}(\mathscr{M})$) satisfies $da = 1$ if and only if a is an automorphism of $\mathscr{M}$ (or more precisely an element in $\mathrm{Out}(\mathscr{M}) = \mathrm{Aut}(\mathscr{M})/\mathrm{Int}(\mathscr{M})$). Notice that automorphisms (i.e., the one-dimensional sectors) in $\sqcup_k (\rho_0\bar{\rho}_0)^k$ form a group. This is the group of non-strongly outer automorphisms in the sense of [**3**], [**35**], [**36**]. It is obviously a conjugacy invariant for a subfactor and plays important roles in recent study of subfactors as mentioned in §1. The condition $a \prec (\rho_0\bar{\rho}_0)^n$ (for some n) with $da = 1$ does not necessarily guarantee that a can be adjusted (after inner perturbation) to an automorphism leaving $\mathcal{N}$ invariant, i.e., $\mathrm{Ad}\, u \circ a(\mathcal{N}) = \mathcal{N}$ for some unitary $u \in \mathcal{M}$. (See **7.2** and p. 284 of [**40**] for typical non-adjustable ones). We denote the group of "adjustable" non-strongly outer automorphisms by $\Gamma = \Gamma(\mathcal{M}, \mathcal{N})$. An automorphism a ($\in \mathrm{Aut}(\mathcal{M})$) is adjustable (relative to $\mathcal{N} = \rho_0(\mathcal{M})$) if and only if there exists an automorphism b (i.e., $db = 1$) satisfying $a\rho_0 = \rho_0 b$ in $\mathrm{Sect}(\mathcal{M})$. Furthermore, when ρ_0 is irreducible, $a \prec (\rho_0\bar{\rho}_0)^n$ if and only if $b \prec (\bar{\rho}_0\rho_0)^n$ (see [**35**], [**36**]). We point out that the automorphisms in Γ are the centrally trivial automorphisms (for a subfactor) in the sense of [**30**] (see also [**53**]) when an inclusion is strongly amenable ([**54**]). A criterion for the strong amenability (for special subfactors treated in this article) was worked out in [**2**].

Let $\mathcal{A} \supseteq \mathcal{B}$ be a finite index inclusion of (AFD II_1) factors with finite depth. Kawahigashi's relative χ group $\chi(\mathcal{A}, \mathcal{B})$ ([**30**]) is defined by

$$\chi(\mathcal{A}, \mathcal{B}) = \frac{\mathrm{Ct}(\mathcal{A}, \mathcal{B}) \cap \overline{\mathrm{Int}(\mathcal{A}, \mathcal{B})}}{\mathrm{Int}(\mathcal{A}, \mathcal{B})},$$

where $\mathrm{Ct}(\mathcal{A}, \mathcal{B})$ means the group of centrally trivial automorphisms and $\mathrm{Int}(\mathcal{A}, \mathcal{B}) = \{\mathrm{Ad}\, u \in \mathrm{Aut}(\mathcal{A}, \mathcal{B});\ u \text{ is a unitary in } \mathcal{B}\}$. The closure $\overline{\mathrm{Int}(\mathcal{A}, \mathcal{B})}$ is characterized as the group of automorphisms (in $\mathrm{Aut}(\mathcal{A}, \mathcal{B})$) with trivial Loi invariant (see [**43**]). On the other hand, as mentioned above, the sector technique enables us to determine what $\mathrm{Ct}(\mathcal{A}, \mathcal{B})$ is. In **7.1** we will determine the relative χ group for inclusions $\mathcal{A} \supseteq \mathcal{B}$ of index 4 with Coxeter-Dynkin graphs $D_{2n}^{(1)}$.

In the above we assumed that $\mathcal{M} \supseteq \mathcal{N}$ are isomorphic type III factors. However, thanks to the standard tensoring trick in [**46**] this assumption is completely irrelevant. In the rest of the article we will use both sectors and bimodules almost synonymously (as explained in **2.1**).

2.4. Composition of subfactors. Let $\mathcal{L}$ be a factor equipped with outer actions α, β of finite groups H and K respectively. We assume $\alpha_H \cap \beta_K = \{1\}$ in $\mathrm{Out}(\mathcal{L}) = \mathrm{Aut}(\mathcal{L})/\mathrm{Int}(\mathcal{L})$ so that

$$\mathcal{N} = \mathcal{L}^{(\beta, K)} \subseteq \mathcal{M} = \mathcal{L} \rtimes_\alpha H$$

is an irreducible inclusion of factors of index $\sharp H \times \sharp K$, and this is the inclusion investigated in the article.

Let $\mathscr{L}^{(\alpha,H)} = \rho_1(\mathscr{L})$ and $\mathscr{L}^{(\beta,K)}$ $(= \mathscr{N}) = \rho_2(\mathscr{L})$ with $\rho_i \in \mathrm{Sect}(\mathscr{L})$. Since $\bar{\rho}_1 = \rho_1^{-1} \circ \mathrm{Ad}\, J_{\rho_1(\mathscr{L})} J_{\mathscr{L}}$, we have $\bar{\rho}_1^{-1} = \mathrm{Ad}\, J_{\mathscr{L}} J_{\rho_1(\mathscr{L})} \circ \rho_1$. (As was seen in **2.1**, we have $\rho = \mathrm{Ad}\, U$ and hence $\rho^{-1} = \mathrm{Ad}\, U^*$. Note that in this way $\rho_1^{\pm 1}, \bar{\rho}_1^{\pm 1}$ and so on are all defined on the whole $B(L^2(\mathscr{L}))$.) Therefore, we compute

$$\bar{\rho}_1^{-1}(\mathscr{L}) = J_{\mathscr{L}} J_{\rho_1(\mathscr{L})} \rho_1(\mathscr{L}) J_{\rho_1(\mathscr{L})} J_{\mathscr{L}} = J_{\mathscr{L}} \rho_1(\mathscr{L})' J_{\mathscr{L}}.$$

This is the basic extension of $\mathscr{L} \supseteq \rho_1(\mathscr{L})$, and it is $\mathscr{L} \rtimes_\alpha H$ in our present setting. Therefore, we have seen $\bar{\rho}_1(\mathscr{L} \rtimes_\alpha H) = \mathscr{L}$. However, remark that this $\bar{\rho}_1$ is no longer an element in $\mathrm{Sect}(\mathscr{L})$ and it is indeed an endomorphism from $\mathscr{L} \rtimes_\alpha H$ onto $\mathscr{L}$.

Notations. To avoid possible confusion, $\bar{\rho}_1$ considered as an element in $\mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$ will be denoted by $\bar{\varrho}_1$ throughout the article. Since $\bar{\varrho}_1(\mathscr{L} \rtimes_\alpha H) = \mathscr{L}$ as an $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L}$ sector, the dimension of $\bar{\varrho}_1$ is 1 and the conjugate of $\bar{\varrho}_1$ is $\bar{\varrho}_1^{-1} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\alpha H)$. Similarly, $\bar{\rho}_2$ considered as an element in $\mathrm{Sect}(\mathscr{L} \rtimes_\beta K, \mathscr{L})$ will be denoted by $\bar{\varrho}_2$. We have $\bar{\varrho}_2(\mathscr{L} \rtimes_\beta K) = \mathscr{L}$ and the conjugate of $\bar{\varrho}_2$ is $\bar{\varrho}_2^{-1} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\beta K)$.

Via the "identification map" $\bar{\varrho}_1$, we have the conjugacy

$$\mathscr{L} \rtimes_\alpha H \supseteq \mathscr{L} \supseteq \mathscr{L}^{(\beta,K)} \cong \mathscr{L} \supseteq \bar{\varrho}_1(\mathscr{L}) \supseteq \bar{\varrho}_1(\mathscr{L}^{(\beta,K)}).$$

But, $\bar{\varrho}_1 = \bar{\rho}_1$ on $\mathscr{L}$, and hence $\bar{\varrho}_1(\mathscr{L}^{(\beta,K)}) = \bar{\rho}_1(\mathscr{L}^{(\beta,K)}) = \bar{\rho}_1 \rho_2(\mathscr{L})$.

Discussions so far suggest the following definition:

Definition. We set

$$\rho_0 = \bar{\varrho}_1^{-1} \bar{\rho}_1 \rho_2 \bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H)$$

with $\bar{\varrho}_1^{-1} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\alpha H), \bar{\rho}_1, \rho_2 \in \mathrm{Sect}(\mathscr{L})$ and $\bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$.

We have $\rho_0(\mathscr{L} \rtimes_\alpha H) = \mathscr{L}^{(\beta,K)}$ so that ρ_0 is our basic sector from which relevant irreducible components (descendant sectors), higher relative commutants, fusion rules, and so on can be computed (see **2.2**).

3. Alternating products.

In the present paper we investigate the inclusion

$$\mathscr{N} = \mathscr{L}^{(\beta,K)} \subseteq \mathscr{M} = \mathscr{L} \rtimes_\alpha H$$

of factors explained in **2.4**, and we set $\rho_0 = \bar{\varrho}_1^{-1} \bar{\rho}_1 \rho_2 \bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H)$ as in **2.4** (so that $\rho_0(\mathscr{L} \rtimes_\alpha H) = \mathscr{L}^{(\beta,K)}$). In this section we will compute higher relative commutants of the inclusion $\mathscr{M} \supseteq \mathscr{N}$ based on sector technique. (Related analysis can be found in [**2**], [**29**], [**42**].)

We compute

$$\rho_0\bar{\rho}_0 = \bar{\varrho}_1^{-1}\bar{\rho}_1\rho_2\bar{\varrho}_1\bar{\varrho}_1^{-1}\bar{\rho}_2\rho_1\bar{\varrho}_1 = \bar{\varrho}_1^{-1}\bar{\rho}_1\rho_2\bar{\rho}_2\rho_1\bar{\varrho}_1$$

since $\bar{\varrho}_1\bar{\varrho}_1^{-1} \in \mathrm{Sect}(\mathscr{L})$ is id. We carefully look at the products $\bar{\varrho}_1^{-1}\bar{\rho}_1 \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\alpha H)$ and $\rho_1\bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$. At first, we obviously have $\bar{\varrho}_1^{-1}\bar{\rho}_1 = \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\alpha H)$, the inclusion map. On the other hand, $\rho_1\bar{\varrho}_1$ is the conjugate of $\bar{\varrho}_1^{-1}\bar{\rho}_1$ and $\rho_1\bar{\varrho}_1 = \mathrm{Ad}(J_{\rho_1(\mathscr{L})}J_{\mathscr{L}}) \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$. Since $\mathscr{L} \rtimes_\alpha H = J_{\mathscr{L}}\rho_1(\mathscr{L})'J_{\mathscr{L}}$, the basic extension, we have $J_{\mathscr{L} \rtimes_\alpha H} = J_{\mathscr{L}}J_{\rho_1(\mathscr{L})'}J_{\mathscr{L}} = J_{\mathscr{L}}J_{\rho_1(\mathscr{L})}J_{\mathscr{L}}$, and

$$\rho_1\bar{\varrho}_1 = \mathrm{Ad}(J_{\mathscr{L}}J_{\mathscr{L} \rtimes_\alpha H}) = \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}).$$

Therefore, we have

$$\rho_0\bar{\rho}_0 = \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} = \sum_{k \in K}{}^{\oplus} \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \beta_k \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}.$$

From this expression, we also have

$$(\rho_0\bar{\rho}_0)^2 = \sum_{k_1, k_2, h_1}{}^{\oplus} \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \beta_{k_1}\alpha_{h_1}\beta_{k_2} \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}},$$

$$(\rho_0\bar{\rho}_0)^3 = \sum_{k_1, k_2, k_3, h_1, h_2}{}^{\oplus} \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \beta_{k_1}\alpha_{h_1}\beta_{k_2}\alpha_{h_2}\beta_{k_3} \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}},$$

and so on, where k_i's run over K and h_j's run over H. In fact, we just notice

$$\overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} = \rho_1\bar{\varrho}_1\bar{\varrho}_1^{-1}\bar{\rho}_1 = \rho_1\bar{\rho}_1 = \sum_{h \in H}{}^{\oplus} \alpha_h.$$

On the other hand, we have

$$\rho_0\bar{\rho}_0\rho_0 = \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \bar{\varrho}_1^{-1}\bar{\rho}_1\rho_2\bar{\varrho}_1.$$

Since

$$\overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \bar{\varrho}_1^{-1}\bar{\rho}_1 = \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} = \rho_1\bar{\rho}_1 \in \mathrm{Sect}(\mathscr{L}),$$

we have

$$\rho_0\bar{\rho}_0\rho_0 = \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \rho_1\bar{\rho}_1 \circ \rho_2\bar{\varrho}_1.$$

The last product can be rewritten as

$$\rho_2\bar{\varrho}_1 = \rho_2\bar{\varrho}_2\bar{\varrho}_2^{-1}\bar{\varrho}_1 = \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1.$$

Therefore, we conclude

$$\rho_0\bar{\rho}_0\rho_0 = \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1.$$

The decomposition of this sector is the same as that of

$$\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \in \mathrm{Sect}(\mathscr{L}\rtimes_\beta K, \mathscr{L}\rtimes_\alpha H),$$

(and $\bar{\varrho}_2^{-1}\bar{\varrho}_1$ just specifies an identification map between $\mathscr{L}\rtimes_\alpha H$ and $\mathscr{L}\rtimes_\beta K$). It is also easy to see

$$\begin{aligned}
(\rho_0\bar{\rho}_0)^2\rho_0 &= \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2\bar{\rho}_2 \circ \rho_1\bar{\rho}_1 \circ \rho_2\bar{\rho}_2 \circ \rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1 \\
&= \sum_{k_1,k_2,h_1,h_2}{}^{\oplus}\, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \beta_{k_1}\alpha_{h_1}\beta_{k_2}\alpha_{h_2} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1, \\
(\rho_0\bar{\rho}_0)^3\rho_0 &= \sum_{k_1,k_2,k_3,h_1,h_2,h_3}{}^{\oplus}\, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \beta_{k_1}\alpha_{h_1}\beta_{k_2}\alpha_{h_2}\beta_{k_3}\alpha_{h_3} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1,
\end{aligned}$$

and so on.

Lemma 1. *Let $\theta = \beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}, \theta' = \beta_{k_1'}\alpha_{h_1'}\beta_{k_2'}\cdots\beta_{k_m'}$ be alternating products. Then, $\mathscr{L}\rtimes_\alpha H$-$\mathscr{L}\rtimes_\alpha H$ sectors $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ and $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta' \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ are not disjoint if and only if $\alpha_H\theta\alpha_H = \alpha_H\theta'\alpha_H$ in $\mathrm{Out}(\mathscr{L})$.*

Proof. The two sectors in question are not disjoint if and only if

$$\dim \mathrm{Hom}(\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta' \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}) \neq 0.$$

But this dimension is equal to

$$\begin{aligned}
&\dim \mathrm{Hom}(\theta \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}, \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta') \\
&\quad = \sum_{h_1,h_2\in H} \dim \mathrm{Hom}([\theta\alpha_{h_1}], [\alpha_{h_2}\theta'])
\end{aligned}$$

thanks to the Frobenius reciprocity (see **2.1**). Therefore, the condition is satisfied if and only if $[\theta\alpha_{h_1}] = [\alpha_{h_2}\theta']$ as sectors (i.e., they are the same in $\mathrm{Out}(\mathscr{L})$) for some $h_1, h_2 \in H$. This condition is obviously equivalent to $\alpha_H\theta\alpha_H = \alpha_H\theta'\alpha_H$ in $\mathrm{Out}(\mathscr{L})$. □

Lemma 2. *If $\alpha_H\theta\alpha_H = \alpha_H\theta'\alpha_H$ in $\mathrm{Out}(\mathscr{L})$, then we have*

$$\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} = \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta' \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}.$$

Proof. Since $\theta' = \alpha_h\theta\alpha_{h'}$ in $\mathrm{Out}(\mathscr{L})$ for some $h, h' \in H$, the dimension in the proof of the previous lemma is equal to

$$\sum_{h_1,h_2\in H}\dim\mathrm{Hom}([\alpha_{h_1}],[\theta^{-1}\alpha_{h_2}\theta']) = \sum_{h_1,h_2\in H}\dim\mathrm{Hom}([\alpha_{h_1}],[\theta^{-1}\alpha_{h_2}\alpha_h\theta\alpha_{h'}])$$

$$= \sum_{h_1,h_2\in H}\dim\mathrm{Hom}([\alpha_{h_1h'^{-1}}],[\theta^{-1}\alpha_{h_2h}\theta]) = \sum_{h_1,h_2\in H}\dim\mathrm{Hom}([\alpha_{h_1}],[\theta^{-1}\alpha_{h_2}\theta])$$

$$= \sharp[\alpha_H\cap\theta^{-1}\alpha_H\theta] = \frac{(\sharp H)^2}{\sharp[\alpha_H\theta\alpha_H]}.$$

From this computation, it is clear that the dimensions of the spaces of self-intertwiners of $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ and $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta'\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ are also identical to the above cardinality. Since the above two $\mathscr{L}\rtimes_\alpha H$-$\mathscr{L}\rtimes_\alpha H$ sectors have the same dimension $(=\sqrt{\sharp H}\times 1\times\sqrt{\sharp H}=\sharp H)$, we get the lemma from the result in Appendix A. $\square$

The referee kindly pointed out the following alternative (and simpler) proof: Notice $\mathrm{Ad}\,\lambda_h\circ\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}=\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\alpha_h(h\in H)$ as endomorphisms with the canonical implementing unitary λ_h in $\mathscr{L}\rtimes_\alpha H$. This means that we have $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}=\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\alpha_h$ and $\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}=\alpha_h\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ as sectors. Therefore, $\theta'=\alpha_h\theta\alpha_{h'}$ (in $\mathrm{Out}(\mathscr{L})$) clearly implies the conclusion $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta'\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}=\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$.

Definition. We set $\theta\sim\theta'$ if $\alpha_H\theta\alpha_H=\alpha_H\theta'\alpha_H$ in $\mathrm{Out}(\mathscr{L})$.

We just consider the alternating products $\theta=\beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}$ of length $2n-1$. Dividing those alternating products into the classes (the above definition), we choose representatives $\{\theta_i\}_{i=1,2,\dots,\ell}$, and for each i let n_i be the number of θ's (of length $2n-1$) equivalent to θ_i. The discussions so far obviously mean

Proposition 3. *We have the following decomposition*:

$$(\rho_0\bar{\rho}_0)^n=\sum_{i=1}^{\ell}{}^{\oplus}n_i(\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta_i\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}).$$

Of course the decomposition of $(\rho_0\bar{\rho}_0)^n\rho_0$ can be described in an analogous fashion.

4. Self-intertwiners of $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$.

A sector of the form $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}\in\mathrm{Sect}(\mathscr{L}\rtimes_\alpha H)$ appearing in Proposition 3 is not irreducible. Its irreducible decomposition is known if the algebra of self-intertwines is determined. In this section this will be carried out by writing down explicitly the $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ bimodule corresponding to $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}$. (See also [**2**] for computation of the principal and dual principal graphs.)

The sector $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H)$ corresponds to the following $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ bimodule:

$$ {}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}^{\theta} L^2(\mathscr{L})_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L} \rtimes_\alpha H} $$

(see **2.1**). Here, $\mathscr{L}$-$\mathscr{L}$ bimodule ${}_{\mathscr{L}}^{\theta} L^2(\mathscr{L})_{\mathscr{L}}$ means the standard Hilbert space $L^2(\mathscr{L})$ equipped with the (mutually commuting) $\mathscr{L}$-$\mathscr{L}$ action $\theta(\ell_1) J_{\mathscr{L}} \ell_2^* J_{\mathscr{L}} (\ell_1, \ell_2 \in \mathscr{L})$. Note that ${}_{\mathscr{L}} L^2(\mathscr{L})_{\mathscr{L}}^{\theta}$ can be defined analogously and that ${}_{\mathscr{L}}^{\theta} L^2(\mathscr{L})_{\mathscr{L}}$ and ${}_{\mathscr{L}} L^2(\mathscr{L})_{\mathscr{L}}^{\theta^{-1}}$ are the same (more precisely unitarily equivalent) bimodule via $\mathrm{Ad}\, U_\theta$. Here and throughout the article, U_θ denotes the canonical implementation of θ. At first we notice

$$ \begin{aligned} {}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}^{\theta} L^2(\mathscr{L})_{\mathscr{L}} &= {}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}} L^2(\mathscr{L})_{\mathscr{L}}^{\theta^{-1}} \\ &= {}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}}^{\theta^{-1}}. \end{aligned} $$

Here, the notation ${}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}}^{\theta^{-1}}$ is self-explanatory. Namely, this is a Hilbert space $L^2(\mathscr{L}) \otimes \ell^2(H)$, which is the standard Hilbert space of $\mathscr{L} \rtimes_\alpha H$, equipped with the left $\mathscr{L} \rtimes_\alpha H$-action

$$ \sum_{h' \in H} \alpha_{h'^{-1}}(\ell) \otimes e_{h',h'} $$

$$ 1 \otimes \lambda(h) $$

(the ordinary generators of $\mathscr{L} \rtimes_\alpha H$) and the right $\mathscr{L}$-action

$$ J_{\mathscr{L}} \theta^{-1}(\ell^*) J_{\mathscr{L}} \otimes 1 $$

(a part of the ordinary generators of the commutant of $\mathscr{L} \rtimes_\alpha H$ twisted by θ^{-1}). Here and in what follows, $e_{h,h'} \in B(\ell^2(H))$ will denote the obvious matrix unit. Via $\mathrm{Ad}(U_\theta \otimes 1)$ these operators are transformed to

$$ \sum_{h' \in H} \theta\alpha_{h'^{-1}}(\ell) \otimes e_{h',h'}, \tag{1} $$

$$ 1 \otimes \lambda(h), \tag{2} $$

$$ J_{\mathscr{L}} \ell^* J_{\mathscr{L}} \otimes 1. \tag{3} $$

Hence, the bimodule ${}_{\mathscr{L} \rtimes_\alpha H} L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}}^{\theta^{-1}}$ can be also thought of $L^2(\mathscr{L}) \otimes \ell^2(H)$ equipped with the left $\mathscr{L} \rtimes_\alpha H$-action (1), (2) and the right $\mathscr{L}$-action (3). In what follows, we will use this realization.

On the other hand, the bimodule ${}_{\mathscr{L}}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L} \rtimes_\alpha H}$ is $L^2(\mathscr{L}) \otimes \ell^2(H)$ equipped with the left $\mathscr{L}$-action

$$\ell \otimes 1 \tag{4}$$

and the right $\mathscr{L} \rtimes_\alpha H$-action

$$\sum_{h'' \in H} U_{\alpha_{h''^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h''}} \otimes e_{h'',h''} \tag{5}$$

$$1 \otimes \lambda(h). \tag{6}$$

Note that the "amplification picture" of the crossed product $\mathscr{L} \rtimes_\alpha H$ is used here, and the reason of doing so will become clear shortly.

We now consider

$$
\begin{aligned}
&{}_{\mathscr{L} \rtimes_\alpha H}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}} \otimes_{\mathscr{L}} {}^{\theta}_{\mathscr{L}}L^2(\mathscr{L})_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L} \rtimes_\alpha H} \\
&= {}_{\mathscr{L} \rtimes_\alpha H}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}}^{\theta^{-1}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L} \rtimes_\alpha H}
\end{aligned}
$$

(based on the realizations discussed so far). The actions (3), (4) together with the obvious fact $L^2(\mathscr{L})_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}L^2(\mathscr{L}) = L^2(\mathscr{L})$ show that the above relative tensor product is the Hilbert space $L^2(\mathscr{L}) \otimes \ell^2(H) \otimes \ell^2(H)$ with the left (resp. right) $\mathscr{L} \rtimes_\alpha H$-action naturally induced by (1), (2) (resp. (5), (6)). Or more precisely, the left $\mathscr{L} \rtimes_\alpha H$-action is

$$\sum_{h' \in H} \theta\alpha_{h'^{-1}}(\ell) \otimes 1 \otimes e_{h',h'} \tag{7}$$

$$1 \otimes 1 \otimes \lambda(h) \tag{8}$$

while the right $\mathscr{L} \rtimes_\alpha H$-action is

$$\sum_{h'' \in H} U_{\alpha_{h''^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h''}} \otimes e_{h'',h''} \otimes 1 \tag{9}$$

$$1 \otimes \lambda(h) \otimes 1. \tag{10}$$

The intertwiners of the right actions (9), (10) are obviously in $(\mathscr{L} \rtimes_\alpha H) \otimes B(\ell^2(H))$. Here, we should note that $\mathscr{L} \rtimes_\alpha H$ is acting on the first two tensor product $L^2(\mathscr{L}) \otimes \ell^2(H)$ and that it is in the amplification picture, i.e., generated by $\ell \otimes 1$ and $U_{\alpha_h} \otimes \rho(h)$, ($\rho(\cdot)$ being the right regular representation).

We next require compatibility between the left action described by (7), (8). So let us take $x = \sum_{h_1, h_2 \in H} x(h_1, h_2) \otimes e_{h_1,h_2}$ from the intertwiners of the right actions, i.e.,

$$x(h_1, h_2) = \sum_{h \in H} (x_h \otimes 1)(U_{\alpha_h} \otimes \rho(h)). \tag{11}$$

(Of course x_h also depends on h_1, h_2.) Notice $\lambda(h) = \sum_{h' \in H} e_{hh',h'}$ and $\rho(h) = \sum_{h' \in H} e_{h',h'h}$.

Lemma 4. (i) *The above operator* x *commutes with the operators defined by* (8) *if and only if we have the following invariance*:

$$x(h'h_1, h'h_2) = x(h_1, h_2) \quad \textit{for each } h_1, h_2, h' \in H.$$

(ii) *The operator* x *commutes with the operators defined by* (7) *if and only if for each* $h_1, h_2 \in H$ *the coefficients* x_h $(\in \mathscr{L})$ *in* (11) *satisfy*

$$\theta\alpha_{h_1^{-1}}(\ell)x_h = x_h\alpha_h\theta\alpha_{h_2^{-1}}(\ell) \quad \textit{for each } \ell \in \mathscr{L}.$$

It is elementary to get (i) from the above matrix expression of $\lambda(h)$ while (ii) comes from the fact that the commutativity between x and (7) means

$$x(h_1, h_2)(\theta\alpha_{h_1^{-1}}(\ell) \otimes 1) = (\theta\alpha_{h_2^{-1}}(\ell) \otimes 1)x(h_1, h_2).$$

Remark. Assume that $\tilde{G}$ is a discrete group with an outer action α^0 on the factor $\mathscr{L}$ and $H, K \subseteq \tilde{G}$ with $\alpha = \alpha^0|_H$ and $\beta = \alpha^0|_K$. In this special case, since $\theta = \alpha^0_g$ with $g = k_1h_1k_2 \cdots k_n \in \tilde{G}$, the condition in Lemma 4, (ii) means

$$\alpha^0_{gh_1^{-1}}(\ell)x_h = x_h\alpha^0_{hgh_2^{-1}}(\ell) \quad \text{for each } \ell \in \mathscr{L}.$$

Hence, x_h can be non-zero only when $h = gh_1^{-1}h_2g^{-1}$, and in this case x_h must be a scalar because α^0 is outer. Notice that $x(h_1, h_2)$ is either zero or contains at most one $U_{\alpha_h} \otimes \rho(h)$ (with a scalar coefficient). The latter case cannot happen unless $h \in H \cap gHg^{-1}$. We start from $h \in H \cap gHg^{-1}$. Then, $\tilde{h} = g^{-1}hg \in H$ and $h_1 = e, h_2 = \tilde{h}$ certainly satisfy the above condition (and corresponding to $U_{\alpha_h} \otimes \rho_h \otimes e_{e,g^{-1}hg}$). The invariance in Lemma 4, (i) means that

$$(U_{\alpha_h}) \otimes \rho(h) \otimes \left(\sum_{h'} e_{h',h'g^{-1}hg}\right) = U_{\alpha_h} \otimes \rho(h) \otimes \rho(g^{-1}hg)$$

is an intertwiner (for the $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ action). Therefore, the algebra of intertwiners in question is obviously

$$\sum_{h \in H \cap gHg^{-1}} \boldsymbol{C}(U_{\alpha_h} \otimes \rho(h) \otimes \rho(g^{-1}hg)),$$

which is of course isomorphic to the group ring $\boldsymbol{C}(H \cap gHg^{-1})$.

From the discussions in the above remark, it is clear that the relevant group we should look at in the general setting is $[\alpha_H \cap \theta\alpha_H\theta^{-1}]$ in $\mathrm{Out}(\mathscr{L})$. In fact, the condition in Lemma 4, (ii) means that x_h can be non-zero if and only if $\alpha_h = \theta\alpha_{h_1^{-1}h_2}\theta^{-1}$ in $\mathrm{Out}(\mathscr{L})$. In particular, this means $[\alpha_h] \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]$. Also, introduction of certain cocycles is now unavoidable for more detailed analysis.

Definition. For $[\alpha_h] \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]$, $\alpha_h = \theta\alpha_{\tilde{h}}\theta^{-1}$ in $\mathrm{Out}(\mathscr{L})$ with a unique $\tilde{h} \in H$. This $\tilde{h}$ will be denoted by h^θ. Let $v(h,\theta)$ be a unitary in $\mathscr{L}$ such that

$$\alpha_h = \mathrm{Ad}\, v(h,\theta)\theta\alpha_{h^\theta}\theta^{-1}.$$

We may and do assume $v(e,\theta) = 1$ in what follows. Note also that $(h_1h_2)^\theta = h_1^\theta h_2^\theta$ and $(h^{\theta'})^\theta = h^{(\theta'\theta)}$. With these unitaries the identical reasoning as in the above remark obviously shows that the algebra of self-intertwiners of the $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ sector $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}$ is

$$\sum_{h \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]} \boldsymbol{C}(v(h,\theta)^* U_{\alpha_h} \otimes \rho(h) \otimes \rho(h^\theta)).$$

Here, the sum actually means all h's in H such that $[\alpha_h] \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]$, i.e., $h \in H$ and $[\alpha_h]$ are identified, and this convention will be used throughout.

To write down explicitly the algebra structure of the above intertwiners, we should investigate the cocycle naturally attached to the above unitaries $v(h,\theta) (h \in [\alpha_H \cap \theta\alpha_H\theta^{-1}])$. From the definition of $v(\cdot,\cdot)$'s we compute

$$\begin{aligned}\mathrm{Ad}\, v(h_1h_2,\theta)\theta\alpha_{(h_1h_2)^\theta}\theta^{-1} &= \alpha_{h_1h_2} = \alpha_{h_1}\alpha_{h_2}\\ &= \mathrm{Ad}\, v(h_1,\theta)\theta\alpha_{h_1^\theta}\theta^{-1}\mathrm{Ad}\, v(h_2,\theta)\theta\alpha_{h_2^\theta}\theta^{-1}\\ &= \mathrm{Ad}(v(h_1,\theta)\theta\alpha_{h_1^\theta}\theta^{-1}(v(h_2,\theta)))\theta\alpha_{h_1^\theta h_2^\theta}\theta^{-1}.\end{aligned}$$

Lemma 5. *For each $h_1, h_2 \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]$ we have*

$$\begin{aligned}v(h_1h_2,\theta) &= \xi_\theta(h_1,h_2)v(h_1,\theta)\theta\alpha_{h_1^\theta}\theta^{-1}(v(h_2,\theta))\\ &= \xi_\theta(h_1,h_2)\alpha_{h_1}(v(h_2,\theta))v(h_1,\theta)\end{aligned}$$

for some scalar $\xi_\theta(h_1,h_2) \in \boldsymbol{T}$.

To determine the algebra structure of the intertwiners, we compute the product of two generators. (But notice that $h \to \rho(h)$ and $h \to \rho(h^\theta)$ are homomorphisms.) For

$h_1, h_2 \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]$ we compute

$$\begin{aligned}(v(h_1,\theta)^* U_{\alpha_{h_1}})(v(h_2,\theta)^* U_{\alpha_{h_2}}) &= v(h_1,\theta)^*\alpha_{h_1}(v(h_2,\theta)^*)U_{\alpha_{h_1h_2}} \\ &= \xi_\theta(h_1,h_2)v(h_1h_2,\theta)^* U_{\alpha_{h_1h_2}}\end{aligned}$$

thanks to the above lemma.

Summarizing the arguments so far, we have

THEOREM 6. *The algebra of self-intertwiners of the* $\mathscr{L} \rtimes_\alpha H$ *sector* $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$, *or equivalently, the one for the* $\mathscr{L} \rtimes_\alpha H$ *bimodule* ${}_{\mathscr{L}\rtimes_\alpha H}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}^{\theta}L^2(\mathscr{L})_{\mathscr{L}} \otimes_{\mathscr{L}} {}_{\mathscr{L}}L^2(\mathscr{L} \rtimes_\alpha H)_{\mathscr{L}\rtimes_\alpha H}$ *is the twisted group ring* $\boldsymbol{C}_{\xi_\theta}([\alpha_H \cap \theta\alpha_H\theta^{-1}])$.

REMARK. Note the associativity of the algebra means that $\xi_\theta(\cdot,\cdot)$ is a ($\boldsymbol{T}$-valued) 2-cocycle on $[\alpha_H \cap \theta\alpha_H\theta^{-1}]$ (which can be of course shown directly). In later sections, we will deal with θ normalizing α_H in $\mathrm{Out}(\mathscr{L})$ so that ξ_θ is a 2-cocycle on H in that case. When the unitary $v(h,\theta)$ is changed to $\tilde{v}(h,\theta) = c(h,\theta)v(h,\theta)$ (with $c(h,\theta) \in \boldsymbol{T}$), the new $\tilde{\xi}_\theta$ becomes

$$\tilde{\xi}_\theta(h_1,h_2) = \xi_\theta(h_1,h_2) \times \frac{c(h_1h_2,\theta)}{c(h_1,\theta)c(h_2,\theta)}$$

(see Lemma 5). Therefore, the 2-cocycle ξ_θ is changed by just a coboundary.

The above theorem and Proposition 3 imply

THEOREM 7. *With the notations right before Proposition* 3 *we have*

$$M_{2n} \cap M' = \sum_{i=1}^{\ell} {}^{\oplus} M_{n_i}(\boldsymbol{C}) \otimes \boldsymbol{C}_{\xi_{\theta_i}}([\alpha_H \cap \theta_i\alpha_H\theta_i^{-1}]).$$

Even in the depth 2 case the algebra structure of $M_{2n} \cap M'$ can be easily deformed due to the presence of ξ_{θ_i}'s as was seen in [**26**]. However, if H is a cyclic group for example, then so are subgroups and hence ξ_{θ_i}'s are all coboundaries. Therefore, deformation does not occur in this case (although fusion rule might be deformed as will be seen in **7.1**).

5. One-dimensional bimodules.

In this section we will determine the group of automorphisms (i.e., sectors with statistical dimension 1) appearing in $\sqcup_n(\rho_0\bar{\rho}_0)^n$ (see **2.3**) for our inclusion $\mathscr{L} \rtimes_\alpha H \supseteq \mathscr{L}^{(\beta,K)}$. Thanks to Proposition 3, to get complete information on those automorphisms, it suffices to study one-dimensional sectors appearing in the irreducible decomposition of $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}\in \mathrm{Sect}(\mathscr{L}\rtimes_\alpha H)$.

Recall that the corresponding $\mathscr{L}\rtimes_\alpha H$-$\mathscr{L}\rtimes_\alpha H$ bimodule is $\mathscr{A}=L^2(\mathscr{L})\otimes \ell^2(H)\otimes\ell^2(H)$ equipped with the action described by (7), (8), (9), (10). Let p be a minimal projection in the twisted group ring $\boldsymbol{C}_{\xi_\theta}([\alpha_H\cap\theta\alpha_H\theta^{-1}])$ (see Theorem 6), and $\mathscr{A}_p=p(L^2(\mathscr{L})\otimes\ell^2(H)\otimes\ell^2(H))$ be the corresponding irreducible component. As explained in **2.1**, $\dim\mathscr{A}_p$ is the square root of the minimal index of the reduced system (by p) of the inclusion

the algebra generated by the operators (7), (8)
$\subseteq$ the commutant of the algebra generated by the operators (9), (10).

As was seen in the previous section (discussions before Lemma 4), this inclusion is actually

$$(\theta\otimes 1\otimes 1)(\mathscr{L}\rtimes_\alpha H)\subseteq(\mathscr{L}\rtimes_\alpha H)\otimes B(\ell^2(H)).$$

Here, the left $\mathscr{L}\rtimes_\alpha H$ is generated by

$$\sum_{h'\in H}\alpha_{h'^{-1}}(\ell)\otimes 1\otimes e_{h',h'},\ 1\otimes 1\otimes\lambda(h)$$

while the right $\mathscr{L}\rtimes_\alpha H$ is in the amplification picture and acts on the first and second Hilbert spaces. The minimal index of this inclusion is $(\sharp H)^2$ since the statistical dimension of $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ is $\sharp H$. Therefore, by the local index formula, we have

$$\dim\mathscr{A}_p=\sqrt{(\sharp H)^2}\times\mathrm{tr}(p)=(\sharp H)\times\mathrm{tr}(p).$$

We remark that the trace (on the relative commutant, i.e., the twisted group ring) here is the restriction of the minimal conditional expectation.

Lemma 8. *The trace on the twisted group ring is computed by*

$$\mathrm{tr}\left(\sum_{h\in[\alpha_H\cap\theta\alpha_H\theta^{-1}]}c_h(v(h,\theta)^*U_{\alpha_h}\otimes\rho(h)\otimes\rho(h^\theta))\right)=c_e\quad(\textit{with } c_h\in\boldsymbol{C}).$$

Proof. The above inclusion can be regarded as the following two step inclusions:

$$(\theta\otimes 1\otimes 1)(\mathscr{L}\rtimes_\alpha H)\subseteq\mathscr{L}\otimes\boldsymbol{C}1\otimes B(\ell^2(H))\subseteq(\mathscr{L}\rtimes_\alpha H)\otimes B(\ell^2(H)).$$

The first inclusion is conjugate to $\mathscr{L}\rtimes_\alpha H\subseteq\mathscr{L}\otimes B(\ell^2(H))$ and irreducible. (This is actually the one coming from the dual coaction of H, but this fact is not necessary here.) On the other hand, the second inclusion is also irreducible and we have the natural (and unique) conditional expectation, i.e., the usual one from $\mathscr{L} \rtimes_\alpha H$ in the amplification picture onto $\mathscr{L} \otimes \boldsymbol{C}1$ tensored with the identity map of $B(\ell^2(H))$. Note that this expectation behaves like

$$\sum_{h \in [\alpha_H \cap \theta\alpha_H\theta^{-1}]} c_h(v(h,\theta)^* U_{\alpha_h} \otimes \rho(h) \otimes \rho(h^\theta)) \to c_e(1 \otimes 1 \otimes 1)$$

since $v(h,\theta)^* U_{\alpha_h} \otimes \rho(h)$ $(h \neq e)$ is killed. The value here is already a scalar, and hence the conditional expectation of the first inclusion acts trivially and the result is proved. □

Lemma 9. *When an* $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ *sector* $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}$ *contains a one-dimensional component*, θ *normalizes* α_H *in* $\mathrm{Out}(\mathscr{L})$. *Furthermore, in this case the* 2-*cocyle* ξ_θ *on* $[\alpha_H \cap \theta\alpha_H\theta^{-1}] = [\alpha_H] \cong H$ *is a coboundary.*

Proof. Assume that the $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ sector $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}$ contains an automorphism $\tilde{\theta} \in \mathrm{Aut}(\mathscr{L} \rtimes_\alpha H)$. Then, the Frobenius reciprocity implies

$$\begin{aligned} 1 &= \dim \mathrm{Hom}(\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}}, \tilde{\theta}) \\ &= \dim \mathrm{Hom}(\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta, \tilde{\theta} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}). \end{aligned}$$

Notice

$$(\mathscr{L} \rtimes_\alpha H) \cap (\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta(\mathscr{L}))' = (\mathscr{L} \rtimes_\alpha H) \cap \theta(\mathscr{L})' = (\mathscr{L} \rtimes_\alpha H) \cap \mathscr{L}' = \boldsymbol{C}1,$$

$$(\mathscr{L} \rtimes_\alpha H) \cap (\tilde{\theta} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}(\mathscr{L}))' = (\mathscr{L} \rtimes_\alpha H) \cap \tilde{\theta}(\mathscr{L})' = \tilde{\theta}((\mathscr{L} \rtimes_\alpha H) \cap \mathscr{L}') = \boldsymbol{C}1,$$

showing the irreducibility of the two $\mathscr{L}$-$\mathscr{L} \rtimes_\alpha H$ sectors $\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta, \tilde{\theta} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}$. Therefore, the above computation on the dimension of intertwiners guarantees

$$\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta = \tilde{\theta} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}.$$

By taking the conjugate, we then see

$$\begin{aligned} \theta^{-1} \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \circ \theta &= \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \tilde{\theta}^{-1} \circ \tilde{\theta} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H} \\ &= \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}} \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\alpha H}, \end{aligned}$$

which means

$$\theta^{-1}\left(\sum_{h\in H}{}^{\oplus}\alpha_h\right)\theta = \sum_{h\in H}{}^{\oplus}\alpha_h \quad (\text{as } \mathscr{L}\text{-}\mathscr{L} \text{ sectors}).$$

This of course means that θ normalizes α_H in $\mathrm{Out}(\mathscr{L})$.

Since $\tilde{\theta}$ appears in $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ with multiplicity one, the algebra $\boldsymbol{C}_{\xi_\theta}([\alpha_H\cap\theta\alpha_H\theta^{-1}]=H)$ of self-intertwiners of $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$ contains a one-dimensional summand. The projection p is the one corresponding to this summand. Hence, if we set $xp=\chi(x)p$, then $x\to\chi(x)$ gives rise to a one-dimensional representation of the twisted group ring $\boldsymbol{C}_{\xi_\theta}(H)$. By taking the trace values of the both sides and recalling $\mathrm{tr}(p)=1/\sharp H$, we get $\chi(x)=(\sharp H)\times\mathrm{tr}(xp)$. Therefore, the map

$$x\to\mathrm{tr}'(xp)=(\sharp H)\,\mathrm{tr}(xp)$$

is multiplicative. Let $p=\sum_{h\in H}(c_h/\sharp H)(v(h,\theta)^*U_{\alpha_h}\otimes\rho(h)\otimes\rho(h^\theta))$, and hence

$$\begin{aligned}&(v(h_1,\theta)^*U_{\alpha_{h_1}}\otimes\rho(h_1)\otimes\rho(h_1^\theta))p\\&\quad=\sum_{h\in H}\frac{c_h}{\sharp H}\xi_\theta(h_1,h)(v(h_1h,\theta)^*U_{\alpha_{h_1h}}\otimes\rho(h_1h)\otimes\rho((h_1h)^\theta)).\end{aligned}$$

The coefficient at the unit e is $c_{h_1^{-1}}/\sharp H$ (note that $\xi_\theta(h_1,h_1^{-1})=1$ can be assumed as usual) so that Lemma 8 implies

$$\mathrm{tr}'((v(h_1,\theta)^*U_{\alpha_{h_1}}\otimes\rho(h_1)\otimes\rho(h_1^\theta))p)=c_{h_1^{-1}}.$$

Thus, the multiplicativity of tr' shows $c_{h_1^{-1}}c_{h_2^{-1}}=\xi_\theta(h_1,h_2)c_{(h_1h_2)^{-1}}$, that is, ξ_θ is a coboundary. $\square$

We now assume that a one-dimensional sector $\mathscr{A}_p$ appears in $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}\circ\theta\circ\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}}$. Lemma 9 shows that $\theta=\beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}$ normalizes α_H in $\mathrm{Out}(\mathscr{L})$ and $\xi_\theta\in Z^2([\alpha_H\cap\theta\alpha_H\theta^{-1}],\boldsymbol{T})=Z^2(H,\boldsymbol{T})$ is a coboundary. We can of course further assume that unitaries $v(\cdot,\cdot)$ have been chosen in such a way that $\xi_\theta(h_1,h_2)=1$ for $h_1,h_2\in H$. The argument in the above proof shows that p corresponds to a character (i.e., one-dimensional representation) χ on H, that is, p is of the form

$$p_\chi=\frac{1}{\sharp H}\sum_{h''\in H}\chi(h'')(v(h'',\theta)^*U_{\alpha_{h''}}\otimes\rho(h'')\otimes\rho((h'')^\theta)).$$

In the rest of the section, we will explicitly write down one-dimensional sectors $\mathscr{A}_{p_\chi}(\chi\in\mathrm{Hom}(H,\boldsymbol{T}))$, i.e., automorphisms of $\mathscr{L}\rtimes_\alpha H$, and determine their product rule.

Lemma 10. *A vector* $y=\sum_{h_1,h_2\in H} y(h_1,h_2)\otimes\delta_{h_1}\otimes\delta_{h_2}$ *in* $L^2(\mathscr{L})\otimes\ell^2(H)\otimes\ell^2(H)$ *is in the range of the projection* p_χ *if and only if*

$$y(h_1h,h_2h^\theta)=\overline{\chi(h)}v(h^{-1},\theta)^*U_{\alpha_{h^{-1}}}y(h_1,h_2)$$

$$(=\overline{\chi(h)}U_{\alpha_{h^{-1}}}v(h,\theta)y(h_1,h_2))$$

for each $h\in H$.

Proof. Let $x=\sum_{h_1,h_2\in H}x(h_1,h_2)\otimes\delta_{h_1}\otimes\delta_{h_2}$, and we set $y=p_\chi x$. We then compute

$$y=\frac{1}{\sharp H}\sum_{h_1,h_2,h''\in H}\chi(h'')v(h'',\theta)^*U_{\alpha_{h''}}x(h_1,h_2)\otimes\delta_{h_1(h''^{-1})}\otimes\delta_{h_2(h''^{-1})^\theta}$$

$$=\frac{1}{\sharp H}\sum_{h_1,h_2,h''\in H}\chi(h'')v(h'',\theta)^*U_{\alpha_{h''}}x(h_1h'',h_2(h'')^\theta)\otimes\delta_{h_1}\otimes\delta_{h_2}.$$

Therefore, the (h_1,h_2)-component of y is

$$y(h_1,h_2)=\frac{1}{\sharp H}\sum_{h''\in H}\chi(h'')v(h'',\theta)^*U_{\alpha_{h''}}x(h_1h'',h_2(h'')^\theta),$$

and we compute

$$y(h_1h,h_2h^\theta)=\frac{1}{\sharp H}\sum_{h''\in H}\chi(h'')v(h'',\theta)^*U_{\alpha_{h''}}x(h_1hh'',h_2(hh'')^\theta)$$

$$=\frac{1}{\sharp H}\sum_{h''\in H}\chi(h^{-1}h'')v(h^{-1}h'',\theta)^*U_{\alpha_{h^{-1}h''}}x(h_1h'',h_2(h'')^\theta)$$

$$=\frac{\overline{\chi(h)}}{\sharp H}\sum_{h''\in H}\chi(h'')v(h^{-1},\theta)^*\alpha_{h^{-1}}(v(h'',\theta)^*)U_{\alpha_{h^{-1}}}U_{\alpha_{h''}}x(h_1h'',h_2(h'')^\theta)$$

(by the second expression of Lemma 5)

$$=\frac{\overline{\chi(h)}}{\sharp H}\sum_{h''\in H}\chi(h'')v(h^{-1},\theta)^*U_{\alpha_{h^{-1}}}v(h'',\theta)^*U_{\alpha_{h''}}x(h_1h'',h_2(h'')^\theta)$$

$$=\overline{\chi(h)}v(h^{-1},\theta)^*U_{\alpha_{h^{-1}}}x(h_1,h_2).$$

Note that by the second expression in Lemma 5 and $v(e,\theta)=1$ the last quantity is also equal to

$$\overline{\chi(h)}U_{\alpha_{h^{-1}}}\alpha_h(v(h^{-1},\theta)^*)y(h_1,h_2)=\overline{\chi(h)}U_{\alpha_{h^{-1}}}v(h,\theta)y(h_1,h_2).$$

Conversely, when $y(h_1, h_2)$'s satisfy this invariance condition, it is straight-forward to see $p_\chi y = y$ (use the second expression of the invariance). $\square$

Thanks to the above lemma, $p_\chi(L^2(\mathscr{L}) \otimes \ell^2(H) \otimes \ell^2(H))$ consists of vectors $y = \sum_{h_1, h_2 \in H} y(h_1, h_2) \otimes \delta_{h_1} \otimes \delta_{h_2}$ satisfying the invariance condition in the lemma. By taking one "row" of this vector, we define the surjective isometry

$$y \in p_\chi(L^2(\mathscr{L}) \otimes \ell^2(H) \otimes \ell^2(H)) \to (\sharp H) \sum_{h' \in H} y(h', e) \otimes \delta_{h'} \in \mathscr{H} = L^2(\mathscr{L}) \otimes \ell^2(H).$$

(Note that thanks to the invariance $y(h_1, h_2)$'s can be recovered from $y(h', e)$'s.) To get the one dimensional bimodule corresponding to p_χ, we have to see how the operators (7), (8), (9), (10) look like on $\mathscr{A}_{p_\chi} = p_\chi(L^2(\mathscr{L}) \otimes \ell^2(H) \otimes \ell^2(H))$, i.e., on $\mathscr{H}$ via the above isometry. Let us denote the operators (7), (8), (9), (10) by A, B, C, D respectively. We compute

$$\begin{aligned}
Ay &= \sum_{h_1, h_2, h'' \in H} \theta\alpha_{h''^{-1}}(\ell) y(h_1, h_2) \otimes \delta_{h_1} \otimes e_{h'', h''}\delta_{h_2} \\
&= \sum_{h_1, h_2 \in H} \theta\alpha_{h_2^{-1}}(\ell) y(h_1, h_2) \otimes \delta_{h_1} \otimes \delta_{h_2}, \\
By &= \sum_{h_1, h_2 \in H} y(h_1, h_2) \otimes \delta_{h_1} \otimes \delta_{hh_2} \\
&= \sum_{h_1, h_2 \in H} y(h_1, h^{-1}h_2) \otimes \delta_{h_1} \otimes \delta_{h_2}, \\
Cy &= \sum_{h_1, h_2, h'' \in H} U_{\alpha_{h''^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h''}} y(h_1, h_2) \otimes e_{h'', h''}\delta_{h_1} \otimes \delta_{h_2} \\
&= \sum_{h_1, h_2 \in H} U_{\alpha_{h_1^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h_1}} y(h_1, h_2) \otimes \delta_{h_1} \otimes \delta_{h_2}, \\
Dy &= \sum_{h_1, h_2 \in H} y(h_1, h_2) \otimes \delta_{hh_1} \otimes \delta_{h_2} \\
&= \sum_{h_1, h_2 \in H} y(h^{-1}h_1, h_2) \otimes \delta_{h_1} \otimes \delta_{h_2}.
\end{aligned}$$

Setting $h_1 = h', h_2 = e$, we have the following (h', e)-components:

$$\begin{aligned}
(Ay)(h', e) &= \theta(\ell) y(h', e), & (By)(h', e) &= y(h', h^{-1}), \\
(Cy)(h', e) &= U_{\alpha_{h'^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h'}} y(h', e), & (Dy)(h', e) &= y(h^{-1}h', e).
\end{aligned}$$

By Lemma 10 (the first expression) we have

$$(By)(h',e) = \chi(h^{(\theta^{-1})})v(h^{(\theta^{-1})},\theta)^* U_{\alpha_{h^{(\theta^{-1})}}} y(h'h^{(\theta^{-1})},e).$$

Therefore, at the level of $\mathscr{H} = L^2(\mathscr{L}) \otimes \ell^2(H)$, the left $\mathscr{L} \rtimes_\alpha H$-action is

$$A = \theta(\ell) \otimes 1$$

$$B = \chi(h^{(\theta^{-1})})v(h^{(\theta^{-1})},\theta)^* U_{\alpha_{h^{(\theta^{-1})}}} \otimes \rho(h^{(\theta^{-1})})$$

while the right $\mathscr{L} \rtimes_\alpha H$-action is

$$C = \sum_{h' \in H} U_{\alpha_{h'^{-1}}} J_{\mathscr{L}} \ell^* J_{\mathscr{L}} U_{\alpha_{h'}} \otimes e_{h',h'}$$

$$D = 1 \otimes \lambda(h).$$

Note that this right action is the commutant of $\mathscr{L} \rtimes_\alpha H$ in the amplification picture. Therefore, the one dimensional bimodule we have constructed so far is the one attached to the following automorphism:

$$\ell \otimes 1 \to \theta(\ell) \otimes 1,$$

$$U_{\alpha_h} \otimes \rho(h) \to \chi(h^{(\theta^{-1})})(v(h^{(\theta^{-1})},\theta)^* \otimes 1)(U_{\alpha_{h^{(\theta^{-1})}}} \otimes \rho(h^{(\theta^{-1})})).$$

Going back to the usual picture of the crossed product $\mathscr{L} \rtimes_\alpha H$ generated by $\pi_\alpha(\ell) \cong \ell$ and $\lambda(h)$, we set

Definition. For a character $\chi \in \mathrm{Hom}(H, \boldsymbol{T})$, we define $\Pi = \Pi_{\chi,\theta} \in \mathrm{Aut}(\mathscr{L} \rtimes_\alpha H)$ by

$$\Pi(\ell) = \theta(\ell) \quad (\ell \in \mathscr{L}),$$

$$\Pi(\lambda(h)) = \chi(h^{(\theta^{-1})})v(h^{(\theta^{-1})},\theta)^* \lambda(h^{(\theta^{-1})}) \quad (h \in H).$$

Since we have seen $\mathscr{A}_{p_\chi} = {}^{\Pi}L^2(\mathscr{L} \rtimes_\alpha H)$, this Π is an automorphism appearing in $\sqcup_n(\rho_0\bar{\rho}_0)^n$. (More precisely it appears in $(\rho_0\bar{\rho}_0)^n$ if $\theta = \beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}$ is of length $2n-1$.) Thanks to the cocycle equation one can check that the above formula indeed gives rise to an automorphism, which is left to the reader as an exercise.

We now determine the product rule. So let us start from two automorphisms $\Pi_{\chi,\theta}, \Pi_{\chi',\theta'}$. At first

$$\Pi_{\chi',\theta'}\Pi_{\chi,\theta}(\ell) = \theta'\theta(\ell)$$

is obvious. We then compute

$$\begin{aligned}\Pi_{\chi',\theta'}\Pi_{\chi,\theta}(\lambda(h)) &= \chi(h^{(\theta^{-1})})\Pi_{\chi',\theta'}(v(h^{(\theta^{-1})},\theta)^*\lambda(h^{(\theta^{-1})}))\\ &= \chi(h^{(\theta^{-1})})\chi'((h^{(\theta^{-1})})^{(\theta'^{-1})})\theta'(v(h^{(\theta^{-1})},\theta)^*)\\ &\quad\times v((h^{(\theta^{-1})})^{(\theta'^{-1})},\theta')^*\lambda((h^{(\theta^{-1})})^{(\theta'^{-1})})\\ &= \chi(h^{(\theta^{-1})})\chi'(h^{((\theta'\theta)^{-1})})\theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*\lambda(h^{((\theta'\theta)^{-1})})\end{aligned}$$

because of $(h^{(\theta^{-1})})^{(\theta'^{-1})} = h^{((\theta^{-1})(\theta'^{-1}))} = h^{((\theta'\theta)^{-1})}$. Notice that the product of the two v's in the last quantity can be rewritten as

$$v(h^{((\theta'\theta)^{-1})},\theta'\theta)^* \times \{v(h^{((\theta'\theta)^{-1})},\theta'\theta)\theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*\}.$$

To deal with the above product of three v's in the parenthesis, we note

$$\mathrm{Ad}(v(h^{(\theta^{-1})},\theta)^*)\alpha_{h^{(\theta^{-1})}} = \theta\alpha_h\theta^{-1},$$

and compute

$$\begin{aligned}\mathrm{Ad}(v(h^{((\theta'\theta)^{-1})},\theta'\theta)^*)\alpha_{h^{((\theta'\theta)^{-1})}} &= (\theta'\theta)\alpha_h(\theta'\theta)^{-1} = \theta'(\theta\alpha_h\theta^{-1})\theta'^{-1}\\ &= \theta'(\mathrm{Ad}(v(h^{(\theta^{-1})},\theta)^*)\alpha_{h^{(\theta^{-1})}})\theta'^{-1}\\ &= \mathrm{Ad}(\theta'(v(h^{(\theta^{-1})},\theta)^*))\theta'\alpha_{h^{(\theta^{-1})}}\theta'^{-1}\\ &= \mathrm{Ad}(\theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*)\alpha_{h^{((\theta'\theta)^{-1})}}\end{aligned}$$

because of $(h^{(\theta^{-1})})^{(\theta'^{-1})} = h^{((\theta'\theta)^{-1})}$. Therefore, we have

$$v(h^{((\theta'\theta)^{-1})},\theta'\theta)^* = \overline{\hat{\chi}_{\theta',\theta}(h)}\theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*$$

for some scalar $\hat{\chi}_{\theta',\theta}(h) \in \boldsymbol{T}$.

Definition. The scalar $\hat{\chi}_{\theta',\theta}(h) \in \boldsymbol{T}$ is defined by

$$\begin{aligned}\hat{\chi}_{\theta',\theta}(h) &= \theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*v(h^{((\theta'\theta)^{-1})},\theta'\theta)\\ &= v(h^{((\theta'\theta)^{-1})},\theta'\theta)\theta'(v(h^{(\theta^{-1})},\theta)^*)v(h^{((\theta'\theta)^{-1})},\theta')^*.\end{aligned}$$

The above second expression is exactly the product of three v's in the previous paragraph. Therefore, by going back to the product of two automorphisms, we have

$$\Pi_{\chi',\theta'}\Pi_{\chi,\theta}(\lambda(h)) = \chi(h^{(\theta^{-1})})\chi'(h^{((\theta'\theta)^{-1})})\hat{\chi}_{\theta',\theta}(h)v(h^{((\theta'\theta)^{-1})},\theta'\theta)^*\lambda(h^{((\theta'\theta)^{-1})}).$$

Notice $\chi(h^{(\theta^{-1})}) = \chi((h^{((\theta'\theta)^{-1})})^{\theta'})$. Since the composition $\Pi_{\chi',\theta'}\Pi_{\chi,\theta}$ is an automorphism of $\mathscr{L} \rtimes_\alpha H$, we know $h \to \hat{\chi}_{\theta',\theta}(h)$ is a character (see also Appendix B) and we conclude

LEMMA 11. *The following product rule holds*:

$$\Pi_{\chi',\theta'}\Pi_{\chi,\theta} = \Pi_{\chi'\chi(\cdot^{\theta'})\hat{\chi}_{\theta',\theta},(\theta'\theta)}.$$

The associativity of composition shows that $(\theta',\theta) \to \hat{\chi}_{\theta',\theta}(\cdot) \in \mathrm{Hom}(H,\boldsymbol{T})$ is a 2-cocycle (which can be also shown directly).

REMARK. Assume that unitaries $v(h,\theta)$ are changed to new unitaries $\tilde{v}(h,\theta) = c(h,\theta)v(h,\theta)$ (see the remark after Theorem 6). Since the corresponding $\tilde{\xi}_\theta$ is also assumed to be identity, $c(\cdot,\theta)$ is a character on H (and $c(\cdot,1) = 1$ after $v(h,1) = \tilde{v}(h,1) = 1$ is assumed). From the definition, we easily observe that $\hat{\chi}_{\theta',\theta}$ will be changed to

$$\hat{\chi}_{\theta',\theta}(h) \times \frac{c(h^{((\theta'\theta)^{-1})},\theta'\theta)}{c(h^{(\theta^{-1})},\theta)c(h^{((\theta'\theta)^{-1})},\theta')}.$$

Thus, the group structure obtained in Lemma 11 and that determined by the above new cocycle are isomorphic via $(\chi,\theta) \to (\chi c(\cdot^{(\theta^{-1})},\theta),\theta)$.

DEFINITION. We collect all alternating products $\theta = \beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}$ of arbitrary (odd) length such that (i) θ normalizes α_H in $\mathrm{Out}(\mathscr{L})$, and (ii) ξ_θ is a coboundary on H. (By adjusting $v(\cdot,\theta)$'s, we do assume $\xi_\theta = 1$.) These θ's form a group (see Appendix B). Recall that we defined $\theta \sim \theta'$ by $[\alpha_H\theta\alpha_H] = [\alpha_H\theta'\alpha_H]$ for general θ,θ' (the definition before Proposition 3). However, thanks to the normalizing condition (i), in the present setting we have $\theta \sim \theta'$ if and only if $[\theta\alpha_H] = [\theta'\alpha_H]$ ("coset" condition). The quotient is a group in the obvious way (the product is well-defined by the normalizing property again), and this group will be denoted by G_0. (We point out here that when $\theta \sim \theta'$ ξ_θ is a coboundary if and only if so is $\xi_{\theta'}$ as will be seen shortly.)

When $\theta_1 \sim \theta$, then $\theta_1 = \theta\alpha_{h_1}$ in $\mathrm{Out}(\mathscr{L})$ for some $h_1 \in H$. Notice $[\alpha_h] = [\theta_1\alpha_{h^{\theta_1}}\theta_1^{-1}] = [\theta\alpha_{h_1h^{\theta_1}h_1^{-1}}\theta^{-1}]$ so that $h^\theta = h_1h^{\theta_1}h_1^{-1}$, or equivalently,

$$h^{\theta_1} = h_1^{-1}h^\theta h_1 \quad \text{and} \quad (h_1hh_1^{-1})^{(\theta^{-1})} = h^{(\theta_1^{-1})} \quad (\text{i.e., } h_1^{(\theta^{-1})}h^{(\theta^{-1})} = h^{(\theta_1^{-1})}h_1^{(\theta^{-1})}).$$

In particular, for a character $\chi \in \mathrm{Hom}(H,\boldsymbol{T})$, $\chi(h^\theta)$ and $\chi(h^{(\theta^{-1})})$ depend only on the class of θ. (For example, $\chi(h^{\theta_1}) = \overline{\chi(h_1)}\chi(h^\theta)\chi(h_1) = \chi(h^\theta)$.) In this way, the quotient group G_0 acts on $\mathrm{Hom}(H,\boldsymbol{T})$.

Let us more precisely assume $\theta_1 = \mathrm{Ad}\, u_\theta \theta \alpha_{h_1}$. We compute

$$\begin{aligned}\alpha_h &= \mathrm{Ad}\, v(h,\theta_1)\theta_1 \alpha_{h^{\theta_1}} \theta_1^{-1} = \mathrm{Ad}(v(h,\theta_1)u_\theta)\theta \alpha_{h_1 h^{\theta_1} h_1^{-1}} \theta^{-1} \mathrm{Ad}\, u_\theta^* \\ &= \mathrm{Ad}(v(h,\theta_1)u_\theta)\theta\alpha_{h^\theta}\theta^{-1}\mathrm{Ad}\, u_\theta^* = \mathrm{Ad}(v(h,\theta_1)u_\theta v(h,\theta)^*)\alpha_h \mathrm{Ad}\, u_\theta^* \\ &= \mathrm{Ad}(v(h,\theta_1)u_\theta v(h,\theta)^* \alpha_h(u_\theta^*))\alpha_h.\end{aligned}$$

This means

$$v(h,\theta_1) = \nu(h;\theta_1,\theta)\alpha_h(u_\theta)v(h,\theta)u_\theta^* \tag{12}$$

with a scalar $\nu(h;\theta_1,\theta) \in \boldsymbol{T}$. (Note that even if we change u_θ to cu_θ $(c \in \boldsymbol{T})$ the above $\nu(h;\theta_1,\theta)$ is not effected.) From the second expression of Lemma 5 (for $v(\cdot,\theta_1)$) we see

$$\begin{aligned}\nu(h_1h_2;\theta_1,\theta)\alpha_{h_1h_2}(u_\theta)v(h_1h_2,\theta)u_\theta^* &= v(h_1h_2,\theta_1) \\ &= \xi_{\theta_1}(h_1,h_2)\alpha_{h_1}(v(h_2,\theta_1))v(h_1,\theta_1) \\ &= \xi_{\theta_1}(h_1,h_2)\nu(h_2;\theta_1,\theta)\nu(h_1;\theta_1,\theta) \\ &\quad \times \alpha_{h_1}(\alpha_{h_2}(u_\theta)v(h_2,\theta)u_\theta^*) \times \alpha_{h_1}(u_\theta)v(h_1,\theta)u_\theta^*.\end{aligned}$$

By canceling $u_\theta, \alpha_{h_1h_2}(u_\theta)$ from outside and $\alpha_{h_1}(u_\theta)$ inside and once again using the cocycle equation for $v(\cdot,\theta)$, we conclude

$$\xi_\theta(h_1,h_2) = \xi_{\theta_1}(h_1,h_2) \times \frac{\nu(h_1;\theta_1,\theta)\nu(h_2;\theta_1,\theta)}{\nu(h_1h_2;\theta_1,\theta)}.$$

Firstly, the property that ξ_θ is a coboundary depends on just the class of θ as was pointed out above. Secondly, when $\xi_{\theta_1} = \xi_\theta = 1$ is assumed (as was done), then $h \in H \to \nu(h;\theta_1,\theta) \in \boldsymbol{T}$ is a character.

When θ_1, θ are not equivalent, then $\Pi_{\chi_1,\theta_1} \neq \Pi_{\chi,\theta}$ in $\mathrm{Out}(\mathscr{L} \rtimes_\alpha H)$ thanks to Lemma 1. We generally cannot choose representatives for the quotient group G_0 in such a way that they form a group. Therefore, the following criterion is necessary:

Lemma 12. *Assume θ_1, θ are equivalent, and $\theta_1 = \mathrm{Ad}\, u_\theta\theta\alpha_{h_1}$ as above. Then, $\Pi_{\chi_1,\theta_1} = \Pi_{\chi,\theta}$ in $\mathrm{Out}(\mathscr{L} \rtimes_\alpha H)$ if and only if $\chi_1 = \chi \times \nu(\cdot\,;\theta_1,\theta)$.*

Proof. It is elementary to see

$$\Pi_{\chi,\theta}^{-1}(\ell) = \theta^{-1}(\ell) \quad \text{and} \quad \Pi_{\chi,\theta}^{-1}(\lambda(h)) = \overline{\chi(h)}\theta^{-1}(v(h,\theta))\lambda(h^\theta).$$

Therefore, the composition $\Pi = \Pi_{\chi_1,\theta_1} \circ \Pi_{\chi,\theta}^{-1}$ satisfies

$$\Pi(\ell) = \theta_1\theta^{-1}(\ell) = \operatorname{Ad} u_\theta \theta \alpha_{h_1} \theta^{-1}(\ell) = \operatorname{Ad}(u_\theta v(h_1^{(\theta^{-1})}, \theta)^*)\alpha_{h_1^{(\theta^{-1})}}(\ell),$$

and hence

$$\operatorname{Ad}(v(h_1^{(\theta^{-1})}, \theta)u_\theta^*) \circ \Pi \circ \operatorname{Ad} \lambda(h_1^{(\theta^{-1})})^*(\ell) = \ell.$$

Since $(\mathscr{L} \rtimes_\alpha H) \cap \mathscr{L}' = \boldsymbol{C}1$, it is easy to see that the above composition is inner if and only if it is the identity, or equivalently,

$$\operatorname{Ad}(v(h_1^{(\theta^{-1})}, \theta)u_\theta^*) \circ \Pi_{\chi_1, \theta_1} = \operatorname{Ad} \lambda(h_1^{(\theta^{-1})}) \circ \Pi_{\chi, \theta}.$$

The left side applied to $\lambda(h)$ is

$$\chi_1(h^{(\theta_1^{-1})})v(h_1^{(\theta^{-1})}, \theta)u_\theta^* v(h^{(\theta_1^{-1})}, \theta_1)^* \lambda(h^{(\theta_1^{-1})})u_\theta v(h_1^{(\theta^{-1})}, \theta)^*$$

while the right side applied to $\lambda(h)$ is

$$\begin{aligned}
&\chi(h^{(\theta^{-1})})\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta)^* \lambda(h^{\theta^{-1}}))\\
&\quad = \chi(h^{(\theta^{-1})})\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta)^*)\lambda((h_1 h h_1^{-1})^{\theta^{-1}})\\
&\quad = \chi(h^{(\theta^{-1})})\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta)^*)\lambda(h^{(\theta_1^{-1})})
\end{aligned}$$

(because of $(h_1 h h_1^{-1})^{(\theta^{-1})} = h^{(\theta_1^{-1})}$). Therefore, Π is inner if and only if

$$\begin{aligned}
&\chi_1(h^{(\theta_1^{-1})})v(h_1^{(\theta^{-1})}, \theta)u_\theta^* v(h^{(\theta_1^{-1})}, \theta_1)^* \alpha_{h^{(\theta_1^{-1})}}(u_\theta v(h_1^{(\theta^{-1})}, \theta)^*)\\
&\quad = \chi(h^{(\theta^{-1})})\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta)^*)
\end{aligned}$$

for each $h \in H$. (Note that we hit $\lambda(h^{(\theta_1^{-1})})^*$ from the right.) By using (12) (to eliminate $v(\cdot, \theta_1)$) and canceling $u_\theta, \alpha_{h^{(\theta_1^{-1})}}(u_\theta)$, we see that this is the same as

$$\begin{aligned}
&\chi_1(h^{(\theta_1^{-1})})\overline{v(h^{(\theta_1^{-1})}; \theta_1, \theta)}v(h_1^{(\theta^{-1})}, \theta)v(h^{(\theta_1^{-1})}, \theta)^* \alpha_{h^{(\theta_1^{-1})}}(v(h_1^{(\theta^{-1})}, \theta)^*)\\
&\quad = \chi(h^{(\theta^{-1})})\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta)^*).
\end{aligned}$$

By the cocycle property (Lemma 5) the product of the last two unitaries in the above left side is $v(h^{(\theta_1^{-1})}h_1^{(\theta^{-1})}, \theta)^*$ while $v(h_1^{(\theta^{-1})}, \theta)$ (the first unitary in the left side) multiplied by $\alpha_{h_1^{(\theta^{-1})}}(v(h^{(\theta^{-1})}, \theta))$ (the unitary in the right side) from the left is $v(h_1^{(\theta^{-1})}h^{(\theta^{-1})}, \theta)$. Since $h^{(\theta_1^{-1})}h_1^{(\theta^{-1})} = h_1^{(\theta^{-1})}h^{(\theta^{-1})}$, the above equality is equivalent to $\chi_1(h^{(\theta_1^{-1})}) = \chi(h^{(\theta^{-1})}) \cdot$

$\nu(h^{(\theta_1^{-1})};\theta_1,\theta)$. Since $h^{(\theta^{-1})}$ and $h^{(\theta_1^{-1})}$ give rise to identical characters as remarked before, we are done. □

COROLLARY 13. (i) *If* $\theta_1' \sim \theta'$ *and* $\theta_1 \sim \theta$ (*so that* $\theta_1'\theta_1 \sim \theta'\theta$), *then we have*

$$\hat{\chi}_{\theta_1',\theta_1} = \hat{\chi}_{\theta',\theta} \times \frac{\nu(\cdot;\theta_1'\theta_1,\theta'\theta)}{\nu(\cdot;\theta_1',\theta')\nu(\cdot^{\theta'};\theta_1,\theta)}.$$

(ii) *When* $\theta_2 \sim \theta_1 \sim \theta$, *we have*

$$\nu(\cdot;\theta_2,\theta_1)\nu(\cdot;\theta_1,\theta) = \nu(\cdot;\theta_2,\theta).$$

PROOF. By Lemma 11 and the above lemma, we compute (in $\mathrm{Out}(\mathscr{L} \rtimes_\alpha H)$)

$$\begin{aligned}\Pi_{\chi'\chi(\cdot^{\theta'})\hat{\chi}_{\theta',\theta},\theta'\theta} &= \Pi_{\chi',\theta'}\Pi_{\chi,\theta} = \Pi_{\chi'\nu(\cdot;\theta_1',\theta'),\theta_1'}\Pi_{\chi\nu(\cdot;\theta_1,\theta),\theta_1} \\ &= \Pi_{\chi'\chi(\cdot^{\theta'})\nu(\cdot;\theta_1',\theta')\nu(\cdot^{\theta'};\theta_1,\theta)\hat{\chi}_{\theta_1',\theta_1},\theta_1'\theta_1} \\ &= \Pi_{\chi'\chi(\cdot^{\theta'})\nu(\cdot;\theta_1',\theta')\nu(\cdot^{\theta'};\theta_1,\theta)\overline{\nu(\cdot;\theta_1'\theta_1,\theta'\theta)}\hat{\chi}_{\theta_1',\theta_1},\theta'\theta}.\end{aligned}$$

The map $\tilde{\chi} \in \mathrm{Hom}(H,\boldsymbol{T}) \to [\Pi_{\tilde{\chi},\theta'\theta}] \in \mathrm{Out}(\mathscr{L} \rtimes_\alpha H)$ being injective from the construction, we obtain (i). Obviously, (ii) can be shown similarly from Lemma 12. □

Let us fix representatives $\{\theta_i\}_{i=1,2,\ldots,\ell}$ for the quotient group G_0 (with $\theta_1 = 1$) (the definition after Lemma 11).

DEFINITION. When $[\theta_i\theta_j] = [\theta_k]$, we set

$$\hat{\chi}_{[\theta_i],[\theta_j]} = \hat{\chi}_{\theta_i,\theta_j} \times \nu(\cdot;\theta_k,\theta_i\theta_j).$$

When we have different representatives $\{\theta_i'\}$, from the above corollary we easily have $\hat{\chi}_{\theta_i',\theta_j'} \times \nu(\cdot;\theta_k',\theta_i'\theta_j') = \hat{\chi}_{\theta_i,\theta_j} \times \nu(\cdot;\theta_k,\theta_i\theta_j) \times \nu(\cdot;\theta_k',\theta_k)/\nu(\cdot;\theta_i',\theta_i)\nu(\cdot^{\theta_i};\theta_j',\theta_j)$. Therefore, the class $[\hat{\chi}] \in H^2(G_0,\mathrm{Hom}(H,\boldsymbol{T}))$ is well-defined independent of the choice of representatives for G_0.

Lemma 1, Lemma 2, and the discussions so far in the present section imply

THEOREM 14. *The group* G *of all one-dimensional bimodules in* $\sqcup_n(\rho_0\bar{\rho}_0)^n$ (*with* $\mathscr{L}^{(\beta,K)} = \rho_0(\mathscr{L} \rtimes_\alpha H)$, $\rho_0 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H)$) *is the extension*

$$1 \to \mathrm{Hom}(H,\boldsymbol{T}) \to G \to G_0 \to 1$$

corresponding to the cohomology class $[\hat{\chi}] \in H^2(G_0,\mathrm{Hom}(H,\boldsymbol{T}))$.

Notice that when $\theta = \beta_k$ (of length 1, $\beta_k \alpha_H \beta_{k^{-1}} = \alpha_H$ in $\mathrm{Out}(\mathscr{L})$ and ξ_{β_k} is a coboundary) the corresponding automorphism Π_{χ,β_k} is in $\rho_0\bar{\rho}_0$, i.e., in the Galois group (see [**29**] and also [**40**]) for $\mathscr{M} = \mathscr{L} \rtimes_\alpha H \supseteq \mathscr{N} = \mathscr{L}^{(\beta,K)}$. Also notice when H is abelian $\{\Pi_{\chi,1}\}_{\chi \in \hat{H}}$ is nothing but the dual action of (α, H).

6. Non-strongly outer automorphisms for a subfactor.

We will determine the group $\Gamma = \Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ of "adjustable" one-dimensional bimodules in $\sqcup_n (\rho_0\bar{\rho}_0)^n$ (recall the discussion in **2.3**). Let $\Pi_{\chi,\theta}$ be the automorphism in $(\rho_0\bar{\rho}_0)^n$ constructed in the previous section, and we assume that θ is an alternating product of length $2n-1$. Then, this $\Pi_{\chi,\theta}$ belongs to Γ if and only if

$$\Pi_{\chi,\theta}\rho_0 = \rho_0\Phi \quad (\text{as an } \mathscr{L} \rtimes_\alpha H\text{-}\mathscr{L} \rtimes_\alpha H \text{ sector}) \tag{13}$$

for some automorphism Φ appearing in $(\bar{\rho}_0\rho_0)^n$ (thanks to the general criterion in **2.3**). Therefore, we need to investigate what this condition means in the present setting.

Let $\bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$, $\bar{\varrho}_2 \in \mathrm{Sect}(\mathscr{L} \rtimes_\beta K, \mathscr{L})$, and $\rho_i \in \mathrm{Sect}(\mathscr{L})$ be as in §3. As in §3 one can easily show

$$\bar{\rho}_0\rho_0 = \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K} \circ \rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1,$$

$$(\bar{\rho}_0\rho_0)^2 = \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K} \circ \rho_1\bar{\rho}_1\rho_2\bar{\rho}_2\rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K}} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1,$$

and so on.

In fact, at first we have

$$\bar{\rho}_0\rho_0 = \bar{\varrho}_1^{-1}\bar{\rho}_2\rho_1\bar{\rho}_1\rho_2\bar{\varrho}_1.$$

We notice

$$\bar{\varrho}_1^{-1}\bar{\rho}_2 = \bar{\varrho}_1^{-1}\bar{\varrho}_2\bar{\varrho}_2^{-1}\bar{\rho}_2 = \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K},$$

and the last product $\rho_2\bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$ in $\bar{\rho}_0\rho_0$ is the conjugate of $\bar{\varrho}_1^{-1}\bar{\rho}_2$. Therefore, we should have $\Phi = \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \Phi' \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1$ with an automorphism Φ' appearing in the following product

$$\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K} \circ \rho_1\bar{\rho}_1\rho_2\bar{\rho}_2\rho_1\bar{\rho}_1 \cdots \rho_1\bar{\rho}_1 \circ \overline{\iota_{\mathscr{L} \hookrightarrow \mathscr{L} \rtimes_\beta K}}.$$

This means that to express Φ' the roles of (H, α) and (K, β) should be exchanged. Thus, we should have an alternating product $\psi = \alpha_{h_1'}\beta_{k_1'}\alpha_{h_2'} \cdots \alpha_{h_n'}$ (of length $2n-1$ but in the different order) such that (i) ψ normalizes β_K in $\mathrm{Out}(\mathscr{L})$, and (ii) the cocycle ξ'_ψ (defined similarly to ξ_θ) on K is a coboundary. ($\beta_k = \psi\beta_{k^\psi}\psi^{-1}$ in $\mathrm{Out}(\mathscr{L})$, $\beta_k =$

$\mathrm{Ad}(v'(k,\psi))\psi\beta_{k^\psi}\psi^{-1}$, and $v'(k_1k_2,\psi) = \xi'_\psi(k_1,k_2)v'(k_1,\psi)\psi\beta_{k_1^\psi}\psi^{-1}(v'(k_2,\psi))$ as before.) For each $\pi \in \mathrm{Hom}(K,\boldsymbol{T})$, we have the automorphism $\Phi' = \Phi'_{\pi,\psi}$ of $\mathscr{L} \rtimes_\beta K$ as in §5 (but the roles of (α,H) and (β,K) are switched) and $\Phi = \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \Phi'_{\pi,\psi} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1$.

The left side of (13) is

$$\Pi_{\chi,\theta}\bar{\varrho}_1^{-1}\bar{\rho}_1\rho_2\bar{\varrho}_1 = \Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2\bar{\varrho}_1.$$

It is also elementary to see that the right side of (13) is equal to

$$\bar{\varrho}_1^{-1}\bar{\rho}_1\rho_2\bar{\varrho}_1 \circ \bar{\varrho}_1^{-1}\bar{\varrho}_2 \circ \Phi'_{\pi,\psi} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1 = \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1$$

by the usual calculation. Noticing that the both sides of (13) are irreducible $\mathscr{L} \rtimes_\alpha H$-$\mathscr{L} \rtimes_\alpha H$ sectors, we see that (13) is equivalent to

$$\begin{aligned} 1 &= \dim \mathrm{Hom}(\Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2\bar{\varrho}_1, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \bar{\varrho}_2^{-1}\bar{\varrho}_1) \\ &= \dim \mathrm{Hom}(\Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \rho_2, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \bar{\varrho}_2^{-1}) \end{aligned}$$

since the conjugate of the last $\bar{\varrho}_1 \in \mathrm{Sect}(\mathscr{L} \rtimes_\alpha H, \mathscr{L})$ is $\bar{\varrho}_1^{-1} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\alpha H)$ and $\bar{\varrho}_1\bar{\varrho}_1^{-1} = id \in \mathrm{Sect}(\mathscr{L})$. Then, since $\bar{\varrho}_2^{-1} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\beta K)$ composed with $\bar{\rho}_2 \in \mathrm{Sect}(\mathscr{L})$ is $\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K} \in \mathrm{Sect}(\mathscr{L}, \mathscr{L} \rtimes_\beta K)$, the above dimension is equal to

$$\begin{aligned} &\dim \mathrm{Hom}(\Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}, \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}) \\ &\quad = \dim \mathrm{Hom}(\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} \circ \Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}, \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}). \end{aligned}$$

Notice that $\Pi_{\chi,\theta} \in \mathrm{Aut}(\mathscr{L} \rtimes_\alpha H)$ leaves $\mathscr{L}$ invariant and its restriction to $\mathscr{L}$ is θ from the construction. Therefore, we see

$$\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} \circ \Pi_{\chi,\theta} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} = \overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H}} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\alpha H} \circ \theta = \sum_{h\in H}{}^{\oplus}[\alpha_h\theta].$$

Exactly in the same way we also have

$$\overline{\iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K}} \circ \Phi'_{\pi,\psi} \circ \iota_{\mathscr{L}\hookrightarrow\mathscr{L}\rtimes_\beta K} = \sum_{k\in K}{}^{\oplus}[\beta_k\psi].$$

Thus, the condition (13) is equivalent to $\alpha_h\theta = \beta_k\psi$ in $\mathrm{Out}(\mathscr{L})$ for some $h \in H, k \in K$.

Summarizing the arguments so far, we have proved

Lemma 15. *We can adjust* $\Pi_{\chi,\theta}$ *to an automorphism in* $\mathrm{Aut}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ *after inner perturbation if and only if one can find* $k \in K$ *and* $h \in H$ *such that* (i) $\beta_k\alpha_h\theta = \alpha_{h'_1}\beta_{k'_1}\alpha_{h'_2}\cdots\alpha_{h'_n}$ $(=\psi)$ *in* $\mathrm{Out}(\mathscr{L})$ *for some* $h'_1, k'_1, h'_2, \ldots, h'_n$, (ii) $\psi = \beta_k\alpha_h\theta$ *(or equiv-*

alently, $\alpha_h\theta$*) normalizes* β_K *in* $\mathrm{Out}(\mathscr{L})$, *and* (iii) *the* 2*-cocycle* ξ'_ψ *on* K *attached to* ψ *is a coboundary.*

Let us point out here that the above condition (i) is actually irrelevant. Assume that the condition (ii) is satisfied with $\theta = \beta_{k_1}\alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}$. Then, since θ normalizes α_H in $\mathrm{Out}(\mathscr{L})$, we observe $\alpha_h\theta = \theta\theta^{-1}\alpha_h\theta = \theta\alpha_{\tilde{h}}$ (in $\mathrm{Out}(\mathscr{L})$) for some $\tilde{h} \in H$. Therefore, by choosing $k = k_1^{-1} \in K$, we see that

$$\beta_k\alpha_h\theta = \alpha_{h_1}\beta_{k_2}\cdots\beta_{k_n}\alpha_{\tilde{h}} \quad (\text{in } \mathrm{Out}(\mathscr{L}))$$

and it is indeed of the form described in (i).

We also remark that the triviality of the cocycle (on K) attached to $\alpha_h\theta$ and that attached to $\psi = \beta_k\alpha_h\theta$ are equivalent. To prove this, (by switching the roles of (β, K) and (α, H)) it suffices to show the following: When θ normalizes α_H in $\mathrm{Out}(\mathscr{L})$, ξ_θ and $\xi_{(\alpha_h,\theta)}$ (both 2-cocycles on H) are equivalent, i.e., the same up to a coboundary. However, we have already known this fact. (See the relation between ξ_θ and ξ_{θ_1} right before Lemma 12.)

The above discussions together with Lemma 15 show

Theorem 16. *Let* $\Pi_{\chi,\theta}$ *be the automorphism of* $\mathscr{L} \rtimes_\alpha H$ *defined in the previous section. Then,* $[\Pi_{\chi,\theta}]$ *belongs to* $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ *if and only if one can find* $h \in H$ *such that* (i) $\varphi = \alpha_h\theta$ *normalizes* β_K *in* $\mathrm{Out}(\mathscr{L})$, *and* (ii) *the* 2*-cocycle* ξ'_φ *on* K *is a coboundary.*

Note that the above φ also normalizes α_H in $\mathrm{Out}(\mathscr{L})$ and $\theta \sim \varphi$. Thus, the above requirement can be also rephrased in the following symmetric fashion: one can find φ equivalent to θ satisfying (i) φ normalizes β_K in $\mathrm{Out}(\mathscr{L})$ (and normalizes α_H in $\mathrm{Out}(\mathscr{L})$ from the beginning) and (ii) the attached cocycle ξ'_φ on K is a coboundary (and so is ξ_φ on H from the beginning since $\theta \sim \varphi$ and $\xi_\theta \in B^2(H, \boldsymbol{T})$).

We shall come back to a certain consequence of this symmetric characterization later. But before this, to see the meaning of the conditions in the above theorem, we at first go back to the situation in the remark after Lemma 4. In this case, the automorphism $\Pi = \Pi_{\chi,\theta}$ is

$$\Pi\left(\sum_{h\in H} x_h\lambda(h)\right) = \sum_{h\in H} \chi(ghg^{-1})\alpha_g^0(x_h)\lambda(ghg^{-1})$$

with $g = k_1h_1k_2\cdots k_n$. Therefore, when $x \in \mathscr{L}^K$, we have $\Pi(x) = \alpha_g^0(x)$. The condition (i) in the theorem ((ii) is automatic) means that $g' = hg$ normalizes K for some $h \in H$. Then, with the inner perturbation $\mathrm{Ad}\,\lambda(h)$, we get $\mathrm{Ad}\,\lambda(h) \circ \Pi(x) = \lambda(h)\alpha_g^0(x)\lambda(h)^* = \alpha_{g'}^0(x)$. For each $k \in K$, we have $\alpha_k^0(\alpha_{g'}^0(x)) = \alpha_{kg'}^0(x) = \alpha_{g'(g'^{-1}kg')}^0(x) = \alpha_{g'}^0(x)$ because of

$g'^{-1}kg' \in K$ and $x \in \mathscr{L}^K$, and hence $\mathrm{Ad}\,\lambda(h) \circ \Pi$ belongs to $\mathrm{Aut}(\mathscr{L} \rtimes H, \mathscr{L}^K)$. (Note that $\mathrm{Ad}\,\lambda(h)$ is not necessary if φ instead of θ is used from the beginning.)

Let us next consider the meaning in the general setting (and in fact the following gives us an alternative proof for the one direction of Theorem 16). Since $\varphi = \alpha_h\theta$ normalizes β_K in $\mathrm{Out}(\mathscr{L})$, we have $\beta_k = \mathrm{Ad}(v'(k,\varphi))\varphi\beta_{k^\varphi}\varphi^{-1}$. By the coboundary condition, (after changing phases of $v'(\cdot,\cdot)$'s) we may and do assume the cocycle property

$$v'(k_1k_2,\varphi) = v'(k_1,\varphi)\varphi\beta_{k_1^\varphi}\varphi^{-1}(v'(k_2,\varphi)) \tag{14}$$

(Lemma 5 with the roles of (α, H) and (β, K) switched). Since $k \to \varphi\beta_{k^\varphi}\varphi^{-1} = \mathrm{Ad}(v'(k,\varphi)^*)\beta_k$ is an outer action, by the result of Connes-Takesaki ([**5**]) there exists a unitary w in $\mathscr{L}$ such that $v'(k,\varphi) = w\varphi\beta_{k^\varphi}\varphi^{-1}(w^*)$, and hence we have

$$\beta_k = \mathrm{Ad}(w\varphi\beta_{k^\varphi}\varphi^{-1}(w^*))\varphi\beta_{k^\varphi}\varphi^{-1} = \mathrm{Ad}\,w \circ \varphi\beta_{k^\varphi}\varphi^{-1} \circ \mathrm{Ad}\,w^*.$$

Therefore, the perturbed automorphism $\mathrm{Ad}(w\lambda(h)) \circ \Pi$ sends $\ell \in \mathscr{L}$ to

$$\mathrm{Ad}(w\lambda(h))\theta(\ell) = \mathrm{Ad}\,w \circ \alpha_h \circ \theta(\ell) = w\varphi(\ell)w^*.$$

Thus, when ℓ belongs to $\mathscr{L}^{(\beta,K)}$ furthermore, with the above expression of β_k in mind we compute

$$\beta_k(w\varphi(\ell)w^*) = \mathrm{Ad}\,w \circ \varphi\beta_{k_\varphi}(\ell) = w\varphi(\ell)w^* \quad (\text{for each } k \in K),$$

which shows that $\mathscr{L}^{(\beta,K)}$ is left invariant under $\mathrm{Ad}(w\lambda(h)) \circ \Pi$ (although the unitary w above is not so explicit).

Generally the dual principal graph of $\mathscr{L} \rtimes_\alpha H \supseteq \mathscr{L}^{(\beta,K)}$ and that of $\mathscr{L} \rtimes_\beta K \supseteq \mathscr{L}^{(\alpha,H)}$ (i.e., the principal graph of the former inclusion) are completely different. In fact, even when $K = \{e\}$ (so that the depth of the inclusion is 2), we have quite different groups $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}) = \mathrm{Hom}(H, \boldsymbol{T})$ and $\Gamma(\mathscr{L}, \mathscr{L}^{(\alpha,H)}) = H$. Hence, it appears that $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ and $\Gamma(\mathscr{L} \rtimes_\beta K, \mathscr{L}^{(\alpha,H)})$ could be totally unrelated. However, we will point out that these two groups are somewhat related.

We introduce the following three groups in $\mathrm{Out}(\mathscr{L})$:

(i) Γ_0 consists of all elements $[\varphi]$ generated by $[\alpha_H]$ and $[\beta_K]$ such that $[\varphi]$ normalizes both of $[\alpha_H]$ and $[\beta_K]$, $\xi_\varphi \in B^2(H, \boldsymbol{T})$, and $\xi'_\varphi \in B^2(K, \boldsymbol{T})$.

(ii) H_0 consists of all $[\alpha_h]$'s such that $[\alpha_h]$ normalizes $[\beta_K]$ and $\xi'_{\alpha_h} \in B^2(K, \boldsymbol{T})$.

(iii) K_0 consists of all $[\beta_k]$'s such that $[\beta_k]$ normalizes $[\alpha_H]$ and $\xi_{\beta_k} \in B^2(H, \boldsymbol{T})$.
A few remarks are in order. Firstly, the properties $\xi_\varphi \in B^2(H, \boldsymbol{T})$ and $\xi'_\varphi \in B^2(K, \boldsymbol{T})$ depend on just the class $[\varphi]$ of φ (as was seen in the previous section). Therefore, the

above definition of Γ_0 is legitimate. Secondly, since α, β are actions, we can take $v(h', \alpha_h) = v'(k', \beta_k) = 1$ so that $\xi_{\alpha_h} = 1$ and $\xi'_{\beta_k} = 1$ are automatic. We also remark that H_0 and K_0 are normal in Γ_0.

Let $\mathrm{Gal}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ be the Galois group. It is obviously a normal subgroup in $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ and was described in the paragraph after Theorem 14. On the other hand, it is an easy exercise to see that $H_0K_0 = K_0H_0$ is a normal subgroup in Γ_0 (thanks to Appendix B again). From Theorem 16 (actually the paragraph after the theorem) we get the next result, and details are left to the reader.

THEOREM 17. *The quotient group* $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})/\mathrm{Gal}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ *is isomorphic to* Γ_0/H_0K_0.

The roles of H_0 and K_0 are completely symmetric here. Hence, the quotient group $\Gamma(\mathscr{L} \rtimes_\beta K, \mathscr{L}^{(\alpha,H)})/\mathrm{Gal}(\mathscr{L} \rtimes_\beta K, \mathscr{L}^{(\alpha,H)})$ is also isomorphic to Γ_0/K_0H_0 $(= \Gamma_0/H_0K_0)$ so that $\Gamma(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})/\mathrm{Gal}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ and $\Gamma(\mathscr{L} \rtimes_\beta K, \mathscr{L}^{(\alpha,H)})/$ $\mathrm{Gal}(\mathscr{L} \rtimes_\beta K, \mathscr{L}^{(\alpha,H)})$ are isomorphic. A completely general result will be obtained in Appendix C.

7. Examples.

In this section we will compute the relative χ group for inclusions of index 4 with the Coxeter-Dynkin graphs $D_{2n}^{(1)}$ and also present typical examples of non-adjustable automorphisms appearing in $\sqcup_k(\rho_0\bar{\rho}_0)^k$ (see **2.3**).

7.1. Let a, b be period 2 automorphisms of $\mathscr{L}$, and we assume that the outer period of ab is $2n$. Therefore, $(ab)^{2n} = \mathrm{Ad}\, u$ with $ab(u) = \omega u$ and ω, the Connes obstruction, satisfies $\omega^{2n} = 1$. Let $H = K = \boldsymbol{Z}_2 = \{0, 1\}$ and $\alpha_1 = a$ and $\beta_1 = b$ (details of this case is studied in [**25**]). It is straight-forward to see $G_0 = \boldsymbol{Z}_2$ consisting of *id* and

$$\theta = (ba)^{n-1}b = \beta_1\alpha_1\beta_1 \cdots \beta_1 \ \ (= \theta^{-1}).$$

Note $\theta a\theta^{-1} = (ba)^n(ba)^{n-1}b = (ba)^{2n-1}b$. Since $(ba)^{2n} = b \circ \mathrm{Ad}\, u \circ b = \mathrm{Ad}(b(u))$, we have $(ba)^{2n-1}b = \mathrm{Ad}(b(u)) \circ a$ and $\theta a\theta^{-1} = \mathrm{Ad}(b(u)) \circ a$. (Of course $\varphi = a\theta$ also normalizes α_H in $\mathrm{Out}(\mathscr{L})$, but $\varphi^{-1} \neq \varphi$ and with this choice we will have to worry about the character $v(\cdot; id, (\varphi)^2)$ in Lemma 12.) This means $1^\theta = 1$ and we can take $v(1, \theta) = b(u)^*$ (and $v(0, \theta) = 1$). From Lemma 5 we have

$$1 = v(0, \theta) = \xi_\theta(1, 1)a(b(u^*))b(u^*) = \bar{\omega}\xi_\theta(1, 1)u^*b(u^*)$$

so that $\xi_\theta(1, 1) = \omega b(u)u$. We know that $b(u)u$ is a scalar (which can be also seen directly from

$$\mathrm{Ad}(b(u)u) = b \circ \mathrm{Ad}\, u \circ b^{-1} \circ \mathrm{Ad}\, u = b \circ (ab)^n \circ b^{-1} \circ (ab)^n = (ba)^n \circ (ab)^n = id).$$

We may and do assume $b(u)u = \bar{\varpi}$ by changing u to a suitable $e^{ix_0}u$ so that we have $b(u) = \bar{\varpi}u^*$ and $\xi_\theta = 1$. We then compute

$$\begin{aligned}\hat{\chi}_{\theta,\theta}(1) &= \theta(v(1,\theta)^*)v(1,\theta)^* = (ba)^{n-1}b(b(u)) \times b(u) = (ba)^{n-1}(u) \times b(u)\\ &= (\bar{\varpi})^{n-1}ub(u) \quad (\text{since } ba(u) = \bar{\varpi}u)\\ &= (\bar{\varpi})^n.\end{aligned}$$

The group G of all one-dimensional bimodules in this case is the extension

$$1 \to \mathrm{Hom}(\boldsymbol{Z}_2, \boldsymbol{T}) \cong \boldsymbol{Z}_2 \to G \to G_0 \cong \boldsymbol{Z}_2 \to 1$$

corresponding to the above $\hat{\chi}$ (Theorem 14). Note that the case $\omega^n = -1$ corresponds to the non-trivial cocycle $\hat{\chi} \in H^2(G_0, \mathrm{Hom}(H, \boldsymbol{T})) = H^2(\boldsymbol{Z}_2, \boldsymbol{Z}_2) = \boldsymbol{Z}_2$. Therefore, we conclude

$$G = \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 & \text{when } \omega^n = 1, \\ \boldsymbol{Z}_4 & \text{when } \omega^n = -1. \end{cases}$$

This result shows that the group structure (fusion rule) depends upon the Connes obstruction although the shapes of the graphs do not. Note that this happens because $H^2(\boldsymbol{Z}_2, \boldsymbol{Z}_2) = \boldsymbol{Z}_2$ and $H^2(\boldsymbol{Z}_2, \boldsymbol{T}) = 0$ (see the paragraph after Theorem 7).

In the rest we assume that $\mathscr{L}$ is the AFD II_1 factor and determine the relative χ group $\chi(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ explained in **2.3**. At first notice that $\varphi = a\theta = (ab)^n$ possesses the normalization property (for K)

$$\varphi b \varphi^{-1} = (ab)^n b (ba)^n = (ab)^{2n-1}a = \mathrm{Ad}\, u \circ b \ \ (= b \quad (\text{in } \mathrm{Out}(\mathscr{L}))).$$

It follows from Theorem 16 that all of the four automorphisms can be assumed to be in $\mathrm{Aut}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ after suitable inner perturbation, which can be of course also seen from the (well-known) principal and dual principal graphs.

To be more explicit, we now repeat the discussions right after Theorem 16. The above computation means $v'(1,\varphi) = u^*$ (and $1^\varphi = 1$ for $1 \in K = \boldsymbol{Z}_2$), and the attached cocycle ξ'_φ on K is computed by

$$1 = \xi'_\varphi(1,1)u^*\varphi\beta_{1^\varphi}\varphi^{-1}(u^*) = \xi'_\varphi(1,1)u^*ub(u^*)u^* = \xi'_\varphi(1,1)b(u^*)u^* = \xi'_\varphi(1,1)\omega.$$

Thus, by changing u to $e^{ix_0}u(e^{2ix_0} = \omega)$, the new implementing unitary $v'(1,\varphi) = e^{-ix_0}u^*$

(giving rise to $\xi'_\varphi = 1$) satisfies the cocycle equation (14) (for the automorphism $\varphi\beta_{1^\varphi}\varphi^{-1} = \mathrm{Ad}\, u \circ b$). Therefore, as remarked before we can find a unitary $w \in \mathscr{L}$ such that

$$e^{-ix_0}u^* = wub(w^*)u^*, \quad \text{i.e.,} \quad b(w) = e^{ix_0}wu. \tag{15}$$

We take $\Pi = \Pi_{\chi,\theta}$ and set $\Pi' = \mathrm{Ad}(w\lambda(1)) \circ \Pi$, where $\lambda(1)$ means the canonical self-adjoint unitary corresponding to a in the crossed product $\mathscr{L} \rtimes_\alpha H = \mathscr{L} \rtimes_a \boldsymbol{Z}_2$. Then, we have $\Pi' \in \mathrm{Aut}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ as was seen before. We also note

$$\Pi'(\ell) = w\lambda(1)\theta(\ell)\lambda(1)^*w^* = wa \circ \theta(\ell)w^* = w\varphi(\ell)w^* \quad (\ell \in \mathscr{L})$$

$$\begin{aligned}\Pi'(\lambda(1)) &= w\lambda(1)(\chi(1)v(1,\theta)^*\lambda(1))\lambda(1)^*w^* = w\lambda(1)(\chi(1)b(u)\lambda(1))\lambda(1)^*w^* \\ &= \chi(1)wab(u)\lambda(1)w^* = \omega\chi(1) \times wu\lambda(1)w^*\end{aligned}$$

because of $a \circ \theta = \varphi, v(1,\theta)^* = b(u)$ and $ab(u) = \omega u$.

Since the inclusion $\mathscr{L} \supseteq \mathscr{L}^{(\beta,K)}$ is of index 2 and the Goldman theorem ([**11**]) can be used, we can find a unitary $v \in \mathscr{L}$ satisfying $v\mathscr{L}^{(\beta,K)}v^* = \mathscr{L}^{(\beta,K)}$ (hence $\mathrm{Ad}\, v \in \mathrm{Aut}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$) and $b(v) = -v$. Notice $\Pi' \circ \mathrm{Ad}\, v = \mathrm{Ad}(\Pi'(v)) \circ \Pi'$ and the unitary $\Pi'(v)$ (in $\mathscr{L}$) normalizes $\mathscr{L}^{(\beta,K)}$ because of $\Pi'(\mathscr{L}^{(\beta,K)}) = \mathscr{L}^{(\beta,K)}$. We now claim $b(\Pi'(v)) = -\Pi'(v)$. At first we have $\Pi'(v) = w\varphi(v)w^*$ as was seen above. Then, by (15) we compute

$$b(w\varphi(v)w^*) = b(w)(b \circ \varphi(v))b(w)^* = wu(b \circ \varphi(v))u^*w^* = wu(b(ab)^n(v))u^*w^*.$$

Since the inside of $\mathrm{Ad}\, w$ in the last expression can be computed as

$$u(b(ab)^n(v))u^* = (ab)^{2n} \circ b(ab)^n(v) = (ab)^n(b(v)) = -(ab)^n(v) = -\varphi(v),$$

we have $b(\Pi'(v)) = -\Pi'(v)$ as desired.

The Weyl group of $\mathscr{L} \supseteq \mathscr{L}^{(\beta,K)}$ is $\boldsymbol{Z}_2$ (see [**51**]), and from $b(v) = -v$ and $b(\Pi'(v)) = -\Pi'(v)$ we conclude that $\Pi'(v)$ and v are identical modulo $\mathscr{U}(\mathscr{L}^{(\beta,K)})$, the unitary group. Therefore, we see that $\mathrm{Ad}\, v$ and Π' indeed commute up to $\mathrm{Int}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})$ (as was expected since the relative χ group is always commutative). On the other hand, $\Pi_{\chi,1}$ (the dual action for $\mathscr{L} \rtimes_\alpha H \supseteq \mathscr{L}$) obviously commutes with $\mathrm{Ad}\, v$.

In the present case, the principal and dual principal graphs are quite simple (both the Coxeter-Dynkin graph $D_{2n}^{(1)}$) and G is abelian. Hence, it is plain to observe that none of the above automorphisms admits a non-trivial Loi invariant (see [**13**]). Therefore, all of our automorphisms fall into $\overline{\mathrm{Int}(\mathscr{L} \rtimes_\alpha H, \mathscr{L}^{(\beta,K)})}$. Also, notice that

$v \in (\mathscr{L} \subseteq)\mathscr{L} \rtimes_\alpha H$ is the only non-trivial normalizer for $\mathscr{L}^{(\beta,K)}$ (up to $\mathscr{U}(\mathscr{L}^{(\beta,K)})$) as long as $n \geq 2$ (otherwise the depth is $2 \times 1 = 2$).

Summing up the discussions so far, we conclude

$$\chi(\mathscr{L} \rtimes_a \boldsymbol{Z}_2, \mathscr{L}^{(b,\boldsymbol{Z}_2)}) = \begin{cases} \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 \times \boldsymbol{Z}_2 & \text{when } \omega^n = 1, \\ \boldsymbol{Z}_4 \times \boldsymbol{Z}_2 & \text{when } \omega^n = -1 \end{cases}$$

(when the outer period is $2n$ with $n \geq 2$). Note that when $n = 1$ (i.e., $(ab)^2$ is inner) we are in the crossed product situation by the abelian group G obtained before. Therefore, the relative χ group is $G \times G$. On the other hand, when the outer period of ab is $2n+1$ (or $+\infty$), just the Galois group ($= \boldsymbol{Z}_2$) appears in $\sqcup_k (\rho_0\bar{\rho}_0)^k$ so that the relative χ group is simply $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$.

7.2. Let $\mathscr{A}_4 = (\boldsymbol{Z}_2 \times \boldsymbol{Z}_2) \rtimes \boldsymbol{Z}_3$ be the alternating group, where the generator $\varepsilon \in \boldsymbol{Z}_3 = K = \{e, \varepsilon, \varepsilon^2\}$ permutes the three non-trivial elements $(1,0), (0,1), (1,1) \in \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$. We fix an outer action α^0 of $\mathscr{A}_4$ on $\mathscr{L}$. Let $H = \{(0,0), (1,0)\} \cong \boldsymbol{Z}_2$ and α, β be just the restrictions of α^0 to H and K respectively (so that the index is 6). In this case it is easy to see that $G_0 \cong \boldsymbol{Z}_2$ and we can take representatives $(0,0)$ and $(0,1)$ (more precisely, *id* and $\alpha^0_{(0,1)}$). Since $\mathrm{Hom}(H, \boldsymbol{T}) \cong \boldsymbol{Z}_2$ and no cocycle is around, we have $G = \boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ (Theorem 14). Notice that $(0,1) = \varepsilon(1,0)\varepsilon^{-1} \in KHK$ normalizes H since $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ is abelian (and $(0,1), (1,1) = (1,0)(0,1)$ are equivalent). Therefore, the two automorphisms $\Pi_{\chi, \alpha^0_{(0,1)}}$ ($\chi \in \mathrm{Hom}(H, \boldsymbol{T}) \cong \boldsymbol{Z}_2$) appear in $(\rho_0\bar{\rho}_0)^2$ (the depth of the inclusion is actually 4 since KHK exhausts the total group $\mathscr{A}_4$). These automorphisms cannot be adjusted to ones in $\mathrm{Aut}(\mathscr{M}, \mathscr{N})$. In fact, none of $(0,1)$ and $(1,1)$ in $\boldsymbol{Z}_2 \times \boldsymbol{Z}_2$ ($\subseteq \mathscr{A}_4$) normalizes K (Theorem 16).

Notice that the same reasoning as above works (for $\mathscr{L}^K \subseteq \mathscr{L} \rtimes H$) as long as a group admits two subgroups $\tilde{H}$ and K with for example the following properties: (i) $\tilde{H} \cap K = \{e\}$, (ii) for each $a \neq e \in \tilde{H}$ we have $aKa^{-1} \neq K$, and (iii) we can choose a subgroup H in $\tilde{H}$ in such a way that some element in $\tilde{H} \backslash H$ is generated by H, K and normalizes H.

Appendix A. Cauchy-Schwarz inequality.

Let $\mathscr{A}, \mathscr{B}, \ldots$ be bimodules (or sectors) of finite dimension throughout.

Lemma 18. *We have*

$$(\dim \mathrm{Hom}(\mathscr{A}, \mathscr{B}))^2 \leq \dim \mathrm{Hom}(\mathscr{A}, \mathscr{A}) \times \dim \mathrm{Hom}(\mathscr{B}, \mathscr{B}).$$

Furthermore, the equality holds if and only if one of $\mathscr{A}, \mathscr{B}$ is a multiple of the other.

PROOF. Let

$$\mathscr{A} = n_1\mathscr{X}_1 \oplus n_2\mathscr{X}_2 \oplus \cdots \oplus n_k\mathscr{X}_k \oplus n_{k+1}\mathscr{Y}_{k+1} \oplus \cdots \oplus n_{\ell_1}\mathscr{Y}_{\ell_1},$$

$$\mathscr{B} = m_1\mathscr{X}_1 \oplus m_2\mathscr{X}_2 \oplus \cdots \oplus m_k\mathscr{X}_k \oplus m_{k+1}\mathscr{Z}_{k+1} \oplus \cdots \oplus m_{\ell_2}\mathscr{Z}_{\ell_2}$$

be the irreducible decomposition, and here we can obviously assume that all $\mathscr{X}$'s, $\mathscr{Y}$'s, and $\mathscr{Z}$'s are mutually distinct. We compute

$$\begin{aligned}(\dim \mathrm{Hom}(\mathscr{A},\mathscr{B}))^2 &= \left(\sum_{i=1}^{k} n_i m_i\right)^2 \le \left(\sum_{i=1}^{k} n_i^2\right) \times \left(\sum_{i=1}^{k} m_i^2\right) \le \left(\sum_{i=1}^{\ell_1} n_i^2\right) \times \left(\sum_{i=1}^{\ell_2} m_i^2\right) \\ &= \dim \mathrm{Hom}(\mathscr{A},\mathscr{A}) \times \dim \mathrm{Hom}(\mathscr{B},\mathscr{B})\end{aligned}$$

thanks to the Cauchy-Schwarz inequality.

On the other hand, if we have the equality, then at first we have $\ell_1 = \ell_2 = k$ from the second inequality, that is, $\mathscr{Y}$'s and $\mathscr{Z}$'s are absent. The rest follows from the well-known equality condition for the Cauchy-Schwarz inequality. □

COROLLARY 19. *Assume that bimodules $\mathscr{A},\mathscr{B}$ have the same dimension. Then, $\mathscr{A} = \mathscr{B}$ if and only if*

$$(\dim \mathrm{Hom}(\mathscr{A},\mathscr{B}))^2 = \dim \mathrm{Hom}(\mathscr{A},\mathscr{A}) \times \dim \mathrm{Hom}(\mathscr{B},\mathscr{B}).$$

Appendix B. Identity for cocycles.

We assume that θ',θ normalize α_H in $\mathrm{Out}(\mathscr{L})$ (as in §5). Recall

$$\hat{\chi}_{\theta',\theta}(h_1h_2) = v((h_1h_2)^{((\theta'\theta)^{-1})},\theta'\theta)\theta'(v((h_1h_2)^{(\theta^{-1})},\theta)^*)v((h_1h_2)^{((\theta'\theta)^{-1})},\theta')^*.$$

By making use of the second expression in Lemma 5 for the first two v's here, we see that the above product of the three v's is equal to

$$\begin{aligned}&\xi_{\theta'\theta}(h_1^{((\theta'\theta)^{-1})},h_2^{((\theta'\theta)^{-1})})\alpha_{h_1^{((\theta'\theta)^{-1})}}(v(h_2^{((\theta'\theta)^{-1})},\theta'\theta))v(h_1^{((\theta'\theta)^{-1})},\theta'\theta)\\ &\quad\times \theta'(\overline{\xi_\theta(h_1^{(\theta^{-1})},h_2^{(\theta^{-1})})}v(h_1^{(\theta^{-1})},\theta)^*\alpha_{h_1^{(\theta^{-1})}}(v(h_2^{(\theta^{-1})},\theta)^*))\\ &\quad\times v((h_1h_2)^{((\theta'\theta)^{-1})},\theta')^*.\end{aligned}$$

By regrouping terms, we thus have

$$\hat{\chi}_{\theta',\theta}(h_1h_2) = \xi_{\theta'\theta}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})\overline{\xi_\theta(h_1^{(\theta^{-1})}, h_2^{(\theta^{-1})})}\alpha_{h_1^{((\theta'\theta)^{-1})}}(v(h_2^{((\theta'\theta)^{-1})}, \theta'\theta))$$
$$\times v(h_1^{((\theta'\theta)^{-1})}, \theta'\theta)\theta'(v(h_1^{(\theta^{-1})}, \theta)^*)$$
$$\times \theta'\alpha_{h_1^{(\theta^{-1})}}(v(h_2^{(\theta^{-1})}, \theta)^*)v((h_1h_2)^{((\theta'\theta)^{-1})}, \theta')^*.$$

The product of the middle two v's here is $\hat{\chi}_{\theta',\theta}(h_1)v(h_1^{((\theta'\theta)^{-1})}, \theta')$, and hence we have

$$\hat{\chi}_{\theta',\theta}(h_1h_2) = \xi_{\theta'\theta}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})\overline{\xi_\theta(h_1^{(\theta^{-1})}, h_2^{(\theta^{-1})})}\hat{\chi}_{\theta',\theta}(h_1)$$
$$\times \alpha_{h_1^{((\theta'\theta)^{-1})}}(v(h_2^{((\theta'\theta)^{-1})}, \theta'\theta))v(h_1^{((\theta'\theta)^{-1})}, \theta')$$
$$\times \theta'\alpha_{h_1^{(\theta^{-1})}}(v(h_2^{(\theta^{-1})}, \theta)^*)v((h_1h_2)^{((\theta'\theta)^{-1})}, \theta')^*.$$

Notice that

$$\mathrm{Ad}(v(h_1^{((\theta'\theta)^{-1})}, \theta'))\theta'\alpha_{h_1^{(\theta^{-1})}}\theta'^{-1} = \alpha_{h_1^{((\theta'\theta)^{-1})}}$$

because of $(h_1^{((\theta'\theta)^{-1})})^{\theta'} = h_1^{(\theta^{-1})}$, and hence

$$v(h_1^{((\theta'\theta)^{-1})}, \theta')\theta'\alpha_{h_1^{(\theta^{-1})}}(v(h_2^{(\theta^{-1})}, \theta)^*) = \alpha_{h_1^{((\theta'\theta)^{-1})}}(\theta'(v(h_2^{(\theta^{-1})}, \theta)^*))v(h_1^{((\theta'\theta)^{-1})}, \theta').$$

We substitute this expression to the right side consisting of four v's (and three scalars). By combining two $\alpha_{h_1^{((\theta'\theta)^{-1})}}$ factors (and by ignoring the three scalars at the front for a moment) the product of four v's can be rewritten as

$$\alpha_{h_1^{((\theta'\theta)^{-1})}}(v(h_2^{((\theta'\theta)^{-1})}, \theta'\theta)\theta'(v(h_2^{(\theta^{-1})}, \theta)^*)) \times v(h_1^{((\theta'\theta)^{-1})}, \theta')v((h_1h_2)^{((\theta'\theta)^{-1})}, \theta')^*.$$

The inside of $\alpha_{h_1^{((\theta'\theta)^{-1})}}$ here is $\hat{\chi}_{\theta',\theta}(h_2)v(h_2^{((\theta'\theta)^{-1})}, \theta')$ so that by lemma 5 we have shown

$$\hat{\chi}_{\theta',\theta}(h_1h_2) = \xi_{\theta'\theta}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})\overline{\xi_\theta(h_1^{(\theta^{-1})}, h_2^{(\theta^{-1})})}\hat{\chi}_{\theta',\theta}(h_1) \times$$
$$\hat{\chi}_{\theta',\theta}(h_2)\alpha_{h_1^{((\theta'\theta)^{-1})}}(v(h_2^{((\theta'\theta)^{-1})}, \theta'))v(h_1^{((\theta'\theta)^{-1})}, \theta')v((h_1h_2)^{((\theta'\theta)^{-1})}, \theta')^*$$

(by recalling the three relevant scalars). Notice that the product of the three v's in the right side is $\overline{\xi_{\theta'}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})}$, and hence we have shown

LEMMA 20. *When θ', θ normalize α_H in $\mathrm{Out}(\mathscr{L})$, we have*

$$\frac{\hat{\chi}_{\theta',\theta}(h_1)\hat{\chi}_{\theta',\theta}(h_2)}{\hat{\chi}_{\theta',\theta}(h_1h_2)} = \frac{\xi_{\theta'}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})\xi_\theta(h_1^{(\theta^{-1})}, h_2^{(\theta^{-1})})}{\xi_{\theta'\theta}(h_1^{((\theta'\theta)^{-1})}, h_2^{((\theta'\theta)^{-1})})}.$$

Compare this with relations in [**18**], [**26**]. From this identity, the following corollary is immediate:

COROLLARY 21. *Assume in addition that $\xi_{\theta'}$, ξ_θ are coboundaries (on H).* (i) *Then, $\xi_{\theta'\theta}$ is also a coboundary.* (ii) *When $v(\cdot,\cdot)$'s are chosen in such a way that $\xi_{\theta'} = \xi_\theta = \xi_{\theta'\theta} = 1$, $h \in H \to \hat{\chi}_{\theta',\theta}(h) \in \boldsymbol{T}$ is a character.*

Appendix C. The group $\Gamma(\mathscr{M},\mathscr{N})/\mathrm{Gal}(\mathscr{M},\mathscr{N})$.

The two quotient groups $\Gamma(\mathscr{M},\mathscr{N})/\mathrm{Gal}(\mathscr{M},\mathscr{N})$ and $\Gamma(\mathscr{M}_1,\mathscr{M})/\mathrm{Gal}(\mathscr{M}_1,\mathscr{M})$ are isomorphic for the inclusion $\mathscr{M} = \mathscr{L} \rtimes_\alpha H \supseteq \mathscr{N} = \mathscr{L}^{(\beta,K)}$ dealt in the present article (thanks to Theorem 17). Let us examine another typical inclusion. Namely, let α be an outer action on a factor $\mathscr{L}$ of a finite group G, and we set

$$\mathscr{M} = \mathscr{L} \rtimes_\alpha G \supseteq \mathscr{N} = \mathscr{L} \rtimes_\alpha H$$

with a subgroup H. Then, (as was shown in [**40**]) we have

$$\Gamma(\mathscr{M},\mathscr{N})/\mathrm{Gal}(\mathscr{M},\mathscr{N}) = \mathrm{Hom}(G,\boldsymbol{T})/\mathrm{Hom}_0(G,\boldsymbol{T}),$$

$$\Gamma(\mathscr{M}_1,\mathscr{M})/\mathrm{Gal}(\mathscr{M}_1,\mathscr{M}) = \mathrm{Hom}^0(H,\boldsymbol{T}).$$

Here, $\mathrm{Hom}_0(G,\boldsymbol{T})$ is the set of characters on G vanishing on H while $\mathrm{Hom}^0(H,\boldsymbol{T})$ is the set of all extendable (to G) characters on H. The surjective homomorphism $\chi \in \mathrm{Hom}(G,\boldsymbol{T}) \to \chi|_H \in \mathrm{Hom}^0(H,\boldsymbol{T})$ induces the isomorphism between $\mathrm{Hom}(G,\boldsymbol{T})/\mathrm{Hom}_0(G,\boldsymbol{T})$ and $\mathrm{Hom}^0(H,\boldsymbol{T})$, and hence once again the quotient groups $\Gamma(\mathscr{M},\mathscr{N})/\mathrm{Gal}(\mathscr{M},\mathscr{N})$ and $\Gamma(\mathscr{M}_1,\mathscr{M})/\mathrm{Gal}(\mathscr{M}_1,\mathscr{M})$ are isomorphic.

In this appendix we show that this is a completely general phenomenon. So let $\mathscr{M} \supseteq \mathscr{N} = \rho_0(\mathscr{M})$ be an irreducible inclusion of finite index with $\rho_0 \in \mathrm{Sect}(\mathscr{M})$ (after the tensoring tick in [**46**]), and $a \in \mathrm{Aut}(\mathscr{M})$ be an automorphism. We can find a unitary $u \in M$ such that $\mathrm{Ad}\,u \circ a \in \mathrm{Aut}(\mathscr{M},\mathscr{N})$ if and only if there is an automorphism $b \in \mathrm{Aut}(\mathscr{M})$ such that $\rho_0 b = a\rho_0$ (as a sector). Furthermore, we have $a \prec (\rho_0\bar{\rho}_0)^n$ if and only if $b \prec (\bar{\rho}_0\rho_0)^n$ (see **2.3**).

We start from $a \prec (\rho_0\bar{\rho}_0)^n$ (for some n) satisfying $\rho_0 b = a\rho_0$ for some automorphism b (i.e., $a \in \Gamma(\mathscr{M},\mathscr{N})$). Then, $b \prec (\bar{\rho}_0\rho_0)^n$ and $b^{-1}\bar{\rho}_0 = \bar{\rho}_0 a^{-1}$. The second fact here implies that b^{-1} (and hence b) can be assumed $b^{-1} \in \mathrm{Aut}(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))$ after inner perturbation. Assume $a\rho_0 = \rho_0 b = \rho_0 b'$ with another automorphism b'. Then, $b'b^{-1}\bar{\rho}_0 = \bar{\rho}_0$, and hence after inner perturbation we could assume $b'b^{-1}(\bar{\rho}_0(m)) = \bar{\rho}_0(m), m \in M$, i.e., $b'b^{-1} \in \mathrm{Gal}(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))$. We thus get the well-defined map $a \in \Gamma(\mathscr{M}, \mathscr{N}) \to [b] \in \Gamma(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))/\mathrm{Gal}(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))$ specified by $a\rho_0 = \rho_0 b$. This map is obviously a homomorphism with the kernel $\mathrm{Gal}(\mathscr{M}, \mathscr{N})$ (by the same reason). Therefore, $\Gamma(\mathscr{M}, \mathscr{N})/\mathrm{Gal}(\mathscr{M}, \mathscr{N})$ and $\Gamma(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))/\mathrm{Gal}(\mathscr{M}, \bar{\rho}_0(\mathscr{M}))$ are isomorphic. Since $\mathscr{M} \supseteq \bar{\rho}_0(\mathscr{M})$ and $\mathscr{M}_1 \supseteq \mathscr{M}$ are conjugate, we have shown

Theorem 22. *For an irreducible inclusion $\mathscr{M} \supseteq \mathscr{N}$ of finite index, the two quotient groups $\Gamma(\mathscr{M}, \mathscr{N})/\mathrm{Gal}(\mathscr{M}, \mathscr{N})$ and $\Gamma(\mathscr{M}_1, \mathscr{M})/\mathrm{Gal}(\mathscr{M}_1, \mathscr{M})$ are isomorphic.*

References

[1] S. Baaj and G. Skandalis, Unitaires multiplicatifs et dualité pour les produits croisé de C^*-algèbres, Ann. Sci. École Norm. Sup., **26** (1993), 425–488.
[2] D. Bisch and U. Haagerup, Composition of subfactors: new examples of infinite depth subfactors, Ann. Sci. École Norm. Sup., **26** (1996), 329–383.
[3] M. Choda and H. Kosaki, Strongly outer actions for an inclusion of factors, J. Funct. Anal., **122** (1994), 315–332.
[4] A. Connes, Non-commutative Geometry, Academic Press, 1994.
[5] A. Connes and M. Takesaki, The flow of weights on factors of type III, Tôhoku Math. J., **29** (1977), 473–575.
[6] M. C. David, Paragroupe d'Adrian Ocneanu et algèbre de Kac, Pacific J. Math., **172** (1996), 331–363.
[7] M. C. David, Coupe assorti de systèmes de Kac et inclusions de facteurs de type II_1, J. Funct. Anal., **159** (1998), 1–42.
[8] S. Doplicher, R. Haag, and J. Roberts, Local observables and particle physics I, II, Commun. Math. Phys., **23** (1971), 199–239 and **35** (1974), 49–85.
[9] D. Evans and Y. Kawahigashi, Orbifold subfactors from Hecke algebras, Commun. Math. Phys., **165** (1994), 445–458.
[10] D. Evans and Y. Kawahigashi, Quantum Symmetries on Operator Algebras, Oxford Univ. Press, 1998.
[11] M. Goldman, On subfactors of type II_1, Michigan Math. J., **7** (1960), 167–172.
[12] S. Goto, Orbifold construction for non-AFD factors, Internat. J. Math., **5** (1994), 721–746.
[13] S. Goto, Commutativity of automorphisms of subfactors modulo inner automorphisms, Proc. Amer. Math. Soc., **124** (1996), 3391–3398.
[14] J. F. Havet, Espérance conditionnelle minimale, J. Operator Theory, **24** (1990), 33–55.
[15] F. Hiai, Minimizing indices of conditional expectations onto a subfactor, Publ. RIMS, Kyoto Univ., **24** (1988), 673–678.
[16] J. H. Hong, Subfactors of principal graph $E_6^{(1)}$, Acta Appl. Math., **40** (1995), 255–264.
[17] J. H. Hong and W. Szymański, On finite subfactors with principal graph $\boldsymbol{D}_{2n+1}/\boldsymbol{Z}_2$, J. Funct. Anal., **141** (1996), 294–300.
[18] J. H. Hong and W. Szymański, Composition of subfactors and twisted bicrossed product, J. Operator Theory, **37** (1997), 281–302.
[19] M. Izumi, Applications of fusion rules to classification of subfactors, Publ. RIMS, Kyoto Univ., **27** (1991), 953–994.
[20] M. Izumi, Goldman's type theorem for index 3, Publ. RIMS, Kyoto Univ., **28** (1992), 833–843.
[21] M. Izumi, Subalgebras of infinite C^*-algebras with finite Watatani index, Commun. Math. Phys., **155** (1993), 157–282.
[22] M. Izumi, On the type II and type III principal graphs for subfactors, Math. Scand., **73** (1994), 307–319.
[23] M. Izumi, Goldman's type theorems in index theory, in "Operator Algebras and Quantum Field Theory" (Proc. of Rome Conference), International Press, 1997 (p. 249–269).

[24] M. Izumi, Subalgebras of infinite C^*-algebras with finite Watatani index II: Cuntz-Krieger algebras, Duke Math. J., **91** (1998), 409–461.
[25] M. Izumi and Y. Kawahigashi, Classification of subfactors with the principal graph $D_n^{(1)}$, J. Funct. Anal., **112** (1993), 257–286.
[26] M. Izumi and H. Kosaki, Finite-dimensional Kac algebras arising from certain group actions on a factor, IMRN, No. 8 (1996), 357–370.
[27] V. F. R. Jones, Index for subfactors, Invent. Math., **72** (1983), 1–25.
[28] G. I. Kac and V. G. Paljutkin, Finite group rings, Trans. Moscow Math. Soc., **15** (1966), 251–294.
[29] T. Kajiwara and S. Yamagami, Irreducible bimodules associated with crossed product algebras II, Pacific J. Math., **171** (1995), 209–229.
[30] Y. Kawahigashi, Centrally trivial automorphisms and an analogue of Connes' $\chi(M)$ for subfactors, Duke Math. J., **71** (1993), 93–118.
[31] Y. Kawahigashi, Orbifold subfactors, central sequences, and relative Jones invariant κ, IMRN, No. 3 (1995), 129–140.
[32] Y. Kawahigashi, Classification of approximately inner automorphisms of subfactors, Math. Ann., **308** (1997), 425–438.
[33] S. Kawakami and Y. Watatani, The multiplicativity for the minimal index of simple C^*-algebras, Proc. Amer. Math. Soc., **123** (1995), 2809–2813.
[34] S. Kawakami and H. Yoshida, The constituents of Jones' index analyzed from the structure of the Galois theory, Math. Japonica, **33** (1988), 551–117.
[35] H. Kosaki, Automorphisms in the irreducible decomposition of sectors, in "Quantum and Non-commutative Analysis" (Proc. of Oji Seminar), Kluwer Academic Publishers, 1993 (p. 305–316).
[36] H. Kosaki, Sector theory and automorphisms for factor-subfactor pairs, J. Math. Soc. Japan, **48** (1996), 427–454.
[37] H. Kosaki, Automorphisms arising from composition of subfactors, in "Operator Algebras and Quantum Field Theory" (Proc. of Rome Conference), International Press, 1997 (p. 236–248).
[38] H. Kosaki and P. H. Loi, A remark on non-splitting inclusions of type III_1 factors, Internat. J. Math., **6** (1995), 581–586.
[39] H. Kosaki and R. Longo, A remark on the minimal index of subfactors, J. Funct. Anal., **107** (1992), 458–470.
[40] H. Kosaki, A. Munemasa and S. Yamagami, On fusion algebras associated to finite group actions, Pacific J. Math., **177** (1997), 269–290.
[41] H. Kosaki and T. Sano, Non-splitting inclusions of factors of type III_0, Pacific J. Math., **178** (1997), 95–125.
[42] H. Kosaki and S. Yamagami, Irreducible bimodules associated with crossed product algebras, Internat. J. Math., **3** (1992), 661–676.
[43] P. H. Loi, Automorphisms for subfactors, J. Funct. Anal., **141** (1996), 275–293.
[44] R. Longo, Simple injective factors, Adv. Math., **63** (1987), 152–172.
[45] R. Longo, Index of subfactors and statistics of quantum fields I, II, Commun. Math. Phys., **126** (1989), 217–247, and **130** (1990), 285–309.
[46] R. Longo, Minimal index and braided subfactors, J. Funct. Anal., **109** (1992), 98–112.
[47] R. Longo, A duality for Hopf algebras and subfactors I, Commun. Math. Phys., **159** (1994), 133–155.
[48] S. Majid, Hopf-von Neumann algebra bicrossed products, Kac algebra bicrossed products, and the classical Yang-Baxter equation, J. Funct. Anal., **95** (1991), 291–319.
[49] A. Ocneanu, Quantized groups, string algebras, and Galois theory, in Operator Algebras and Applications Vol. II (London Math. Soc. lecture Note Series 136), Cambrige Univ. Press, 1988.
[50] A. Ocneanu, Quantum Symmetry, Differential Geometry of Finite Graphs, and Classification of Subfactors, Univ. of Tokyo Seminary Notes (recorded by Y. Kawahigashi), 1991.
[51] M. Pimsner and S. Popa, Entropy and index of subfactors, Ann. Sci. École Norm. Sup., **19** (1986), 57–106.
[52] S. Popa, Correspondences, preprint.
[53] S. Popa, Classification of subfactors: reduction to commuting squares, Invent. Math., **101** (1990), 19–43.
[54] S. Popa, Classification of amenable subfactors of type II, Acta Math., **172** (1994), 352–445.
[55] S. Popa, Classification of actions of discrete amenable groups on amenable subfactors of type II, preprint.
[56] J. Roberts, Crossed products of von Neumann algebras by group duals, Symposia Math., **20** (1976), 335–363.

[57] T. Sano, Commuting co-commuting squares and finite dimensional Kac algebras, Pacific J. Math., **172** (1996), 243–253.
[58] N. Sato, Fourier transform for irreducible inclusions of type II_1 factors with finite index and its application to depth two case, Publ. RIMS, Kyoto Univ., **33** (1997), 189–222.
[59] J. L. Sauvageot, Sur le produit tensoriel relatif d'espaces de hilbert, J. Operator Theory, **9** (1983), 237–252.
[60] W. Szymański, Finite index subfactors and Hopf algebra crossed products, Proc. Amer. Math. Soc., **120** (1994), 519–528.
[61] M. Takeuchi, Matched pairs of groups and bismashed product of Hopf algebras, Commun. Algebra, **9** (1981), 841–882.
[62] S. Yamagami, A note on Ocneanu's approach to Jones' index theory, Internat. J. Math., **4** (1993), 859–871.
[63] S. Yamagami, Modular theory in bimodules, J. Funct. Anal., **125** (1994), 327–357.
[64] S. Yamagami, Group symmetry in tensor categories, preprint.
[65] T. Yamanouchi, The intrinsic group of Majid's bicossed product Kac algebra, Pacific J. Math., **159** (1993), 185–199.

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