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The completions of metric ANR's and homotopy dense subsets

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Abstract. In this paper, considering the problem when the completion of a metric ANR X is an ANR and X is homotopy dense in the completion, we give some sufficient conditions. It is also shown that each uniform ANR is homotopy dense in any metric space containing X isometrically as a dense subset, and that a metric space X is a uniform ANR if and only if the metric completion of X is a uniform ANR with X a homotopy dense subset. Furthermore, introducing the notions of densely (local) hyper-connectedness and uniformly (local) hyper-connectedness, we characterize of AR's (ANR's) and uniform AR's (uniform ANR's), respectively.

Introduction.

A subset Y of a space X is said to be *homotopy dense* in X if there exists a homotopy $h : X \times [0,1] \to X$ such that $h_0 = \mathrm{id}$ and $h_t(X) \subset Y$ for $t > 0$. This concept is very important in ANR Theory and Infinite-Dimensional Topology. When X is an ANR, the concept of the homotopy denseness is dual to the one of local homotopy negligibility introduced by Toruńczyk in [**To**$_3$]. Actually, $Y \subset X$ is homotopy dense in X if and only if the complement $X \backslash Y$ is locally homotopy negligible in X (cf. [**To**$_3$, Theorem 2.4]). As well-known, every homotopy dense subset of an ANR is also an ANR and a metrizable space is an ANR if it contains an ANR as a homotopy dense subset. The lack of the homotopy denseness of a metric ANR in its completion often destroys the ANR property of the completion. For instance, the $\sin 1/x$-curve in the plane $\mathbf{R}^2$ is an ANR but the completion of this curve (= the closure in $\mathbf{R}^2$) is not an ANR. Moreover, even if the completion is an ANR, it is very different from the original ANR. The circle $\mathbf{S}^1$ is the completion of the space $\mathbf{S}^1 \backslash \{\mathrm{pt}\}$ and the both spaces are ANR but they are topologically very different from each other. It should be remarked that $\mathbf{S}^1 \backslash \{\mathrm{pt}\}$ is not homotopy dense in $\mathbf{S}^1$. It is an interesting problem when a metric ANR is homotopy dense in the metric completion and, in particular, the completion is an ANR.

In [**N**], Nguyen To Nhu gave a characterization of ANR's, a variation of which was given in [**NS**]. In §1 of this paper, we give its alternative proof and apply the technique involved in the proof to find conditions that the completion of a metric space X is an ANR with X a homotopy dense subset. In [**Mi**$_2$], E. Michael introduced

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uniform AR's and uniform ANR's, and studied them. The concept of uniform ANR's is useful since the metric completion of every uniform ANR is also a uniform ANR. In §2, we show that each uniform ANR is homotopy dense in any metric space which contains X isometrically as a dense subset, and that a metric space X is a uniform ANR if and only if the metric completion of X is a uniform ANR with X a homotopy dense subset. By using the notion of (local) hyper-connectedness, C. R. Borges [**Bo**] and R. Cauty [**Ca**] characterized AR's and ANR's, respectively. It is shown in §3 that a little weaker notion also characterizes AR's (or ANR's). Furthermore, we give a characterization of uniform AR's (or uniform ANR's) which is similar to the one of [**Bo**] (or [**Ca**]).

The n-skeleton of a simplicial complex K is denoted by $K^{(n)}$ and the polyhedron $|K|$ is the space $|K| = \bigcup_{\sigma \in K} \sigma$ endowed with the Whitehead topology. For each simplex $\sigma \in K$, we denote $\sigma^{(n)} = \sigma \cap |K^{(n)}|$, which is the union of all n-faces of σ. The nerve of an open cover $\mathscr{U}$ of a space X is denoted by $N(\mathscr{U})$. Note that $\mathscr{U}$ is the set of vertices of $N(\mathscr{U})$, i.e., $\mathscr{U} = N(\mathscr{U})^{(0)}$. Recall a canonical map $\varphi : X \to |N(\mathscr{U})|$ for $\mathscr{U}$ is a map which sends each $x \in X$ into a simplex $\sigma \in N(\mathscr{U})$, all vertices of which contain x. The star of $\mathscr{U}$ is denoted by $\operatorname{st}\mathscr{U} = \{\operatorname{st}(U, \mathscr{U}) \mid U \in \mathscr{U}\}$, where $\operatorname{st}(U, \mathscr{U}) = \bigcup\{V \in \mathscr{U} \mid U \cap V \neq \emptyset\}$. For a collection $\mathscr{A}$ of subsets of X, $\mathscr{A} \prec \mathscr{U}$ means that each $A \in \mathscr{A}$ is contained in some $U \in \mathscr{U}$. In case $X = (X, d)$ is a metric space, the open ball in X centered at $x \in X$ with radius $r > 0$ is denoted by $B_X(x, r)$ (or $B(x, r)$). For $a \in X$ and $C \subset X$, let $\operatorname{dist}(a, C) = \inf\{d(a, x) \mid x \in C\}$ and $\operatorname{diam} C = \sup\{d(x, y) \mid x, y \in C\}$. For a collection $\mathscr{A}$ of subsets of X, let $\operatorname{mesh}\mathscr{A} = \sup\{\operatorname{diam} A \mid A \in \mathscr{A}\}$.

1. A characterization of metric ANR's.

A sequence $\mathscr{U} = (\mathscr{U}_n)_{n \in \mathbf{N}}$ of open covers of a metric space X is called a *zero-sequence* if $\lim_{n \to \infty} \operatorname{mesh}\mathscr{U}_n = 0$. For such a sequence, we define the simplicial complex

$$TN(\mathscr{U}) = \bigcup_{n \in \mathbf{N}} N(\mathscr{U}_n \cup \mathscr{U}_{n+1}),$$

where we regard $\mathscr{U}_n \cap \mathscr{U}_m = \emptyset$ $(n \neq m)$ as sets of vertices of $TN(\mathscr{U})$ even if $\mathscr{U}_n \cap \mathscr{U}_m \neq \emptyset$ as collections of open sets,[1] whence

$$N(\mathscr{U}_n \cup \mathscr{U}_{n+1}) \cap N(\mathscr{U}_{n+1} \cup \mathscr{U}_{n+2}) = N(\mathscr{U}_{n+1}).$$

For each $\sigma \in TN(\mathscr{U})$, let $n(\sigma) = \max\{n \in \mathbf{N} \mid \sigma \in N(\mathscr{U}_n \cup \mathscr{U}_{n+1})\}$. Observe that, for a map $f : |TN(\mathscr{U})| \to X$,

$$\lim_{n \to \infty} \operatorname{mesh}\{f(\sigma) \mid \sigma \in N(\mathscr{U}_n \cup \mathscr{U}_{n+1})\} = 0$$

if and only if $\operatorname{diam} f(\sigma_i) \to 0$ for any sequence $(\sigma_i)_{i \in \mathbf{N}}$ in $TN(\mathscr{U})$ with $n(\sigma_i) \to \infty$. The following is the characterization of ANR's obtained in [**NS**, Theorem 1]. Here is given an alternative proof without the assumption that X has no isolated points.

[1] In [**NS**], we did not regard like this. Considering the set $\bigcup_{n \in \mathbf{N}}\{(U, n) \mid U \in \mathscr{U}_n\}$ as the set of vertices of $NT(\mathscr{U})$, this is reasonable.

Theorem 1. *A metric space $X = (X, d)$ is an ANR if and only if X has a zero-sequence $\mathcal{U} = (\mathcal{U}_n)_{n\in\mathbf{N}}$ of open covers with a map $f : |TN(\mathcal{U})| \to X$ satisfying the following conditions*:

(i) $f(U) \in U$ *for each* $U \in TN(\mathcal{U})^{(0)} = \bigcup_{n\in\mathbf{N}}\mathcal{U}_n$, *and*

(ii) $\lim_{n\to\infty} \operatorname{mesh}\{f(\sigma) \,|\, \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} = 0$.

Under the above circumstances, if the image $f(|TN(\mathcal{U})|)$ is contained in $Y \subset X$, then Y is homotopy dense in X.

Proof. The "only if" part is proved by the same way as [**N**, Theorem 1-1, (i) $\Rightarrow$ (ii)], but we give the proof for the reader's convenience and to make an observation which will be discussed later. Suppose that X is an ANR. By Arens–Eells' embedding theorem [**AE**] (cf. [**To**$_1$]), X can be isometrically embedded in a normed linear space E as a closed set. Then, there is a retraction $r : V \to X$ of an open neighborhood V of X in E. For each $n \in \mathbf{N}$, let $\mathcal{W}_n$ be a convex open cover of V such that $\operatorname{mesh} r(\mathcal{W}_n) < 2^{-n}$. We can construct a zero-sequence $\mathcal{U} = (\mathcal{U}_n)_{n\in\mathbf{N}}$ of open covers of X so that $\operatorname{st}\mathcal{U}_n \prec \mathcal{W}_n$ and $\mathcal{U}_{n+1} \prec \mathcal{U}_n$. By choosing a point $f_0(U) \in U$ for each $U \in TN(\mathcal{U})^{(0)} = \bigcup_{n\in\mathbf{N}}\mathcal{U}_n$, we define a map $f_0 : TN(\mathcal{U})^{(0)} \to X$. For each $\sigma \in TN(\mathcal{U})$, let $U_\sigma \in \sigma^{(0)} \cap \mathcal{U}_{n(\sigma)}$. Then $f_0(\sigma^{(0)}) \subset \operatorname{st}(U_\sigma, \mathcal{U}_{n(\sigma)})$, which is contained in some $W_\sigma \in \mathcal{W}_{n(\sigma)}$. Note that W_σ is convex and $\operatorname{diam} r(W_\sigma) < 2^{-n(\sigma)}$. By using the linear structure of E, we can extend f_0 to a map $f : |TN(\mathcal{U})| \to V$ such that $f(\sigma) \subset W_\sigma$ for each $\sigma \in TN(\mathcal{U})$, whence $\operatorname{diam} rf(\sigma) < 2^{-n(\sigma)}$. The map $rf : |TN(\mathcal{U})| \to X$ clearly satisfies the conditions (i) and (ii).

To prove the "if" part, let $\mathcal{U} = (\mathcal{U}_n)_{n\in\mathbf{N}}$ be a zero-sequence of open covers of X with a map $f : |TN(\mathcal{U})| \to X$ satisfying the conditions (i) and (ii). Then,

$$\alpha_n = \operatorname{mesh}\{f(\sigma) | \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} + \operatorname{mesh}\mathcal{U}_n \to 0 \quad \text{as } n \to \infty.$$

For each $n \in \mathbf{N}$, let $\varphi_n : X \to |N(\mathcal{U}_n)|$ be a canonical map. Observe that, for each $x \in X$, we have $\sigma_x \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})$ such that $\varphi_n(x), \varphi_{n+1}(x) \in \sigma_x$. Then, there is a homotopy $g^{(n)} : X \times [0,1] \to |N(\mathcal{U}_n \cup \mathcal{U}_{n+1})|$ such that $g_0^{(n)} = \varphi_n, g_1^{(n)} = \varphi_{n+1}$ and $g^{(n)}(\{x\} \times [0,1]) \subset \sigma_x$ for each $x \in X$, whence

$$\operatorname{diam} fg^{(n)}(\{x\} \times [0,1]) \le \operatorname{mesh}\{f(\sigma) \,|\, \sigma \in N(\mathcal{U}_n \cup \mathcal{U}_{n+1})\} < \alpha_n.$$

On the other hand, since $\varphi_n(x) \in \sigma_x$ and $x, f(U) \in U$ for some $U \in \sigma_x^{(0)}$, it follows that

$$\begin{aligned} d(f\varphi_n(x), x) &\le d(f\varphi_n(x), f(U)) + d(f(U), x) \\ &\le \operatorname{diam} f(\sigma_x) + \operatorname{diam} U \le \alpha_n. \end{aligned}$$

Now, we can define a homotopy $h : X \times [0,1] \to X$ as follows:

$$h(x,t) = \begin{cases} x & \text{if } t = 0; \\ fg^{(n)}(x, 2 - 2^n t) & \text{if } 2^{-n} \le t \le 2^{-n+1}. \end{cases}$$

The restriction $h|X \times (0,1]$ is clearly continuous. For each $\varepsilon > 0$, we have $n \in \mathbf{N}$ such that $\operatorname{diam} h(\{x\} \times [0, 2^{n+1}]) < \varepsilon$ for every $x \in X$. In fact, choose $n \in \mathbf{N}$ so that $\alpha_m < \varepsilon/2$ for all $m \ge n$. For $0 < t \le 2^{-n+1}$, we have $2^{-m} < t \le 2^{-m+1}$ for some $m \ge n$, whence

$$d(h(x,t),x) \le d(fg^{(m)}(x, 2-2^m t), fg_0^{(m)}(x)) + d(f\varphi_m(x), x)$$
$$\le \operatorname{diam} fg^{(m)}(\{x\} \times [0,1]) + \alpha_m < 2\alpha_m < \varepsilon.$$

This implies that h is continuous at each $(x,0)$. Moreover, $f\varphi_n = h_{2^{-n+1}}$ is ε-homotopic to id_X, which means that X is ε-homotopy dominated by the simplicial complex $TN(\mathscr{U})$. Therefore, X is an ANR.

In the above argument, if $f(|TN(\mathscr{U})|) \subset Y$ then the homotopy h constructed above satisfies that $h(X \times (0,1]) \subset Y$, hence Y is homotopy dense in X. Thus, we have the additional statement. □

Remark. In the above theorem, if $\mathscr{U}_1 = \{X\}$ then X is an AR. In fact, X is contractible because $f\varphi_1$ is constant.

Corollary 1. *Let X be an ANR* (*resp. AR*) *contained in a metric space M. Then, there exists a G_δ-set $Z \subset M$ such that Z is an ANR* (*resp. AR*) *and X is homotopy dense in Z.*

Proof. By Theorem 1, X has a zero-sequence $\mathscr{U} = (\mathscr{U}_n)_{n\in\mathbf{N}}$ of open covers with a map $f : |TN(\mathscr{U})| \to X$ satisfying the conditions (i) and (ii) of Theorem 1. For each open set U in X, we define

$$E(U) = \{x \in M \mid \operatorname{dist}(x, U) < \operatorname{dist}(x, X\setminus U)\},$$

where $\operatorname{dist}(x, \varnothing) = \infty$, so $E(\varnothing) = \varnothing$ and $E(X) = M$. Then, $E(U)$ is open in M, $E(U) \cap X = U$ and $E(U) \cap E(V) = E(U \cap V)$. The desired G_δ-set in M is defined by

$$Z = \operatorname{cl} X \cap \bigcap_{n\in\mathbf{N}} \bigcup_{U\in\mathscr{U}_n} E(U).$$

In fact, for each $n \in \mathbf{N}$, let $\tilde{\mathscr{U}}_n = \{Z \cap E(U) \mid U \in \mathscr{U}_n\}$. Since $\operatorname{mesh} \tilde{\mathscr{U}}_n = \operatorname{mesh} \mathscr{U}_n$, $\tilde{\mathscr{U}} = (\tilde{U}_n)_{n\in\mathbf{N}}$ is a zero-sequence of open covers of Z. The correspondence $Z \cap E(U) \mapsto U$ induces the isomorphism from $TN(\tilde{\mathscr{U}})$ onto $TN(\mathscr{U})$. By the additional statement of Theorem 1, we have the result. □

We can also apply Theorem 1 to find conditions such that the metric completion of a metric space X is an ANR with X a homotopy dense subset. A subset D of a metric space X is said to be *δ-dense* in X if $\operatorname{dist}(x, D) < \delta$ for every $x \in X$.

Corollary 2. *Let X be a metric space which has a zero-sequence $\mathscr{U} = (\mathscr{U}_n)_{n\in\mathbf{N}}$ of open covers with a map $f : |TN(\mathscr{U})| \to X$ satisfying the conditions* (i) *and* (ii) *of Theorem* 1, *where suppose $\mathscr{U}_n = \{B_X(x, \gamma_n) \mid x \in D_n\}$ for some δ_n-dense subset $D_n \subset X$ and* $0 < \delta_n < \gamma_n$. *Then, any metric space Z containing X isometrically as a dense subset is an ANR and X is homotopy dense in Z. In particular*, *the metric completion $\tilde{X}$ of X is an ANR and X is homotopy dense in $\tilde{X}$.*

Proof. In this case, each $\mathscr{U}_n$ extends to the open cover $\tilde{\mathscr{U}}_n = \{B_Z(x, \gamma_n) \mid x \in D_n\}$ of Z. Thus Z has a zero-sequence $\tilde{\mathscr{U}} = (\tilde{\mathscr{U}}_n)_{n\in\mathbf{N}}$. Since $TN(\tilde{\mathscr{U}})$ can be identified with $TN(\mathscr{U})$, the result follows from the additional statement of Theorem 1. □

In the above, note that the γ_n-dense subset D_n of X may not be δ_n-dense in Z. For example, $D_n = \{i/n \mid 1 \le i < n\}$ is $1/n$-dense in (0,1) but it is not $1/n$-dense in $[0,1]$.

Now, we consider the following extension property:

$(e)_k$ There exist constants $\alpha > 0$ and $\beta > 1$ such that every map $f : |K^{(k)}| \to X$ of the k-skeleton of an arbitrary simplicial complex K with $\operatorname{mesh}\{f(\sigma^{(k)}) \mid \sigma \in K\} < \alpha$ extends to a map $\tilde{f} : |K| \to X$ such that $\operatorname{diam} \tilde{f}(\sigma) \le \beta \operatorname{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

The following corollary is motivated by the proof of AR property of hyperspaces (cf. [**vM**, §5.3]).

Corollary 3. *Every LC^{k-1} metric space X with the property $(e)_k$ is an ANR.*

Proof. Without loss of generality, we may assume that X has no isolated points. Since X is LC^{k-1}, X has open covers $\mathscr{V}_{(i,n)}$, $0 \le i \le k$, $n \in \mathbf{N}$, such that $\operatorname{mesh} \operatorname{st} \mathscr{V}_{(k,n)} < 2^{-n}\alpha$, $\mathscr{V}_{(i,n+1)} \prec \mathscr{V}_{(i,n)}$ and each $W \in \operatorname{st} \mathscr{V}_{(i,n)}$ is contained in some $V \in \mathscr{V}_{(i+1,n)}$ such that every map $f : \mathbf{S}^i \to W$ extends to a map $\tilde{f} : \mathbf{B}^{i+1} \to V$. For each $n \in \mathbf{N}$, let $\mathscr{U}_n = \mathscr{V}_{(0,n)}$. Then, $\mathscr{U} = (\mathscr{U}_n)_{n \in \mathbf{N}}$ is a zero-sequence of open covers of X. Let $f_0 : TN(\mathscr{U})^{(0)} \to X$ be a map such that $f_0(U) \in U$ for each $U \in TN(\mathscr{U})^{(0)} = \bigcup_{n \in \mathbf{N}} \mathscr{U}_n$. For each $\sigma \in TN(\mathscr{U})$, $f(\sigma^{(0)})$ is contained in some member of $\operatorname{st} \mathscr{U}_{n(\sigma)} = \operatorname{st} \mathscr{V}_{(0,n(\sigma))}$. By the induction, we can extend f_0 to a map $f_k : |TN(\mathscr{U})^{(k)}| \to X$ such that $f(\sigma^{(k)})$ is contained in some member of $\operatorname{st} \mathscr{V}_{(k,n(\sigma))}$ for each $\sigma \in TN(\mathscr{U})$, hence

$$\operatorname{mesh}\{f_k(\sigma^{(k)}) \mid \sigma \in N(\mathscr{U}_n \cup \mathscr{U}_{n+1})\} \le 2^{-n}\alpha.$$

By the hypothesis, f_k extends to a map $f : |TN(\mathscr{U})| \to X$ such that

$$\operatorname{mesh}\{f(\sigma) \mid \sigma \in N(\mathscr{U}_n \cup \mathscr{U}_{n+1})\} \le 2^{-n}\alpha\beta.$$

Then, the result follows from Theorem 1. □

Remark. The following extension property is stronger than $(e)_k$:

$(\tilde{e})_k$ there exists a constant $\beta > 1$ such that every map $f : |K^{(k)}| \to X$ of the k-skeleton of an arbitrary simplicial complex K extends to a map $\tilde{f} : |K| \to X$ such that $\operatorname{diam} \tilde{f}(\sigma) \le \beta \operatorname{diam} f(\sigma^{(k)})$ for each $\sigma \in K$.

It can be proved that every C^{k-1} and LC^{k-1} metric space X with the property $(\tilde{e})_k$ is an AR. Cf. Remark after Theorem 1.

2. Uniform ANR's.

Let $X = (X, d_X)$ and $Y = (Y, d_Y)$ be metric spaces and $A \subset X$. A map $f : X \to Y$ is said to be *uniformly continuous* at A if, for any $\varepsilon > 0$, there exists $\delta > 0$ such that if $a \in A$, $x \in X$ and $d_X(a, x) < \delta$ then $d_Y(f(a), f(x)) < \varepsilon$. A neighborhood U of A in X is called a *uniform neighborhood* if $\bigcup_{a \in A} B_X(a, \delta) \subset U$ for some $\delta > 0$.

A *uniform* ANR is defined in [**Mi**$_2$] as a metric space Y such that, for an arbitrary metric space X and a closed set $A \subset X$, every uniformly continuous map $f : A \to Y$ extends to a map $\tilde{f} : U \to Y$ from some uniform neighborhood U of A in X which is uniformly continuous at A. When f always extends over X (i.e., $U = X$), Y is called a *uniform* AR. By virtue of [**Mi**$_2$, Theorem 1.2], a metric space Y is a uniform ANR (resp. a uniform AR) if and only if, for an arbitrary metric space Z which contains Y isometrically as a closed subset, there exists a retraction $r : U \to Y$ for some uniform neighborhood U in Y in Z (resp. $r : Z \to Y$) which is uniformly continuous at Y.[2]

Lemma 1. *Every uniform ANR X has a zero-sequence $\mathscr{U} = (\mathscr{U}_n)_{n \in \mathbf{N}}$ of open covers with a map $f : |TN(\mathscr{U})| \to X$ such as Corollary 2.*

Proof. In the proof of the "only if" part of Theorem 1, since the retraction $r : V \to X$ can be assumed to be a retraction of a uniform open neighborhood of X in E which is uniformly continuous at X, we can take as $\mathscr{W}_n$ the open cover $\{B_E(x, r_n) \mid x \in X\}$ for some $r_n > 0$. Let $\delta_n = r_n/3$ and $\gamma_n = r_n/2$. Take a δ_n-dense subset D_n of X and define $\mathscr{U}_n = \{B_X(x, \gamma_n) \mid x \in D_n\}$. By the same argument, we have the result. □

By using this lemma, we can strengthen Proposition 1.4 in [**Mi**$_2$] as follows:

Theorem 2. *For an arbitrary metric space X, the following conditions are equivalent:*
(a) *X is a uniform ANR;*
(b) *Every metric space Z containing X isometrically as a dense subset is a uniform ANR and X is homotopy dense in Z;*
(c) *X is isometrically embedded in some uniform ANR Z as a homotopy dense subset.*

Proof. The implications (a) $\Rightarrow$ (c) and (b) $\Rightarrow$ (a) are obvious.

(a) $\Rightarrow$ (b): By Proposition 1.4 in [**Mi**$_2$], Z is a uniform ANR. Combining Lemma 1 with Corollary 2, it follows that X is homotopy dense in Z.

(c) $\Rightarrow$ (a): By Arens–Eells' embedding theorem [**AE**] (cf. [**To**$_1$]), Z can be isometrically embedded in a normed linear space $E = (E, \|\cdot\|)$ as a closed set which is linearly independent. Let F be the linear subspace of E spanned by X. Then $X = Z \cap F$ is closed in F. Since Z is a uniform ANR, we have a uniform open neighborhood U of Z in E and a retraction $r : U \to Z$ which is uniformly continuous at Z. On the other hand, we have a homotopy $h : Z \times [0,1] \to Z$ such that $h_0 = \mathrm{id}$ and $h_t(Z) \subset X$ for all $t > 0$. It is easy to construct maps $\alpha_n : Z \to (0,1)$, $n \in \mathbf{N}$, such that $\alpha_{n+1}(z) < \alpha_n(z)$ $(\leq 2^{-n})$ and $\operatorname{diam} h(\{z\} \times [0, \alpha_n(z)]) < 2^{-n}$. Then we have a homeomorphism $\varphi : Z \times [0,1] \to Z \times [0,1]$ such that $\varphi | Z \times \{0,1\} = \mathrm{id}$ and $\varphi(z, 2^{-n}) = (z, \alpha_n(z))$ for each $z \in Z$. Observe that $d(z, h\varphi(z,t)) < 2^{-n}$ if $t < 2^{-n}$. We define a retraction $r' : U \to Z$ by $r'(x) = h\varphi(r(x), \operatorname{dist}(x, Z))$ for each $x \in U$. Note that $r'(U \backslash Z) \subset X$. For each $\varepsilon > 0$, choose $n \in \mathbf{N}$ so that $2^{-n+1} < \varepsilon$. Since r is uniformly continuous at Z, there is $\delta > 0$ such that if $x \in U$, $z \in Z$ and $\|x - z\| < \delta$, then $d(r(x), z) < 2^{-n}$. Now, let $x \in U$ and $z \in Z$ with $\|x - z\| < \min\{2^{-n}, \delta\}$. Since $\operatorname{dist}(x, Z) \leq \|x - z\| < 2^{-n}$, it follows that

$$d(r'(x), z) \leq d(h\varphi(r(x), \operatorname{dist}(x, Z)), r(x)) + d(r(x), z) < 2^{-n} + 2^{-n} < \varepsilon.$$

Therefore, r' is also uniformly continuous at Z. The restriction $r'|U \cap F : U \cap F \to X = Z \cap F$ is a retraction which is uniformly continuous at X. By [**Mi**$_2$, Theorem 1.2], X is a uniform ANR. □

[2] Such a retraction is called a *regular retraction* by H. Toruńczyk in [**To**$_2$].

Theorem 2 above means that a metric space X is a uniform ANR if and only if the metric completion $\tilde{X}$ of X is a uniform ANR and X is homotopy dense in $\tilde{X}$. However, in order that the metric completion of a metric ANR X is an ANR with X a homotopy dense subset, it is not necessary that X is a uniform ANR.

Example. The following subspace X of Euclidean plane $\mathbf{R}^2$ is not a uniform ANR but the metric completion of X is an ANR with X a homotopy dense subset:

$$X = \mathbf{R} \times \{0\} \cup \mathbf{N} \times [0,1) \cup \bigcup_{n \in \mathbf{N}} \{n + 2^{-n}\} \times [0,1) \subset \mathbf{R}^2.$$

In fact, X is not a uniform neighborhood retract of $\mathbf{R}^2$, but X and the closure of X in $\mathbf{R}^2$ are ANR's and X is homotopy dense in the closure.

In case X is totally bounded, we have the following:

Proposition 1. *A totally bounded metric space X a uniform ANR if and only if the metric completion $\tilde{X}$ of X is an ANR with X a homotopy dense subset.*

Proof. It suffices to show the "if" part. Assume that $\tilde{X}$ is an ANR and X is homotopy dense in $\tilde{X}$. Since $\tilde{X}$ is also totally bounded, it is a compact ANR, hence it is a uniform ANR. By Theorem 2, X is also a uniform ANR. □

Now, we prove the following theorem:

Theorem 3. *Every metric space Y with the property $(e)_0$ is a uniform ANR.*

Proof. This can be shown by an alteration of the proof of [**Mi**$_2$, Theorem 7.1 (c) ⇒ (a)] as follows: Let $s_1 > s_2 > \cdots > 0$ be any sequence such that $8s_1 < \alpha$, $\lim_{n\to\infty} s_n = 0$ and $\mathscr{V}_m \cap \mathscr{V}_n = \emptyset$ if $m \neq n$, where $\mathscr{V}_n$ is defined in [**Mi**$_2$, p. 135]. Then, the map f in the Michael's proof satisfies the following condition:

$$\operatorname{diam} f(\sigma^{(0)}) < 8s_n \quad \text{for each } \sigma \in \mathscr{U}_n.$$

Here, instead of extending f step by step, we can apply the property $(e)_0$ to extend f to a map $h : \bigcup_{n\in\mathbf{N}} |N(\mathscr{U}_n)| \to Y$ such that $\operatorname{diam} h(\sigma) < 8s_n\beta$ for each $\sigma \in N(\mathscr{U}_n)$. For each $n \in \mathbf{N}$, let $h_n = h||N(\mathscr{U}_n)|$. By the same definition as in the proof, we can obtain a uniform neighborhood W of Y in Z and a retraction $r : W \to Y$ which is uniformly continuous at Y. □

By Theorems 2 and 3, we have the following corollary (cf. [**SU**, Lemma 2]):

Corollary 4. *Let X be a metric space and Y a dense subset of X. If Y has the property $(e)_0$, then X and Y are ANR's and Y is homotopy dense in X.* □

Remark. In Theorem 3 and Corollary 4, if the property $(e)_0$ is replaced by $(\tilde{e})_0$, then "ANR" can be "AR".

A metric space Y is said to be *uniformly LC^k* if, for each $\varepsilon > 0$, there exists $\delta > 0$ such that any map $f : \mathbf{S}^i \to Y$ with $\operatorname{diam} f(\mathbf{S}^i) < \delta$ extends to a map $\tilde{f} : \mathbf{B}^{i+1} \to Y$ with $\operatorname{diam} \tilde{f}(\mathbf{B}^{i+1}) < \varepsilon$ for every $i \leq k$. In stead of "uniformly LC^0", we also say "uniformly locally path-connected". The subspace of $\mathbf{R}^2$ in the example above is not uniformly locally path-connected.

Proposition 2. *Every uniformly LC^{k-1} metric space Y with the property $(e)_k$ is a uniform ANR.*

Proof. This is also shown by an alteration of the proof of [**Mi**$_2$, Theorem 7.1 (c) $\Rightarrow$ (a)]. Here, we can apply the condition (c) of [**Mi**$_2$, Theorem 7.1] to a simplicial complex K with $\dim K \leq k$. In the Michael's proof, replacing $1/n$ by $\alpha/3n$, the map $f|N(\mathscr{V}_n)^{(0)}$ extends to a map $h'_n : |N(\mathscr{U}_n)^{(k)}| \to Y$ such that $\operatorname{diam} h'_n(\sigma) < \alpha/3n$ for each $\sigma \in N(\mathscr{U}_n)^{(k)}$. For each $\sigma \in N(\mathscr{U}_n)$, since $\operatorname{diam} h'_n(\sigma^{(0)}) < \alpha/3n$, we have $\operatorname{diam} h'_n(\sigma^{(k)}) < \alpha/n$. Now, by using the property $(e)_k$, each h'_n can be extended to a map $h_n : |N(\mathscr{U}_n)| \to Y$ such that $\operatorname{diam} h_n(\sigma) < \alpha\beta/n$ for each $\sigma \in N(\mathscr{U}_n)$. Then, by the same definition as in the proof, we can obtain a uniform neighborhood W of Y in Z and a retraction $r : W \to Y$ which is uniformly continuous at Y. □

Combining of Proposition 2 with Theorem 2, we have the following variation of Corollary 3.

Corollary 5. *Let X be a metric space and Y a dense subset of X. If Y is uniformly LC^{k-1} and has the property $(e)_k$, then X and Y are uniformly ANR's and Y is homotopy dense in X.* □

Remark. In Proposition 2 and Corollary 5, by replacing the property $(e)_k$ with $(\tilde{e})_k$ and adding the condition that Y is C^{k-1}, "uniform ANR" can be "uniform AR".

3. Dense (or uniform) local hyper-connectedness.

By Δ^{n-1}, we denote the standard $(n-1)$-simplex in $\mathbf{R}^n$, that is,

$$\Delta^{n-1} = \left\{ (t_1, \dots, t_n) \in \mathbf{R}^n \;\middle|\; t_i \geq 0, \sum_{i=1}^{n+1} t_i = 1 \right\}.$$

For an open cover $\mathscr{U}$ of a space X and $Y \subset X$, we denote

$$Y^n(\mathscr{U}) = \{(y_1, \dots, y_n) \in Y^n \mid \exists U \in \mathscr{U} \text{ such that } \{y_1, \dots, y_n\} \subset U\}.$$

It is said that a space X is *densely locally hyper-connected* if X has an open cover $\mathscr{W}$, a dense subset D and functions $h_n : D^n(\mathscr{W}) \times \Delta^{n-1} \to X$, $n \in \mathbf{N}$, which satisfy the following conditions:

(i) if $t_i = 0$ then

$$h_n(y_1, \dots, y_n; t_1, \dots, t_n) = h_{n-1}(y_1, \dots, y_{i-1}, y_{i+1}, \dots, y_n; t_1, \dots, t_{i-1}, t_{i+1}, \dots, t_n);$$

(ii) $\Delta^{n-1} \ni (t_1, \dots, t_n) \mapsto h_n(y_1, \dots, y_n; t_1, \dots, t_n) \in X$ is continuous for each $(y_1, \dots, y_n) \in D^n(\mathscr{W})$;

(iii) every open cover $\mathscr{U}$ of X has an open refinement $\mathscr{V}$ such that $\mathscr{V} \prec \mathscr{W}$ (hence $D^n(\mathscr{V}) \subset D^n(\mathscr{W})$) and

$$\{h_n((D \cap V)^n \times \Delta^{n-1}) | V \in \mathscr{V}\} \prec \mathscr{U} \quad \text{for each } n \in \mathbf{N}.$$

It should be noticed that each h_n need not be continuous. If $\mathscr{W}$ can be taken as $\mathscr{W} = \{X\}$ (i.e., $D^n(\mathscr{W}) = D^n$), we say that X is *densely hyper-connected*. In case $D = X$ (resp. $D = X$ and $\mathscr{W} = \{X\}$), X is *locally hyper-connected*[3] (resp. *hyper-connected*). This concept is very similar to Michael's convex structure in [**Mi**$_1$]. In [**Bo**] and [**Ca**], AR's and ANR's are characterized by the hyper-connectedness and the local hyper-connectedness, respectively. A similar characterization was obtained by Himmelberg [**Hi**] (cf. Curtis [**Cu**]). These characterizations can be generalized in terms of the dense hyper-connectedness as follows:

Theorem 4. *A metrizable space X is an ANR if and only if X is densely locally hyper-connected. Moreover, X is an AR if and only if X is densely hyper-connected.*

Proof. By the characterization of ANR's in [**Ca**] (or AR's in [**Bo**]), it suffices to prove the "if" part only. (Or see the proof of Theorem 5 below.)

Assume that X is a densely locally hyper-connected metric space, that is, X has an open cover $\mathscr{W}$, a dense subset D and functions $h_n : D^n(\mathscr{W}) \times \Delta^{n-1} \to X$, $n \in \mathbf{N}$, which satisfy the conditions (i), (ii) and (iii). By the condition (iii), we obtain a sequence $\mathscr{U} = (\mathscr{U}_n)_{n\in\mathbf{N}}$ of open covers of X such that $\operatorname{st}\mathscr{U}_1 \prec \mathscr{W}$, $\mathscr{U}_{n+1} \prec \mathscr{U}_n$, $\operatorname{mesh}\mathscr{U}_n < 2^{-n}$ and

$$\operatorname{mesh}\{h_k((D \cap \operatorname{st}(U, \mathscr{U}_n))^k \times \Delta^{k-1}) | k \in \mathbf{N}, U \in \mathscr{U}_n\} < 2^{-n}.$$

By choosing a point $f_0(U) \in D \cap U$ for each $U \in TN(\mathscr{U})^{(0)} = \bigcup_{n\in\mathbf{N}} \mathscr{U}_n$, we define a map $f_0 : TN(\mathscr{U})^{(0)} \to D$. For each $\sigma \in TN(\mathscr{U})$, let $\sigma^{(0)} = \{U_1, \dots, U_k\} \subset \mathscr{U}_n \cup \mathscr{U}_{n+1}$, where we can assume $U_1 \in \mathscr{U}_n$. Then $f_0(\sigma^{(0)}) \subset \operatorname{st}(U_1, \mathscr{U}_n)$ because $\mathscr{U}_{n+1} \prec \mathscr{U}_n$. By using h_k, we can define $f_\sigma : \sigma \to X$ by

$$f_\sigma\left(\sum_{i=1}^{k} t_i U_i\right) = h_k(f_0(U_1), \dots, f_0(U_k); t_1, \dots, t_k).$$

Then $\operatorname{diam} f_\sigma(\sigma) < 2^{-n}$. Observe that $f_\sigma|\sigma \cap \tau = f_\tau|\sigma \cap \tau$ for each $\sigma, \tau \in TN(\mathscr{U})$. Therefore, we can define a map $f : |TN(\mathscr{U})| \to X$ by $f|\sigma = f_\sigma$ for each $\sigma \in TN(\mathscr{U})$. It is easy to verify that $\mathscr{U}$ and f satisfy the conditions (i) and (ii) of Theorem 1, which implies that X is an ANR.

In the above, we may assume that $\operatorname{diam} X < 2^{-1}$. In case X is densely hyper-connected, $\mathscr{W} = \{X\}$, hence we can take $\mathscr{U}_1 = \{X\}$. Then X is an AR by the remark of Theorem 1. □

Remark. In the definition of densely local hyper-connectedness, if the images of functions h_n are contained in Y, then Y is homotopy dense in X. In fact, if the images of functions h_n are contained in Y, then $f(|TN(\mathscr{U})|) \subset Y$, hence Y is homotopy dense in X by the additional statement of Theorem 1.

For a metric space X and $\eta > 0$, we denote

$$X^n(\eta) = \{(x_1, \dots, x_n) \in X^n \,|\, \operatorname{diam}\{x_1, \dots, x_n\} < \eta\}.$$

A metric space X is said to be *uniformly locally hyper-connected* if there are $\eta > 0$ and

[3] The local hyper-connectedness is in the sense of [**Ca**] but not in the sense of [**Bo**].

functions $h_n : X^n(\eta) \times \Delta^{n-1} \to X$, $n \in \mathbf{N}$, which satisfy the same conditions as (i) and (ii) above, and the following (iii′) instead of (iii):

(iii′) for each $\varepsilon > 0$, there is $0 < \delta < \varepsilon$ such that

$$\operatorname{diam} h_n(\{x\} \times \Delta^{n-1}) < \varepsilon \quad \text{for every } n \in \mathbf{N} \text{ and } x \in X^n(\delta).$$

When every h_n is defined on the whole space $X^n \times \Delta^{n-1}$, it is said that X is *uniformly hyper-connected.*

Now, we give a characterization of uniform ANR's and uniform AR's.

Theorem 5. *A metric space $X = (X, d)$ is a uniform ANR if and only if X is uniformly locally hyper-connected. Moreover, X is a uniform AR if and only if X is uniformly hyper-connected.*

Proof. First, we see the "only if" part. By Arens–Eells' embedding theorem [**AE**] (cf. [**To**$_1$]), X can be isometrically embedded in a normed linear space $E = (E, \|\cdot\|)$ as a closed set. If X is a uniform ANR, there is a uniform open neighborhood U of X in E with a retraction $r : U \to X$ which is uniformly continuous at X. Choose $\eta > 0$ so that $\bigcup_{x \in X} B_E(x, \eta) \subset U$. For each $n \in \mathbf{N}$, we can define a map $h_n : X^n(\eta) \times \Delta^{n-1} \to X$ as follows:

$$h_n(x_1, \dots, x_n; t_1, \dots, t_n) = r\left(\sum_{i=1}^{n} t_i x_i\right).$$

It is clear that the maps h_n's satisfy the conditions (i) and (ii). Since the retraction r is uniformly continuous at X, for each $\varepsilon > 0$, there is $0 < \delta < \eta$ such that if $x \in X$, $z \in U$ and $\|x - z\| < \delta$ then $d(x, r(z)) < \varepsilon$. For $(x_1, \dots, x_n) \in X^n(\delta)$ and $(t_1, \dots, t_n) \in \Delta^{n-1}$, let $z = \sum_{i=1}^{n} t_i x_i \in U$. Since $\operatorname{diam}\{x_1, \dots, x_n\} < \delta$, it follows that $\|x_1 - z\| \le \sum_{i=1}^{n} t_i \|x_1 - x_i\| < \delta$, which implies that

$$d(x_1, h_n(x_1, \dots, x_n; t_1, \dots, t_n)) = d(x_1, r(z)) < \varepsilon.$$

Hence, $\operatorname{diam} h_n(\{x\} \times \Delta^{n-1}) < \varepsilon$ for every $n \in \mathbf{N}$ and $x \in X^n(\delta)$. Thus the condition (iii′) is also satisfied. Therefore, X is uniformly locally hyper-connected.

In case X is a uniform AR, since $X^n(\eta)$ can be replaced by X^n in the above, X is uniformly hyper-connected.

Next, to show the "if" part, assume that X is uniformly locally hyper-connected, that is, there are $\eta > 0$ and functions $h_n : X^n(\eta) \times \Delta^{n-1} \to X$, $n \in \mathbf{N}$, which satisfy the conditions (i), (ii) and (iii′). For each $\varepsilon > 0$, we have $\gamma, \delta > 0$ such that $\operatorname{diam} h_n(\{x\} \times \Delta^{n-1}) < \varepsilon/3$ for every $n \in \mathbf{N}$ and $x \in X^n(\gamma)$ and $\operatorname{diam} h_n(\{x\} \times \Delta^{n-1}) < \gamma/2$ for every $n \in \mathbf{N}$ and $x \in X^n(\delta)$. Note that $\delta \le \gamma/2$ and $\gamma \le \varepsilon/3$. Let K be a simplicial complex, L a subcomplex of K with $K^{(0)} \subset L$ and $f : |L| \to X$ be a map such that $f(\sigma \cap |L|) < \delta$ for each $\sigma \in K$. Then, by using h_n, we can extend $f|K^{(0)}$ to a map $f' : |K| \to X$ such that $f'(\sigma) < \gamma/2$ for each $\sigma \in K$. Each $x \in |L|$ is contained in $\sigma \in L$, whence

$$\begin{aligned} d(f(x), f'(x)) &\le d(f(x), f(v)) + d(f'(v), f'(x)) \\ &< \delta + \gamma/2 < \gamma, \end{aligned}$$

where $v \in \sigma^{(0)}$. By using h_1, we define a homotopy $h : |L| \times [0,1] \to X$ by $h(x,t) = h_1(f(x), f'(x); t, (1-t))$. Then h is an $\varepsilon/3$-homotopy from f to $f'||L|$, that is, $\operatorname{diam} h(\{x\} \times [0,1]) < \varepsilon/3$ for each $x \in |L|$. Since X is an ANR, we can apply the homotopy extension theorem to extend f to a map $\tilde{f} : |K| \to X$ which is $\varepsilon/3$-homotopic to f'. Then $\operatorname{diam} \tilde{f}(\sigma) < \varepsilon$ for each $\sigma \in K$. In fact, for each $x, x' \in \sigma$,

$$d(\tilde{f}(x), \tilde{f}(x')) \le d(\tilde{f}(x), f'(x)) + d(f'(x), f'(x')) + d(f'(x'), \tilde{f}(x'))$$

$$< \varepsilon/3 + \gamma/2 + \varepsilon/3 < \varepsilon/2 + \varepsilon/6 < \varepsilon.$$

By [**Mi**$_2$, Theorem 7.1], this means that X is a uniform ANR.

In case X is uniformly hyper-connected, since it is an AR and a uniform ANR, X is a uniform AR by [**Mi**$_2$, Proposition 1.3]. □

The following is a combination of Theorems 2 and 5:

Corollary 6. *Let X be a uniformly (locally) hyper-connected metric space and Z a metric space which contains X isometrically as a dense subset. Then, X and Z are uniform AR's (uniform ANR's) and X is homotopy dense in Z. In particular, the metric completion $\tilde{X}$ of X is a uniform AR (uniform ANR) and X is homotopy dense in $\tilde{X}$.* □

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