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Convergence rates to viscous shock profile for general scalar viscous conservation laws with large initial disturbance

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Abstract. This paper is concerned with the convergence rates to viscous shock profile for general scalar viscous conservation laws. Compared with former results in this direction, the main novelty in this paper lies in the fact that the initial disturbance can be chosen arbitrarily large. This answers positively an open problem proposed by A. Matsumura in [**12**] and K. Nishihara in [**16**]. Our analysis is based on the L^1-stability results obtained by H. Freistühler and D. Serre in [**1**].

1. Introduction and the statement of our main results.

This paper is concerned with the convergence rates to viscous shock profile of solutions to the Cauchy problem for general scalar viscous conservation laws

$$u_t + f(u)_x = u_{xx}, \quad x \in \boldsymbol{R}, \quad t > 0 \tag{1.1}$$

with initial data

$$u(t,x)|_{t=0} = u_0(x), \quad x \in \boldsymbol{R}, \tag{1.2}$$

where $f(u) \in C^2(\boldsymbol{R})$ on the domain under our consideration and the initial data $u_0(x)$ is asymptotically constant as $x \to \pm\infty$:

$$u_0(x) \to u_\pm \quad \text{as } x \to \pm\infty. \tag{1.3}$$

The traveling wave $u(x - st) \equiv \phi(\xi)$ is called a viscous shock profile to (1.1)–(1.3) if it satisfies

$$-s\phi_\xi + f(\phi)_\xi = \phi_{\xi\xi}, \quad \phi(\xi) \to u_\pm \quad \text{as } \xi \to \pm\infty. \tag{1.4}$$

Here the constants $u_\pm$ and s (shock speed) satisfy the Rankine-Hugoniot condition

$$-s(u_+ - u_-) + f(u_+) - f(u_-) = 0 \tag{1.5}$$

and the generalized entropy condition

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$$h(u) \equiv -s(u - u_\pm) + f(u) - f(u_\pm) \begin{cases} < 0, & \text{if } u_+ < u < u_-, \\ > 0, & \text{if } u_- < u < u_+. \end{cases} \tag{1.6}$$

That is, the viscous shock profile ϕ is a solution to

$$\phi_\xi = h(\phi), \quad \phi(\pm\infty) = u_\pm.$$

It is noted that the condition (1.6) implies

$$f'(u_+) \le s \le f'(u_-) \tag{1.7}$$

which includes the following cases: *the nondegenerate shock condition*

$$f'(u_+) < s < f'(u_-) \tag*{$(1.8)_1$}$$

and *the degenerate shock condition*

$$f'(u_+) = s < f'(u_-), \quad f'(u_+) < s = f'(u_-) \quad \text{or} \quad s = f'(u_\pm). \tag*{$(1.8)_2$}$$

We call the shock satisfying $(1.8)_1$ *Lax shock* (regardless of viscous or inviscid case) while those satisfying $(1.8)_2$ are called *marginal shock*. In what follows, for the marginal shock, we only pay our attention to the case $f'(u_+) = s < f'(u_-)$ since the other cases can be treated similarly.

Stability results have a long history starting with the paper of A. M. Il'in and O. A. Oleinik [**4**], in which they proved that the viscous shock profile in the case of a convex flux function is indeed stable. Since then, a lot of good results have been obtained by employing various methods (All references [**1**]–[**18**] are on this line. Especially, see the survey paper [**12**]).

To go directly to the main point of this paper, we only review two results which are closely related to ours. The most general result on the nonlinear stability of the viscous shock profile is given by H. Freistühler and D. Serre in [**1**].

Theorem 1.1 (L^1-stability). *Let $\phi(\xi) : \boldsymbol{R} \to \boldsymbol{R}$ be a bounded viscous shock profile of* (1.1), (1.2). *Then for any $u_0(x)$ satisfying $u_0(x) - \phi(x) \in L^1(\boldsymbol{R})$, the Cauchy problem* (1.1), (1.2) *admits a unique solution $u(t,x)$ satisfying*

$$\lim_{t\to+\infty} \|u(t,x) - \phi(x - st + \delta)\|_{L^1} = 0, \tag{1.9}$$

where

$$\delta := \frac{\int_{\boldsymbol{R}} (u_0(x) - \phi(x))\,dx}{u_+ - u_-}. \tag{1.10}$$

Although the results obtained in Theorem 1.1 are quite perfect, no decay rates have been obtained. On the other hand, A. Matsumura and K. Nishihara [**13**], M. Nishikawa [**17**] have obtained the following decay properties via the L^2-energy method. Notations are given in Remark 1.1 below.

THEOREM 1.2 (Decay rates). (I) *When* $f'(u_+) < s < f'(u_-)$, *suppose that* $u_0(x) - \phi(x)$ *is integrable and that*

$$U_0(x) := \int_{-\infty}^{x} \{u_0(z) - \phi(z+\delta)\}\, dz \in H^2 \cap L^2_\alpha(\boldsymbol{R}).$$

Then there exists a sufficiently small positive constant ε_1 *such that if* $\|U_0(x)\|_2 < \varepsilon_1$, *the Cauchy problem* (1.1), (1.2) *has a unique global solution* $u(t,x)$ *satisfying*

$$\sup_{x\in\boldsymbol{R}} |u(t,x) - \phi(x - st + \delta)| \le O(1)(1+t)^{-\alpha/2}(\|u_0 - \phi\|_1 + |U_0|_\alpha). \tag{1.11}$$

(II) *When* $f'(u_+) = s < f'(u_-)$, *suppose that* $f(u) \in C^{n+1}(\boldsymbol{R})$ *such that*

$$f''(u_+) = \cdots = f^{(n)}(u_+) = 0 \ \text{and}\ f^{(n+1)}(u_+) \neq 0 \quad \text{for some } n \ge 1 \tag{1.12}$$

and that $u_0(x) - \phi(x)$ *is integrable and* $U_0(x) \in H^2 \cap L^2_{\alpha,\langle\xi\rangle_+}$ $(0 < \alpha < 2/n)$, *then there exists a sufficiently small positive constant* $\varepsilon_1 > 0$ *such that if* $\|U_0\|_2 + |U_0|_{\langle\xi\rangle_+} < \varepsilon_1$, *the Cauchy problem* (1.1), (1.2) *has a unique global solution* $u(t,x)$ *satisfying*

$$\sup_{x\in\boldsymbol{R}} |u(t,x) - \phi(x - st + \delta)| \le O(1)(1+t)^{-\alpha/4}(\|u_0 - \phi\|_1 + |U_0|_{\alpha,\langle\xi\rangle_+}). \tag{1.13}$$

Here

$$\langle\xi\rangle_+ := \begin{cases} \sqrt{1+\xi^2}, & \xi \ge 0, \\ 1, & \xi < 0. \end{cases} \tag{1.14}$$

REMARK 1.1 (Notations). Here in the above and in what follows, by C or $O(1)$, we denote several generic constants and for each $\tau \ge 0$, $C(t-\tau)$ (or $C_i(t-\tau)$ for some $i \in \boldsymbol{Z}^+$) will be used to denote some generic function which is continuous with respect to t on $[\tau, \infty)$. For two functions $f(x)$ and $g(x)$, $f(x) \sim g(x)$ as $x \to a$ means

$$C^{-1}f(x) \le g(x) \le Cf(x) \tag{1.15}$$

in the neighborhood of a. $H^l(\boldsymbol{R})$ $(l \ge 0)$ denotes the usual Sobolev space with norm $\|\cdot\|_l$ and $\|\cdot\|_0 = \|\cdot\|$ will denote the usual L^2-norm. For the weighted function $w(x) > 0$, $L^2_w(\boldsymbol{R})$ denotes the space of measurable functions $f(x)$ satisfying $\sqrt{w(x)}f(x) \in L^2(\boldsymbol{R})$ with the norm

$$|f|_w := \left(\int_{\boldsymbol{R}} w(x)|f(x)|^2\, dx\right)^{1/2}. \tag{1.16}$$

When $C^{-1} \le w(x) \le C$, we note that $L^2_w(\boldsymbol{R}) = L^2(\boldsymbol{R})$ with $|\cdot|_w = \|\cdot\|$. When $w(x) \sim \langle x \rangle^\alpha = (1+x^2)^{\alpha/2}$, we write $L^2_w(\boldsymbol{R}) = L^2_\alpha(\boldsymbol{R})$ and $|\cdot|_w = |\cdot|_\alpha$ without confusion. Moreover, if $w(x)$ is replaced by $\langle x \rangle^\alpha w(x)$, we denote the space by $L^2_{\alpha,w}(\boldsymbol{R})$ with the norm

$$|f|_{\alpha,w} := \left(\int_{\boldsymbol{R}} \langle x \rangle^\alpha w(x) |f(x)|^2 \, dx \right)^{1/2}. \tag{1.17}$$

From the above two results, it is easy to find that in Theorem 1.1, the initial disturbance can be chosen arbitrarily large but no decay rates can be obtained. In Theorem 1.2, some decay rates have been obtained but, due to the limitation of their arguments, its initial disturbance should be small in certain Sobolev space. Thus it is of interest how to get the decay without smallness condition. In fact, such a problem is one of the open problems proposed by A. Matsumura in [**12**] and K. Nishihara in [**16**]. Our main purpose of this paper is to give a positive answer to this problem.

THEOREM 1.3 (Main results). *Let the initial data* $u_0(x) - \phi(x) \in L^1 \cap L^\infty(\boldsymbol{R})$ *and* $U_0(\xi) \in L^2(\boldsymbol{R})$, *then the following attsertions hold.*

(I) *When* $f'(u_+) < s < f'(u_-)$, *the estimate* (1.11) *holds provided* $U_0(\xi) \in L^2_\alpha(\boldsymbol{R})$;

(II) *When* $f'(u_+) = s < f'(u_-)$, *the estimate* (1.13) *holds provided that the assumption* (1.12) *holds and* $U_0(\xi) \in L^2_{\alpha,\langle\xi\rangle_+}(\boldsymbol{R})$ *with* $0 < \alpha < 2/n$.

REMARK 1.2. When $f'(u_+) < s = f'(u_-)$ or $s = f'(u_\pm)$, then $L^2_{\alpha,\langle\xi\rangle_+}(\boldsymbol{R})$ in (II) of Theorem 1.3 should be replaced by $L^2_{\alpha,\langle\xi\rangle_-}(\boldsymbol{R})$ or $L^2_{\alpha,\langle\xi\rangle}(\boldsymbol{R}) \equiv L^2_{\alpha+1}(\boldsymbol{R})$ respectively while the same results also hold. Here

$$\langle \xi \rangle_- := \begin{cases} \sqrt{1+\xi^2}, & \xi \le 0, \\ 1, & \xi > 0. \end{cases}$$

REMARK 1.3. Compared with the results obtained in [**13**], [**17**], the regularity assumptions on the initial data is also weaker than those in [**13**], [**17**].

REMARK 1.4. As pointed out by A. Matsumura and K. Nishihara in [**13**], for the Lax shock, the decay rates obtained in Theorem 1.3 is expected to be optimal in the L^2-setting. In fact, when $f(u) = u^2/2$, by exploiting an explicit formula, K. Nishihara showed in [**15**] that if

$$|U_0(x)| \le O(1)|x|^{-\alpha/2} \quad \text{as } |x| \to +\infty,$$

then

$$\sup_{x\in\boldsymbol{R}} |u(t,x) - \phi(x - st + \delta)| \le O(1)t^{-\alpha/2},$$

which is an optimal decay rate in general.

Before concluding this section, we give main ideas in deducing our main result, Theorem 1.3. Decay rates (1.11), (1.13) in Theorem 1.2 have been obtained by the weighted energy method developed by Kawashima, Matsumura and Nishihara *etc.* in [**6**], [**7**], [**13**]. In their method, to obtain the *a priori* estimates is a key point under the *a priori* assumption

$$N(t) := \sup_{0\le s\le t} \|U(s,\cdot)\|_2 \le \varepsilon$$

for sufficiently small positive constant ε, so that the initial disturbance $U_0(\xi)$ should be small. However, we found that the *a priori* estimates are available provided that $\|U(t,\cdot)\|_{L^\infty}$ is small. The L^1-stability theorem, Theorem 1.1, by H. Freistüler and D. Serre in [**1**] also shows that $\|U(t,\cdot)\|_{L^\infty} \to 0$ as $t \to \infty$. Therefore we can apply the weighted energy method on $[T_1,\infty)\times\boldsymbol{R}$ for some large T_1.

Our plan is as follows. In Section 2, we give some preliminary results. The proof of our main results will be given in Section 3.

2. Preliminary lemmas.

In this section, we give some preliminary lemmas which will be used in proving our main results in the next section.

First, the existence of viscous shock profiles $\phi(\xi)$ follows from Kawashima and Matsumura [**5**].

Lemma 2.1 (Existence of the viscous shock profile). (i) *If the Cauchy problem* (1.1), (1.2) *admits viscous shock profile* $\phi(x - st)$ *connecting* u_- *and* u_+, *then* u_-, u_+ *and* s *must satisfy the Rankine-Hugoniot condition* (1.5) *and the generalized entropy condition* (1.6);

(ii) *Conversely, suppose that* (1.5) *and* (1.6) *hold, then there exists a viscous shock profile* $\phi(x - st)$ *of* (1.1), (1.2) *which connects* u_- *and* u_+ *and is unique up to a shift in* $\xi = x - st$ *and is monotone in* ξ. *Moreover, if*

$$h(\phi) \sim |\phi - u_\pm|^{1+k_\pm} \tag{2.1}$$

as $\phi \to u_\pm$ *with* $k_\pm \ge 0$, *then it holds*

$$\begin{cases} |\phi(\xi) - u_\pm| \sim \exp(-C_\pm|\xi|) & \text{as } \xi \to \pm\infty \text{ if } k_\pm = 0, \\ |\phi(\xi) - u_\pm| \sim |\xi|^{-1/k_\pm} & \text{as } \xi \to \pm\infty \text{ if } k_\pm \ne 0, \end{cases} \tag{2.2}$$

for some positive constant $C_\pm$.

Note that $k_\pm = n$ in (2.1) if $h'(u_\pm) = \cdots = h^{(n)}(u_\pm) = 0$ and $h^{(n+1)}(u_\pm) \neq 0$ which are corresponding to (1.12).

We now define the shift δ of the viscous shock profile $\phi(x - st)$ as

$$\int_{\boldsymbol{R}} (u_0(x) - \phi(x + \delta))\, dx = 0 \tag{2.3}$$

and set

$$U_0(x) := \int_{-\infty}^{x} (u_0(z) - \phi(z + \delta))\, dz. \tag{2.4}$$

It is easy to see that δ satisfies (1.10) and, without loss of generality, we may take $\delta = 0$. Following A. Matsumura and K. Nishihara [**13**], we put the perturbation

$$u(t, x) = \phi(\xi) + U_\xi(t, \xi), \quad \xi = x - st, \tag{2.5}$$

then the problem (1.1), (1.2) is reformulated to

$$U_t - U_{\xi\xi} + h'(\phi) U_\xi = F(t, \xi), \tag{2.6}$$

$$U(t, \xi)|_{t=0} = U_0(\xi) \equiv \int_{-\infty}^{\xi} (u_0(z) - \phi(z))\, dz, \tag{2.7}$$

where

$$F(t, \xi) := -\{f(\phi + U_\xi) - f(\phi) - f'(\phi) U_\xi\}. \tag{2.8}$$

Note that $\phi(\xi) \in L^\infty(\boldsymbol{R})$ and $U_{0\xi}(\xi) \in L^\infty(\boldsymbol{R})$. From the well-known result on the global solvability of the Cauchy problem to scalar parabolic equations [**1**], we have that

LEMMA 2.2 (Global existence to the Cauchy problem (2.6), (2.7)). *Suppose that $f(u) \in C^1(\boldsymbol{R})$, $u_0(x) - \phi(x) \in L^1 \cap L^\infty(\boldsymbol{R})$, then the Cauchy problem* (2.6), (2.7) *admits a unique global smooth solution $U(t, \xi)$ satisfying*

$$|U_\xi(t, \xi)| \le C_1, \tag{2.9}$$

where

$$C_1 := \|u_0(x) - \phi(x)\|_{L^\infty} + \|\phi(x)\|_{L^\infty}. \tag{2.10}$$

From the Duhamel principle, the solution $U(t, \xi)$ to the Cauchy problem (2.6), (2.7) has the following integral representation

$$U(t,\xi) = K(t,\xi) * U_0(\xi) + \int_0^t K(t-s,\xi) * G(s,\xi)\,ds. \tag{2.11}$$

Here $*$ denotes the convolution in space and

$$\begin{cases} K(t,\xi) = \dfrac{1}{\sqrt{4\pi t}} \exp\left(-\dfrac{|\xi|^2}{4t}\right), \\ G(t,\xi) = -(f(\phi(\xi) + U_\xi(t,\xi)) - f(\phi(\xi)) - sU_\xi). \end{cases} \tag{2.12}$$

Having obtained the above integral representation, we can deduce that

LEMMA 2.3. *In addition to the assumptions stated in Lemma* 2.2, *we assume further that* $U_0(\xi) \in L^2(\boldsymbol{R})$, $f(u) \in C^1(\boldsymbol{R})$, *then we have for each* $\tau > 0$, $T > 0$ *and* $i = 0, 1$ *that*

$$\left\| \frac{\partial^i}{\partial \xi^i} U(t,\xi) \right\|_{L^2} \le C_2(t) t^{-1/2}, \quad \tau \le t \le T. \tag{2.13}$$

PROOF. Notice that

$$\frac{\partial^i}{\partial \xi^i} U(t,\xi) = \frac{\partial^i}{\partial \xi^i} K(t,\xi) * U_0(\xi) + \int_0^t \frac{\partial^i}{\partial \xi^i} K(t-s,\xi) * G(s,\xi)\,ds, \quad i = 0, 1 \tag{2.14}$$

and

$$G(t,\xi) = O(1)|U_\xi(t,\xi)|. \tag{2.15}$$

By Hausdorff-Young's inequality and (2.9) we have that, for $i = 0, 1$,

$$\begin{aligned} \left\| \frac{\partial^i}{\partial \xi^i} U(t,\xi) \right\|_{L^2} &\le \left\| \frac{\partial^i}{\partial \xi^i} K(t,\xi) \right\|_{L^1} \|U_0(\xi)\|_{L^2} + \int_0^t \left\| \frac{\partial^i}{\partial \xi^i} K(t-s,\xi) \right\|_{L^1} \|G(s,\xi)\|_{L^2}\,ds \\ &\le O(1) t^{-i/2} \|U_0(\xi)\|_{L^2} + O(1) \int_0^t (t-s)^{-i/2} \|U_\xi(s,\xi)\|_{L^2}\,ds, \end{aligned} \tag{2.16}$$

and hence

$$\begin{aligned} \sum_{i=0}^{1} \left\| \frac{\partial^i}{\partial \xi^i} U(t,\xi) \right\|_{L^2} &\le O(1)(1 + t^{-1/2}) \|U_0(\xi)\|_{L^2} \\ &\quad + O(1) \int_0^t (1 + (t-s)^{-1/2}) \|U_\xi(s,\xi)\|_{L^2}\,ds. \end{aligned} \tag{2.17}$$

Thus the singular Gronwall inequality gives

$$\left\|\frac{\partial^i}{\partial\xi^i}U(t,\xi)\right\|_{L^2} \le C(t)t^{-1/2}\|U_0(\xi)\|_{L^2}, \tag{2.17}$$

which is the desired estimates (2.13). This completes the proof of Lemma 2.3. □

REMARK 2.1. Since the viscous shock profile $\phi(\xi)$ satisfies

$$\phi_{\xi\xi} = (f(\phi) - s\phi)_\xi \equiv h'(\phi)\phi_\xi, \tag{2.18}$$

we can deduce that, if $f(u) \in C^k(\boldsymbol{R})$ for some positive integer $k > 0$, then

$$\left|\frac{\partial^{k+1}}{\partial\xi^{k+1}}\phi(\xi)\right| \le O(1). \tag{2.19}$$

Combining the above observation with the technique used in the proof of Lemma 2.3, we have the following lemma.

LEMMA 2.4. *In addition to the assumptions listed in Lemma* 2.3, *we assume further that* $f(u) \in C^k(\boldsymbol{R})$ *for some positive integer* k, *then we have*

$$\left\|\frac{\partial^i}{\partial\xi^i}U(t,\xi)\right\|_{L^2} \le C_3\left(\frac{\tau}{2}, t-\frac{\tau}{2}\right)\|U_0(\xi)\|_{L^2}, \quad \tau \le t \le T, \ i = 0,1,\dots,k. \tag{2.20}$$

PROOF. We only treat the case $k = 2$. The case $k > 2$ can be shown by employing the induction method. In the case $k = 2$, from Lemma 2.3, we only need to estimate $\|U_{\xi\xi}(t,\xi)\|_{L^2}$. We first have that for each $\tau_1 > 0$

$$U(t,\xi) = K(t-\tau_1,\xi) * U(\tau_1,\xi) + \int_{\tau_1}^t K(t-s,\xi) * G(s,\xi)\,ds, \tag{2.21}$$

and hence

$$U_{\xi\xi}(t,\xi) = K_{\xi\xi}(t-\tau_1,\xi) * U(\tau_1,\xi) + \int_{\tau_1}^t K_\xi(t-s,\xi) * G_\xi(s,\xi)\,ds. \tag{2.22}$$

Since

$$G_\xi(t,\xi) = O(1)|\phi(\xi)|\,|U_\xi(t,\xi)| + O(1)|U_{\xi\xi}(t,\xi)|, \tag{2.23}$$

we have from (2.9), (2.19), Lemma 2.3 and Hausdorff-Young's inequality that for $t > \tau_1$

$$
\begin{aligned}
\|U_{\xi\xi}(t,\xi)\|_{L^2} &\le O(1)(t-\tau_1)^{-1}\|U(\tau_1,\xi)\|_{L^2} \\
&\quad + O(1)\int_{\tau_1}^{t}(t-s)^{-1/2}(\|U_\xi(s,\xi)\|_{L^2} + \|U_{\xi\xi}(s,\xi)\|_{L^2})\,ds \\
&\le O(1)(t-\tau_1)^{-1}\tau_1^{-1/2}C_2(\tau_1)\|U_0(\xi)\|_{L^2} \\
&\quad + O(1)\int_{\tau_1}^{t} C_2(s)(t-s)^{-1/2}s^{-1/2}\|U_0(\xi)\|_{L^2}\,ds \\
&\quad + O(1)\int_{\tau_1}^{t}(t-s)^{-1/2}\|U_{\xi\xi}(s,\xi)\|_{L^2}\,ds \\
&\le O(1)(t-\tau_1)^{-1}C(t,\tau_1)\|U_0(\xi)\|_{L^2} \\
&\quad + O(1)\int_{\tau_1}^{t}(t-s)^{-1/2}\|U_{\xi\xi}(s,\xi)\|_{L^2}\,ds. \qquad (2.24)
\end{aligned}
$$

Thus the singular Gronwall's inequality deduces

$$\|U_{\xi\xi}(t,\xi)\|_{L^2} \le (t-\tau_1)^{-1}C(t,t-\tau_1,\tau_1)\|U_0(\xi)\|_{L^2}. \qquad (2.25)$$

Here $C(t,t-\tau_1,\tau_1)$ is a continuous, monotonically increasing function of t and $t-\tau_1$.

By (2.25), if we take $\tau_1 = \tau/2$ for each given $\tau > 0$, then we have

$$\|U_{\xi\xi}(t,\xi)\|_{L^2} \le \left(t-\frac{\tau}{2}\right)^{-1} C\left(t, t-\frac{\tau}{2}, \frac{\tau}{2}\right)\|U_0(\xi)\|_{L^2}, \qquad (2.26)$$

which shows (2.20) with $k = 2$ and completes the proof of Lemma 2.4.

Our final result in this section is concerned with the weighted energy estimate on the solution $U(t,\xi)$ obtained in Lemma 2.2.

Lemma 2.5. *In addition to the assumptions in Lemma* 2.2, *suppose further that* $U_0(\xi) \in L^2_{\overline{w}}(\boldsymbol{R})$, *then the solution* $U(t,\xi)$ *obtained in Lemma* 2.2 *satisfies*

$$\|\sqrt{\overline{w}(\xi)}\,U(t,\xi)\|_{L^2} \le C_4(t)\|\sqrt{\overline{w}(\xi)}\,U_0(\xi)\|_{L^2} \qquad (2.27)$$

provided that the weighted function $\overline{w}(\xi)$ *satisfies*

$$\left|\frac{\overline{w}'(\xi)}{\overline{w}(\xi)}\right| \le O(1)\overline{w}(\xi). \qquad (2.28)$$

Multiplying (2.6) by $\overline{w}(\xi)U(t,\xi)$ and integrating the resultant equation with respect to t and ξ over $[0,t]\times\boldsymbol{R}$. If (2.28) holds, then we can employ the Gronwall inequality and obtain (2.27). Since this is a standard way, we omit the details.

Remark 2.2. It is easy to check that all the weighted functions used in our subsequent analysis satisfying (2.28).

3. The proof of Theorem 1.3.

In this section we devote ourselves to the proof of our main result, Theorem 1.3. The non-degenerate shock case can be treated easier than the degenerate shock case. Hence we deal with the case $s = f'(u_+) < f'(u_-)$. Without loss of generality, we assume $u_+ < u_-$ and $h(\phi) < 0$ for $\phi \in (u_+, u_-)$. Consequently, there is a unique number $\xi_* \in \boldsymbol{R}$ such that

$$\phi(\xi_*) = \bar{u} := \frac{u_+ + u_-}{2}. \tag{3.1}$$

To overcome the nonconvexity of $f(u)$, as in [**13**], the weight $w(\xi)$ is chosen as

$$w(\phi) := \frac{(\phi - u_+)(\phi - u_-)}{h(\phi)}. \tag{3.2}$$

It is easy to find that

$$w(\phi(\xi)) \sim \begin{cases} C, & \text{if } f'(u_+) < s < f'(u_-) \\ |\xi| & \text{if } f'(u_\pm) = s \end{cases} \tag{3.3}$$

as $\xi \to \pm\infty$ and

$$\frac{d^2}{d\phi^2}(h(\phi)w(\phi)) = 2. \tag{3.4}$$

For the weight function $w(\xi)$ chosen above, we have the following basic energy estimates.

Lemma 3.1. *Let $U(t,\xi)$ be the solution of the Cauchy problem* (2.6), (2.7) *obtained in Lemma* 2.5, *then it follows that*

$$\frac{1}{2}|U(t)|^2_{w(\phi)} + \int_{T_1}^t \|\sqrt{-\phi_\xi}U(s)\|^2\,ds + \left(1 - C_5 \sup_{[T_1,t]}\|U(t)\|_{L^\infty}\right)\int_{T_1}^t |U_\xi(s)|^2_{w(\phi)}\,ds \le C_6(T_1). \tag{3.5}$$

Proof. Multiplying (2.6) by $w(\phi(\xi))U(t,\xi)$, we have

$$\left(\frac{1}{2}w(\phi)U^2(t)\right)_t+\left(\frac{1}{2}(wh)'(\phi)U^2(t)-w(\phi)U(t)U_\xi(t)\right)_\xi$$
$$+w(\phi)U_\xi^2(t)-\frac{1}{2}(wh)''(\phi)\phi_\xi U^2(t)=w(\phi)U(t)F(t). \tag{3.6}$$

Here we have used the fact that $\phi_\xi(\xi)=h(\phi(\xi))$.

Noticing $\phi_\xi(\xi)<0$ and $F(t,\xi)=O(1)|U_\xi(t,\xi)|^2$, we can get (3.5) from (2.27) and (3.4) immediately by integrating (3.6) with respect to t and ξ over $[T_1,t]\times\boldsymbol{R}$. This completes the proof of Lemma 3.1. □

The next lemma is concerned with the improvement of the estimate (3.5).

Lemma 3.2. *For* $0<\beta\le\alpha<2/n$ $(n\ge1)$, *we have that the solution* $U(t,\xi)$ *of the Cauchy problem* (2.6), (2.7) *satisfies*

$$\int_{\boldsymbol{R}}w(\phi)^{1+\beta}U^2(t)\,d\xi+\int_{T_1}^t\int_{\xi>0}w(\phi)^{\beta-1}U^2(s)\,d\xi ds$$
$$+\left(1-C_7\sup_{[T_1,t]}\|U(t)\|_{L^\infty}\right)\int_{T_1}^t\int_{\boldsymbol{R}}w(\phi)^\beta|\phi_\xi|U^2(s)\,d\xi ds$$
$$+\int_{T_1}^t\int_{\boldsymbol{R}}w(\phi)^{1+\beta}U_\xi^2(s)\,d\xi ds$$
$$\le C_8(T_1)+C_9\sup_{[T_1,t]}\|U(t)\|_{L^\infty}\int_{T_1}^t|U_\xi(s)|^2_{w(\phi)}\,ds. \tag{3.7}$$

Proof. The proof of Lemma 3.2 follows essentially the arguments developed by A. Matsumura and K. Nishihara in [**13**]. Thus, we only give a sketch of the proof, and the difference between our arguments and those in [**13**] will be emphasized.

Multiplying (2.6) by $2w(\phi)^{1+\beta}U(t)$, we have similar to the proof of Lemma 6.1 in [**13**] that

$$(w(\phi)^{1+\beta}U^2(t))_t+(\cdots)_\xi+2(1-\varepsilon)w(\phi)^{1+\beta}U_\xi^2(t)$$
$$+2\{-2w(\phi)^\beta\phi_\xi+\beta w(\phi)^{\beta-1}h(\phi)(2(\bar{u}-\phi)-\beta w'(\phi)h(\phi)/2\varepsilon)\}U^2(t)$$
$$\le 2w(\phi)^{1+\beta}|U(t)F(t)|. \tag{3.8}$$

Here $\varepsilon\in(0,1)$ is an arbitrarily chosen constant.

On the other hand, if $\delta=\phi(\xi)-u_+>0$ and $\tilde{u}=u_--u_+>0$, then

$$I(\xi) := \beta w(\phi(\xi))^{\beta-1} w'(\phi(\xi)) h(\phi(\xi))(2(\bar{u} - \phi(\xi)) - \beta w'(\phi(\xi)) h(\phi(\xi))/2\varepsilon)$$
$$= \beta w(\phi(\xi))^{\beta-1}(\tilde{u}n + O(\delta))(\tilde{u}(1 - \beta n/2\varepsilon) + O(\delta)) \tag{3.9}$$

as $\xi \to +\infty$.

Since $\beta \le \alpha < 2/n$, we can always choose $\varepsilon \in (0,1)$ such that $1 - \beta n/2\varepsilon > 0$. Consequently, there are positive constants C_{10} and R_1 such that

$$I(\xi) \ge C_{10} \quad \text{for } \xi \ge R_1. \tag{3.10}$$

Noticing also $C^{-1} \le w(\phi(\xi)) \le C$, $C^{-1} \le w'(\phi(\xi)) \le C$ as $\xi \to -\infty$, we have from (3.5) that

$$\int_{T_1}^{t} \int_{\xi \le R_1} 2I(\xi) U^2(s,\xi)\, d\xi ds \le O(1) \int_{T_1}^{t} |\phi_\xi(\xi)| U^2(s,\xi)\, d\xi ds$$
$$\le C(T_1) + O(1) \sup_{[T_1,t]} \|U(t,\xi)\|_{L^\infty} \int_{T_1}^{t} |U_\xi(s,\xi)|^2_{w(\phi)}\, ds \tag{3.11}$$

and

$$\int_{T_1}^{t} \int_{\boldsymbol{R}} w(\phi(\xi))^{1+\beta} |U(s,\xi) F(s,\xi)|\, d\xi ds$$
$$\le O(1) \sup_{[T_1,t]} \|U(t,\xi)\|_{L^\infty} \int_{T_1}^{t} \int_{\boldsymbol{R}} w(\phi(\xi))^{1+\beta} U_\xi^2(s,\xi)\, d\xi ds. \tag{3.12}$$

Integrating (3.8) with respect to t and ξ over $[T_1, t] \times \boldsymbol{R}$, we can immediately get (3.7) from (3.10)–(3.12). This completes the proof of Lemma 3.2. □

Lemma 3.3. *For each given $\alpha > 0$, the solution $U(t,\xi)$ to the Cauchy problem* (2.6), (2.7) *satisfies for $\beta \in [0,\alpha]$*

$$(1+t)^\gamma |U(t)|^2_{\beta, w(\phi)} + \left(1 - C_{11} \sup_{[T_1,t]} \|U(t,\xi)\|_{L^\infty}\right) \int_{T_1}^{t} (1+s)^\gamma |U_\xi(s)|^2_{\beta, w(\phi)}\, ds$$
$$+ \beta \int_{T_1}^{t} (1+s)^\gamma |U(s)|^2_{\beta-1}\, ds$$
$$\le C_{12}(T_1) \left\{ 1 + \gamma \int_{T_1}^{t} (1+s)^{\gamma-1} |U(s)|^2_{\beta, w(\phi)}\, ds \right.$$
$$\left. + \beta \int_{T_1}^{t} (1+s)^\gamma \int_{\boldsymbol{R}} \langle \xi - \xi_* \rangle^{\beta-1} w(\phi(\xi)) |U(s,\xi) U_\xi(s,\xi)|\, d\xi ds \right\}. \tag{3.13$_{\gamma,\beta}$}$$

Proof. Putting $\langle \xi - \xi_* \rangle := \sqrt{1 + (\xi - \xi_*)^2}$ and multiplying (2.6) by $2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi(\xi)) U(t, \xi)$, we get

$$
\begin{aligned}
&((1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi) U^2(t))_t + (2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta U(t) U_\xi(t)(w(\phi) + (wh)'(\phi)))_\xi \\
&\quad + 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi) U_\xi^2 - \gamma (1+t)^{\gamma-1} \langle \xi - \xi_* \rangle^\beta w(\phi) U^2(t) \\
&\quad + (1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-1} A_\beta(\xi) U^2(t) \\
&\quad + 2\beta (1+t)^\gamma \langle \xi - \xi_* \rangle^{\beta-2} (\xi - \xi_*) w(\phi) U(t) U_\xi(t) \\
&= 2(1+t)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi) U(t) F(t). \qquad (3.14)
\end{aligned}
$$

Here

$$
\begin{aligned}
A_\beta(\xi) &:= -\langle \xi - \xi_* \rangle \phi_\xi(\xi)(wh)''(\phi(\xi)) - (\beta(\xi - \xi_*)/\langle \xi - \xi_* \rangle)(wh)'(\phi(\xi)) \\
&= -2\langle \xi - \xi_* \rangle \phi_\xi(\xi) - (2\beta(\xi - \xi_*)/\langle \xi - \xi_* \rangle)(\phi(\xi) - \bar{u}). \qquad (3.15)
\end{aligned}
$$

Due to (3.1), there exists a positive constant C_0 independent of β such that

$$
A_\beta(\xi) \ge C_0 \beta \quad \text{for any } \xi \in \boldsymbol{R}. \qquad (3.16)
$$

Integrating (3.14) with respect to t and ξ over $[T_1, t] \times \boldsymbol{R}$, it is deduced by (3.16) that

$$
\begin{aligned}
&(1+t)^\gamma |U(t)|_{\beta, w(\phi)}^2 + 2\int_{T_1}^t (1+s)^\gamma |U_\xi(s)|_{\beta, w(\phi)}^2 \, ds + C_0 \beta \int_{T_1}^t (1+s)^\gamma |U(s)|_{\beta-1}^2 \, ds \\
&\quad \le (1+T_1)^\gamma |U(T_1)|_{\beta, w(\phi)}^2 + \gamma \int_{T_1}^t (1+s)^{\gamma-1} |U(s)|_{\beta, w(\phi)}^2 \, ds \\
&\quad + 2\beta \int_{T_1}^t \int_{\boldsymbol{R}} (1+s)^\gamma \langle \xi - \xi_* \rangle^{\beta-1} w(\phi) |U(s) U_\xi(s)| \, d\xi ds \\
&\quad + 2\int_{T_1}^t \int_{\boldsymbol{R}} (1+s)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi) |U(s) F(s)| \, d\xi ds. \qquad (3.17)
\end{aligned}
$$

Due to

$$
\begin{aligned}
&\int_{T_1}^t \int_{\boldsymbol{R}} (1+s)^\gamma \langle \xi - \xi_* \rangle^\beta w(\phi) |U(s) F(s)| \, d\xi ds \\
&\qquad \le C_{11} \sup_{[T_1, t]} \|U(t, \xi)\|_{L^\infty} \int_{T_1}^t (1+s)^\gamma |U_\xi(s)|_{\beta, w(\phi)}^2 \, ds, \qquad (3.18)
\end{aligned}
$$

we can get $(3.13)_{\gamma,\beta}$ immediately by substituting (3.18) into (3.17), which completes the proof of Lemma 3.3. □

Now from the L^1-stability result, Theorem 1.1, by H. Freistühler and D. Serre in [1], we conclude that

$$\lim_{t\to\infty} \|U(t,\xi)\|_{L^\infty} \le \lim_{t\to\infty} \int_{\boldsymbol{R}} |u(t,x) - \phi(x - st)|\, dx = 0. \tag{3.19}$$

Thus if we choose $\bar{T}_1$ sufficiently large such that

$$\sup_{[\bar{T}_1,\infty]} \|U(t,\xi)\|_{L^\infty} < \frac{1}{2} \min\left\{\frac{1}{C_5}, \frac{1}{C_7}, \frac{1}{C_{11}}\right\}, \tag{3.20}$$

then we have from (3.20) and Lemma 3.1–Lemma 3.3 that

Corollary 3.1. *For $\bar{T}_1$ chosen as above and $0 < \beta \le \alpha < 2/n$ $(n \ge 1)$, the solution $U(t,\xi)$ to the Cauchy problem* (2.6), (2.7) *satisfies for $t \ge \bar{T}_1$*

$$\frac{1}{2}|U(t)|^2_{w(\phi)} + \int_{\bar{T}_1}^t \|\sqrt{-\phi_\xi}\, U(s)\|^2\, ds + \int_{\bar{T}_1}^t |U_\xi(s)|^2_{w(\phi)}\, ds \le C_6(\bar{T}_1), \tag{3.21}$$

$$\int_{\boldsymbol{R}} w(\phi)^{1+\beta} U^2(t)\, d\xi + \int_{\bar{T}_1}^t \int_{\xi>0} w(\phi)^{\beta-1} U^2(s)\, d\xi ds$$
$$+ \int_{\bar{T}_1}^t \int_{\boldsymbol{R}} w(\phi)^\beta |\phi_\xi| U^2(s)\, d\xi ds + \int_{\bar{T}_1}^t \int_{\boldsymbol{R}} w(\phi)^{1+\beta} U_\xi^2(s)\, d\xi ds \le C_8(\bar{T}_1) \qquad (3.22)_\beta$$

and

$$(1+t)^\gamma |U(t)|^2_{\beta, w(\phi)} + \int_{\bar{T}_1}^t (1+s)^\gamma |U_\xi(s)|^2_{\beta, w(\phi)}\, ds + \beta \int_{\bar{T}_1}^t (1+s)^\gamma |U(s)|^2_{\beta-1}\, ds$$
$$\le C_{12}(\bar{T}_1)\left\{1 + \gamma \int_{\bar{T}_1}^t (1+s)^{\gamma-1} |U(s)|^2_{\beta, w(\phi)}\, ds\right.$$
$$\left. + \beta \int_{\bar{T}_1}^t (1+s)^\gamma \int_{\boldsymbol{R}} \langle \xi - \xi_* \rangle^{\beta-1} w(\phi(\xi)) |U(s,\xi) U_\xi(s,\xi)|\, d\xi ds\right\}. \qquad (3.23)_{\gamma,\beta}$$

The proof of (II) of Theorem 1.3 follows from (3.21), $(3.22)_\beta$ and $(3.23)_{\gamma,\beta}$, in a similar fashion to that in [**13**], [**17**]. For completeness, we give the outline.

First, letting $\gamma = 0$ and $\beta \le \alpha$ in $(3.23)_{\gamma,\beta}$, we can estimate the corresponding last term as in the following

$$|\text{last term in } (3.23)_{0,\beta}| \le \frac{\beta}{2}\int_{\bar{T}_1}^t |U(s)|^2_{\beta-1}\,ds$$

$$+ O(1)\int_{\bar{T}_1}^t \int_{\boldsymbol{R}} \langle \xi - \xi_* \rangle^{\beta-1} w(\phi(\xi))^2 U_\xi^2(s,\xi)\,d\xi ds := I_1 + I_2. \tag{3.24}$$

Noticing

$$w(\phi(\xi))\begin{cases} \sim \xi & \text{as } \xi \to +\infty, \\ \sim \text{Const.} & \text{as } \xi \to -\infty, \end{cases} \tag{3.25}$$

we can find two positive constants $R_2 > 0$ and $R_3 > 0$ such that

$$\begin{aligned} I_2 &\le \frac{1}{2}\int_{\bar{T}_1}^t \int_{\xi<-R_3} \langle \xi - \xi_* \rangle^{\beta} w(\phi) U_\xi^2(s)\,d\xi ds \\ &\quad + O(1)\int_{\bar{T}_1}^t \int_{\xi>R_2} w(\phi)^{\beta+1} U_\xi^2(s)\,d\xi ds + O(1)\int_{\bar{T}_1}^t \int_{-R_3\le\xi\le R_2} U_\xi^2(s)\,d\xi ds \\ &\le \frac{1}{2}\int_{\bar{T}_1}^t |U_\xi(s)|^2_{\beta,w(\phi)}\,ds + O(1)\int_{\bar{T}_1}^t \int_{\boldsymbol{R}} w(\phi)^{1+\beta} U_\xi^2(s)\,d\xi ds \\ &\quad + O(1)\int_{\bar{T}_1}^t \int_{\boldsymbol{R}} U_\xi^2(s)\,d\xi ds \\ &\le C(\bar{T}_1) + \frac{1}{2}\int_{\bar{T}_1}^t |U_\xi(s)|^2_{\beta,w(\phi)}\,ds. \end{aligned} \tag{3.26}$$

Here we have used (3.21) and $(3.22)_\beta$.

Substituting (3.26) and (3.24) into $(3.23)_{0,\beta}$ and letting $\beta = \alpha$, we have for $\alpha < 2/n$ $(n \ge 1)$ that

$$|U(t)|^2_{\alpha,w(\phi)} + \int_{\bar{T}_1}^t (|U(s)|^2_{\alpha-1} + |U_\xi(s)|^2_{\alpha,w(\phi)})\,ds \le C_{13}(\bar{T}_1) \tag{3.27}$$

provided that $t \ge \bar{T}_1$.

Next, we consider $(3.23)_{\gamma,\beta}$ with $\gamma = \alpha/2 + \varepsilon$ and $\beta = 0$

$$(1+t)^{\alpha/2+\varepsilon}|U(t)|^2_{w(\phi)} + \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon}|U_\xi(s)|^2_{w(\phi)}\,ds$$

$$\le C_{12}(\bar{T}_1)\left(1 + \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1}|U(s)|^2_{w(\phi)}\,ds\right). \tag*{$(3.23)_{\alpha/2+\varepsilon,0}$}$$

Here $\varepsilon > 0$ is chosen sufficiently small such that

$$\varepsilon < \frac{\alpha}{2}, \quad \frac{\alpha}{2} + \varepsilon < \frac{1}{n} \le 1.$$

Since

$$\begin{aligned}
\int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} |U(s)|_{w(\phi)}^2 \, ds &\le \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} \left(\int_{\xi>0} + \int_{\xi\le 0} \right) w(\phi) U^2(s,\xi) \, d\xi ds \\
&\le O(1) \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} \int_{\xi>0} w(\phi) U^2(s) \, d\xi ds \\
&\quad + O(1) \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} \int_{\xi\le 0} U^2(s) \, d\xi ds \\
&:= J_1 + J_2, \qquad (3.28)
\end{aligned}$$

we have from (3.25) and (3.22) that

$$\begin{aligned}
J_1 &\le O(1) \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} \left(\int_{\xi>0} w(\phi)^{1+\alpha} U^2(s) \, d\xi \right)^{(2-\alpha)/2} \\
&\quad \times \left(\int_{\xi>0} w(\phi)^{\alpha-1} U^2(s) \, d\xi \right)^{\alpha/2} ds \\
&\le C(\bar{T}_1) \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1} \left(\int_{\xi>0} w(\phi)^{\alpha-1} U^2(s) \, d\xi \right)^{\alpha/2} ds \\
&\le C(\bar{T}_1) \left(\int_{\bar{T}_1}^t (1+s)^{-1+2\varepsilon/(2-\alpha)} \, ds \right)^{(2-\alpha)/2} \\
&\quad \times \left(\int_{\bar{T}_1}^t \int_{\xi>0} w(\phi)^{\alpha-1} U^2(s) \, d\xi ds \right)^{\alpha/2} \\
&\le C(\bar{T}_1)(1+t)^{\varepsilon}. \qquad (3.29)
\end{aligned}$$

As to J_2, if $\alpha \ge 1$ (consequently $n = 1$), we have from $\alpha/2 + \varepsilon < 1$ that

$$J_2 \le O(1) \int_{\bar{T}_1}^t \int_{\xi<0} U^2(s,\xi) \, d\xi ds \le O(1) \int_{\bar{T}_1}^t |U(s)|_{\alpha-1}^2 \, ds \le C(\bar{T}_1). \qquad (3.30)$$

When $n \ge 2$ (consequently $\alpha < 1$), we have from (3.25) and (3.27) that

$$J_2 \le O(1)\int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1}\left(\int_{\xi<0}\langle\xi-\xi_*\rangle^{\alpha}U^2(s,\xi)\,d\xi\right)^{1-\alpha}$$
$$\times\left(\int_{\xi<0}\langle\xi-\xi_*\rangle^{\alpha-1}U^2(s,\xi)\,d\xi\right)^{\alpha}ds$$
$$\le O(1)\int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1}\left(\int_{\xi<0}\langle\xi-\xi_*\rangle^{\alpha}w(\phi)U^2(s,\xi)\,d\xi\right)^{1-\alpha}$$
$$\times\left(\int_{\xi<0}\langle\xi-\xi_*\rangle^{\alpha-1}U^2(s,\xi)\,d\xi\right)^{\alpha}ds$$
$$\le C(\bar{T}_1)\int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon-1}|U(s)|_{\alpha-1}^{2\alpha}\,ds$$
$$\le C(\bar{T}_1)\left(\int_{\bar{T}_1}^t (1+s)^{-(1-\alpha/2-\varepsilon)/(1-\alpha)}\,ds\right)^{1-\alpha}\left(\int_{\bar{T}_1}^t |U(s)|_{\alpha-1}^2\,ds\right)^{\alpha}\le C(\bar{T}_1) \tag{3.31}$$

since $\varepsilon<\alpha/2$.

Inserting (3.28)–(3.31) into $(3.23)_{\alpha/2+\varepsilon,0}$ deduces

$$(1+t)^{\alpha/2+\varepsilon}|U(t)|_{w(\phi)}^2+\int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon}|U_\xi(s)|_{w(\phi)}^2\,ds\le C_{14}(\bar{T}_1)(1+t)^{\varepsilon} \tag{3.32}$$

provided that $t\ge\bar{T}_1$. Thus we have the following lemma.

Lemma 3.4. *Under the conditions* (II) *in Theorem* 1.3, *the solution $U(t,\xi)$ to the Cauchy problem* (2.6), (2.7) *satisfies* (3.32) *for any $t\ge\bar{T}_1$ and some sufficiently small $\varepsilon>0$.*

Now we turn to get the decay rates for derivatives of $U(t,\xi)$. We first have

Lemma 3.5. *In additional to the assumptions listed in Theorem* 1.3, *suppose that $f(u)\in C^k(\boldsymbol{R})$ for some $k\in\boldsymbol{Z}^+$, then, for each fixed $\tau>0$, the solution $U(t,\xi)$ to the Cauchy problem* (2.6), (2.7) *satisfies*

$$\sup_{[\tau,\infty)}\left\|\frac{\partial^j}{\partial\xi^j}U(t,\xi)\right\|_{L^\infty}\le C_{15}(\tau)<\infty,\quad j=1,\dots,k. \tag{3.33}$$

Proof. We only prove (3.33) for the case $k=2$ since the rest can be treated similarly. For each $0<\tau_1<\tau_2<\tau\le t\le T$, we have

$$\begin{cases} U_\xi(t,\xi)=K_\xi(t-\tau_1,\xi)*U(\tau_1,\xi)+\displaystyle\int_{\tau_1}^t K_\xi(t-s,\xi)*G(s,\xi)\,ds,\\ U_{\xi\xi}(t,\xi)=K_{\xi\xi}(t-\tau_2,\xi)*U(\tau_2,\xi)+\displaystyle\int_{\tau_2}^t K_\xi(t-s,\xi)*G_\xi(s,\xi)\,ds.\end{cases} \tag{3.34}$$

On the other hand, we have from the L^1-stability result obtained in [**1**] that

$$\|U(t,\xi)\|_{L^\infty} \le O(1). \tag{3.35}$$

Consequently from (2.9), (3.34) and (3.35), we have by the iteration arguments used in Lemma 2.4 that for $\tau \le t \le T$

$$\left\|\frac{\partial^j}{\partial \xi^j} U(t,\xi)\right\|_{L^\infty} \le C_{16}(t-\tau_j; \tau_1, \dots, \tau_{j-1}), \quad j = 1, 2. \tag{3.36}$$

Having obtained (3.36), we now turn to prove (3.33).

First we notice that (3.36) holds for each given τ_1, τ_2, T. Hence, for each fixed $\tau > 0$, letting $\bar{\tau}_2 = 2\bar{\tau}_1 = \tau/2$, $T = 2t_1$ (where $t_1 > \tau$ is an arbitrarily given positive constant), we have from (3.36) that

$$\sup_{[\tau, 2t_1]}\left\|\frac{\partial^j}{\partial \xi^j} U(t,\xi)\right\|_{L^\infty} \le C_{17}(2t_1 - \bar{\tau}_j; \bar{\tau}_1, \dots, \bar{\tau}_{j-1}), \quad j = 1, 2. \tag{3.37}$$

Now suppose that for some $1 < m \in \boldsymbol{Z}^+$

$$\sup_{[\tau, (m+1)t_1]}\left\|\frac{\partial^j}{\partial \xi^j} U(t,\xi)\right\|_{L^\infty} \le C_{17}(2t_1 - \bar{\tau}_j; \bar{\tau}_1, \dots, \bar{\tau}_{j-1}), \quad j = 1, 2, \tag{3.38}$$

then it holds that

$$\sup_{[mt_1+\tau, (m+2)t_1]}\left\|\frac{\partial^j}{\partial \xi^j} U(t,\xi)\right\|_{L^\infty} \le C_{17}(2t_1 - \bar{\tau}_j; \bar{\tau}_1, \dots, \bar{\tau}_{j-1}), \quad j = 1, 2. \tag{3.39}$$

In fact, letting T, τ_1, τ_2 in (3.36) be equal to $(m+2)t_1, mt_1 + \bar{\tau}_1, mt_1 + \bar{\tau}_2$ respectively, we can get (3.39). By setting $t_1 = 2\tau$ and $C_{15}(\tau) = C_{17}(4\tau - j\tau/4; \tau/4, \dots, (j-1)\tau/4)$, (3.33) follows easily. This completes the proof of Lemma 3.5. □

Since

$$\begin{aligned}|U_\xi(t,\xi)|^2 &= |u(t,\xi) - \phi(\xi)|^2 \le \|(u(t,\xi) - \phi(\xi))_\xi\|_{L^\infty}\|u(t,\xi) - \phi(\xi)\|_{L^1} \\ &= \|U_{\xi\xi}(t,\xi)\|_{L^\infty}\|U(t,\xi)\|_{L^\infty},\end{aligned}$$

we have from Lemma 3.5 and Theorem 1.1 that

$$\lim_{t\to\infty}\|U_\xi(t,\xi)\|_{L^\infty} = 0. \tag{3.40}$$

Furthermore, from Lemma 2.4, under the assumption that $U_0(\xi) \in L^2(\boldsymbol{R})$, we have

$$\|U(T_1,\xi)\|_{H^2} \le C_{18}(T_1) \tag{3.41}$$

for each given $T_1 > 0$. With (3.40) and (3.41), we also have the following lemma.

LEMMA 3.6. *Let $l = 1, 2$ and assume that the conditions listed in Lemma 3.4 are satisfied, then it holds for any $t \ge \bar{T}_1$ and some sufficiently small $\varepsilon > 0$ that*

$$(1+t)^{\alpha/2+\varepsilon}\left\|\frac{\partial^l}{\partial\xi^l}U(t)\right\|^2 + \int_{\bar{T}_1}^t (1+s)^{\alpha/2+\varepsilon}\left\|\frac{\partial^l}{\partial\xi^l}U_\xi(s)\right\|^2 ds \le C_{19}(\bar{T}_1)(1+t)^{\varepsilon}. \tag{3.42}$$

Combining Lemma 3.4 with Lemma 3.6, we can deduce that

$$\begin{aligned}\sup_{x\in\mathbf{R}}|u(t,x)-\phi(x-st)| &= \sup_{x\in\mathbf{R}}|U_\xi(t,\xi)| \\ &\le C(\bar{T}_1)\|U_\xi(t)\|^{1/2}\|U_{\xi\xi}(t)\|^{1/2} \le C(\bar{T}_1)(1+t)^{-\alpha/4},\end{aligned}$$

which proves (II) of Theorem 1.3.

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