J. Math. Soc. Japan
Vol. 54, No. 2, 2002

On the boundary of self affine tilings generated by Pisot numbers

By Shigeki Akiyama

(Received Feb. 25, 2000)
(Revised Sept. 14, 2000)

Abstract. Definition and fundamentals of tilings generated by Pisot numbers are shown by arithmetic consideration. Results include the case that a Pisot number does not have a finitely expansible property, i.e. a sofic Pisot case. Especially we show that the boundary of each tile has Lebesgue measure zero under some weak condition.

1. Introduction.

First we explain notations used in this paper. The rational integers is denoted by $\boldsymbol{Z}$, the rational numbers by $\boldsymbol{Q}$, the complex numbers by $\boldsymbol{C}$ and the positive integers by $\boldsymbol{N}$. We denote by $\boldsymbol{Z}[u]$, the ring generated by $\boldsymbol{Z}$ and $u \in \boldsymbol{C}$, and by $\boldsymbol{Q}(u)$, the minimum field containing $\boldsymbol{Q}$ and u. We write A_{-} for the subset of A with constraints by its subscript '$_{-}$', when the subscript is a conditional term. For example, $\boldsymbol{Z}_{\geq \ell}$ is the integers not less than ℓ. Let $\beta > 1$ be a real number which is not an integer. A *greedy expansion* of a positive real x in base β is an expansion of a form:

$$x = \sum_{i=N_0}^{\infty} a_{-i}\beta^{-i} = a_{-N_0}a_{-N_0-1}\cdots$$

with $a_i \in [0, \beta) \cap \boldsymbol{Z}$ and a 'greedy condition'

$$\left| x - \sum_{N_0}^{N} a_{-i}\beta^{-i} \right| < \beta^{-N}$$

for all $N \geq N_0$. Throughout this paper, we identify $a_{-N_0}a_{-N_0-1}\cdots$ with the corresponding word generated by $\mathscr{A} = [0, \beta) \cap \boldsymbol{Z}$ for the sake of simplicity. For a greedy expansion:

2000 *Mathematics Subject Classification*. Primary 11K26, 11A63; Secondary 37B50, 52C23, 28A80.

Key Words and Phrases. Pisot number, tiling, fractal.

This research was partially supported by Grant-in-Aid for Scientific Research (No. 12640017), The Ministry of Education, Culture, Sports, Science and Technology, Japan.

$$x = \sum_{i=-k}^{\infty} a_{-i}\beta^{-i} = a_k a_{k-1} \cdots a_0 . a_{-1} a_{-2} \cdots,$$

$.a_{-1}a_{-2}\cdots$ is called the *fractional part* of x. If $x < \beta^{-M}$ then put $a_{-i} = 0$ for $i \le M$ to extend definition of the fractional parts to $x < 1$. Similarly the *integer part* of x is defined to be $a_k a_{k-1} \cdots a_0$. As an example of symbolic dynamical system and ergodic theory, this expansion was called 'beta expansion' and extensively studied in A. Rényi [**14**], W. Parry [**12**] and S. Ito and Y. Takahashi [**9**]. In [**7**], F. Blanchard gave a necessary and sufficient condition that the corresponding symbolic dynamical system belongs to some important classes of subshifts, that is, *finite type* subshifts and *sofic* subshifts. For precise properties of such subshifts, see [**11**].

Let us assume further that $\beta > 1$ be a Pisot number, that is, a real algebraic integer whose conjugates other than itself have modulus less than one. An important feature of this restriction is that the corresponding dynamical system is always sofic. We say this expansion is '*eventually periodic*', if there exists a positive integer L that $a_{-N} = a_{-N-L}$ for sufficiently large N. Rather surprisingly, this kind of expansion by Pisot numbers has analogous properties with usual decimal or binary expansions. In fact, any greedy expansion of $x \in \boldsymbol{Q}(\beta)_{\ge 0}$ is eventually periodic provided β is a Pisot number (see [**16**], [**6**]). A greedy expansion of $x \in \boldsymbol{Q}(\beta) \cap [0,1)$ is *purely periodic* if there exists a positive integer L that $a_{-N} = a_{-N-L}$ for $N \in \boldsymbol{N}$, i.e. the period start from a_{-1}. If there exists M that $a_N = 0$ for all $N \ge M$, the greedy expansion is said to be *finite*. Let $\mathrm{Fin}(\beta)$ be the set of all finite greedy expansions. A *Pisot unit* is a Pisot number which is also an algebraic unit. If we assume that β is a Pisot unit which has a *finitely expansible property*:

$$\text{(F)} \quad \mathrm{Fin}(\beta) = \boldsymbol{Z}[1/\beta]_{\ge 0},$$

then the expansion of a sufficiently small rational number is purely periodic (see [**1**]). The condition (F) sufficiently implies that the corresponding dynamical system is of finite type, i.e. a finite greedy word is obtained by the prohibition of a finite set of words. See [**8**] and [**4**] for characterizations of (F). If a Pisot number β does not satisfy (F), then the corresponding dynamical system is at least sofic, i.e. any finite greedy word can be recognized by a finite automaton. Thus we designate a Pisot number without (F) as a *sofic Pisot number*.

The main object of this paper is to study the topological structure of tilings generated by Pisot units. In [**18**], W. P. Thurston proposed a method to construct a tiling of some Euclidean space by a Pisot unit with (F). Remark that this kind of tile was first introduced by G. Rauzy [**15**] in a different approach closely related to substitutions. Fundamental properties of this tiling are studied in S. Akiyama [**3**] and B. Praggastis [**13**]. Although the strategies of these two papers are quite different, it is shown in both of them that the origin is an inner point of the 'central tile'. As shown in [**3**], this fact has several important consequences:

1. Each tile is the closure of its interior,
2. The boundary of each tile is nowhere dense,

being fit for the name 'tiling'. Note that sometimes the first property is employed as a definition of a tile but here we use a definition after the equation (3) in §1, i.e., a tile is the closure of the image by Φ of greedy expansions having a fixed fractional part, where Φ is a standard embedding map defined in §2 to some Euclidean space.

In this paper, we wish to generalize these tilings to a wider class of Pisot units having a *finite difference* property:

(W): *For any element x of $\boldsymbol{Z}[1/\beta]_{\geq 0}$ and any positive ε, there exist two elements y, z in* $\mathrm{Fin}(\beta)$ *with $|z| < \varepsilon$ such that $x = y - z$,*

which is obviously weaker than (F). It is shown in Proposition 3 that a class of sofic Pisot units treated in [**8**] satisfies this property (W). It is likely that all Pisot unit has the property (W)[1].

Definition of tilings for such general Pisot units will be given in §2 by a straightforward generalization of Thurston's idea in [**18**] and mine in [**3**]. Under this assumption (W), our goal is to show fundamental properties of this tiling and to generalize the results shown under the stronger assumption (F). In §2, it will be shown that the Euclidean space is covered by these tiles, there are only finitely many tiles up to translation and the number of tiles coincides with that of different tails of the characteristic sequence attached to β. Also an 'inflation-subdivision principle' is established which says that any tile is subdivided into arbitrary small affine images of the tiles. Though the origin may no longer be an inner point of a single tile but of a collection of tiles which correspond to purely periodic expansions (Theorem 1), the tile is shown to be a closure of its interior (Theorem 2). Moreover it is shown that the boundary has Lebesgue measure zero (Theorem 3) clearly improving the results in [**3**] saying that it is nowhere dense.

Hardest task of this paper is to show that this tiling has little overlaps. An idea of an *exclusive* inner point in §3 will play an essential role to settle this problem. For example, see Corollaries 1 and 2. It is interesting to see that the purely algebraic criterion (W) is equivalent to the existence of an exclusive inner

[1] The author would like to express his deep gratitude to N. Sidorov for informing this conjecture with applications to a different subject and for stimulating discussions.

point (Proposition 2). As a result, we could show Theorem 3 by using number theoretical arguments. Briefly speaking on the proof, we count greedy words under some restrictions in §4, and show in §5 that sufficiently many tiles, which do not touch the boundary, can be ignored. Results in this paper, some part of which was announced in [**4**], seems at least fundamental. Starting from these observations, the author hopes there would be further topological studies on this tiling.

Note that in [**10**], R. Kenyon and A. Vershik constructed a quite general sofic partition related to some tiling by different approach. At present, the relationship between their results and mine is not so clear. At least we may say that here in this paper we only used integer digits to construct a sofic Pisot tiling, which was a desirable fact in their paper. See Remark 1 of [**10**].

2. Construction and fundamental properties of the tiling.

Let β be a Pisot number of degree n and $\beta^{(j)}$ $(j = 1, \dots, n)$ be its conjugates. We also assume $\beta^{(1)} = \beta$, $\beta^{(j)}$ is real for $j = 1, \dots, r_1$ and $\beta^{(j)}$ is complex for $j = r_1 + 1, \dots, r_1 + 2r_2 = n$ with

$$\beta^{(r_1+j)} = \text{complex conjugate of } \beta^{(r_1+r_2+j)} \quad \text{for } j = 1, \dots, r_2.$$

Denote $x^{(j)}$ $(j = 1, \dots, n)$ the corresponding conjugates of $x \in \boldsymbol{Q}(\beta)$. Consider a map $\Phi : \boldsymbol{Q}(\beta) \to \boldsymbol{R}^{n-1}$ defined by

$$\Phi(x) = (x^{(2)}, \dots, x^{(r_1)}, \Re(x^{(r_1+1)}), \Im(x^{(r_1+1)}), \cdots, \Re(x^{(r_1+r_2)}), \Im(x^{(r_1+r_2)})),$$

that is, the 'non trivial part' of the standard embedding. Then we have

Lemma 1. *Let β be a Pisot number. Then $\Phi(\boldsymbol{Z}[\beta]_{\geq 0})$ is dense in $\boldsymbol{R}^{n-1}$.*

Proof. This is Proposition 1 of [**3**]. □

Lemma 2. *The number of purely periodic elements in $\boldsymbol{Z}[\beta]_{\geq 0}$ is finite.*

Proof. Each purely periodic element has the form:

$$0 \leq \frac{\sum_{i=0}^{N-1} a_{-i}\beta^{-i}}{1 - \beta^{-N}} = \frac{\sum_{i=0}^{N-1} a_{-i}\beta^{N-i}}{\beta^N - 1} < 1.$$

Then we have

$$\left| \frac{\sum_{i=0}^{N-1} a_{-i}(\beta^{(j)})^{N-i}}{(\beta^{(j)})^N - 1} \right| \leq \frac{\sum_{i=0}^{N-1} \beta|\beta^{(j)}|^{N-i}}{1 - |\beta^{(j)}|^N} \leq \frac{\beta}{(1 - |\beta^{(j)}|)^2},$$

for $j = 2, \dots, n$. This means each purely periodic element has bounded absolute values on every conjugates. The assertion follows immediately. □

The finite set that consists of all purely periodic expansions in $\boldsymbol{Z}[\beta]_{\geq 0}$ is denoted by $\mathscr{P}$. Now we classify $\boldsymbol{Z}[\beta]_{\geq 0}$ by the fractional parts. Every element x of $\boldsymbol{Q}(\beta)_{\geq 0}$ has eventually periodic expansion by [**16**] or [**6**]. Write

$$x = \sum_{N_0} a_{-i}\beta^{-i} = a_{-N_0}a_{-N_0-1}\cdots a_{-M}[a_{-M-1}\cdots a_{-M-L}], \tag{1}$$

when $a_{-i} = a_{-i-L}$ for $i \geq M+1$. Also we require that this expansion is minimal, that is, we assume both $a_{-M} \neq a_{-M-L}$ and L is minimal. For $x \in \boldsymbol{Q}(\beta)_{\geq 0}$, we define functions $M(x)$ and $L(x)$ respectively by the values M and L in (1). In other words, $M(x)$ is the last index of the non periodic part of x and $L(x)$ is the length of the period of x. We say the expansion is *finite* when $a_i = 0$ for $i > M = M(x)$. Then every fractional part ω can be written as

$$\omega = \sum_{i=1}^{M(\omega)} a_{-i}\beta^{-i} + \beta^{-M(\omega)}u \tag{2}$$

with $u \in \mathscr{P}$. One can show

LEMMA 3. *$\mathscr{P} = \{0\}$ is equivalent to* (F).

PROOF. It suffices to show that $\mathscr{P} = \{0\}$ implies (F). Let $x \in \boldsymbol{Z}[1/\beta]_{>0}$. Take a sufficiently large integer K that $\beta^K x \in \boldsymbol{Z}[\beta]_{>1}$ and write

$$\beta^K x = a_{-N_0}a_{-N_0-1}\cdots a_{-M}[a_{-M-1}\cdots a_{-M-L}].$$

Then we have a purely periodic element

$$[a_{-M-1}\cdots a_{-M-L}] \times \beta^M = \beta^{M+K}x - \sum_{i=N_0}^{M} a_{-i}\beta^{M-i} \in \boldsymbol{Z}[\beta]_{\geq 0}$$

which is clearly less than one. So $\mathscr{P} = \{0\}$ implies that the expansion of x is finite. □

Let **Fr** be the set of all fractional parts of $\boldsymbol{Z}[\beta]_{\geq 0}$ and S_ω be the subset of $\boldsymbol{Z}[\beta]_{\geq 0}$ consists of elements whose fractional part coincides with ω. Of course the set **Fr** is countable. It is convenient to define S_x for a general right infinite (or finite) word x generated by $\mathscr{A} = [0, \beta) \cap \boldsymbol{Z}$. If

$$x = a_p a_{p-1}\cdots a_0.a_{-1}a_{-2}\cdots$$

then S_x is the set of greedy expansions of the form:

$$a_q a_{q-1}\cdots a_{p+1}a_p a_{p-1}\cdots a_0.a_{-1}a_{-2}\cdots$$

for $q \geq p$. Consider that the empty word λ is an element of **Fr** and S_λ is just the set of greedy expansions with no fractional parts. According to the beginning convention, the element 0 in $\mathscr{P}$ is identified with this empty word λ. Now we have

$$\boldsymbol{Z}[\beta]_{\geq 0} = \bigcup_{\omega \in \mathbf{Fr}} S_\omega.$$

Applying Φ to both sides,

$$\Phi(\boldsymbol{Z}[\beta]_{\geq 0}) = \bigcup_{\omega \in \mathbf{Fr}} \Phi(S_\omega). \tag{3}$$

It is easily seen that $\Phi(S_\omega)$ is a bounded set, since β is a Pisot number. Put $T_\omega = \overline{\Phi(S_\omega)}$, where $\bar{A}$ is the closure of a set A in the Euclidean topology of $\boldsymbol{R}^{n-1}$. Hereafter we call T_ω a tile. Let $\mathscr{B}(x,r) = \{z \in \boldsymbol{R}^{n-1} : |z - x| < r\}$, the open ball of radius r centered at x. A family $\{A_i\}_{i \in \Lambda}$ of sets in $\boldsymbol{R}^{n-1}$ is *locally finite* if for all $x \in \boldsymbol{R}^{n-1}$, there exists a positive r that the set $\{i \in \Lambda : A_i \cap \mathscr{B}(x,r) \neq \emptyset\}$ is finite.

Lemma 4. *The family* $\{T_\omega\}_{\omega \in \mathbf{Fr}}$ *is locally finite.*

Proof. Let $\omega \in \mathbf{Fr}$. It suffices to show

$$\lim_{M(\omega) \to \infty} \operatorname{dist}(\{0\}, \Phi(S_\omega)) = \infty, \tag{4}$$

where $\operatorname{dist}(A,B) = \inf_{a \in A, b \in B} |a - b|$. Here $|x|$ is the Euclidean norm of x in $\boldsymbol{R}^{n-1}$. As the set $\Phi(S_\omega)$ is bounded, it is sufficient to show

$$\lim_{M(x) \to \infty} |\Phi(x)| = \infty$$

for $x \in \boldsymbol{Z}[\beta]_{\geq 0}$. Now we employ the idea in the proof of Lemma 1 in [**3**]. Suppose the contrary. Then there exist a constant $C > 0$ and a sequence $(x_i)_{i=1}^{\infty}$ in $\boldsymbol{Z}[\beta]_{\geq 0}$ with

$$|\Phi(x_i)| < C \quad \text{and} \quad M(x_i) \to \infty.$$

Without loss of generality, we may assume $x_i < 1$, since otherwise we can replace x_i by its fractional part. Then $|\Phi(x_i)| < C$ means that every conjugate of x_i has bounded absolute value. So $\{x_i : i = 1, 2, \ldots\}$ is a finite set. On the other hand, by the definition of $M(\cdot)$, it is obvious that $\{x_i : i = 1, 2, \ldots\}$ is an infinite set, which is a contradiction. $\square$

Combining Lemmas 1, 4 and (3), we have

$$\boldsymbol{R}^{n-1} = \bigcup_{\omega \in \mathbf{Fr}} T_\omega, \tag{5}$$

when β be a Pisot number. Indeed, if $\{A_i\}_{i=1}^{\infty}$ is locally finite then

$$\overline{\bigcup_{i \in \Lambda} A_i} = \bigcup_{i \in \Lambda} \overline{A_i}.$$

Now we recall some fundamental results of [**12**]. Denote by $[x]$ the greatest integer not greater than x. Expand $1 - [\beta]/\beta$ into the greedy form:

$$1 - [\beta]/\beta = \sum_{i=2}^{\infty} c_{-i}\beta^{-i}.$$

So formally we have

$$1 = \sum_{i=1}^{\infty} c_{-i}\beta^{-i} = .c_{-1}c_{-2}\cdots,$$

with $c_{-1} = [\beta]$. This expansion is called the *characteristic sequence*.

Let $a \oplus b$ be the concatenation of two words a and b. A word b is a *tail* of w if there exists a non empty word a that $w = a \oplus b$. Let $\preceq$ be the partial order of Parry, i.e. the lexicographical order generated by absolute values of elements of $\mathscr{A}$ in the right direction. We say a is less than b when $a \preceq b$ and $a \neq b$ and write $a \prec b$. A finite word ω generated by $\mathscr{A}$ is realized by the greedy expansion on β if and only if any sub word ω' of ω satisfy

$$\omega' \prec c_{-1}c_{-2}\cdots,$$

i.e. the word ω is less than the characteristic sequence at any starting point. For an infinite word, we have a similar characterization, if we forbid some special periodic expansion in the tail. To be more precise, when the characteristic sequence is *finite*:

$$1 = .c_{-1}c_{-2}\cdots c_{-M},$$

with $c_{-M} \neq 0$, we exclude infinite words whose tail can be

$$[c_{-1}c_{-2}\cdots c_{-M+1}(c_{-M} - 1)]$$

from our consideration. Under this restriction, an infinite word generated by $\mathscr{A}$ is a greedy expansion on β if and only if such word is lexicographically less than the characteristic sequence at any starting point. Let us define $M(1)$ and $L(1)$ similarly by the characteristic sequence. If the characteristic sequence is finite, then let $L(1) = 0$.

From now on, we assume that β be a Pisot unit, so $\boldsymbol{Z}[\beta] = \boldsymbol{Z}[1/\beta]$.

LEMMA 5. *There are exactly $M(1) + L(1)$ tiles up to translation.*

Proof. Let $\boldsymbol{T}$ be the set of all tails of the characteristic sequence $.c_{-1}c_{-2}\cdots$. Since the characteristic sequence is periodic, $\boldsymbol{T}$ is a finite set. So we write this set $\boldsymbol{T} = \{u_1, u_2, \dots, u_\ell\}$ with $u_i \prec u_{i+1}$ for $i = 1, \dots, \ell - 1$. Now we consider the set S_ω with $\omega \in \mathbf{Fr}$. If $u_i \preceq \omega$ for some i and $u_i = .c_{-q_i}c_{-q_i-1}\cdots$ with $q_i \geq 2$, then the word $a_N a_{N-1}\cdots a_0 \oplus \omega \in S_\omega$ has a necessary restriction

$$a_{q_i-2}a_{q_i-3}\cdots a_0 \prec c_{-1}c_{-2}\cdots c_{-q_i+1}.$$

First we consider the case that the characteristic sequence is finite. Then $000\cdots \in \boldsymbol{T}$ and $\ell = M(1) + 1 = M(1) + L(1) + 1$. Subdivide the set $\mathbf{Fr}$ into

$$\mathbf{Fr} = \bigcup_{i=1}^{M(1)+L(1)} \mathcal{Q}_i, \quad \mathcal{Q}_i = \mathbf{Fr} \cap [u_i, u_{i+1}).$$

If $i \geq 2$ and $\omega \in \mathcal{Q}_i$, then any element $x \in S_\omega$ has above mentioned restriction on integer parts by $u_1, \dots, u_i$. Conversely, if we take such restricted integer part y then $y \oplus \omega$ is a greedy expansion. If $i = 1$, then there are no restriction on the integer part. Thus we have shown

$$S_\omega = S_{u_i} + \omega - u_i, \quad \text{for } \omega \in \mathcal{Q}_i.$$

This shows that

$$T_\omega = T_{u_i} + \Phi(\omega - u_i),$$

which shows the assertion. Second assume that the characteristic sequence is not finite. In this case, the word $000\cdots \notin \boldsymbol{T}$ and $\ell = M(1) + L(1)$. Let $u_0 = 000\cdots$ and subdivide $\mathbf{Fr}$ into

$$\mathbf{Fr} = \bigcup_{i=0}^{M(1)+L(1)-1} \mathcal{Q}_i, \quad \mathcal{Q}_i = \mathbf{Fr} \cap [u_i, u_{i+1}).$$

Then we can show the assertion similarly as above. □

We now have a locally finite tiling of the Euclidean space $\boldsymbol{R}^{n-1}$ by finite kind of tiles and their translations. Define, for any $K \in \boldsymbol{N}$, an affine map G_K from $\boldsymbol{R}^{n-1}$ to itself by

$$G_K(x_2, x_3, \dots, x_n) = (x_2, x_3, \dots, x_n)A_K,$$

where A_K is a $(n-1) \times (n-1)$ matrix:

$$A_K = \operatorname{diag}((\beta^{(2)})^{-K}, \dots, (\beta^{(r_1)})^{-K}) \otimes B_1 \otimes \cdots \otimes B_{r_2}$$

with

$$B_j = \begin{pmatrix} \Re((\beta^{(r_1+j)})^{-K}) & \Im((\beta^{(r_1+j)})^{-K}) \\ -\Im((\beta^{(r_1+j)})^{-K}) & \Re((\beta^{(r_1+j)})^{-K}) \end{pmatrix},$$

for $j = 1, \dots, r_2$. Here $\mathrm{diag}(d_1, \dots, d_s)$ is the $s \times s$ diagonal matrix of diagonal elements $d_1, \dots, d_s$ and we define

$$A \otimes B = \begin{pmatrix} A & 0 \\ 0 & B \end{pmatrix},$$

for any square matrices A and B. As β is a Pisot number, G_K must be an expanding map. For a set $A \subset \boldsymbol{R}^{n-1}$, we denote by $\mathrm{Inn}(A)$ the set of inner points of A, and by $\partial(A) = A \backslash \mathrm{Inn}(A)$, the boundaries of A. Since G_K is a homeomorphism from $\boldsymbol{R}^{n-1}$ onto itself, we have $G_K(\bar{A}) = \overline{G_K(A)}$ and $G_K(\partial(A)) = \partial(G_K(A))$ for any subset A in $\boldsymbol{R}^{n-1}$. We can show a commutative diagram:

$$\text{(6)} \qquad \begin{array}{ccc} \boldsymbol{Q}(\beta) & \xrightarrow{\times \beta^{-K}} & \boldsymbol{Q}(\beta) \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Phi} \\ \boldsymbol{R}^{n-1} & \xrightarrow[G_K]{} & \boldsymbol{R}^{n-1}, \end{array}$$

and $G_{K_2} \circ G_{K_1} = G_{K_1+K_2}$. Now $\beta^{-K} S_\omega$ is subdivided into a disjoint sum of $S_{\omega'}$ with $M(\omega') = K + M(\omega)$ and

$$\omega' = .d_{-1} \cdots d_{-K} \oplus \omega.$$

This subdivision gives rise to a relation:

$$\text{(7)} \qquad G_K(T_\omega) = \bigcup_{\omega'} T_{\omega'},$$

since we can confirm

$$G_K(T_\omega) = \overline{\Phi(\beta^{-K} S_\omega)},$$

by (6). We call this property of our tiling an *inflation-subdivision principle*. Especially when every conjugates $\beta^{(j)}$ $(j = 2, \dots, n)$ has a same absolute value, the expanding map G_K is just a similitude. This case occurs when and only when β is a cubic Pisot unit which is not totally real. See Theorem 8.1.3 of [**5**].

First we show a generalization of Theorem 2 of [**3**].

Theorem 1. *Suppose that β is a Pisot unit and consider the tiling generated by β. Then the origin 0 belongs to T_ω for any $\omega \in \mathscr{P}$ and 0 is an inner point of $\bigcup_{\omega \in \mathscr{P}} T_\omega$.*

When the Pisot unit β has property (F) then this Theorem implies that the origin is an inner point of the central tile T_λ, by Lemma 3.

Proof. First we show that the origin is an inner point of $\bigcup_{\omega \in \mathscr{P}} T_\omega$. The essential idea was found in the proof of Theorem 2 of [**3**]. It was shown in the proof of Lemma 4 that

$$\lim_{M(x)\to\infty} |\Phi(x)| = \infty,$$

for $x \in \boldsymbol{Z}[\beta]_{\geq 0}$. By this formula, for any $C > 0$ there exists m that if $M(x) \geq m$ then $|\Phi(x)| > C$. Note that multiplication of $1/\beta$ causes a bijection from $\boldsymbol{Z}[\beta]_{\geq 0}$ to itself, since β is a unit. Replacing x with $x\beta^{-m+1}$, we see that for any $C > 0$ there exists m that if $M(x) > 0$ then $|\Phi(x)| > C(\min_{j=2}^{n}|\beta^{(j)}|)^{m-1}$. Note that C and m are independent of the choice of x with $M(x) > 0$. This implies that $\operatorname{dist}(\{0\}, T_\omega) > 0$ when $M(\omega) > 0$. As we have already shown (5), we see that the origin $\Phi(0) = (0, 0, \ldots, 0) \in \boldsymbol{R}^{n-1}$, denoted by 0, is an inner point of $\bigcup_{\omega \in \mathscr{P}} T_\omega$, since $\mathscr{P} = \{\omega \in \mathbf{Fr} : M(\omega) = 0\}$.

Second, let $\omega \in \mathscr{P}$ and $L = L(\omega)$. Then we have

$$\omega = .[a_{-1} \cdots a_{-L}].$$

So we see $\Phi(\beta^{kL}\omega) \in T_\omega$, for $k = 1, 2, \ldots$ and $\lim_{k\to\infty} \Phi(\beta^{kL}\omega) = 0$. This shows that 0 is an accumulation point of the closed set T_ω. This proves the assertion. □

3. Existence of inner points.

We say an inner point x in T_ω is *exclusive* when x is not contained in other tiles $T_{\omega'}$ $(\omega' \neq \omega)$. We consider the following property:

(Ex): There exists an exclusive inner point in T_λ.

Reviewing the proof of Theorem 1, if β has property (F) then the origin is an exclusive inner point of T_λ. So (F) implies (Ex). However in a general case, (Ex) seems no longer trivial. Assuming the contrary, if every inner point of T_λ is not exclusive then we can show

$$\boldsymbol{R}^{n-1} = \bigcup_{\omega \neq \mu} T_\omega$$

for any fixed $\mu \in \mathbf{Fr}$. Indeed, if there exists an exclusive inner point of T_μ then one can find such an exclusive inner point of the form $\Phi(x) \in T_\mu$ with $x \in S_\mu$, since $\Phi(S_\mu)$ is dense in T_μ. Thus there exists a positive $g \in \boldsymbol{R}$ that

$$|\Phi(x) - \Phi(y)| > g, \quad y \in \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\mu.$$

Since $S_\mu \subset S_\lambda + \mu$, substituting y by $y + \mu$, we have

$$|\Phi(x - \mu) - \Phi(y)| > g, \quad y \in \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\lambda.$$

Thus $\Phi(x - \mu)$ is an exclusive inner point of T_λ. So rather curiously, if we pick out any single tile then it is dispensable to cover the whole space, i.e. a family $\{T_\omega\}$ forms a double covering of $\boldsymbol{R}^{n-1}$. Hence it is likely that all tiling generated by Pisot units have the property (Ex). Hereafter we wish to consider a relationship between (Ex) and (W).

REMARK 1. It is not hard to show that T_λ has an inner point, which is not necessary exclusive. Indeed, by using the theorem of Baire-Hausdorff, there exists $\omega \in \mathbf{Fr}$ that T_ω contain an inner point. Since $T_\omega \subset T_\lambda + \omega$, we can find an inner point of T_λ, similarly as above.

Now we prepare some Lemmas. Recall that $1 = .c_{-1}c_{-2}\cdots$ is the characteristic sequence.

LEMMA 6. *Let x and y be two elements of $\boldsymbol{Z}[\beta]_{\geq 0}$ with greedy expansions*:

$$x = x_{-M}x_{-M-1}x_{-M-2}\cdots, \quad y = y_{-M}y_{-M-1}y_{-M-2}\cdots$$

and $x > y$. Here we permit $x_{-M} = 0$ or $y_{-M} = 0$ to simplify the notation. Assume $x_{-M} \neq y_{-M}$. Then for any $N \in \boldsymbol{N}$ there exists a positive ε that if $x - y < \varepsilon$ then we have

$$x_{-M} - y_{-M} = 1$$

and

$$x_{-M-i} = 0, \quad y_{-M-i} = c_{-i} \quad \textit{for } i = 1, 2, \ldots, N.$$

PROOF. This lemma follows easily from the definition of the characteristic sequence. □

LEMMA 7. *The characteristic sequence can not be purely periodic. In other words, we always have $M(1) > 0$.*

PROOF. Assume that the characteristic sequence has the form

$$1 = .[c_{-1}c_{-2}\cdots c_{-L}]. \tag{8}$$

This shows

$$\beta^L - 1 = \sum_{i=1}^{L} c_{-i}\beta^{L-i}.$$

Thus β is a root of a polynomial:

$$x^L - c_{-1}x^{L-1} - c_{-2}x^{L-2} - \cdots - c_{-L+1}x - c_{-L} - 1.$$

It can be shown that $L \geq 2$. Indeed, if $L = 1$ then β is an integer which is excluded at the beginning. Calculating the L-th digit of the characteristic sequence by definition, we have

$$\left[\beta^L\left(1 - \sum_{i=1}^{L-1} c_{-i}\beta^{-i}\right)\right] = c_{-L} + 1 < \beta.$$

So we have another characteristic sequence (!):

$$1 = .c_{-1}\cdots c_{-L+1}(c_{-L}+1).$$

Since characteristic sequence is unique, we get a desired contradiction. □

The next lemma, as well as its proof, will be frequently used later on.

Lemma 8. *Let $z \in \boldsymbol{Z}[\beta]_{\geq 0}$ whose fractional part is purely periodic. Then the integer part of z coincides with that of $z + x$ when we take a sufficiently small $x \in \boldsymbol{Z}[\beta]_{>0}$.*

Proof. Let $z \in \boldsymbol{Z}[\beta]_{\geq 0}$ whose fractional part u is contained in $\mathscr{P}$. Let $x < 1 - u$ be a sufficiently small element of $\boldsymbol{Z}[\beta]_{>0}$ and compare the expansions $u + x = a_{-1}a_{-2}\cdots$ and $u = b_{-1}b_{-2}\cdots$. By using Lemma 6, if $a_{-1} > b_{-1}$ then there exists a large M_0 that $b_{-1-i} = c_{-i}$ for $i = 1, \dots, M_0$. Let L_1 the least common multiple of $L(1)$ and $L(u)$ and suppose $M_0 \geq L_1 + M(1)$. Then we have

$$b_{-1-M(1)} = b_{-1-L_1-M(1)} = c_{-L_1-M(1)} \neq c_{-M(1)} = b_{-1-M(1)},$$

which is a contradiction. So we see that $a_{-1} = b_{-1}$. Next we compare the expansions of $u + x - a_{-1}\beta^{-1}$ and $u - a_{-1}\beta^{-1}$. By a similar argument we see $a_{-2} = b_{-2}$. Repeating this, one may assume that there exists a sufficiently large M_1 that $a_{-j} = b_{-j}$ for $j = 1, \dots, M_1$. Let g be the least common multiple of $L(u)$ for $u \in \mathscr{P} \cup \{1\}$ and consider a formal expansion:

$$z + x = z - u + (u + x) = a_N a_{N-1} \cdots a_0 . a_{-1} a_{-2} \cdots,$$

with two greedy expansions $z - u = a_N \cdots a_0$ and $u + x = .a_{-1}a_{-2}\cdots$. Our aim is to show that if we take a sufficiently small x, i.e. a sufficiently large $M_1 > g$, then this expression itself is a greedy expansion. Indeed, if the expansion $a_N a_{N-1} \cdots a_0 . a_{-1} a_{-2} \cdots$ is not a greedy expansion, by Lemma 6, there exist $N_1 > 0$ and a sufficiently large $N_2 > 0$ that $a_{N_1 - i} = c_{-i}$ for $i = 1, \dots, N_2$. Let us take $N_2 > N_1 + g$. Then $u = .b_{-1}b_{-2}\cdots \in \mathscr{P}$ is determined uniquely from the expansion of $u + x = .a_{-1}a_{-2}\cdots$ by periodicity and coincides with a tail of the characteristic sequence. This shows that $a_N a_{N-1}\cdots a_0.b_{-1}b_{-2}\cdots$ contains a sub word $c_{-1}c_{-2}\cdots$, i.e. the characteristic sequence itself. But this causes a contradiction, because $z = (z-u)+u = a_N a_{N-1}\cdots a_0.b_{-1}b_{-2}\cdots$ is a greedy expansion. $\square$

Lemma 9. *For any positive A, the set $\Phi(\boldsymbol{Z}[\beta]\cap[0,A])$ is discrete in $\boldsymbol{R}^{n-1}$.*

Proof. It is enough to show that $\Phi(\mathbf{Fr})$ is discrete. Indeed there exist $K \in \boldsymbol{N}$ and a homeomorphism G_K that

$$G_K(\Phi(\boldsymbol{Z}[\beta]\cap[0,A])) = \Phi(\boldsymbol{Z}[\beta]\cap[0,A\beta^{-K}]) \subset \Phi(\boldsymbol{Z}[\beta]\cap[0,1)).$$

Again by $\lim_{M(x)\to\infty}|\Phi(x)| = \infty$, the set $\Phi(\mathbf{Fr})$ must be discrete. $\square$

There exists a concrete and practical way to find an exclusive inner point with the help of Theorem 1:

Proposition 1. *Take an element $x \in S_\lambda$. The point $\Phi(x)$ is an exclusive inner point if and only if for any $K_0 \in \boldsymbol{N}$ there exists $K \geq K_0$ that $\beta^K u + x \in S_\lambda$ for any $u \in \mathscr{P}$.*

Proof. Reviewing the proof of Theorem 1, we see that the origin $0 = \Phi(0) \in \boldsymbol{R}^{n-1}$ is, by an abuse of terminology, an exclusive inner point of $\bigcup_{u\in\mathscr{P}} T_u$. To be exact, there exists a positive g_1 that

$$|\Phi(y)| > g_1, \quad y \in \boldsymbol{Z}[\beta]_{\geq 0}\Big\backslash \bigcup_{u\in\mathscr{P}} S_u.$$

Since $\beta\boldsymbol{Z}[\beta]_{\geq 0} = \boldsymbol{Z}[\beta]_{\geq 0}$, substituting y by $\beta^{-K}(y-x)$, we have

$$|\Phi(x)-\Phi(y)| > g_2, \quad y \in (x+\boldsymbol{Z}[\beta]_{\geq 0})\Big\backslash \bigcup_{u\in\mathscr{P}} (x+\beta^K S_u)$$

with some positive g_2. Now we show that $x + \beta^K S_u \subset S_\lambda$ for sufficiently large K. Let $u = a_{-M}a_{-M-1}\cdots$ and $u + \beta^{-K}x = b_{-M}b_{-M-1}\cdots$ be the greedy expansions as in Lemma 6. By the assumption, $u + \beta^{-K}x \in \mathrm{Fin}(\beta)$. Since we may assume that $(u+\beta^{-K}x) - u$ is sufficiently small when K is large, by the proof of Lemma 8, we see that there exists sufficiently large $M_0 \in \boldsymbol{N}$ that $a_{-M-i} = b_{-M-i}$ for $i \leq M_0$. Proceeding along the same line with the proof of Lemma 8, multiplying β^K and using the assumption $x + \beta^K u \in S_\lambda$, we see that $x + \beta^K S_u \subset S_\lambda$ when K is sufficiently large, as desired. As a result, we see

$$|\Phi(x)-\Phi(y)| > g_2, \quad y \in (x+\boldsymbol{Z}[\beta]_{\geq 0})\backslash S_\lambda.$$

By Lemma 9, we see $\Phi(\boldsymbol{Z}[\beta] \cap [0, x])$ is a discrete set. This shows that there exists a positive constant g_3 that

$$|\Phi(x) - \Phi(y)| > g_3, \quad y \in \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\lambda,$$

which means that $\Phi(x)$ is an exclusive inner point of T_λ. Next we will show the converse. Assume the existence of an exclusive inner point. Since $\Phi(S_\lambda)$ is dense in T_λ, there exists $x \in S_\lambda$ that $\Phi(x)$ be an exclusive inner point. If we take a sufficiently large K, then $\Phi(x + \beta^K u)$ is an exclusive inner point for any $u \in \mathscr{P}$. As $x + \beta^K u \in \boldsymbol{Z}[\beta]_{\geq 0}$, we see that $x + \beta^K u$ is contained in S_λ, since otherwise $\Phi(x + \beta^K u)$ is not exclusive. $\square$

EXAMPLE 1. Let β be a Pisot unit whose irreducible polynomial is $x^3 - 3x^2 + 2x - 1$. Then the characteristic sequence is $1 = .20111\cdots$. One can show $\mathscr{P} = \{0, .111\cdots\}$. If we take $x = 10. = \beta$, then we have

$$1111\cdots 1111.111\cdots + 10. = 1111\cdots 1201.$$

This shows that $\Phi(\beta)$ is an exclusive inner point.

EXAMPLE 2. Let β be a Pisot unit whose irreducible polynomial is $x^3 - 5x^2 + 2x + 1$. Then the characteristic sequence is $1 = .42111\cdots$. One can show that $\mathscr{P} = \{0, .111\cdots, .222\cdots, .333\cdots\}$. If we take $x = 241.$, then we have

$$1111\cdots 11111.111\cdots + 241. = 1111\cdots 11410.$$

$$2222\cdots 22222.222\cdots + 241. = 2222\cdots 23100.$$

$$3333\cdots 33333.333\cdots + 241. = 3333\cdots 40000.$$

This shows that $\Phi(2\beta^2 + 4\beta + 1)$ is an exclusive inner point.

We are now in position to show an important

PROPOSITION 2. *The conditions* (W) *and* (Ex) *are equivalent.*

PROOF. First we show that (W) implies (Ex). By Lemma 2, the set $\mathscr{P}$ is finite. Write $\mathscr{P} = \{u_1, u_2, \ldots, u_s\}$. Then by the assumption (W), there exist $x_1, y_1 \in \mathrm{Fin}(\beta)$ with $u_1 + x_1 = y_1$ and x_1 is small. Expand $u_2 + x_1$ in a greedy form:

$$\begin{aligned} u_2 + x_1 &= a_{-N} a_{-N-1} \cdots a_{-M} [a_{-M-1} \cdots a_{-M-L}] \\ &= a_{-N} a_{-N-1} \cdots a_{-M} + \beta^{-M} u_2' \end{aligned}$$

with $M = M(u_2 + x_1)$, $L = L(u_2 + x_1)$ and $u_2' \in \mathscr{P}$. Using (W), one can find a small $x_2 \in \mathrm{Fin}(\beta)$ and $y_2 \in \mathrm{Fin}(\beta)$ with $u_2' + x_2 = y_2$. Taking a sufficiently small x_2 we have

$$u_i + x_1 + \beta^{-M} x_2 \in \mathrm{Fin}(\beta), \quad \text{for } i = 1, 2.$$

Indeed, $u_1 + x_1 + \beta^{-M} x_2 = y_1 + \beta^{-M} x_2$ is contained in $\mathrm{Fin}(\beta)$ when x_2 is sufficiently small. Also we see

$$u_2 + x_1 + \beta^{-M} x_2 = a_{-N} \cdots a_{-M} + \beta^{-M} y_2,$$

and $|u_2' - y_2|$ is small. By using Lemma 8 and its proof, $u_2 + x_1 + \beta^{-M} x_2 \in \mathrm{Fin}(\beta)$. Repeating this argument, we see that there exists a decreasing sequence $\xi_1, \xi_2, \dots, \xi_s$ in $\mathrm{Fin}(\beta)$ that

$$u_i + \sum_{j=1}^{s} \xi_j \in \mathrm{Fin}(\beta), \quad \text{for } i = 1, 2, \dots, s.$$

Let L_p be the least common multiple of $L(u_j)$ $(j = 1, \dots, s)$. Again by using Lemma 8 and its proof and taking a sufficiently small $\xi_1 = x_1$, we may assume that

$$u_i + \beta^{-KL_p} \sum_{j=1}^{s} \xi_j = u_i(1 - \beta^{-KL_p}) + \beta^{-KL_p} \left(u_i + \sum_{j=1}^{s} \xi_j \right) \in \mathrm{Fin}(\beta)$$

for $i = 1, 2, \dots, s$ and $K \geq 0$. Note that

$$u_i(1 - \beta^{-KL_p}) = \overbrace{\omega_i \oplus \omega_i \oplus \cdots \oplus \omega_i}^{K \text{ times}},$$

with some word ω_i of length L_p. Multiplying β^{KL_p}, the assertion follows immediately from Proposition 1. Second we show the converse. Assume the condition (Ex). So let $\Phi(x)$ be an exclusive inner point of T_λ with $x \in S_\lambda$. Let $v \in \boldsymbol{Z}[\beta]_{\geq 0}$. Then there exists a $K_0 \in \boldsymbol{N}$ that $\Phi(x + \beta^K v)$ is also an exclusive inner point of T_λ for all integer $K \geq K_0$. By the definition of the exclusive inner point, we have $y = x + \beta^K v \in S_\lambda$. Thus we have

$$v = \beta^{-K} y - \beta^{-K} x,$$

and both $\beta^{-K} y$ and $\beta^{-K} x$ are contained in $\mathrm{Fin}(\beta)$. The proposition is proved. □

Here we want to show there exists a finite algorithm to confirm (W). Consider a slightly modified condition:

(W′): For any element x of $\mathscr{P}$ and any positive ε, there exist two elements y, z in $\mathrm{Fin}(\beta)$ with $|z| < \varepsilon$ such that $x = y - z$.

Then we can prove

Lemma 10. *The assumption* (W) *is equivalent to* (W′).

Proof. It suffices to show that (W′) implies (W). Any element $x \in \mathrm{Fin}(\beta)$ has a form:

$$x = a_{-N}a_{-N-1}\cdots a_{-M}[a_{-M-1}\cdots a_{-M-L}]$$
$$= a_{-N}a_{-N-1}\cdots a_{-M} + \beta^{-M}u$$

with $M = M(x)$, $L = L(x)$ and $u \in \mathscr{P}$. Thus by (W′), we can find $b \in \mathrm{Fin}(\beta)$ and a small $a \in \mathrm{Fin}(\beta)$ that $u = b - a$. By using the proof of Lemma 8, sufficiently long leading words of $u + a$ and u coincides if we take a small a. This implies that $\xi = a_{-N}a_{-N-1}\cdots a_{-M} + \beta^{-M}(u+a) \in \mathrm{Fin}(\beta)$, since otherwise the word $a_{-N}a_{-N-1}\cdots a_{-M}[a_{-M-1}\cdots a_{-M-L}]$ itself can not be a greedy expansion. Thus we have $x = \xi - \beta^{-M}a$, as desired. □

It is obvious that (F) implies (W). We want to show here another sufficient condition of (W). Assume for a while that $\beta > 1$ is an arbitrary real number. Let us denote by $\boldsymbol{Z}_{\geq 0}[\beta]$ the set of polynomials in β with non negative integer coefficients. Consider the condition:

(Pf): $\boldsymbol{Z}_{\geq 0}[1/\beta] \subset \mathrm{Fin}[\beta]$.

By Theorem 3 of [**8**], if $\beta > 1$ is a root of the polynomial

$$x^n - a_{n-1}x^{n-1} + a_{n-2}x^{n-2} + a_{n-3}x^{n-3} + \cdots + a_1 x + a_0,$$

with non negative integers a_i $(i = 0, 1, \ldots, n-1)$ with $a_0 > 0$ and $a_{n-1} - \sum_{j=0}^{n-2} a_j \geq 2$ then β has property (Pf). Then we can show

Proposition 3. *The condition* (Pf) *implies* (W).

Proof. By Proposition 1 of [**4**], (Pf) implies that β is a Pisot number. Let $z \in \boldsymbol{Z}[1/\beta]_{\geq 0}$. Then z is expanded into

$$z = a_{-N}a_{-N-1}\cdots a_{-M}[a_{-M-1}\cdots a_{-M-L}]$$
$$= a_{-N}a_{-N-1}\cdots a_{-M} + \beta^{-M}u$$

with $M = M(z)$, $L = L(z)$ and $u \in \mathscr{P}$. Taking a $K \in \boldsymbol{N}$, one can rewrite this expression to

$$z = a_{-N}a_{-N-1}\cdots a_{-M} + \beta^{-M}(1 - \beta^{-KL})u + \beta^{-M-KL}u. \tag{9}$$

Note that

$$(1 - \beta^{-KL})u = .\overbrace{w \oplus w \oplus \cdots \oplus w}^{K\ \text{times}}, \tag{10}$$

with $w = a_{-M-1} \cdots a_{-M-L}$. Since $u \in \boldsymbol{Z}[1/\beta]$, there exists some s and we have an expression:

$$\beta^s u = \sum_{j=0}^{n-1} b_j \beta^j = \sum_{b_j \geq 0} b_j \beta^j - \sum_{b_j < 0} |b_j| \beta^j, \quad b_j \in \boldsymbol{Z}.$$

Thus we can find $x, y \in \boldsymbol{Z}_{\geq 0}[1/\beta]$ with $u = x - y$. Putting this expression into (9), we have

$$z = a_{-N} a_{-N-1} \cdots a_{-M} + \beta^{-M}(1 - \beta^{-KL}) u + \beta^{-M-KL} x - \beta^{-M-KL} y.$$

Using (10), we see that

$$a_{-N} a_{-N-1} \cdots a_{-M} + \beta^{-M}(1 - \beta^{-KL}) u + \beta^{-M-KL} x$$

is an element of $\boldsymbol{Z}_{\geq 0}[1/\beta]$. As one can choose a sufficiently large K, the proposition is proved by using the assumption (Pf). $\square$

Now we go back to the story and assume that β is a Pisot unit again. In Proposition 2, we have shown that a topological assumption (Ex) and an algebraic assumption (W) are equivalent. Thus hereafter in this paper, we assume (W).

LEMMA 11. *For any $\omega \in \mathbf{Fr}$, there exists an exclusive inner point in T_ω.*

PROOF. Since $\Phi(S_\lambda)$ is dense in T_λ and we assumed (W), so there exists $y \in S_\lambda$ so that $\Phi(y)$ be an exclusive inner point of T_λ. Take any $\omega \in \mathbf{Fr}$:

$$\begin{aligned} \omega &= .c_{-1} \cdots c_{-m}[c_{-m-1} \cdots c_{m-L}] \\ &= \beta^{-m} u + .c_{-1} \cdots c_{-m}, \end{aligned}$$

with $u \in \mathscr{P}$, $M(\omega) = m$ and $L = L(\omega)$. Again by the result of Parry, for any $\theta = a_p a_{p-1} \cdots a_0. \in S_\lambda$ and any $M \geq M(1) + L(1)$,

$$\beta^M \theta + \omega = a_p a_{p-1} \cdots a_0 \overbrace{00 \cdots 0}^{M \text{ times}} .c_{-1} \cdots c_{-m}[c_{-m-1} \cdots c_{m-L}]$$

is a greedy expansion by itself. This shows

$$S_\kappa = \beta^M S_\lambda + \omega \quad \text{for } \kappa = \overbrace{00 \cdots 0}^{M \text{ times}} . \oplus \omega. \tag{11}$$

Thus $\beta^M y + \omega \in S_\omega$ and $\Phi(\beta^M y + \omega)$ is an inner point of T_κ, so clearly of T_ω. It remains to prove that this point is exclusive. Since $\Phi(y)$ is exclusive, we have

$$|\Phi(y) - \Phi(x)| > g_1, \quad \text{for } x \in \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\lambda,$$

with a positive g_1. Similarly as in the proof of Proposition 1, there exists a positive g_2 that

$$|\Phi(\beta^M y + \omega) - \Phi(x)| > g_2, \quad \text{for } x \in \omega + \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\kappa.$$

Again by Lemma 9, there exists a positive g_3 that

$$|\Phi(\beta^M y + \omega) - \Phi(x)| > g_3, \quad \text{for } x \in \boldsymbol{Z}[\beta]_{\geq 0} \backslash S_\kappa,$$

which shows that $\Phi(\beta^K y + \omega)$ is an exclusive inner point of T_κ, and consequently of T_ω. □

Let $\mathrm{Inn}^*(T_\omega)$ be the set of all exclusive inner points of T_ω. The *diameter* of the set $A \subset \boldsymbol{R}^{n-1}$ is the value $\sup_{x,y \in A} |x - y|$.

Theorem 2. *Let us assume* (W). *For any* $\omega \in \mathbf{Fr}$, *we have* $T_\omega = \overline{\mathrm{Inn}^*(T_\omega)} = \overline{\mathrm{Inn}(T_\omega)}$.

Proof. It suffices to show $T_\omega = \overline{\mathrm{Inn}^*(T_\omega)}$. The inclusion $T_\omega \supset \overline{\mathrm{Inn}^*(T_\omega)}$ is clear, since T_ω is closed. Let $\xi \in T_\omega$. Then there exists a sequence $(x_i)_{i=1}^\infty$ in S_ω with $\lim \Phi(x_i) = \xi$. Write

$$x_i = \cdots a_{i3} a_{i2} a_{i1} a_{i0}. \oplus \omega.$$

One can find at least one $b_0 \in \mathscr{A}$ and infinite i's with $b_0 = a_{i0}$. Denote the corresponding subsequence by $(x_i^{(0)})_{i=1}^\infty$ where

$$x_i^{(0)} = \cdots a_{i3}^{(0)} a_{i2}^{(0)} a_{i1}^{(0)} b_0. \oplus \omega.$$

Next we find at least one $b_1 \in \mathscr{A}$ and infinite i's with $b_1 = a_{i1}^{(0)}$. Denote the corresponding subsequence by $(x_i^{(1)})_{i=1}^\infty$. Repeating this process, we get b_j and $(x_i^{(j)})_{i=1}^\infty$ for $j = 0, 1, \ldots$. Putting $d_K = b_K b_{K-1} \cdots b_0. \oplus \omega$, we easily see that $\lim_{K \to \infty} \Phi(d_K) = \xi$. By Lemma 11, there exists an exclusive inner point v_K of T_ω which belongs to its subdivided tile T_{d_K}, since $G_K(T_{d_K}) = T_{\beta^{-K} d_K}$ $(\beta^{-K} d_K \in \mathbf{Fr})$ has an exclusive inner point. Noting the diameter of T_{d_K} tends to 0 as $K \to \infty$, we see $|\Phi(d_K) - v_K| \to 0$. Thus $\lim v_K = \xi$, as desired. □

Corollary 1. *For any distinct* $\omega, \omega' \in \mathbf{Fr}$, *we have* $\mathrm{Inn}(T_\omega) \cap T_{\omega'} = \emptyset$.

In other words, we finally know that every inner point of a tile is exclusive, so $\mathrm{Inn}(T_\omega) = \mathrm{Inn}^*(T_\omega)$.

Proof. Let $x \in \mathrm{Inn}(T_\omega) \cap T_{\omega'}$ with $\omega \neq \omega'$. Then there exists a sequence $(\xi_i)_{i=1}^\infty$ consists of exclusive inner points of $T_{\omega'}$ which converge to x. So we can find some i that $\xi_i \in \mathrm{Inn}(T_\omega) \cap \mathrm{Inn}^*(T_{\omega'})$, which is impossible by the definition of an exclusive inner point. $\square$

Corollary 2. *For any $\omega \in \mathbf{Fr}$, there exist finite elements $v_1, v_2, \ldots, v_m$ of $\mathbf{Fr}$ that*

$$\partial(T_\omega) = \bigcup_{i=1}^{m} (T_\omega \cap T_{v_i}).$$

Proof. By using Lemma 4, there exist only finite elements $v_j \neq \omega$ $(j = 1, 2, \ldots, m)$ such that $T_\omega \cap T_{v_j} \neq \emptyset$. Let $x \in \partial(T_\omega) = T_\omega \backslash \mathrm{Inn}(T_\omega)$. Then the shrinking balls $\mathscr{B}(x, 2^{-n})$ $(n = 0, 1, 2, \ldots)$ contain points x_n $(n = 0, 1, \ldots)$ which do not belong to T_ω. Thus we have $x_n \in \bigcup_{j=1}^{m} T_{v_j}$. One can find $j_0 \in \{1, 2, \ldots, m\}$ and an infinite subsequence $(x'_k)_{k=1}^{\infty}$ of $(x_n)_{i=0}^{\infty}$ that $x'_k \in T_{v_{j_0}}$. This shows that $x \in T_{v_{j_0}}$. So we have proved

$$\partial(T_\omega) \subset \bigcup_{i=1}^{m} (T_\omega \cap T_{v_i}).$$

We want to show the converse inclusion. Let $x \in T_\omega \cap T_{v_i}$. Then by Corollary 1, the point x is not an inner point of T_ω. So $x \in T_\omega \backslash \mathrm{Inn}(T_\omega) = \partial(T_\omega)$, as desired. $\square$

Remark 2. The reader see that the clue to show Corollaries 1 and 2 is the existence of an exclusive inner point in each tile. If β has property (F) then we have no problems in finding such points. However, this fact should have been used more explicitly in the proof of Corollary 2 of [**3**].

4. Counting the number of subdivisions.

Let $\omega \in \mathbf{Fr}$ and denote $N_\omega(r)$ be the number of greedy expansions of the form:

$$a_{r-1} a_{r-2} \cdots a_0. \oplus \omega.$$

Then we have a formula:

Lemma 12. *There exist constants $g_i > 0$ $(i = 1, 2)$ depending only on β and ω such that*

$$g_1 \beta^r \leq N_\omega(r) \leq g_2 \beta^r.$$

Proof. By Lemma 11, we see $\mu_{n-1}(T_\omega) > 0$ for any $\omega \in \mathbf{Fr}$. By using Lemma 5, there exists a maximum of $\mu_{n-1}(T_\omega)$ for $\omega \in \mathbf{Fr}$. Let a be the maximum and choose $\gamma \in \mathbf{Fr}$ with $\mu_{n-1}(T_\gamma) = a$. By (5), we have

$$\begin{aligned} \beta^K \mu_{n-1}(T_\omega) &= \mu_{n-1}(G_K(T_\omega)) \\ &\le \sum_{\omega'} \mu_{n-1}(T_{\omega'}) \\ &\le aN_\omega(K). \end{aligned} \tag{12}$$

This shows the existence of $g_1 > 0$.

Let $1 = .c_{-1}c_{-2}\cdots$ be the characteristic sequence. Consider a function $\mathscr{F}(z) = 1 - \sum_{i=1}^{\infty} c_{-i}z^{-i}$. Since characteristic sequence is eventually periodic and the function $\mathscr{F}(z)$ can be expressed as a rational function of z, the function $\mathscr{F}(z)$ has a meromorphic continuation to the whole z-plane. Again by the result of [**12**], apart from trivial solution β, any other solutions of $\mathscr{F}(z) = 0$ have modulus less than $\min\{2, \beta\}$. Let η be the maximum modulus of such non trivial solutions.

First we consider the case that the characteristic sequence is *finite*, that is, $c_{-i} = 0$ for sufficiently large i. So we put $1 = .c_{-1}\cdots c_{-M}$. By using the result of Parry, we read $N_\lambda(r+M) - c_{-1}N_\lambda(r+M-1)$ as a number of greedy words of length $r+M$ whose leading word is c_{-1}, since $c_{-1}N_\lambda(r+M-1)$ is the number of greedy words whose head word is contained in $\{0, 1, \ldots, c_{-1}-1\}$. Extending this idea similarly, we see

$$N_\omega(r+M) - c_{-1}N_\omega(r+M-1) - c_{-2}N_\omega(r+M-2)\cdots - c_{-M+1}N_\omega(r+1)$$

is the number of elements θ in **Fr** of the form

$$c_{-1}c_{-2}\cdots c_{-M+1}\overbrace{\cdots\cdots\cdots}^{r+1\ \text{letters}}.\oplus\omega.$$

This number coincides with $c_{-M}N_\omega(r)$. Indeed the letter after c_{-M+1} must be in $\{0, 1, \ldots, c_{-M}-1\}$, and there are no additional restrictions on the remaining greedy word $c_{-M-1}\cdots\oplus\omega$. Thus we have a linear recurrence formula:

$$N_\omega(r+M) = \sum_{i=1}^{M} c_{-i}N_\omega(r+M-i).$$

So for any $\tau > \eta$, there exists a non negative c such that

$$N_\omega(r) = c\beta^r + O(\tau^r), \tag{13}$$

since β is a root of $\mathscr{F}(z) = 0$ of maximum modulus. In fact we see $c > 0$, as we have shown there exists $g_1 > 0$. This proves the assertion for the finite characteristic sequence. Next we assume that the characteristic sequence in not finite:

$$1 = .c_{-1}\cdots c_{-M}[c_{-M-1}\cdots c_{-M-L}].$$

By a similar trick as above, putting $N_\omega(0) = 1$, we have

$$\begin{aligned}
N_\omega(1) &\le 1 + c_{-1}N_\omega(0)\\
N_\omega(2) &\le 1 + c_{-1}N_\omega(1) + c_{-2}N_\omega(0)\\
N_\omega(3) &\le 1 + c_{-1}N_\omega(2) + c_{-2}N_\omega(1) + c_{-3}N_\omega(0)\\
&\ \ \vdots\\
N_\omega(K) &\le 1 + \sum_{i=1}^{K} c_{-i}N_\omega(K-i),
\end{aligned}$$

for any $K \in \boldsymbol{N}$. Let $\boldsymbol{T}$ be the set of all tails of the characteristic sequence $.c_{-1}c_{-2}\cdots$ and $u \in \boldsymbol{T}$ be the minimum element with respect to the order $\preceq$. Since characteristic sequence is not finite, we see $u \neq 00\cdots$. Identify the tail word $u = c_{-M}c_{-M-1}\cdots$ with the value

$$.c_{-M}c_{-M-1}\cdots = \sum_{i=1}^{\infty} c_{-M-i+1}\beta^{-i}.$$

Then we have

$$\beta^K\left(1 - \sum_{i=-1}^{K} c_{-i}\beta^{-i}\right) \ge u > 0$$

for any integer $K \ge 2$, by the definition of the characteristic sequence. Take $g_2 > 0$ so that both $N_\omega(1) \le g_2\beta$ and $g_2 u \ge 1$ hold. Assume that $N_\omega(r) \le g_2\beta^r$ for $r \le K-1$, then

$$\begin{aligned}
N_\omega(K) &\le 1 + g_2\beta^K \sum_{i=1}^{K} c_{-i}\beta^{-i}\\
&\le 1 + (\beta^K - u)g_2 \le g_2\beta^K.
\end{aligned}$$

This completes the proof. $\square$

The asymptotic formula (13) might be valid for any non finite characteristic sequence.

5. Lebesgue measure of the boundary.

Lemma 13. *For any $\omega \in \mathbf{Fr}$, there exist a $K \in \boldsymbol{N}$ and $\theta \in \mathbf{Fr}$ that $T_\theta \subset \mathrm{Inn}(G_K(T_\omega))$.*

Proof. There exists an inner point of T_ω, by Lemma 11. So for any $R > 0$, there exists a $K \in \boldsymbol{N}$ that $G_K(T_\omega)$ contains an open ball $\mathscr{B}(x, R)$. By Lemma 5, there exists a maximum $C > 0$ of the diameter of T_ω for $\omega \in \mathbf{Fr}$. Let $R > C$ and let T_θ contains the center x. Then T_θ must be contained in $\mathscr{B}(x, R)$, thus $T_\theta \subset \mathrm{Inn}(G_K(T_\omega))$. $\square$

One of the main purpose of this paper is to show:

Theorem 3. *Let β be a Pisot unit with the property* (W). *Then $\mu(\partial(T_\omega)) = 0$ for $\omega \in \mathbf{Fr}$. Here $\mu = \mu_{n-1}$ is Lebesgue measure of $\boldsymbol{R}^{n-1}$.*

Proof. By Lemma 5, there exists $D = \max_\omega \mu(\partial(T_\omega))$. Choose $\eta \in \mathbf{Fr}$ such that $D = \mu(\partial(T_\eta))$. From the definition of G_K, we see easily that $\mu(G_K(A)) = \beta^K \mu(A)$, for some $K \in \boldsymbol{N}$ and a measurable set A. The main tool of the proof is the inflation-subdivision principle:

$$G_K(T_\omega) = \bigcup_{\omega'} T_{\omega'}. \tag{14}$$

By Lemma 13, there exist some $K \in \boldsymbol{N}$ and θ that $T_\theta \subset \mathrm{Inn}(G_K(T_\omega))$. Applying Corollaries 1 and 2, we see T_θ does not touch the boundary of $G_K(T_\omega)$. By Lemma 5, one may assume that K is independent of the choice of ω. So there exists K such that

$$\beta^K \mu(\partial(T_\omega)) \le (N_\omega(K) - 1)D,$$

for any $\omega \in \mathbf{Fr}$. This inequality seems unsatisfactory at a glance, since $N_\gamma(K) \ge \beta^K$ in (12). However we can get more information by the generalization of this argument. Let

$$c = \min_{\omega, \omega'} \inf_{m=1,2,\dots} \frac{N_\omega((m-1)K)}{N_{\omega'}(mK)}.$$

By Lemmas 5 and 12, we see $c > 0$. Considering the case $\omega = \omega'$, we have $c < 1$. Then we have

$$\begin{aligned} \mu(G_K(\partial(T_\omega))) &\le (N_\omega(K) - 1)D \\ &\le (1 - c)N_\omega(K)D, \end{aligned}$$

since $N_\omega(0) = 1$. Now we prove, for any $m = 1, 2, \dots$ and $\omega \in \mathbf{Fr}$,

$$\mu(G_{mK}(\partial(T_\omega))) \le (1 - c)^m N_\omega(mK)D, \tag{15}$$

by induction. Applying $G_{(m-1)K}$ to (14),

$$G_{mK}(T_\omega) = \bigcup_{\omega'} G_{(m-1)K}(T_{\omega'})$$
$$= \bigcup_{\omega'} \left(\bigcup_{\omega''} T_{\omega''} \right),$$

where ω'' is of the form $.a_{-1} \cdots a_{-(m-1)K} \oplus \omega'$. Since $G_{(m-1)K}(T_\theta)$ does not touch the boundary of $G_{mK}(T_\omega)$,

$$G_{mK}(\partial(T_\omega)) \subset \bigcup_{\omega' \neq \theta} G_{(m-1)K}(\partial(T_{\omega'})).$$

Thus we have

$$\mu(G_{mK}(\partial(T_\omega))) \leq \sum_{\omega' \neq \theta} (1-c)^{m-1} N_{\omega'}((m-1)K)D$$
$$= (1-c)^{m-1}(N_\omega(mK) - N_\theta((m-1)K))D$$
$$\leq (1-c)^m N_\omega(mK)D,$$

so we have shown (15). Putting $\omega = \eta$ in (15)

$$\beta^{mK}\mu(\partial(T_\eta)) \leq (1-c)^m N_\eta(mK)\mu(\partial(T_\eta)), \tag{16}$$

for any $m \in \boldsymbol{N}$. By using Lemma 12, we have

$$\mu(\partial(T_\eta)) \leq g_2(1-c)^m \mu(\partial(T_\eta)).$$

As we can take any large m, we have $D = \mu(\partial(T_\eta)) = 0$. □

6. Examples.

Let $\beta = 2.3247179572\cdots$ be a Pisot number defined by an irreducible polynomial $x^3 - 3x^2 + 2x - 1$. Then $\mathscr{P} = \{0, \omega\}$ with $\omega = .111\cdots$. By Theorem 1, the origin is an inner point of $T_\lambda \cup T_\omega$. The characteristic sequence is $.20111\cdots$. So by Proposition 5, there exist 3 tiles up to translation. The tiling is self similar in this case. (See Figure 1.)

Let $\beta = 2.87938524157\cdots$ be a Pisot number defined by an irreducible polynomial $x^3 - 3x^2 + 1$, which satisfies (Pf). This polynomial is not totally real and so the tiling is not self similar but self affine. We can also show $\mathscr{P} = \{0, \omega\}$ with $\omega = .111\cdots$. Thus the origin is an inner point of $T_\lambda \cup T_\omega$. The characteristic sequence is $.22111\cdots$ so there are 3 tiles up to translation. (See Figure 2.)

Let $\beta = 2.2469796037\cdots$ which corresponds to $x^3 - 2x^2 - x + 1$. The tiling is self affine. Since $\mathscr{P} = \{0, [01], [10], [1]\}$, the origin is an inner point of $T_\lambda \cup T_{[01]} \cup T_{[10]} \cup T_{[1]}$. The characteristic sequence is $.2010101\cdots$ so there are 3 tiles up to translation. (See Figure 3.)

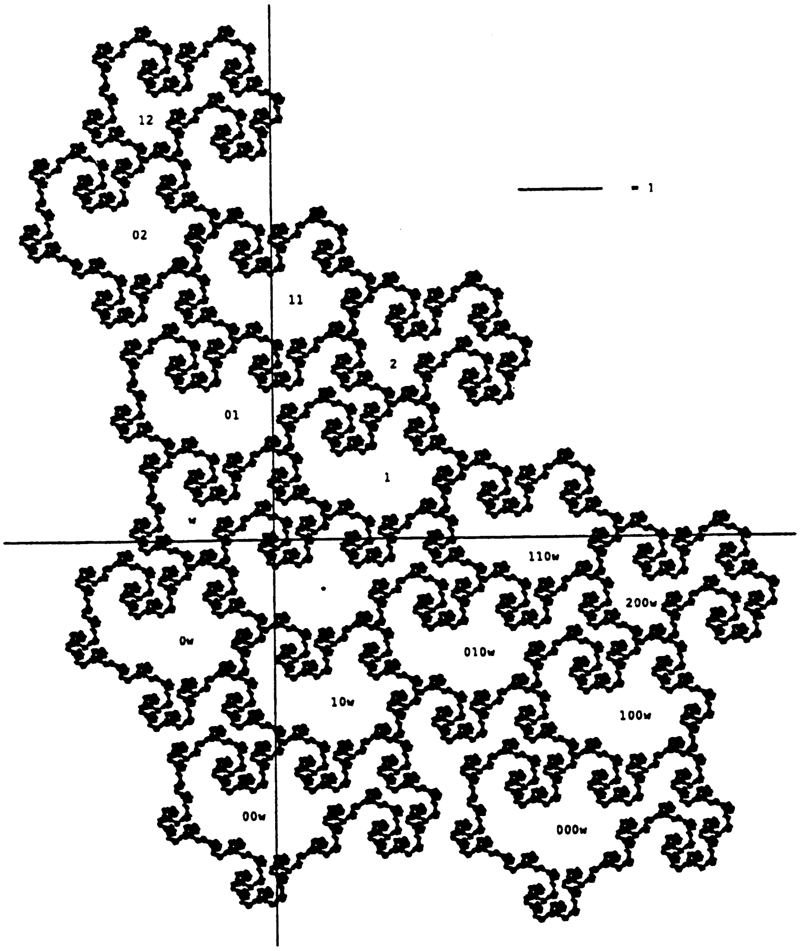

Figure 1. Sofic Pisot tiling for $x^3 - 3x^2 + 2x - 1 = 0$

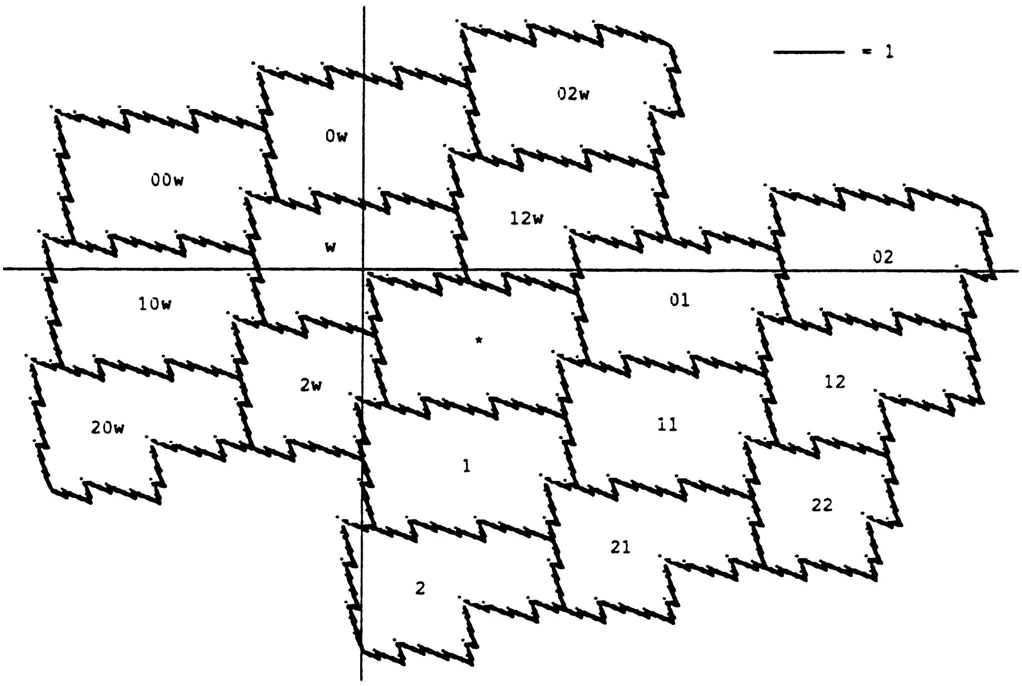

Figure 2. Sofic Pisot tiling for $x^3 - 3x^2 + 1 = 0$

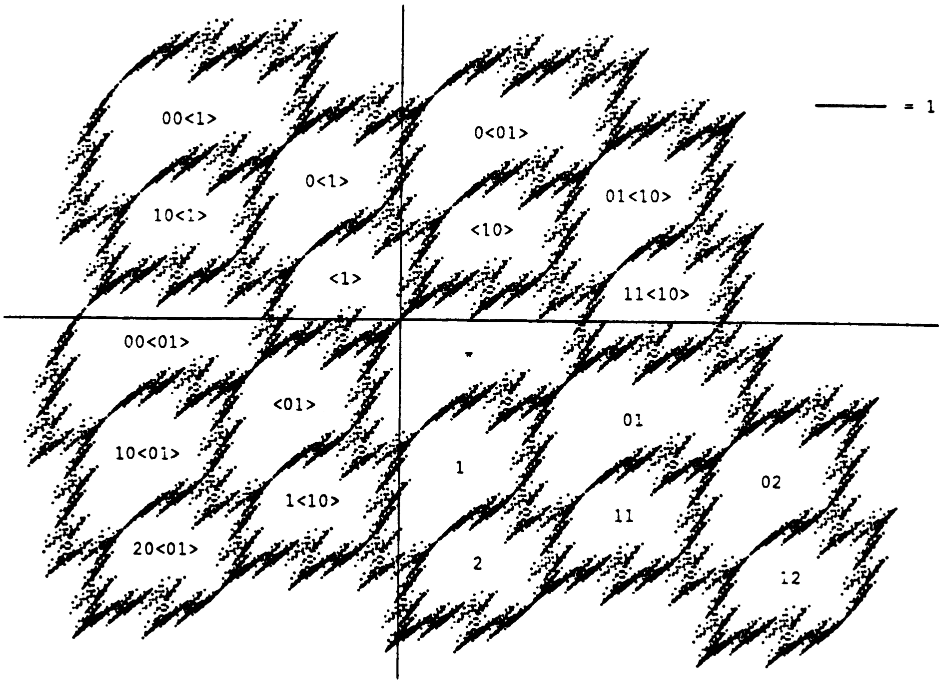

Figure 3. Sofic Pisot tiling for $x^3 - 2x^2 - x + 1 = 0$

References

[1] S. Akiyama, Pisot numbers and greedy algorithm, 'Number Theory, Diophantine, Computational and Algebraic Aspects', ed. by K. Győry, A. Pethö and V. T. Sós, 9–21, de Gruyter 1998.

[2] S. Akiyama and T. Sadahiro, A self-similar tiling generated by the minimal Pisot number, Acta Math. Inform. Univ. Ostraviensis, **6** (1998), 9–26.

[3] S. Akiyama, Self affine tiling and Pisot numeration system, 'Number Theory and its Applications', ed. by K. Győry and S. Kanemitsu, 7–17, Kluwer 1999.

[4] S. Akiyama, Cubic Pisot units with finite beta expansions, 'Algebraic Number Theory and Diophantine Analysis', ed. F. Halter-Koch and R. F. Tichy, 11–26, de Gruyter 2000.

[5] M. J. Bertin, A. Decomps-Guilloux, M. Grandet-Hugot, M. Pathiaux-Delefosse and J. P. Schreiber, Pisot and Salem numbers, Birkhäuser Verlag, Basel-Boston-Berlin, 1992.

[6] A. Bertrand, Dévelopment en base de Pisot et répartition modulo 1, C. R. Acad. Sci., Paris, **385** (1977), 419–421.

[7] F. Blanchard, β-expansion and symbolic dynamics, Theoret. Comput. Sci., **65** (1989), 131–141.

[8] C. Frougny and B. Solomyak, Finite beta-expansions, Ergodic Theory Dynam. Systems, **12** (1992), 713–723.

[9] S. Ito and Y. Takahashi, Markov subshifts and realization of β-expansions, J. Math. Soc. Japan, **26** (1974), no. 1, 33–55.

[10] R. Kenyon and A. Vershik, Arithmetic construction of sofic partitions of hyperbolic toral automorphisms, Ergodic Theory Dynam. Systems, **18** (1998), no. 2, 357–372.

[11] D. Lind and B. Marcus, An introduction to symbolic dynamics and coding, Cambridge Univ. Press 1995.

[12] W. Parry, On the β-expansions of real numbers, Acta Math. Acad. Sci. Hungary, **11** (1960), 269–278.

[13] B. Praggastis, Markov partition for hyperbolic toral automorphism, Ph.D. Thesis, Univ. of Washington, 1992.

[14] A. Rényi, Representation for real numbers and their ergodic properties, Acta Math. Acad. Sci. Hungary, **8** (1957), 477–493.

[15] G. Rauzy, Nombres Algébriques et substitutions, Bull. Soc. Math. France, **110** (1982), 147–178.

[16] K. Schmidt, On periodic expansions of Pisot numbers and Salem numbers, Bull. London Math. Soc., **12** (1980), 269–278.

[17] B. Solomyak, Dynamics of self-similar tilings, Ergodic Theory Dynam. Systems, **17** (1997), no. 3, 695–738.

[18] W. P. Thurston, Groups, Tilings and Finite state automata, AMS Colloq. Lectures, 1989.

Shigeki Akiyama

Department of Mathematics
Faculty of Science Niigata University
Ikarashi-2, 8050
Niigata 950-2181, JAPAN
E-mail: akiyama@math.sc.niigata-u.ac.jp